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The Banach–Saks property in rearrangement invariant spaces

by

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Abstract. This paper studies the Banach–Saks property in rearrangement invariant spaces on the positive half-line. A principal result of the paper shows that a separable rearrangement invariant space E with the Fatou property has the Banach–Saks property if and only if E has the Banach–Saks property for disjointly supported sequences. We show further that for Orlicz and Lorentz spaces, the Banach–Saks property is equivalent to separability although the separable parts of some Marcinkiewicz spaces fail the Banach–Saks property.

1. Introduction. A Banach space X is said to have the *Banach–Saks property* if every weakly null sequence contains a subsequence whose Cesàro averages converge strongly to zero. This property has its roots in the classical work of Banach and Saks [BS] who established its validity in the function spaces $L_p[0,1]$ for $1 < p \leq 2$. The corresponding result for the case $2 < p < \infty$ is due to Kadec and Pełczyński [KP]. Subsequently, it was shown by Kakutani [Di] that every uniformly convex Banach space has the Banach–Saks property. In contrast, it was shown by Szlenk [Sz] that the (non-uniformly convex) space $L_1[0,1)$ also has the Banach–Saks property.

The aim of the present paper is to study Banach–Saks type properties in the setting of rearrangement invariant spaces on a finite (or infinite) interval. We restrict our attention to separable spaces, as any rearrangement invariant space with the Banach–Saks property is necessarily separable. Our approach is partly based on a subsequence splitting property in rearrangement invariant spaces (Proposition 3.2) which states that any norm bounded sequence in a separable rearrangement invariant space with the Fatou property con-

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tains a subsequence which is a perturbation of the sum of an equimeasurable sequence and a bounded, disjointly supported sequence which converges to zero in measure.

The principal results of the paper are given in the fourth section. We show (Theorem 4.5) that for separable rearrangement invariant spaces with the Fatou property, the Banach–Saks property is equivalent to the Banach–Saks property for disjointly supported sequences. It is readily seen that each of the spaces L_p, $1 \le p < \infty$, satisfies this latter condition and so we recover the classical results of [BS] and [KP] in the case $1 < p < \infty$ and of Szlenk [Sz] in the case $p = 1$. We show further (Corollary 4.6) that any separable rearrangement invariant space with the Fatou property which satisfies an upper p-estimate for some $p > 1$ has the Banach–Saks property. This complements a result of Rakov [Ra] who showed that any Banach space with non-trivial type has the Banach–Saks property.

In the fifth section, we show (Theorems 5.5, 5.7) that each separable Orlicz and Lorentz space on an interval has the Banach–Saks property, and that the separable parts of non-separable Orlicz and Lorentz spaces (on an interval) do not have the Banach–Saks property. This contrasts markedly with the results of Rakov [Ra] where it is shown that not only does every separable Orlicz sequence space have the Banach–Saks property but so also does the separable part of every non-separable Orlicz sequence space.

In the case of Orlicz function spaces, our results considerably strengthen earlier results of Alexopoulos [Al]. As well, we show that the separable parts of some Marcinkiewicz spaces fail the Banach–Saks property (Theorem 5.9). Further, we show that if E is a separable rearrangement invariant space, then each weakly null sequence which in addition is E-equi-integrable contains a subsequence for which the Cesàro means of each further subsequence converge in norm to zero (Theorem 4.10). Finally, we give an example of a reflexive rearrangement invariant space E on $[0,1)$ with non-trivial Boyd indices, and having an equivalent rearrangement invariant locally uniformly convex norm, but which does not have the Banach–Saks property. This complements the classical result of Kakutani cited above.

Some of the results of this paper were announced in [DFSS].

2. Definitions and preliminaries. A Banach space $(E, \|\cdot\|_E)$ of real-valued Lebesgue measurable functions on the interval $[0,\alpha)$, $0 < \alpha \le \infty$, (with identification λ-a.e.) will be called *rearrangement invariant* if

(i) E is an ideal lattice, that is, if $y \in E$, and if x is any measurable function on $[0,\alpha)$ with $0 \le |x| \le |y|$ then $x \in E$ and $\|x\|_E \le \|y\|_E$;

(ii) E is rearrangement invariant in the sense that if $y \in E$, and if x is any measurable function on $[0,\alpha)$ with $x^* = y^*$, then $x \in E$ and $\|x\|_E = \|y\|_E$.

Here, λ denotes Lebesgue measure and x^* denotes the non-increasing, right-continuous rearrangement of x given by

$$x^*(t) = \inf\{s \geq 0 : \lambda(\{|x| > s\}) \leq t\}, \qquad t > 0.$$

For basic properties of rearrangement invariant spaces, we refer to the monographs [BeS], [KPS], [LT2]. We note that for any rearrangement invariant space $E = E[0,\alpha)$,

$$L_1[0,\alpha) \cap L_\infty[0,\alpha) \subseteq E[0,\alpha) \subseteq L_1[0,\alpha) + L_\infty[0,\alpha)$$

with continuous embeddings.

The *Köthe dual* $E^\times$ of a rearrangement invariant space E on the interval $[0,\alpha)$ consists of all measurable functions y for which

$$\|y\|_{E^\times} := \sup\Big\{ \int_0^\alpha |x(t)y(t)|\,dt : x \in E,\ \|x\|_E \leq 1 \Big\} < \infty.$$

Basic properties of Köthe duality may be found in [KPS], [BeS] (where the Köthe dual is called the *associate* space). If E^* denotes the Banach dual of E, it is known that $E^\times \subset E^*$ and $E^\times = E^*$ if and only if the norm $\|\cdot\|_E$ is order continuous, i.e. from $\{x_n\} \subseteq E$, $x_n \downarrow_n 0$, it follows that $\|x_n\|_E \to 0$. We note that the norm $\|\cdot\|_E$ of the rearrangement invariant space E on $[0,\alpha)$ is order continuous if and only if E is separable, in which case $\lim_{t\to\infty} x^*(t) = 0$ for all $x \in E$. We denote by $L_0[0,\infty)$ the closure of $L_1[0,\infty) \cap L_\infty[0,\infty)$ in $L_1[0,\infty) + L_\infty[0,\infty)$. The space $L_0[0,\infty)$ is separable and is the largest separable rearrangement invariant subspace of $L_1[0,\infty)+L_\infty[0,\infty)$. Further, $x \in L_0[0,\infty)$ if and only if $\lim_{t\to\infty} x^*(t) = 0$.

If E is a rearrangement invariant space on $[0,\alpha)$, then E is said to have the *Fatou property* if from $\{f_n\}_{n\geq 1} \subseteq E$, $f \in L_1[0,\alpha) + L_\infty[0,\alpha)$, $f_n \to f$ a.e. on $[0,\alpha)$ and $\sup_n \|f_n\|_E < \infty$ it follows that $f \in E$ and $\|f\|_E \leq \liminf_{n\to\infty} \|f_n\|_E$. It is well known that the rearrangement invariant space E has the Fatou property if and only if the natural embedding of E into its Köthe bidual $E^{\times\times}$ is a surjective isometry. Such spaces are called *maximal*. We note that if E is separable but not maximal, then E contains a Banach sublattice isomorphic to c_0. See, for example, [MN, Theorem 2.4.12].

If $x, y \in L_1[0,\alpha) + L_\infty[0,\alpha)$, we will say that x is *submajorized* by y and write $x \prec\prec y$ if

$$\int_0^t x^*(s)\,ds \leq \int_0^t y^*(s)\,ds \qquad \text{for all } t > 0.$$

We shall need the following criterion for weak compactness in rearrangement invariant spaces. See [DSS, Proposition 2.1(v)].

Proposition A. *Let E be a rearrangement invariant space on $[0,\alpha)$, $0 < \alpha \leq \infty$, such that $E, E^\times \subseteq L_0[0,\infty)$. If E has the Fatou property or is* *separable, and if $K \subset E^\times$ is bounded, then K is relatively $\sigma(E^\times, E)$-compact if and only if for every $f \in E$ and sequence $\{f_n\} \subseteq E$ with $f_n \prec\prec f$, $n \in \mathbb{N}$ and $f_n \to 0$ in measure, we have*

$$\sup\Big\{\int_{[0,\alpha)} f_n^*(t) g^*(t)\,dt : g \in K\Big\} \to_n 0.$$

Denote by Ψ the set of increasing concave functions on $[0,\infty)$ with $\psi(0) = \psi(+0) = 0$. If $\psi \in \Psi$, then the *Lorentz space* $(\Lambda_\psi[0,\alpha), \|\cdot\|_{\Lambda_\psi[0,\alpha)})$ on $[0,\alpha)$ is the space of all measurable functions x on $[0,\alpha)$ for which

$$\|x\|_{\Lambda_\psi[0,\alpha)} := \int_0^\alpha x^*(t)\,d\psi(t) < \infty.$$

The rearrangement invariant space $\Lambda_\psi^\times[0,\alpha)$ associated with the Lorentz space $\Lambda_\psi[0,\alpha)$ is the *Marcinkiewicz space* $M_\psi[0,\alpha)$ consisting of those measurable functions x for which

$$\|x\|_{M_\psi[0,\alpha)} := \sup_{0<s<\alpha} \frac{1}{\psi(s)} \int_0^s x^*(t)\,dt < \infty.$$

The space $\Lambda_\psi[0,\alpha)$, $1 \le \alpha < \infty$, is always separable. The space $\Lambda_\psi[0,\infty)$, is separable if and only if $\psi(\infty) = \lim_{t\to\infty} \psi(t) = \infty$, and, in this case, the simple functions are dense in $\Lambda_\psi[0,\infty)$ [KPS, II.5.3]. The space $M_\psi[0,\alpha)$ is non-separable. The closure of $L_1[0,\alpha) \cap L_\infty[0,\alpha)$ in $M_\psi[0,\alpha)$ is separable and is denoted by $M_\psi^0[0,\alpha)$.

By [KPS, Section 2.5.4], the Banach dual $M_\psi^0[0,\alpha)^*$ may be identified with $\Lambda_\psi[0,\alpha)$ and every linear functional $f \in M_\psi^0[0,\alpha)^*$ can be written in the form

$$f(x) = \int_{[0,\alpha)} x(t)y(t)\,dt, \qquad x \in M_\psi^0[0,\alpha),$$

where $y \in \Lambda_\psi[0,\alpha)$ and $\|f\|_{M_\psi^0[0,\alpha)^*} = \|y\|_{\Lambda_\psi[0,\alpha)}$.

Let Φ be an *Orlicz function* on $[0,\infty)$, that is, Φ is a continuous convex increasing function on $[0,\infty)$ satisfying $\Phi(0) = 0$ and $\Phi(\infty) = \infty$. The *Orlicz space* $L_\Phi = L_\Phi[0,\alpha)$, $0 < \alpha \le \infty$, is the space of all Lebesgue measurable functions f on $[0,\alpha)$ for which

$$\int_0^\alpha \Phi\Big(\frac{|f(t)|}{\varrho}\Big)\,dt < \infty$$

for some $\varrho > 0$. The (Luxemburg) norm in $L_\Phi = L_\Phi[0,\alpha)$ is defined by

$$\|f\|_\Phi = \inf\Big\{\varrho > 0 : \int_0^\alpha \Phi\Big(\frac{|f(t)|}{\varrho}\Big)\,dt \le 1\Big\}.$$

The Orlicz space $L_\Phi[0,\alpha)$ is maximal. In the case of $\alpha<\infty$, $L_\Phi[0,\alpha)$ is separable if and only if Φ satisfies the Δ_2-condition at ∞. The space $L_\Phi[0,\infty)$ is separable if and only if Φ satisfies the Δ_2-condition both at ∞ and at 0 (see, for example, [LT2]). We shall denote by $L^0_\Phi[0,\alpha)$ the *separable part* of the Orlicz space $L_\Phi[0,\alpha)$, that is, $L^0_\Phi[0,\alpha)$ is the closure in $L_\Phi[0,\alpha)$ of the linear subspace $L_1[0,\alpha)\cap L_\infty[0,\alpha)$.

3. Subsequence splitting of bounded sequences. In this section, we study bounded sequences in a rearrangement invariant space E. In our variant of the subsequence splitting property we follow the approach of [Su1, Lemma 1.1] (see also [KP] and [We, Corollary 2.6]). We shall need the following result, given in [BeS, Corollary 2.7.6]. See also [KPS, Theorem II.2.1].

LEMMA 3.1. *If $x\in L_1[0,\alpha)+L_\infty[0,\alpha)$ and if $\lim_{t\to\infty}x^*(t)=0$, then there exists a surjective measure-preserving transformation $\sigma:[0,\alpha)\to[0,\alpha)$ such that $|x(t)|=x^*(\sigma(t))$, $t\in[0,\alpha)$.*

Combining the measure-preserving transformation from Lemma 3.1 with multiplication by a unimodular function, we see that if the rearrangement invariant space $E[0,\infty)$ is separable and if $x\in E$ then there exists a rearrangement-preserving transformation $T_x:E\to E$ such that $T_x(x^*)=x$.

The first part of the following proposition was established in [Su1, Lemma 1.1] under the additional assumption that $\alpha<\infty$.

PROPOSITION 3.2. *Let E be a separable rearrangement invariant space on the interval $[0,\alpha)$, $0<\alpha\le\infty$, with the Fatou property.*

(i) *For any sequence $\{x_n\}_{n=1}^\infty\subset E$ with*

$$\sup_{n\in\mathbb{N}}\|x_n\|_E=C<\infty$$

there exists a subsequence $\{x'_n\}_{n=1}^\infty\subseteq\{x_n\}_{n=1}^\infty$ which admits the splitting

$$x'_n=y_n+z_n+d_n,\qquad n\ge1, \tag{3.1}$$

where $\{y_n\}_{n=1}^\infty,\{z_n\}_{n=1}^\infty,\{d_n\}_{n=1}^\infty\subseteq E$ are bounded sequences satisfying

- (a) $y_1^*(t)=y_n^*(t)$, $\forall n\in\mathbb{N}$, $\forall t\in[0,\infty)$ *and* $\|y_1\|_E\le C$;
- (b) $z_nz_m=0$ *for* $n,m\in\mathbb{N}$, $n\ne m$, $z_n\to0$ *in measure and* $\sup_{n\in\mathbb{N}}\|z_n\|_E\le 2C$;
- (c) $\|d_n\|_E\to0$.

(ii) *If, in addition, the sequence $\{x_n\}_{n=1}^\infty$ is weakly null and $E^\times\subseteq L_0[0,\infty)$, then the sequences $\{y_n\}_{n=1}^\infty$, $\{z_n\}_{n=1}^\infty$ from* (3.1) *may be chosen to be weakly null as well.*

Proof. (i) It follows from the inequalities

$$\sup_n x_n^*(t)\le C/\|\chi_{[0,t)}\|_E,\qquad t>0,$$

that the sequence $\{x_n^*\}_{n=1}^\infty$ is uniformly bounded on every interval of the form $[a,b]$ for all $0<a<b<\infty$. By Helly's selection theorem, we choose a subsequence of $\{x_n^*\}$ (which we again denote by $\{x_n^*\}$) such that $x_n^*\to f$ almost everywhere on $[0,\alpha)$ for some right-continuous, non-increasing function $f:(0,\alpha)\to[0,\infty)$. Since E has the Fatou property, it follows that $f\in E$ and that $\|f\|_E\le C$. We set

$$a_n(t):=(x_n')^*(t)-f(t),\qquad \forall t>0,\ \forall n\in\mathbb{N}.$$

Since $\lim_{t\to\infty}f(t)=\lim_{t\to\infty}(x_n')^*(t)=0$ for all $t>0$, and since f,x_n^* are non-increasing for all $n\in\mathbb{N}$, it follows that $a_n\to 0$ in measure.

Since $\lim_{t\to\infty}(x_n')^*(t)=0$, it follows from Lemma 3.1 that there exists a rearrangement-preserving transformation $T_n:E\to E$ such that $T_n((x_n')^*)=x_n'$ for all $n\in\mathbb{N}$. We now set

$$y_n=T_nf,\quad w_n=T_n(a_n)=T_n((x_n')^*-f),\qquad n\in\mathbb{N}.$$

It follows immediately that

$$x_n'=y_n+w_n,\qquad y_n^*=f$$

for all $n\in\mathbb{N}$. In particular, it follows that

$$\|y_n\|_E=\|f\|_E\le C,\quad \|w_n\|_E\le 2C$$

for all $n\in\mathbb{N}$. Since $a_n\to 0$ in measure, we also have $w_n=T_n(a_n)\to 0$ in measure. Finally, since E is separable, the commutative specialization of [CDS, Theorem 2.5] shows that, by passing to a subsequence if necessary and relabelling, there exists a sequence of mutually disjoint elements $\{z_n\}_{n=1}^\infty\subseteq E$ such that $\|z_n\|_E\le 2C$ and $\|z_n-w_n\|_E\to 0$.

(ii) From part (i), we may assume that the decomposition (3.1) holds for $\{x_n\}_{n=1}^\infty$. We set $f=y_1^*$ and let

$$\Omega(f):=\{x\in E: x\prec\prec f\}.$$

Using the assumption that $E^\times\subseteq L_0[0,\infty)$ and the separability of E, it follows from [DSS, Proposition 2.1(v)] (see also [Fr, Section 28], or [CSS]) that $\Omega(f)$ is sequentially compact for the weak topology on E induced by $E^\times$. Passing again to a subsequence if necessary, we may assume that the sequence $\{y_n\}_{n=1}^\infty$ is $\sigma(E,E^\times)$-convergent, and using the assumption that $\{x_n\}_{n=1}^\infty$ is weakly null, we may assume further that the disjoint sequence $\{z_n\}_{n=1}^\infty$ is weakly convergent. To complete the proof of (ii), it will suffice to show that

$$\int_{[0,\alpha)} z_n(s)g(s)\,ds\to 0 \tag{3.2}$$

for every $0\le g\in E^\times$. We observe first that weak compactness of the sequence $\{z_n\}_{n=1}^{\infty}$ together with [DSS, Proposition 2.1(iii)] implies that

$$\lim_{N\to\infty}\sup_{n\in\mathbb{N}}\int\limits_{[0,\alpha)}\chi_{[N,\infty)}(s)z_n(s)g(s)\,ds=0.$$

Accordingly, it suffices to show that

$$\int\limits_{[0,N)} z_n(s)g(s)\,ds\to 0$$

for each $N=1,2,\ldots$ To this end, let

$$A_n=\{s:z_n(s)\neq 0\}\cap[0,N),\qquad n\in\mathbb{N}.$$

Disjointness of the sequence $\{z_n\}_{n=1}^{\infty}$ implies that $\sum_{n=1}^{\infty}\lambda(A_n)<\infty$ and this implies in turn that

$$E_k:=\bigcup_{n\geq k}A_n\downarrow_k\emptyset.$$

Again using the weak compactness of the sequence $\{z_n\}_{n=1}^{\infty}$ together with [DSS, Proposition 2.1(iii)] shows that

$$\lim_{k\to\infty}\sup_{n\in\mathbb{N}}\int\limits_{[0,N)}\chi_{E_k}(s)z_n(s)g(s)\,ds=0.$$

In particular, it follows that

$$\lim_{k\to\infty}\int\limits_{[0,N)} z_k(s)g(s)\,ds=\lim_{k\to\infty}\int\limits_{[0,N)}\chi_{E_k}(s)z_k(s)g(s)\,ds=0,$$

and this suffices to complete the proof. ∎

Definition 3.3. Let E be a rearrangement invariant space on $[0,\alpha),\ 0<\alpha\leq\infty$. A bounded set $M\subseteq E$ is said to be *E-equi-integrable* if

$$\lim_{n\to\infty}\sup_{x\in M}\|x\chi_{A_n}\|_E=0$$

for all sequences $\{A_n\}_{n=1}^{\infty}$ of measurable subsets of $[0,\alpha)$ for which $A_n\downarrow_n\emptyset$.

We make the simple remark that if the bounded set $M\subseteq E$ is E-equi-integrable, and if $\{A_n\}_{n=1}^{\infty}$ is any sequence of measurable subsets of $[0,\alpha)$ for which $\lambda(A_n)\to 0$, then necessarily

$$\lim_{n\to\infty}\sup_{x\in M}\|x\chi_{A_n}\|_E=0.$$

Proposition 3.4. *Let $(E,\|\cdot\|_E)$ be a separable rearrangement invariant space on $[0,\alpha)$, $0<\alpha\leq\infty$, with the Fatou property and let $\{x_n\}_{n=1}^{\infty}\subset E$ be a bounded sequence. If $\{x_n\}_{n=1}^{\infty}$ is E-equi-integrable, and if*

$$x_n'=y_n+z_n+d_n,\quad n\in\mathbb{N},$$

is the decomposition (3.1) *then*

$$\lim_{n\to\infty} \|z_n\|_E = 0. \tag{3.3}$$

If $\alpha < \infty$, *and if in the decomposition* (3.1) *we have* $\lim_{n\to\infty} \|z_n\|_E = 0$, *then the sequence* $\{x'_n\}_{n=1}^{\infty}$ *is* E*-equi-integrable.*

Proof. Assume that $\{x_n\}_{n=1}^{\infty}$ is E-equi-integrable, and observe that since $\|d_n\|_E \to 0$, it follows immediately that $\{x'_n - d_n\}_{n=1}^{\infty}$ is also E-equi-integrable. Making a simple change of notation, we suppose that $\{x_n\}_{n=1}^{\infty}$ is E-equi-integrable and admits the decomposition

$$x_n = y_n + z_n, \qquad n \in \mathbb{N},$$

with $\{z_n\}_{n=1}^{\infty}$ pairwise disjoint and convergent to 0 in measure, and $y_n^* = y_1^*$ for all $n \in \mathbb{N}$. We note first that if e_n, $n = 1, 2, \ldots$, is any sequence of measurable subsets of $[0,\infty)$ for which $m(e_n) \to 0$, then it follows from the inequalities

$$\|z_m\chi_{e_n}\|_E \le \|x_m\chi_{e_n}\|_E + \|y_m\chi_{e_n}\|_E \le \|x_m\chi_{e_n}\|_E + \|y_1^*\chi_{[0,m(e_n))}\|_E$$

and from the remark following Definition 3.3 that

$$\lim_{n\to\infty} \sup_{m\in\mathbb{N}} \|z_m\chi_{e_n}\|_E = 0. \tag{3.4}$$

Using the fact that the sequence $\{z_n\}_{n=1}^{\infty}$ is pairwise disjoint, it follows immediately that if $e \subseteq [0,\infty)$ is any measurable subset for which $\lambda(e) < \infty$, then $\|z_n\chi_e\|_E \to 0$. In particular, this establishes the first assertion of the proposition in the case of $\alpha < \infty$ by taking e to be $\chi_{[0,\alpha)}$. We may now assume that $\alpha = \infty$. If the proposition fails, then we may assume that there exists $\varepsilon > 0$ such that

$$\|z_n\|_E > \varepsilon, \qquad n \in \mathbb{N}.$$

From the first part of the proof and by suitable relabelling, we may assume that there exists a sequence $t_n \uparrow \infty$ such that

$$\|z_n\chi_{[t_n,\infty)}\|_E \ge \varepsilon, \qquad n \in \mathbb{N}.$$

Since

$$\lim_{n\to\infty} \sup_{m\in\mathbb{N}} \|x_m\chi_{[t_n,\infty)}\|_E = 0,$$

we may assume further that

$$\|y_n\chi_{[t_n,\infty)}\|_E \ge 7\varepsilon/8, \qquad n \in \mathbb{N}. \tag{3.5}$$

Since E is separable the norm on E is order continuous and so there exist numbers $0 < s_1 < s_2$ such that

$$\max\{\|y_1^*\chi_{[0,s_1)}\|_E, \|y_1^*\chi_{[s_2,\infty)}\|_E\} < \varepsilon/16. \tag{3.6}$$

It follows from Lemma 3.1 that for each $n \in \mathbb{N}$, there exist measurable sets e_n^i, $i = 1, 2, 3$, with $\lambda(e_n^2) = s_2 - s_1$ such that

$$(y_n \chi_{e_n^1})^* = (y_1^* \chi_{[0,s_1)})^*, \qquad (y_n \chi_{e_n^2})^* = (y_1^* \chi_{[s_1,s_2)})^*,$$
$$(y_n \chi_{e_n^3})^* = (y_1^* \chi_{[s_2,\infty)})^*.$$

By (3.6), this implies that

$$\max\{\|y_n \chi_{e_n^1 \cap [t_n,\infty)}\|_E, \|y_n \chi_{e_n^3 \cap [t_n,\infty)}\|_E\} \le \varepsilon/16, \qquad n \in \mathbb{N},$$

and consequently, it follows from (3.5) that

$$\|y_n \chi_{e_n^2 \cap [t_n,\infty)}\|_E \ge 3\varepsilon/4, \qquad n \in \mathbb{N}. \tag{3.7}$$

We now observe that

$$\|z_n \chi_{e_n^2 \cap [t_n,\infty)}\|_E \to 0.$$

In fact, if this is not so, we may assume that there exists $\delta > 0$ such that

$$\|z_n \chi_{e_n^2}\|_E > \delta, \qquad n \in \mathbb{N}.$$

Since $z_n \to 0$ in measure and since $\lambda(e_n^2) = s_2 - s_1$ for all $n \in \mathbb{N}$, we may assume further that there exist measurable sets $e_n \subseteq e_n^2$, $n = 1, 2, \ldots$, with $\lambda(e_n) \to 0$ and such that

$$\|z_n \chi_{e_n}\|_E > \delta/2$$

for all $n = 1, 2, \ldots$, and this contradicts the assertion of (3.4). Accordingly, there exists $N \in \mathbb{N}$ such that

$$\|z_n \chi_{e_n^2 \cap [t_n,\infty)}\|_E \le \varepsilon/4$$

for all $n \ge N$. From (3.7) it now follows that

$$\|x_n \chi_{[t_n,\infty)}\|_E \ge \|x_n \chi_{e_n^2 \cap [t_n,\infty)}\|_E \ge \|y_n \chi_{e_n^2 \cap [t_n,\infty)}\|_E - \|z_n \chi_{e_n^2 \cap [t_n,\infty}\|_E$$
$$\ge \|y_n \chi_{e_n^2 \cap [t_n,\infty)}\|_E - \varepsilon/4 \ \ge \varepsilon/2$$

for all $n \ge N$. This contradicts the E-equi-integrability of the sequence $\{x_n\}_{n=1}^\infty$, and suffices to establish the first assertion of the proposition.

The final assertion of the proposition is an immediate consequence of the observation that, if $\alpha < \infty$, then the sequence $\{y_n\}_{n=1}^\infty$, being equimeasurable, is necessarily E-equi-integrable. ∎

4. The Banach–Saks property. It will be convenient to adopt the following terminology.

DEFINITION 4.1. Let X be a Banach space.

(a) If $\{x_n\}_{n=1}^\infty$ is a weakly null sequence in X, then the sequence $\{x_n\}_{n=1}^\infty$ is called a *Banach–Saks sequence* if

$$\lim_{n\to\infty} n^{-1}\Big\|\sum_{j=1}^{n} y_j\Big\| = 0$$

for all subsequences $\{y_j\}_{j=1}^{\infty} \subseteq \{x_n\}_{n=1}^{\infty}$.

(b) X is said to have the *Banach–Saks property* if every weakly null sequence in X has a Banach–Saks subsequence.

We remark that the classical formulation of the Banach–Saks property requires that each bounded sequence contain a Cesàro summable subsequence, and any Banach space enjoying this property is necessarily reflexive. See, for example, [Di]. In reflexive spaces, the classical Banach–Saks property is easily seen to be equivalent to the (so-called) *weak* Banach–Saks property which requires that each weakly null sequence should contain a Cesàro summable subsequence. That the apparent strengthening of the weak Banach–Saks property given in Definition 4.1(b) is, in fact, equivalent to the weak Banach–Saks property is due to Erdős and Magidor [EM]. See also [FS] and [Ro].

If X is a Banach lattice and the elements of the sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ are pairwise disjoint, then the preceding definitions yield the corresponding definitions of a *Banach–Saks d-sequence* and the *Banach–Saks d-property.*

We remark that any rearrangement invariant space E which has the Banach–Saks property is necessarily separable. In fact, if E is rearrangement invariant and not separable, then E contains a copy of l_∞, by [LT2, Proposition 1.a.7]. Since l_∞ is universal for separable Banach spaces, it follows that E contains a copy of the separable Banach space which fails the Banach–Saks property given by Baernstein [Ba]. Consequently, E also fails the Banach–Saks property.

PROPOSITION 4.2. *Let E be a separable rearrangement invariant Banach function space on the interval* $[0,\alpha)$, $0<\alpha\le\infty$, *with the Fatou property. If* $a_n, y \in E$ *satisfy* $a_n \prec\prec y$, $n\in\mathbb{N}$, *and if* $a_n \to 0$ *in measure*, *then* $\|a_n\|_E \to 0$.

Proof. Without loss of generality, it may be assumed that $a_n^* = a_n$ for all $n\in\mathbb{N}$. Suppose first that $E^\times \subseteq L_0[0,\infty)$ and let K be the unit ball of $E^\times$. Since K is $\sigma(E^\times, E)$-sequentially compact, Proposition A yields

$$\|a_n\|_E = \sup\Big\{\int_{[0,\infty)} a_n(t)g^*(t)\,dt : g\in K\Big\} \to 0$$

as $n\to\infty$. We may therefore suppose that $E^\times \not\subseteq L_0[0,\infty)$ or, equivalently, that $L_\infty[0,\infty) \subseteq E^\times$. By the maximality of E, it follows that

$$E = E^{\times\times} \subseteq (L_\infty[0,\infty))^\times = L_1[0,\infty).$$

Since $a_n \prec\prec y$, $n\in\mathbb{N}$, it follows from [DSS, Proposition 2.1(v)] that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is relatively $\sigma(L_1, L_\infty)$-compact, and since $a_n \to 0$ in measure, it follows from the well known Vitali convergence theorem that

$$\|a_n\|_{L_1} \to 0. \tag{4.1}$$

Let $\varepsilon > 0$ be given. For each $\alpha > 0$, it follows from the submajorization

$$a_n\chi_{[0,\alpha)} \prec\prec y\chi_{[0,\alpha)}, \qquad n \in \mathbb{N},$$

that

$$\|a_n\chi_{[0,\alpha)}\|_E \le \|y\chi_{[0,\alpha)}\|_E, \qquad n \in \mathbb{N}.$$

By order continuity of the norm on E, it follows that there exists $\alpha > 0$ such that

$$\sup_{n\in\mathbb{N}} \|a_n\chi_{[0,\alpha)}\|_E < \varepsilon. \tag{4.2}$$

Now observe that, for all $n, m \in \mathbb{N}$, the submajorizations

$$a_n(m)\chi_{[0,m)} \prec\prec a_n\chi_{[0,m)} \prec\prec y\chi_{[0,m)}$$

imply that

$$ma_n(m) \le \|y\|_{L_1}.$$

We denote by $C > 0$ any constant for the continuous embedding of $L_1 \cap L_\infty$ into E. Setting $M = C\|y\|_{L_1}/\varepsilon$, we find that, for all $m \ge M$,

$$\sup_{n\in\mathbb{N}} a_n(m) \le \varepsilon/C$$

or, equivalently,

$$\sup_{n\in\mathbb{N}} \|a_n\chi_{[m,\infty)}\|_{L_\infty} \le \varepsilon/C, \qquad m \ge M. \tag{4.3}$$

From (4.1), it follows that there exists $N \in \mathbb{N}$ such that

$$\sup_{m\in\mathbb{N}} \|a_n\chi_{[m,\infty)}\|_{L_1} \le \varepsilon/C, \qquad n \ge N. \tag{4.4}$$

Combining (4.3) and (4.4) shows that

$$\|a_n\chi_{[m,\infty)}\|_E \le C\max\left\{\|a_n\chi_{[m,\infty)}\|_{L_1}, \|a_n\chi_{[m,\infty)}\|_{L_\infty}\right\} \le \varepsilon$$

for all $n \ge N$, $m \ge M$. Finally, using the fact that $a_n \to 0$ in measure we obtain

$$\|a_n\chi_{(\alpha,M]}\|_E \le a_n(\alpha)\|\chi_{(\alpha,M]}\|_E \to 0$$

as $n \to \infty$, and, together with (4.2), this suffices to complete the proof. ■

The proposition which follows is an analogue of the well known theorem of Komlós [Ko]. For convenience, we denote the norm on $L_1[0,\infty)+L_\infty[0,\infty)$ by $\|\cdot\|_+$ and note that (see, for example, [LT2, Proposition 2.a.2])

$$\|x\|_+ = \int_0^1 x^*(s)\,ds, \qquad x \in L_1[0,\infty) + L_\infty[0,\infty).$$

We note that the space $L_0[0,\infty)$ does not have the Banach–Saks property. See the remark following Corollary 5.8 below. Nonetheless, the following proposition shows that each equimeasurable sequence in $L_0[0,\infty)$ is a Banach–Saks sequence.

PROPOSITION 4.3. *Suppose that* $\{x_n\}_{n=1}^\infty \subseteq L_1[0,\infty) + L_\infty[0,\infty)$ *satisfies* $x_n^* = x_1^*$, $n \in \mathbb{N}$. *If* $\lim_{t\to\infty} x_1^*(t) = 0$, *then there exists a subsequence* $\{x_{n(k)}\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty$ *and there exists* $x \in L_1[0,\infty) + L_\infty[0,\infty)$ *such that*

$$\frac{1}{N}\sum_{k=1}^N x_{m(k)} \to x$$

in $L_1[0,\infty)+L_\infty[0,\infty)$ *for all further subsequences* $\{x_{m(k)}\}_{k=1}^\infty \subset \{x_{n(k)}\}_{k=1}^\infty$.

Proof. Since $x_n^* = x_1^*$, $n \in \mathbb{N}$, and by Lemma 3.1, there exist measurable sets $e_n^{(i)} \subseteq [0,\infty)$ with $e_n^{(i)} \cap e_n^{(j)} = \emptyset$ for $i \neq j$ and such that

$$(x_n \chi_{e_n^{(i)}})^* = (x_1^* \chi_{[1/i,1/(i-1))\cup[i-1,i)})^*$$

for all $n, i \in \mathbb{N}$, $i \geq 2$. Since

$$\sup_n \|x_n \chi_{e_n^{(i)}}\|_{L^2[0,\infty)} \leq \|x_1^* \chi_{[1/i,1/(i-1))\cup[i-1,i)}\|_{L^2[0,\infty)}$$

and since $L^2[0,\infty)$ has the Banach–Saks property, it follows from a diagonal argument that there exists a subsequence $\{x_{n(k)}\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty$ such that, for all $2 \leq i \in \mathbb{N}$, there exists $x^{(i)} \in L^2[0,\infty)$ such that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^N x_{m(j)} \chi_{e_{m(j)}^{(i)}} = x^{(i)} \tag{4.5}$$

holds in $L^2[0,\infty)$, and hence also in $L_1[0,\infty) + L_\infty[0,\infty)$, for all further subsequences $\{x_{m(j)}\}_{j=1}^\infty \subset \{x_{n(k)}\}_{k=1}^\infty$. We let $\{x_{m(j)}\}_{j=1}^\infty \subset \{x_{n(k)}\}_{k=1}^\infty$ be a fixed subsequence, set

$$w_N := \frac{1}{N}\sum_{k=1}^N x_{m(k)}, \qquad N = 1, 2, \ldots,$$

and let $\varepsilon > 0$ be given. We observe that, for every $m = 1, 2, \ldots$ and $M, N \in \mathbb{N}$,

$$\|w_N - w_M\|_+ \leq \Big\| \frac{1}{N}\sum_{k=1}^N \Big(x_{m(k)} - \sum_{i=1}^m x_{m(k)} \chi_{e_{m(k)}^{(i)}}\Big)\Big\|_+ + \Big\| \frac{1}{M}\sum_{k=1}^M \Big(x_{m(k)} - \sum_{i=1}^m x_{m(k)} \chi_{e_{m(k)}^{(i)}}\Big)\Big\|_+$$

$$+\Big\|\sum_{i=1}^{m} x^{(i)} - \frac{1}{N}\sum_{k=1}^{N}\sum_{i=1}^{m} x_{m(k)}\chi_{e^{(i)}_{m(k)}}\Big\|_+$$
$$+\Big\|\sum_{i=1}^{m} x^{(i)} - \frac{1}{M}\sum_{k=1}^{M}\sum_{i=1}^{m} x_{m(k)}\chi_{e^{(i)}_{m(k)}}\Big\|_+.$$

Choose $m\in\mathbb{N}$ such that

$$(4.6)\qquad 1/m<\varepsilon,\quad x_1^*(m)<\varepsilon,\quad \int_0^{1/m} x_1^*(s)\,ds<\varepsilon.$$

Observe that

$$\Big(x_{m(k)} - \sum_{i=1}^{m} x_{m(k)}\chi_{e^{(i)}_{m(k)}}\Big)^*(t) = \begin{cases} x_1^*(t) & \text{if } 0<t<1/m,\\ x_1^*(t+m) & \text{if } t\ge 1/m.\end{cases}$$

It follows that

$$\Big\|x_{m(k)} - \sum_{i=1}^{m} x_{m(k)}\chi_{e^{(i)}_{m(k)}}\Big\|_+ = \int_0^1 \Big(x_{m(k)} - \sum_{i=1}^{m} x_{m(k)}\chi_{e^{(i)}_{m(k)}}\Big)^*(s)\,ds \le \int_0^{1/m} x_1^*(s)\,ds + \int_{1/m}^1 \varepsilon\,ds \le 2\varepsilon.$$

Since

$$\frac{1}{N}\sum_{k=1}^{N}\Big(x_{m(k)} - \sum_{i=1}^{m} x_{m(k)}\chi_{e^{(i)}_{m(k)}}\Big) \prec\prec \frac{1}{N}\sum_{k=1}^{N}\Big(x_{m(k)} - \sum_{i=1}^{m} x_{m(k)}\chi_{e^{(i)}_{m(k)}}\Big)^*,$$

it follows further that

$$(4.7)\qquad \Big\|\frac{1}{N}\sum_{k=1}^{N}\Big(x_{m(k)} - \sum_{i=1}^{m} x_{m(k)}\chi_{e^{(i)}_{m(k)}}\Big)\Big\|_+ \le 2\varepsilon,$$

and similarly

$$\Big\|\frac{1}{M}\sum_{k=1}^{M}\Big(x_{m(k)} - \sum_{i=1}^{m} x_{m(k)}\chi_{e^{(i)}_{m(k)}}\Big)\Big\|_+ \le 2\varepsilon,$$

for all $M,N\in\mathbb{N}$. Now if we observe that

$$\Big\|\sum_{i=1}^{m} x^{(i)} - \frac{1}{N}\sum_{k=1}^{N}\sum_{i=1}^{m} x_{m(k)}\chi_{e^{(i)}_{m(k)}}\Big\|_+ = \Big\|\sum_{i=1}^{m}\Big(x^{(i)} - \frac{1}{N}\sum_{k=1}^{N} x_{m(k)}\chi_{e^{(i)}_{m(k)}}\Big)\Big\|_+$$

together with the same equality with N replaced by M, and use (4.5), it follows that there exists $N_0 \in \mathbb{N}$ such that

$$\Big\| \sum_{i=1}^{m} x^{(i)} - \frac{1}{N} \sum_{k=1}^{N} \sum_{i=1}^{m} x_{m(k)} \chi_{e^{(i)}_{m(k)}} \Big\|_+ + \Big\| \sum_{i=1}^{m} x^{(i)} - \frac{1}{M} \sum_{k=1}^{M} \sum_{i=1}^{m} x_{m(k)} \chi_{e^{(i)}_{m(k)}} \Big\|_+ \le \varepsilon$$

for all $M, N \ge N_0$. We obtain

$$\|w_N - w_M\|_+ \le 5\varepsilon$$

for all $M, N \ge N_0$. Consequently, there exists $x \in L_1[0,\infty) + L_\infty[0,\infty)$ such that

$$\frac{1}{N} \sum_{k=1}^{N} x_{m(k)} \to x.$$

To show that x is independent of the subsequence $\{x_{m(k)}\}_{k=1}^{\infty}$, let $\varepsilon > 0$ be given and suppose that m satisfies (4.6). From (4.7), it follows that

$$\Big\| \frac{1}{N} \sum_{k=1}^{N} x_{m(k)} - \sum_{i=1}^{m} \Big(\frac{1}{N} \sum_{k=1}^{N} x_{m(k)} \chi_{e^{(i)}_{m(k)}} \Big) \Big\|_+ = \Big\| \frac{1}{N} \sum_{k=1}^{N} \Big(x_{m(k)} - \sum_{i=1}^{m} x_{m(k)} \chi_{e^{(i)}_{m(k)}} \Big) \Big\|_+ \le 2\varepsilon$$

for all $N \in \mathbb{N}$. Letting $N \to \infty$, we obtain

$$\Big\| x - \sum_{i=1}^{m} x^{(i)} \Big\|_+ < 2\varepsilon$$

for all sufficiently large $m \in \mathbb{N}$. This shows that the equality

$$x = \sum_{i=1}^{\infty} x^{(i)}$$

holds in $L_1[0,\infty) + L_\infty[0,\infty)$, which suffices to complete the proof of the proposition. ∎

The following lemma is given in [PSW, Lemma 5.3].

LEMMA 4.4. *Let $0 < \alpha \le \infty$ and let $\{y_n\}_{n=1}^{\infty}$ be a sequence in $L_1[0,\alpha) + L_\infty[0,\alpha)$. If x is measurable and $y \in L_1[0,\alpha) + L_\infty[0,\alpha)$ are such that $y_n \to x$ locally in measure and $y_n \to y$ for the weak topology on $L_1[0,\alpha) + L_\infty[0,\alpha)$ induced by $L_1[0,\alpha) \cap L_\infty[0,\alpha)$, then $x = y$.*

The following theorem is the principal result of this section concerning the Banach–Saks property.

THEOREM 4.5. *If E is a separable rearrangement invariant space E on $[0,\alpha)$, $0<\alpha\le\infty$, with the Fatou property then the following conditions are equivalent.*

(i) *E has the Banach–Saks property*;
(ii) *E has the Banach–Saks d-property.*

Proof. We only need to prove that (ii) implies (i). Let $\{x_n\}_{n=1}^\infty \subseteq E$ be a weakly null sequence. By Proposition 3.2(ii), and passing to a subsequence if necessary, there exist weakly null sequences $\{y_n\}_{n=1}^\infty \subseteq E$ and $\{z_n\}_{n=1}^\infty \subseteq E$ and a null sequence $\{d_n\}_{n=1}^\infty \subseteq E$ such that

$$x_n = y_n + z_n + d_n$$

and such that

$$y_n^* = y_1^*, \quad z_n z_m = 0 \quad \text{for all } n,m\in\mathbb{N},\ n\neq m,$$

with $z_n \to 0$ in measure. Since E has the Banach–Saks d-property, we may assume that $\{z_n\}_{n=1}^\infty$ is a Banach–Saks sequence. Using Proposition 4.3, Lemma 4.4 and passing to a further subsequence and relabelling if necessary, we may assume that

$$\frac{1}{N}\sum_{n=1}^{N} w_n \to 0$$

in measure as $N\to\infty$ for every subsequence $\{w_n\}_{n=1}^\infty \subseteq \{y_n\}_{n=1}^\infty$. Suppose then that $\{w_n\}_{n=1}^\infty \subseteq \{y_n\}_{n=1}^\infty$ is an arbitrary subsequence. Set

$$a_N := \frac{1}{N}\sum_{n=1}^{N} w_n, \qquad N\in\mathbb{N}.$$

Since

$$w_n^* = y_1^*, \quad n\in\mathbb{N},$$

it follows from [KPS, Chapter II.2] (see also [BeS, Chapter 2, Theorem 3.4]) that

$$a_N \prec\prec y_1^*, \qquad N\in\mathbb{N}.$$

Since $a_N \to 0$ in measure as $N\to\infty$, it now follows from Proposition 4.3 that $\|a_N\|_E \to 0$ as $N\to\infty$. This implies that the sequence $\{y_n\}_{n=1}^\infty$ is a Banach–Saks sequence and this suffices to complete the proof of the Theorem. ∎

We recall ([LT2, Definition 1.f.4]) that, if $1<p<\infty$, then the Banach lattice X is said to satisfy an *upper p-estimate* if there exists a positive constant $M<\infty$ such that, for every choice of pairwise disjoint elements

$\{x_i\}_{i=1}^n \subseteq X$, it follows that

$$\Big\|\sum_{i=1}^n x_i\Big\| \le M\Big(\sum_{i=1}^n \|x_i\|^p\Big)^{1/p}.$$

We obtain the following immediate consequence of Theorem 4.5.

COROLLARY 4.6. *Let E be a separable rearrangement invariant space on $[0,\alpha)$, $0<\alpha\le\infty$, with the Fatou property. If E satisfies an upper p-estimate for some $p>1$, then E has the Banach–Saks property.*

It is a trivial remark that each L_p-space, $1<p<\infty$, satisfies an upper p-estimate. Consequently, the preceding corollary implies that each L_p-space, $1<p<\infty$, has the Banach–Saks property, thus recovering the seminal results of [BS] in the case of $1<p\le 2$ and those of [KP] in the case of $2<p<\infty$.

Before proceeding, we observe that for uniformly bounded sequences in a separable rearrangement invariant space E on $[0,1)$, the notion of weak convergence does not depend on the space E.

LEMMA 4.7. *Let $\{x_n\}_{n=1}^\infty \subseteq L_\infty[0,1)$ with $\sup_n \|x_n\|_{L_\infty} = C<\infty$, and let E_1, E_2 be separable rearrangement invariant spaces on $[0,1)$. If $x_n \to_n 0$ weakly in E_1, then $x_n\to_n 0$ weakly in E_2.*

The proof of the lemma is straightforward and therefore omitted.

We denote by $\{r_n\}_{n=1}^\infty$ the usual Rademacher system on $[0,1)$ defined by setting

$$r_n(t)=\operatorname{sgn}\sin(2^n\pi t),\qquad t\in[0,1).$$

We shall need the following result of S. V. Astashkin [A] which asserts that each uniformly bounded, weakly null sequence in $L_2[0,1)$ contains a subsequence majorized in distribution by the Rademacher system. More precisely, if $\{x_n\}_{n=1}^\infty$ is uniformly bounded in $L_\infty[0,1)$ and weakly null in $L_2[0,1)$, then there exists a constant $C>0$ depending only on the uniform bound of the sequence $\{x_n\}_{n=1}^\infty$ and there exists a subsequence $\{y_n\}_{n=1}^\infty \subseteq \{x_n\}_{n=1}^\infty$ such that

$$\lambda\Big\{t:\Big|\sum_{k=1}^\infty a_k y_k(t)\Big|\ge C\tau\Big\} \le C\lambda\Big\{t:\Big|\sum_{k=1}^\infty a_k r_k(t)\Big|\ge \tau\Big\}$$

for every sequence $\{a_k\}_{k=1}^\infty\in l_2$, $\tau>0$.

LEMMA 4.8. *Let E be a separable rearrangement invariant space on $[0,1)$, let $\{x_n\}_{n=1}^\infty\subseteq E$ be uniformly bounded with $\sup_n\|x_n\|_{L_\infty}=C<\infty$, and suppose that $\{x_n\}_{n=1}^\infty$ is weakly null. Then there exists a sequence*

$a_n \downarrow_n 0$ and a subsequence $\{y_n\}_{n=1}^\infty \subset \{x_n\}_{n=1}^\infty$ such that

$$\sup\left\{\frac{1}{m}\Big\|\sum_{n\in B} y_n\Big\|_E : B\subset\mathbb{N},\ |B|=m\right\}\le a_m$$

for each $m\in\mathbb{N}$.

Proof. By Lemma 4.7, the sequence $\{x_n\}_{n=1}^\infty$ tends to 0 weakly in L_2. By Astashkin's theorem, there exists a constant $C>0$ which depends only on the uniform bound of the sequence $\{x_n\}$ such that for all $m\in\mathbb{N}$ and any set $B\subset\mathbb{N}$ with $|B|=m$, we have

$$\lambda\Big\{t:\Big|\sum_{n\in B} y_k(t)\Big|\ge C\tau\Big\}\le C\lambda\Big\{t:\Big|\sum_{n\in B} r_k(t)\Big|\ge \tau\Big\}. \tag{4.8}$$

It is well known that the value

$$\lambda\Big\{t:\Big|\sum_{n\in B} r_k(t)\Big|\ge \tau\Big\},$$

where $|B|=m$, depends on m only. Using this statement, [KPS, Corollary 2 of Section 2.4.5], and (4.8), we obtain

$$\Big\|\sum_{k\in B} y_k\Big\|_E\le C^2\Big\|\sum_{k=1}^m r_k\Big\|_E,$$

and consequently

$$\sup\left\{\frac{1}{m}\Big\|\sum_{n\in B} y_n\Big\|_E : B\subset\mathbb{N},\ |B|=m\right\}\le C^2\Big\|\frac{1}{m}\sum_{n=1}^m r_n\Big\|_E.$$

The sequence

$$a_m:=\Big\|\frac{1}{m}\sum_{n=1}^m r_n\Big\|_E,\qquad m\in\mathbb{N},$$

tends to 0 for any rearrangement invariant space $E\ne L_\infty[0,1)$ ([LT, 2.c.10]), and this completes the proof of the lemma. ∎

We remark that the subsequence $\{y_n\}_{n=1}^\infty$ given in the preceding lemma does not depend on the space E and the sequence $\{a_n\}$ depends only on E and the uniform bound of $\{x_n\}_{n=1}^\infty$.

LEMMA 4.9. *Let E be a separable rearrangement invariant space on $[0,\infty)$. If $\{x_n\}_{n=1}^\infty$ is an E-equi-integrable, weakly null sequence and if $\varepsilon>0$, then there exists $M>0$, a subsequence $\{y_n\}_{n=1}^\infty\subseteq\{x_n\}_{n=1}^\infty$ and weakly null sequences $\{u_n\}_{n=1}^\infty,\{v_n\}_{n=1}^\infty\subseteq E$ such that*

$$y_n=u_n+v_n,\qquad \sup\|u_n\|_{L_\infty}\le M,\qquad \|v_n\|_E<\varepsilon,$$

and $\operatorname{supp}(u_n)\subseteq[0,M)$, *for all $n\in\mathbb{N}$.*

Proof. For any $k, n \in \mathbb{N}$, set

$$e_n^{(k)} := \{|x_n| > k\}$$

and observe that

$$\|x_n\|_{L_1+L_\infty} \geq k \min\{1, \lambda(e_n^{(k)})\}$$

for all $n, k \in \mathbb{N}$. Since $\{x_n\}_{n=1}^\infty$ is bounded in E and hence in $L_1 + L_\infty$, it follows that

$$\lim_{k\to\infty} \sup_{n\in\mathbb{N}} \lambda(e_n^{(k)}) = 0. \tag{4.9}$$

We now show that for all $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$\sup_{n\in\mathbb{N}} \|x_n \chi_{e_n^{(k)}}\|_E < \varepsilon/4. \tag{4.10}$$

If this is not the case, then by a change of notation if necessary, we may assume that there exists $\varepsilon > 0$ such that

$$\|x_n \chi_{e_n^{(n)}}\|_E \geq \varepsilon, \qquad n \in \mathbb{N}. \tag{4.11}$$

Using (4.9), and passing to a subsequence and relabelling if necessary, we may assume further that

$$\sum_{n=1}^\infty \lambda(e_n^{(n)}) < \infty. \tag{4.12}$$

Set

$$E_n = \bigcup_{k\geq n} e_k^{(k)}, \qquad n \in \mathbb{N},$$

and observe that (4.12) implies that $E_n \downarrow_n \emptyset$. It now follows from the E-equi-integrability of the sequence $\{x_n\}_{n=1}^\infty$ that

$$\|x_n \chi_{e_n^{(n)}}\|_E \leq \|x_n \chi_{E_n^{(n)}}\|_E \to 0$$

as $n \to \infty$, and this clearly contradicts (4.11) and establishes (4.10). If $F_k = [k, \infty), k \in \mathbb{N}$, noting that $F_k \downarrow_k \emptyset$, and again using the E-equi-integrability of the sequence $\{x_n\}_{n=1}^\infty$, we may assume further that

$$\sup_{n\in\mathbb{N}} \|x_n \chi_{F_k}\|_E < \varepsilon/4. \tag{4.13}$$

We now set

$$x_n^{(k)} = x_n - x_n \chi_{e_n^{(k)} \cup F_k}, \qquad n \in \mathbb{N}.$$

It follows from (4.10) and (4.13) that the sequence $\{x_n^{(k)}\}_{n=1}^\infty$ is supported by the interval $[0, k)$ and satisfies

$$\|x_n - x_n^{(k)}\|_E < \varepsilon/2, \quad \|x_n^{(k)}\|_{L_\infty} \leq k, \qquad n \in \mathbb{N}.$$

In particular, the sequence $\{x_n^{(k)}\}_{n=1}^\infty$ is order bounded in E and consequently is relatively weakly compact in E, since separability of E implies that the norm on E is order continuous, and this implies that order intervals in E are weakly compact. Let $\{w_n\}_{n=1}^\infty \subseteq \{x_n^{(k)}\}_{n=1}^\infty$ be a weakly convergent subsequence, with weak limit $w \in E$, and let $\{y_n\}_{n=1}^\infty$ be the corresponding subsequence of $\{x_n\}_{n=1}^\infty$. In particular note that

$$\|w_n - y_n\|_E < \varepsilon/2, \qquad n \in \mathbb{N}.$$

Set

$$u_n = w_n - w, \quad v_n = y_n - u_n = y_n - w_n + w, \qquad n \in \mathbb{N}.$$

It is clear that $u_n, v_n \to 0$ weakly in E. Since

$$\|w\|_{L_\infty} \le \liminf_{n\to\infty} \|w_n\|_{L_\infty} \le k,$$

it follows that

$$\sup_n \|u_n\|_{L_\infty} \le 2k.$$

Further, since $w_n - y_n \underset{n}{\to} w$ weakly in E, it follows that

$$\|w\|_E \le \liminf_{n\to\infty} \|w_n - y_n\|_E < \varepsilon/2.$$

This yields

$$\|v_n\|_E \le \|w_n - y_n\|_E + \|w\|_E < \varepsilon$$

for all $n \in \mathbb{N}$. Since it is clear that $\operatorname{supp}(w_n) \subseteq [0, k)$ for all $n \in \mathbb{N}$ implies that $\operatorname{supp}(w) \subseteq [0, k)$, it also follows that $\operatorname{supp}(u_n) \subseteq [0, k)$ for all $n \in \mathbb{N}$. The assertion of the lemma now follows by taking $M = 2k$. ∎

THEOREM 4.10. *Let E be a separable rearrangement invariant space on $[0,\infty)$. If $\{x_n\}_{n=1}^\infty$ is weakly null and E-equi-integrable, then it contains a Banach–Saks subsequence.*

Proof. It may be assumed that $\|x_n\|_E \le 1$ for all $n \in \mathbb{N}$. We prove first that given $\varepsilon > 0$, we can choose a subsequence $\{z_n\}_{n=1}^\infty \subset \{x_n\}_{n=1}^\infty$ depending on ε such that

$$\lim_{m\to\infty} \frac{1}{m} \sup_{|B|=m} \Big\| \sum_{n\in B} z_n \Big\|_E \le \varepsilon.$$

By Lemma 4.9, there exist a subsequence $\{y_n\}_{n=1}^\infty \subseteq \{x_n\}_{n=1}^\infty$ and weakly null sequences $\{u_n\}_{n=1}^\infty, \{v_n\}_{n=1}^\infty \subseteq E$ such that $y_n = u_n + v_n$, $\sup_n \|u_n\|_{L_\infty} < \infty$, $\sup_n \|v_n\|_E \le \varepsilon$, and $\operatorname{supp}(u_n) \subset [0, M]$ for some $M > 0$ and all $n \in \mathbb{N}$. By Lemma 4.8, we can choose a subsequence $\{u_{k_n}\} \subset \{u_n\}$ such that

$$a_m = \frac{1}{m} \sup_{|B|=m} \Big\| \sum_{n\in B} u_{k_n} \Big\|_E \downarrow_m 0.$$

It follows that

$$\limsup_{m\to\infty}\frac{1}{m}\sup_{|B|=m}\Big\|\sum_{n\in B}y_{k_n}\Big\|_E \le \limsup_{m\to\infty}\frac{1}{m}\sup_{|B|=m}\Big(\Big\|\sum_{n\in B}u_{k_n}\Big\|_E+\sum_{n\in B}\|v_{k_n}\|_E\Big) \le \limsup(a_m+\varepsilon)=\varepsilon.$$

We now complete the proof of the theorem by a diagonal argument. Given an integer k, there exist an integer N_k and a subsequence $\{z_n^{(k)}\}$ such that

$$\{x_n\}\supset\{z_n^{(1)}\}\supset\{z_n^{(2)}\}\supset\ldots\supset\{z_n^{(k)}\}\supset\ldots$$

and

$$\frac{1}{m}\sup_{|B|=m}\Big\|\sum_{n\in B}z_n^{(k)}\Big\|_E\le\frac{1}{k} \tag{4.14}$$

for each $m\ge N_k$. Without loss of generality we may assume that $N_1<N_2<\ldots$ Now we shall prove that

$$\lim_{m\to\infty}\frac{1}{m}\sup_{|B|=m}\Big\|\sum_{n\in B}z_n^{(n)}\Big\|_E=0.$$

Let k be an integer, let $\{w_n\}\subset\{z_n^{(n)}\}$, and let the integer m satisfy

$$m\ge M(k)=\max(k^2,N_k+k). \tag{4.15}$$

Since $z_n^{(n)}\in\{z_n^{(k)}\}_{n=1}^{\infty}$ for every $n\ge k$, we have $w_n\in\{z_n^{(k)}\}$ for $n\ge k$. From (4.15), it is clear that

$$|\{k+1,k+2,\ldots,m\}|=m-k\ge N_k.$$

Using (4.14) and (4.15), we obtain

$$\frac{1}{m}\Big\|\sum_{n=1}^{m}w_n\Big\|_E\le\frac{1}{m}\Big(\sum_{n=1}^{k}\|w_n\|_E+\Big\|\sum_{n=k+1}^{m}w_n\Big\|_E\Big)\le\frac{k}{m}+\frac{m-k}{mk}\le\frac{2}{k}$$

for all $m\ge M(k)$, and this completes the proof of the theorem. ■

The assumption in the preceding theorem that the sequence $\{x_n\}_{n=1}^{\infty}$ is E-equi-integrable cannot be omitted. This is shown in Theorem 5.9 below.

5. The Banach–Saks property in Orlicz, Lorentz and Marcinkiewicz spaces. In this section, we show that an Orlicz space on any interval $[0,\alpha)$ has the Banach–Saks property if and only if it is separable, and that the same result holds also for Lorentz spaces. As noted earlier, any rearrangement invariant space with the Banach–Saks property is necessarily separable. However, we show further that the separable parts of non-separable Orlicz and Lorentz spaces on an arbitrary interval do not have the Banach–Saks property. That contrasts with the sequence space setting: it has been shown by Rakov [Ra] that the separable part of any Orlicz sequence space always has the Banach–Saks property.

We shall need the following theorem which is stated in [Ad] for the Bochner space $L_2(c_0)$. We let $\{r_n\}_{n=1}^\infty$ denote the sequence of Rademacher functions and let (e_i) denote the unit vector basis of c_0. We define the sequence $\{x_n\}_{n=1}^\infty \subseteq L_p(c_0)$, $1 \le p < \infty$, by setting

$$x_n := \sum_{i=1}^{2^n} r_{2^n+i} e_i, \qquad n \in \mathbb{N}.$$

THEOREM 5.1. *For every $1 \le p < \infty$, the sequence $\{x_n\}_{n=1}^\infty$ is a normalized, weakly null sequence in $L_p(c_0)$ such that*

$$\Big\| \sum_{n=1}^{m} x_{k_n} \Big\|_{L_p(c_0)} \ge \frac{m(e-1)}{2e} \tag{5.1}$$

for any subsequence $\{x_{k_n}\}_{n=1}^\infty$ of $\{x_k\}$.

Proof. That the sequence $\{x_n\}_{n=1}^\infty$ is normalized and weakly null in $L_p(c_0)$, $1 < p < \infty$, follows from [Ad, Lemma 7(a)]. The proof of the same assertion in the case of $L_1(c_0)$ follows easily from the same result, by observing that the sequence $\{x_n\}_{n=1}^\infty$ is obviously bounded in $L_1(c_0)$ and that $L_1(c_0)^* = L_\infty(l_1) \subseteq L_2(l_1)$ (see, for example, [DU, p. 98]). It remains to be shown that (5.1) holds for every subsequence $\{x_{k_n}\}_{n=1}^\infty \subseteq \{x_n\}_{n=1}^\infty$.

For notational simplicity, we assume that $k_n = n$ for all $n \in \mathbb{N}$ and let $1 \le m \in \mathbb{N}$ be given. We have

$$\sum_{n=1}^{m} x_n = \sum_{n=1}^{m} \sum_{i=1}^{2^n} r_{2^n+i} e_i = \sum_{i=1}^{2^m} \Big(\sum_{n \in Q_i} r_n \Big) e_i,$$

where $Q_i \subseteq \{1, \ldots, 2^{m+1}\}$, $Q_i \cap Q_j = \emptyset$ for all $i \ne j$ and $|Q_i| \le m$ for all $i = 1, \ldots, 2^m$. It is important to note that

$$m/2 \le |Q_i| \le m, \quad \forall 1 \le i \le 2^{m/2}. \tag{5.2}$$

We shall make use of the following (elementary and well known) inequality

$$\lambda\Big\{ t : \Big| \sum_{k=1}^{n} r_k(t) \Big| \ge m/2 \Big\} \ge 2^{-m/2}, \tag{5.3}$$

which is valid for $m/2 \le n \le m$. Since the system $\{\sum_{n \in Q_i} r_n\}_{i=1}^{2^m}$ consists of independent functions, it follows from (5.2) and (5.3) that

$$\begin{aligned}\lambda\Big\{t : \max_{1\le i\le 2^{m/2}} \Big|\sum_{n\in Q_i} r_n(t)\Big| < \frac{m}{2}\Big\} &= \prod_{1\le i\le 2^{m/2}} \lambda\Big\{t : \Big|\sum_{n\in Q_i} r_n(t)\Big| < \frac{m}{2}\Big\}\\ &= \prod_{1\le i\le 2^{m/2}} \Big(1 - m\Big\{t : \Big|\sum_{n\in Q_i} r_n(t)\Big| \ge \frac{m}{2}\Big\}\Big)\\ &\le (1-2^{-m/2})^{2^{m/2}} \le 1/e.\end{aligned}$$

It follows immediately that

$$\lambda\Big\{t : \max_{1\le i\le 2^{m/2}} \Big|\sum_{n\in Q_i} r_n(t)\Big| \ge m/2\Big\} > 1 - 1/e.$$

Consequently, for all $1 \le p < \infty$, we have

$$\begin{aligned}\Big\|\sum_{n=1}^{m} x_n\Big\|_{L_p(c_0)} \ge \Big\|\sum_{n=1}^{m} x_n\Big\|_{L_1(c_0)} &\ge \frac{m}{2}\,\lambda\Big\{t : \max_{1\le i\le 2^{m/2}} \Big|\sum_{n\in Q_i} r_n(t)\Big| \ge m/2\Big\}\\ &\ge \frac{m(e-1)}{2e}. \ \blacksquare\end{aligned}$$

Let E be a rearrangement invariant space on $[0,\alpha)$, $0 < \alpha \le \infty$, and let $\mathbf{E}$ be the (isometrically isomorphic) space of measurable functions on the rectangle $[0,1) \times [0,\alpha)$ given by

$$\mathbf{E} := \{f \in (L_1 + L_\infty)([0,1)\times[0,\alpha)) : f^* \in E\}, \qquad \|f\|_{\mathbf{E}} := \|f^*\|_E.$$

Here, the decreasing rearrangement f^* is calculated with respect to product Lebesgue measure on the rectangle $[0,1)\times[0,\alpha)$.

Lemma 5.2. *Suppose that E is separable but not maximal and let $\{e_n\}_{n=1}^{\infty} \subseteq E$ be (order) equivalent to the unit vector basis of c_0. If the sequence $\{y_n\}_{n=1}^{\infty} \subseteq \mathbf{E}$ is defined by setting*

$$y_n := \sum_{i=1}^{2^n} r_{2^n+i} \otimes e_i, \qquad n \in \mathbb{N},$$

then $\{y_n\}_{n=1}^{\infty}$ is weakly null in $\mathbf{E}$.

Proof. Note that order continuity of the norm on E implies order continuity of the norm on $\mathbf{E}$, and so the Banach dual $\mathbf{E}^*$ may be identified with the Köthe dual $\mathbf{E}^\times$. Consequently, if $F \in \mathbf{E}^*$ then there exists a uniquely determined $f \in \mathbf{E}^\times$ such that

$$F(x) = \int_{[0,1)\times[0,\alpha)} f(s,t)x(s,t)\,ds\,dt, \qquad x \in \mathbf{E}.$$

We set

$$y_n := \sum_{i=1}^{2^n} r_{2^n+i} \otimes e_i, \qquad n \in \mathbb{N},$$

and suppose that the sequence $\{y_n\}_{n=1}^{\infty}$ is not weakly null in $\mathbf{E}$. We may assume, therefore, that there exists $\eta > 0$, $f \in \mathbf{E}'$ and an increasing sequence $n(j)$, $j \in \mathbb{N}$, of natural numbers such that

$$\int\limits_{[0,1)\times[0,\alpha)} f(s,t)\Big(\sum_{i=1}^{2^{n(j)}} r_{2^{n(j)}+i}(s)e_i(t)\Big)\,ds\,dt > \eta \tag{5.4}$$

for all $j = 1, 2, \ldots$ It follows from Fubini's theorem that the function

$$s \mapsto \int\limits_{[0,\alpha)} f(s,t)e_i(t)\,dt, \qquad s \in [0,1),$$

is integrable on $[0,1)$. Consequently, by the fact that the Rademacher sequence is a uniformly bounded, orthonormal sequence, it follows that

$$\lim_{n\to\infty} \int\limits_{[0,1)\times[0,\alpha)} f(s,t)r_{2^n+i}(s)e_i(t)\,ds\,dt$$
$$= \lim_{n\to\infty} \int\limits_{[0,1)} \Big(\int\limits_{[0,\alpha)} f(s,t)e_i(t)\,dt\Big) r_{2^n+i}(s)\,ds = 0$$

for all $i \in \mathbb{N}$. We may therefore assume further that

$$\int\limits_{[0,1)\times[0,\alpha)} f(s,t)\Big(\sum_{i=1}^{2^{n(j-1)}} r_{2^{n(j)}+i}(s)e_i(t)\Big)\,ds\,dt \le \eta/2 \tag{5.5}$$

for all $j \ge 1$. We set

$$I_j = \{2^{n(j-1)}+1, \ldots, 2^{n(j)}\}, \qquad j \in \mathbb{N}.$$

From (5.4) and (5.5), it follows that

$$\int\limits_{[0,1)\times[0,\alpha)} f(s,t)\Big(\sum_{i\in I_j} r_{2^{n(j)}+i}(s)e_i(t)\Big)\,dt\,ds \ge \eta/2 \tag{5.6}$$

for all $j \in \mathbb{N}$. We set

$$x_k := \sum_{j=1}^{k} \sum_{i\in I_j} r_{2^{n(j)}+i} \otimes e_i, \qquad k \in \mathbb{N}.$$

Noting that the sequence $\{e_i\}_{i=1}^{\infty}$ is disjointly supported and that $I_j \cap I_k = \emptyset$ whenever $j \ne k$, we deduce that

$$|x_k| = \sum_{j=1}^{k} \sum_{i\in I_j} \mathbf{1} \otimes |e_i|, \qquad k \in \mathbb{N},$$

where $\mathbf{1}$ denotes the indicator function of the interval $[0,1)$. Since $\{e_i\}_{i=1}^\infty$ is equivalent to the unit vector basis of c_0, this implies that

$$\sup_{j\in\mathbb{N}} \|x_j\|_{\mathbf{E}} < \infty. \tag{5.7}$$

However, it follows from (5.6) that

$$F(x_k) = \sum_{j=1}^{k} \int_{[0,1)\times[0,\alpha)} f(s,t)\Big(\sum_{i\in I_j} r_{2^{n(j)}+i}(s)e_i(t)\Big)\,dt\,ds \ge k\eta/2$$

for all $k \in \mathbb{N}$. This contradicts (5.7) and suffices to complete the proof of the lemma. ∎

We shall need the following notion introduced in [Su2]. We denote by $\chi_j^{(n)}$ the indicator function of the interval $[j\cdot 2^{-n},(j+1)\cdot 2^{-n})$, $1 \le j \le 2^n$.

Definition 5.3. The rearrangement invariant space E on the interval $[0,\alpha)$, $1<\alpha\le\infty$, is said to have the *L_1-embedding property*, in symbols $E \in (\mathrm{EP}_1)$, if there exists a positive constant K_E such that for any natural number n and family $\{y_i\}_{i=1}^{2^n} \subseteq L_1[0,\alpha)\cap L_\infty[0,\alpha)$,

$$2^{-n}\sum_{i=1}^{2^n}\|y_i\|_E = \Big\|\sum_{i=1}^{2^n}\chi_i^{(n)}(\cdot)y_i\Big\|_{L_1([0,1],E)} \le K_E\Big\|\sum_{i=1}^{2^n}\chi_i^{(n)}\otimes y_i\Big\|_{\mathbf{E}}.$$

We note that if $\{w_j\}_{j=1}^n \subseteq L_1[0,\alpha)\cap L_\infty[0,\alpha)$ is any finite family, then there exists a finite family $\{y_i\}_{i=1}^{2^n} \subseteq L_1[0,\alpha)\cap L_\infty[0,\alpha)$ such that

$$\sum_{j=1}^{n} r_j(\cdot)w_j = \sum_{i=1}^{2^n}\chi_i^{(n)}(\cdot)y_i \quad\text{and}\quad \sum_{j=1}^{n} r_j\otimes w_j = \sum_{i=1}^{2^n}\chi_i^{(n)}\otimes y_i.$$

Consequently, if $E\in(\mathrm{EP}_1)$, and if $\{w_j\}_{j=1}^n \subseteq L_1[0,\alpha)\cap L_\infty[0,\alpha)$ is any finite family, then

$$\Big\|\sum_{j=1}^{n} r_j(\cdot)w_j\Big\|_{L_1([0,1),E)} \le K_E\Big\|\sum_{j=1}^{n} r_j\otimes w_j\Big\|_{\mathbf{E}}.$$

Proposition 5.4. *Let E be a separable rearrangement invariant space on $[0,\alpha)$. If E is not maximal, and if $E\in(\mathrm{EP}_1)$, then E does not have the Banach–Saks property.*

Proof. Using the fact that E is not maximal, let $\{e_i'\}_{i=1}^\infty \subseteq E$ be a mutually disjoint sequence which is K-equivalent to the standard unit vector basis $\{e_i\}_{i=1}^\infty$ of c_0 for some $K<\infty$. Since the norm on E is order continuous, it is easy to see that we may assume, in addition, that $e_i' \in L_1[0,\alpha)\cap L_\infty[0,\alpha)$

for all $i \in \mathbb{N}$. We set

$$y_n := \sum_{i=1}^{2^n} r_{2^n+i} \otimes e_i', \quad x_n := \sum_{i=1}^{2^n} r_{2^n+i} \otimes e_i, \quad n \geq 1.$$

Using now Theorem 5.1, together with the assumption that $E \in (\mathrm{EP}_1)$ and the remark following Definition 5.3, we find that

$$K_E \Big\| \sum_{n=1}^{m} y_{k_n} \Big\|_{\mathbf{E}} \geq \Big\| \sum_{n=1}^{m} \Big(\sum_{i=1}^{2^{k_n}} r_{2^{k_n}+i} e_i' \Big) \Big\|_{L_1(E)} \geq K^{-1} \Big\| \sum_{n=1}^{m} x_{k_n} \Big\|_{L_1(c_0)} \geq K^{-1} \frac{m(e-1)}{2e}$$

for all $m \geq 1$, for any subsequence $\{y_{k_n}\}_{n=1}^{\infty}$ of $\{y_k\}_{k=1}^{\infty}$. It now follows from Lemma 5.2 that $\mathbf{E}$ does not have the Banach–Saks property. Consequently E, being isometrically isomorphic to $\mathbf{E}$, does not have the Banach–Saks property, and this completes the proof of the proposition. ■

THEOREM 5.5. *Let L_Φ be an Orlicz space on* $[0, \alpha)$, $0 < \alpha \leq \infty$.

(i) *If L_Φ is separable, then L_Φ has the Banach–Saks property.*

(ii) *If L_Φ is not separable, then the separable part L_Φ^0 does not have the Banach–Saks property.*

Proof. (i) By Theorem 4.5, it suffices to show that L_Φ has the Banach–Saks d-property. To this end, suppose that $\{x_n\}_{n=1}^{\infty} \subseteq L_\Phi$ is a normalized, weakly null, disjointly supported sequence. Applying [LT3, Proposition 3] in the case of $\alpha < \infty$ and [Ni, Theorem 1.1] in the case of $\alpha = \infty$, and passing to a subsequence if necessary, we may assume that there exists an Orlicz function G such that the sequence $\{x_n\}_{n=1}^{\infty}$ is equivalent to the unit vector basis $\{e_n\}_{n=1}^{\infty}$ of the Orlicz sequence space l_G. In particular, it follows that there exists $C > 0$ such that

$$\frac{1}{C} \Big\| \sum_{k=1}^{n} e_k \Big\|_{l_G} \leq \Big\| \sum_{k=1}^{n} x_k \Big\|_{L_\Phi} \leq C \Big\| \sum_{k=1}^{n} e_k \Big\|_{l_G}$$

for every $m \in \mathbb{N}$. It is well known that

$$\Big\| \sum_{k=1}^{n} e_k \Big\|_{l_G} = 1/G^{-1}(1/m)$$

so that

$$\Big\| \sum_{k=1}^{n} x_k \Big\|_{L_\Phi} \leq C/G^{-1}(1/m) \tag{5.8}$$

for all $m \in \mathbb{N}$. Here G^{-1} denotes the function inverse to G.

Let us now suppose that

$$\liminf_{m\to\infty} mG^{-1}(1/m) < \infty. \tag{5.9}$$

Since the function G^{-1} is concave, it follows that $\{mG^{-1}(1/m)\}_{m=1}^{\infty}$ is increasing and (5.9) then implies that $(G^{-1})'(0+) < \infty$. Consequently, there exist constants $c_1, c_2 > 0$ such that

$$c_1 t \le G(t) \le c_2 t$$

for all $t \in [0,1)$. From this it follows that the sequence $\{x_n\}_{n=1}^{\infty}$ is equivalent to the unit vector basis of l_1. This contradicts the fact that $\{x_n\}_{n=1}^{\infty}$ is weakly null, and consequently (5.9) is not valid. It then follows that there exists a sequence $\beta_m \uparrow_m \infty$ such that

$$G^{-1}(1/m) = \beta_m/m$$

for all $m \in \mathbb{N}$. Now, (5.8) entails that

$$\lim_{n\to\infty} \frac{1}{n}\Big\|\sum_{k=1}^{n} x_k\Big\|_{L_\Phi} \le \lim_{n\to\infty} C/\beta_n = 0$$

and this suffices to establish the assertion of (i).

(ii) If L_Φ is not separable, then it follows that L_Φ^0 is not maximal. Since L_Φ has property (EP_1) by [Su2, Proposition 2.4], it follows also that L_Φ^0 has property (EP_1). Since L_Φ^0 is separable, it now follows from Proposition 5.4 that L_Φ^0 does not have the Banach–Saks property, and this completes the proof of the theorem. ∎

COROLLARY 5.6. *The Orlicz space L_Φ has the Banach–Saks property if and only if it is separable.*

We remark that if an Orlicz space $L_\Phi[0,1)$ is simultaneously a Marcinkiewicz space $M_\psi[0,1)$, then it follows from Theorem 5.5 that the separable part $M_\psi^0[0,1)$ of the Marcinkiewicz space $M_\psi[0,1)$ does not have the Banach–Saks property. This is the case, for example, if the Orlicz function Φ is given by setting

$$\Phi(t) := (e^{t^p} - 1)/(e-1), \qquad t \ge 0,$$

for some $1 \le p < \infty$. See, for example, [Lo]. This remark also serves to show that there are separable rearrangement invariant Banach function spaces on $[0,1)$ for which the Banach–Saks property and the Banach–Saks d-property are not equivalent. Indeed, it is a simple exercise to show that the separable part of *any* Marcinkiewicz space on $[0,1)$ always has the Banach–Saks d-property.

We remark that the preceding Corollary 5.6 substantially extends [Al, Corollary 2.10].

The following theorem extends a well known result of Szlenk [Sz] that L_1-spaces have the Banach–Saks property.

THEOREM 5.7. *Let Λ_ψ be a Lorentz space on $[0,\alpha)$, $0<\alpha\le\infty$.*

(i) *If Λ_ψ is separable, then Λ_ψ has the Banach–Saks property.*

(ii) *If Λ_ψ is not separable, then its separable part Λ_ψ^0 does not have the Banach–Saks property.*

Proof. (i) It is sufficient to check that Λ_ψ has the Banach–Saks d-property. Consider an arbitrary weakly null sequence $\{x_n\}_{n=1}^\infty \subseteq \Lambda_\psi$ of pairwise disjoint elements. If the sequence $\{x_n\}_{n=1}^\infty$ is not norm convergent to 0 then $\{x_n\}_{n=1}^\infty$ contains a basic sequence. However, it is well-known that any basic sequence of pairwise disjoint elements in Λ_ψ contains a subsequence equivalent to the standard unit vector basis of the space l_1. This is shown, for example in [CD1, Lemma 3.1] (see also [CD2, Lemma 2.1]) in the special setting of the Lorentz $L_{p,1}$-spaces and the proof in the more general case is similar. It follows that the sequence $\{x_n\}_{n=1}^\infty$ cannot converge weakly to 0. Consequently, the sequence $\{x_n\}_{n=1}^\infty$ converges in norm to 0 and so is trivially a Banach–Saks sequence.

(ii) The proof is identical to that of Theorem 5.5(ii), by using [Su2, Proposition 2.5] instead of [Su2, Proposition 2.4]. ■

COROLLARY 5.8. *The Lorentz space Λ_ψ has the Banach–Saks property if and only if it is separable.*

We remark that the space $L_1[0,\infty)+L_\infty[0,\infty)$ is a non-separable Lorentz space, with Lorentz function ψ given by

$$\psi(t)=\min\{t,1\}, \qquad t\ge 0.$$

Consequently, it follows from Theorem 5.7 that its separable part, which consists of those elements $x\in L_1[0,\infty)+L_\infty[0,\infty)$ for which $\lim_{t\to\infty} x^*(t)=0$, does not have the Banach–Saks property. This is in contrast to the situation on the unit interval where the Lorentz space $L_1[0,1)+L_\infty[0,1)$ coincides with the (separable) space $L_1[0,1)$, which has the Banach–Saks property via Szlenk's theorem.

THEOREM 5.9. *If $\psi\in\Omega$, $\psi(\infty)=\infty$ and*

$$\liminf_{t\to 0}\frac{\psi(2t)}{\psi(t)}=1,$$

then the separable part M_ψ^0 of the Marcinkiewicz space M_ψ on the interval $[0,\infty)$ does not have the Banach–Saks property. Moreover, given $\eta\in(0,1)$, there exists a disjointly supported, weakly null sequence $\{x_n\}_{n=1}^\infty\subseteq M_\psi^0$ such that

$$\Big\| \sum_{n \in B} x_n \Big\|_{M_\psi} \ge \eta m \tag{5.10}$$

for any $B \subseteq \mathbb{N}$ *with* $|B| = m$.

Proof. We observe first that the function $\tau \mapsto \psi(t\tau)/\psi(t)$, $\tau > 0$, is concave and increasing for each $t > 0$ and set

$$\alpha(\tau) := \liminf_{t \to 0} \frac{\psi(t\tau)}{\psi(t)}, \qquad \tau > 0.$$

The function α is concave and increasing, and from the assumption on ψ, it follows that $\alpha(1) = \alpha(2) = 1$. Consequently,

$$\alpha(\tau) = \liminf_{t \to 0} \frac{\psi(t\tau)}{\psi(t)} = 1 \tag{5.11}$$

for all $\tau \ge 1$. Let $q_n = [n(1 - \sqrt{\eta})]$ and note that

$$1 - q_n/n \ge \sqrt{\eta}.$$

It follows from (5.11) that there exists a sequence $1 > t_n \downarrow_n 0$ such that

$$\psi(t_{q_n}) \ge \sqrt{\eta}\, \psi(n t_{q_n})$$

for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let x_n be any element of M_ψ^0 with $\|x_n\|_{M_\psi} = 1$ which is supported by the interval $(n, n+1)$ and which satisfies

$$\int_0^s x_n^*(t)\, dt = \psi(s), \qquad t_n \le s \le 1.$$

It follows that

$$\begin{aligned}\Big\| \sum_{k=1}^{n} x_k \Big\|_{M_\psi} &\ge \Big\| \sum_{k=q_n}^{n} x_k \Big\|_{M_\psi} \ge \frac{\sum_{k=q_n}^{n} \int_0^{t_{q_n}} x_k^*(t)\, dt}{\psi((n - q_n + 1) t_{q_n})} \\ &\ge \frac{(n - q_n)\psi(t_{q_n})}{\psi(n t_{q_n})} = n\Big(1 - \frac{q_n}{n}\Big) \frac{\psi(t_{q_n})}{\psi(n t_{q_n})} \ge n\sqrt{\eta}\sqrt{\eta} = n\eta\end{aligned}$$

for all $n \in \mathbb{N}$. Since

$$\sum_{k=1}^{n} x_k \prec\prec \sum_{k \in B} x_k$$

for any $B \subset \mathbb{N}$ with $|B| = n$, it follows that

$$\Big\| \sum_{k \in B} x_k \Big\|_{M_\psi} \ge \Big\| \sum_{k=1}^{n} x_k \Big\|_{M_\psi} \ge \eta n$$

for all $n \in \mathbb{N}$ whenever $|B| = n$ and this establishes (5.10).

To show that the sequence $\{x_n\}_{n=1}^\infty$ is weakly null in M_ψ^0, it suffices to show that

$$\int_{[0,\infty)} y(t)x_n(t)\,dt \to 0$$

for every $y \in \Lambda_\psi = (M_\psi^0)^*$. Let $y \in \Lambda_\psi$. By the assumption that $\psi(\infty) = \infty$, the space Λ_ψ is separable and consequently

$$\lim_{n\to\infty} \|y\chi_{(n,\infty)}\|_{\Lambda_\psi} = 0$$

for each $y \in \Lambda_\psi$. Since

$$\begin{aligned}\int_{[0,\infty)} y(t)x_n(t)\,dt &= \Big|\int_n^{n+1} y(t)x_n(t)\,dt\Big| \le \|x_n\|_{M_\psi^0}\|y\chi_{(n,n+1)}\|_{\Lambda_\psi} \\ &\le \|y\chi_{(n,\infty)}\|_{\Lambda_\psi},\end{aligned}$$

it follows that

$$\int_{[0,\infty)} y(t)x_n(t)\,dt \to 0$$

for every $y \in \Lambda_\psi$ and this completes the proof of the theorem. ■

6. Final remarks. In this section we point out that the Banach–Saks property is not, in general, preserved by interpolation. Our discussion relies on the existence of a reflexive Banach space Z with unconditional basis which does not have the Banach–Saks property. Such an example has been constructed by A. Baernstein II [Ba].

Let us recall briefly a special case of the K-method of interpolation. For details we refer to [LT2]. Let (E_1, E_2) be an interpolation pair of Banach spaces, that is, E_1, E_2 are continuously embedded in some Hausdorff topological space. For every choice of positive scalars a, b, let $k(\cdot, a, b)$ denote the equivalent norm on the Banach sum $E_1 + E_2$ defined by setting

$$k(x, a, b) = \inf\{a\|x_1\|_{E_1} + b\|x_2\|E_2 : x = x_1 + x_2,\ x_i \in E_i,\ i = 1, 2\}$$

for all $x \in E_1+E_2$. Let Y be a Banach space with a normalized unconditional basis $\{y_n\}_{n=1}^\infty$ whose unconditional constant is one, and let $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$ be sequences of positive numbers such that

$$\sum_{n=1}^{\infty} \min(a_n, b_n) < \infty.$$

The space $K(E_1, E_2, Y, \{a_n\}, \{b_n\})$ is defined to be the space of all elements $x \in E_1 + E_2$ such that $\sum_{n=1}^\infty k(x, a_n, b_n)y_n$ converges, normed by setting

$$\|x\| = \sup_m \Big\|\sum_{n=1}^{m} k(x, a_n, b_n)y_n\Big\|_Y.$$

For any rearrangement invariant space E on $[0, 1)$, we denote the upper and lower Boyd indices by q_E, p_E respectively. For basic definitions and properties, we refer to [LT2]. We note that if $1 \le p < \infty$ and if $E = L_p[0,1)$, then $p_E = q_E = p$.

PROPOSITION 6.1. *Let $\{m_n\}$ be an increasing sequence of numbers satisfying the conditions*

$$m_n^{-1} \sum_{i=1}^{n-1} m_i + m_n \sum_{i=n+1}^{\infty} m_i^{-1} < 2^{-n-1}, \qquad n = 1, 2, \ldots$$

Let Z be a reflexive Banach space with normalized unconditional basis $\{y_n\}_{n=1}^{\infty}$ but without the Banach–Saks property. If $1 < p < q < \infty$ then the space

$$W := K(L_q[0,1), L_p[0,1), Z, \{m_n^{-1}\}, \{m_n\})$$

is a reflexive rearrangement invariant space on $[0,1)$, has an unconditional basis, and admits an equivalent rearrangement invariant, locally uniformly convex norm, but does not have the Banach–Saks property.

Proof. We remark that the second part of the proof of [LT2, Theorem 2.g.11] shows that the interpolation space $K(X_1, X, l_2, \{a_n\}, \{b_n\})$ is reflexive, provided X_1 is continuously and weakly compactly embedded in X, $\sum_{n=1}^{\infty} a_n < \infty$ and $b_n \uparrow_n \infty$. Noting that $L_p[0,1)$ is weakly compactly embedded in $L_r[0,1)$ if $r < p$, and using the fact that reflexivity of Z implies that the basis in Z is shrinking (see [LT1, Proposition 1.b.1 and Theorem 1.b.5]), it is not difficult to adapt the proof of [LT2, Theorem 2.g.11] to show that W is reflexive. It follows from [LT2, Proposition 2.g.4] that W has non-trivial Boyd indices. Consequently, [LT2, Theorem 2.c.6] implies that the Haar system is an unconditional basis in W. The proof of [LT2, Theorem 2.g.5] now shows that W contains a complemented subspace isomorphic to Z. Since Z fails to have the Banach–Saks property, it follows as well that W fails to have the Banach–Saks property. Finally, that W admits an equivalent rearrangement invariant, locally uniformly convex norm follows from the fact that W is rearrangement invariant and separable, together with [DGL, Corollary 1.2]. ■

We note that the preceding proposition complements the well known theorem of Kakutani [Di] that every uniformly convex Banach space has the Banach–Saks property.

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