STUDIA MATHEMATICA 168 (1) (2005)

Uniform convergence of N-dimensional Walsh–Fourier series

by

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Abstract. We establish conditions on the partial moduli of continuity which guarantee uniform convergence of the N-dimensional Walsh–Fourier series of functions f from the class $C_W(I^N) \cap \bigcap_{i=1}^N BV_{i,\{p(n)\}}$, where $p(n) \uparrow \infty$ as $n \to \infty$.

1. Definitions and notation. Let $I^N = [0,1)^N$ be the unit cube in the N-dimensional Euclidean space $\mathbb{R}^N$. The elements of $\mathbb{R}^N$ are denoted by $\mathbf{x} = (x_1, \ldots, x_N)$. For any $\mathbf{x} = (x_1, \ldots, x_N)$ and $\mathbf{y} = (y_1, \ldots, y_N)$ the vector $(x_1 \oplus y_1, \ldots, x_N \oplus y_N) \in \mathbb{R}^N$ is denoted by $\mathbf{x} \oplus \mathbf{y}$, where $\oplus$ denotes dyadic addition.

Let $M = \{1, \ldots, N\}$, $B = \{s_1, \ldots, s_r\}$, $B_1 = \{s_{r_1}, \ldots, s_{r_j}\}$, $s_k < s_{k+1}$, $s_{r_i} < s_{r_{i+1}}$, $k = 1, \ldots, r-1$, $i = 1, \ldots, j-1$, $B_1 \subset B \subset M$, $B' = M \setminus B$, $B_1' = M \setminus B_1$. For an integer n the vector $(n, \ldots, n) \in \mathbb{R}^N$ is denoted by $\widetilde{\mathbf{n}}$. The cardinality of B is denoted by $|B|$. For any $\mathbf{x} = (x_1, \ldots, x_N)$ and $B \subset M$, let $\mathbf{x}_B$ denote the vector in $\mathbb{R}^N$ whose coordinates with indices from B coincide with the corresponding coordinates of $\mathbf{x}$, and the coordinates with indices from B' are zero. Note that $\mathbf{x}_M = \mathbf{x}$ and $\mathbf{x}_\emptyset = \widetilde{\mathbf{0}}$.

For later convenience we introduce the following notation:

$$\sum_{\mathbf{i}_B = \mathbf{p}_B}^{\mathbf{m}_B} \quad \text{for} \quad \sum_{i_{s_1} = p_{s_1}}^{m_{s_1}} \cdots \sum_{i_{s_r} = p_{s_r}}^{m_{s_r}},$$

$$\frac{\mathbf{q}}{\mathbf{2^k}} \quad \text{for} \quad \left(\frac{q_1}{2^{k_1}}, \ldots, \frac{q_N}{2^{k_N}}\right),$$

$$d\mathbf{u} \quad \text{for} \quad du_1 \cdots du_N.$$

Denote by $C(I^N)$ the space of all real-valued functions continuous on I^N that can be extended to functions 1-periodic in each variable on $\mathbb{R}^N$. If $f \in$

2000 *Mathematics Subject Classification*: Primary 42C10.

Key words and phrases: Walsh function, N-dimensional Fourier series, uniform convergence.

$C(I^N)$ then the function

$$\omega_i(\delta, f) = \sup_{\mathbf{x}} \sup_{|h_i|\le\delta} |f(\mathbf{x}+\mathbf{h}_{\{i\}}) - f(\mathbf{x})|, \qquad i = 1,\dots,N,$$

is called a *partial modulus of continuity* of f.

Denote by $C_W(I^N)$ the space of all real-valued functions uniformly W-continuous on I^N that extend to functions 1-periodic in each variable, with the norm

$$\|f\|_{C_W} = \sup_{\mathbf{x}\in I^N} |f(\mathbf{x})|.$$

Let

$$\dot{\Delta}^{\{s_i\}}(f, \mathbf{x}, \mathbf{h}_{\{s_i\}}) = f(\mathbf{x}\oplus\mathbf{h}_{\{s_i\}}) - f(\mathbf{x}), \qquad i = 1,\dots,r.$$

Successive application of such partial difference operators leads to the definitions:

$$\dot{\Delta}^{B}(f, \mathbf{x}, \mathbf{h}_B) = \dot{\Delta}^{\{s_r\}}(\dot{\Delta}^{B\setminus\{s_r\}}(f, \cdot, \mathbf{h}_{B\setminus\{s_r\}}), \mathbf{x}, \mathbf{h}_{\{s_r\}}),$$

and

$$\dot{\omega}_B(\delta, f) = \sup_{0\le h_i<\delta_i,\, i\in B} \|\dot{\Delta}^B(f, \cdot, \mathbf{h}_B)\|_{C_W}.$$

DEFINITION 1. Suppose that the function f is bounded on I^N and extends to a function 1-periodic in each variable. Let $1 \le p < \infty$. We say that f is of *bounded partial p-variation* (written $f \in PBV_p$) if for any $i = 1,\dots,N$,

$$V_i(f) = \sup_{x_j, j\in M\setminus\{i\}} \sup_{n\ge1} \sup_{\pi^{(i)}} \sum_{k=0}^{n-1} |f(x_1,\dots,x_{i-1},x_i^{(2k)},x_{i+1},\dots,x_N) - f(x_1,\dots,x_{i-1},x_i^{(2k+1)},x_{i+1},\dots,x_N)|^p < \infty,$$

where $\pi^{(i)}$ is an arbitrary partition $0 \le x_i^{(0)} < x_i^{(1)} \le x_i^{(2)} < \dots \le x_i^{(2n-2)} < x_i^{(2n-1)} \le 1$.

Let f be a function defined on $\mathbb{R}^N$ which is 1-periodic relative to each variable. $\Pi^{(i)}$ is said to be a *partition with period* 1 if

$$\Pi^{(i)}: \ \dots < t_{-1}^{(i)} < t_0^{(i)} < t_1^{(i)} < \dots < t_{m_i}^{(i)} < t_{m_i+1}^{(i)} < \dots$$

satisfies $t_{k+m_i}^{(i)} = t_k^{(i)} + 1$ for $k \in \mathbb{Z}$, where m_i is a positive integer.

DEFINITION 2. Let $1 \le p(n) \uparrow p$ as $n \to \infty$, where $1 \le p \le \infty$. We say that a function f on $\mathbb{R}^N$ belongs to the class $BV_{i,\{p(n)\}}$ if

$$V_{i,\{p(n)\}}(f) = \sup_{x_s,\, s\in M\setminus\{i\}} \sup_{n\ge1} \sup_{\Pi^{(i)}} \left\{ \left(\sum_{j=1}^{m_i} |f(x_1,\dots,x_{i-1},t_j^{(i)},x_{i+1},\dots,x_N) - f(x_1,\dots,x_{i-1},t_{j-1}^{(i)},x_{i+1},\dots,x_N)|^{p(n)} \right)^{1/p(n)} : \varrho(\Pi^{(i)}) \ge \frac{1}{2^n} \right\} < \infty,$$

where

$$\varrho(\Pi^{(i)}) = \inf_k |t_k^{(i)} - t_{k-1}^{(i)}|.$$

For $N = 1$ see [7].

When $p(n) = p$ for all n, it is easy to see that $\bigcap_{i=1}^N BV_{i,\{p(n)\}}$ coincides with PBV_p.

Let r_0 be a function on $\mathbb{R}$ defined by

$$r_0(x) = \begin{cases} 1 & \text{if } x \in [0, 1/2), \\ -1 & \text{if } x \in [1/2, 1), \end{cases} \qquad r_0(x+1) = r_0(x).$$

The *Rademacher system* is defined by

$$r_n(x) = r_0(2^n x), \qquad n \geq 1,\ x \in [0, 1).$$

Let $w_0, w_1, \ldots$ represent the *Walsh functions*, i.e. $w_0(x) = 1$ and if $k = 2^{n_1} + \cdots + 2^{n_s}$ is a positive integer with $n_1 > \cdots > n_s$ then

$$w_k(x) = r_{n_1}(x) \cdots r_{n_s}(x).$$

The idea of using products of Rademacher functions to define the Walsh system comes from Paley [9].

The *Walsh–Dirichlet kernel* is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

The rectangular partial sums of the N-dimensional Walsh–Fourier series are defined as follows:

$$S_{\mathbf{m}}(f, \mathbf{x}) = \sum_{\boldsymbol{\nu}=\widetilde{\mathbf{0}}}^{\mathbf{m}-\widetilde{\mathbf{1}}} a_{\boldsymbol{\nu}} \prod_{i \in M} w_{\nu_i}(x_i),$$

where

$$a_{\boldsymbol{\nu}} = a_{\nu_1,\ldots,\nu_N}(f) = \int_{I^N} f(\mathbf{x}) \prod_{i \in M} w_{\nu_i}(x_i)\, d\mathbf{x}.$$

2. Formulation of the problem. Getzadze [1, 2] considered the question of uniform convergence for N-dimensional Walsh–Fourier series in terms of partial moduli of continuity. He proved the following

THEOREM A. (a) *Let $f \in C(I^N)$. If there exists $i_0 \in M$ such that*

$$\omega_{i_0}(\delta, f) = o\bigg(\bigg(\frac{1}{\log(1/\delta)}\bigg)^N\bigg) \qquad \text{as } \delta \to 0+$$

and

$$\omega_i(\delta, f) = O\bigg(\bigg(\frac{1}{\log(1/\delta)}\bigg)^N\bigg) \qquad \text{as } \delta \to 0+,\ 1 \leq i \leq N,\ i \neq i_0,$$

then the N-dimensional Walsh–Fourier series of f converges uniformly in the sense of Pringsheim (1).

(b) *There exists a function $f_0 \in C(I^N)$such that*

$$\omega_i(\delta, f_0) = O\left(\left(\frac{1}{\log(1/\delta)}\right)^N\right) \quad as\ \delta \to 0+,\ i = 1, \dots, N,$$

and the N-dimensional Walsh–Fourier cubic partial sums of f diverge in the metric of C.

In 1881 Jordan [6] introduced a class of functions of bounded variation and, applying it to the theory of trigonometric Fourier series, he proved that if a continuous function has bounded variation, then its trigonometric Fourier series converges uniformly. In 1906 G. Hardy [5] generalized the Jordan criterion to double Fourier series and introduced the notion of bounded variation for functions of two variables. He proved that if a continuous function of two variables has bounded variation (in the sense of Hardy), then its trigonometric Fourier series converges uniformly in the sense of Pringsheim.

Móricz [8] proved that if $f \in C_W(I^2)$ and the function f is of bounded variation in Hardy's sense [5], then its two-dimensional Walsh–Fourier series is uniformly convergent to f.

For N-dimensional Walsh–Fourier series the author [4] proved that if $f \in C_W(I^N)$ and the function f is of bounded partial p-variation ($f \in PBV_p$) for some $p \in [1, \infty)$, then the N-dimensional Walsh–Fourier series is uniformly convergent to f. The analogous result for the N-dimensional trigonometric Fourier series was verifed by the author [3].

On the basis of the above facts we can formulate the following problem:

Let $p(n) \uparrow \infty$ as $n \to \infty$ and $f \in C_W(I^N) \cap \bigcap_{i=1}^N BV_{i,\{p(n)\}}$. What conditions on the partial moduli of continuity ensure the uniform convergence in the Pringsheim sense of the N-dimensional Walsh–Fourier series of the function f?

A solution of this problem is given in Theorems 1 and 2.

3. Formulation of the main results. The main result of this paper is

THEOREM 1. *Let $p(n)\uparrow\infty$ as $n \to \infty$ and $f \in C_W(I^N) \cap \bigcap_{i=1}^N BV_{i,\{p(n)\}}$. If there exists $i_0 \in M$ such that*

$$\dot{\omega}_{\{i_0\}}(1/2^k, f) = o\left(\left(\frac{1}{p(k+1)\log p(k+1)}\right)^N\right) \quad as\ k \to \infty$$

(1) An N-dimensional series is said to converge in the sense of Pringsheim if its rectangular partial sums converge.

and

$$\dot\omega_{\{i\}}(1/2^k, f) = O\biggl(\biggl(\frac{1}{p(k+1)\log p(k+1)}\biggr)^N\biggr)$$
$$\text{as } k\to\infty,\ 1\le i\le N,\ i\ne i_0,$$

then the N-dimensional Walsh–Fourier series of f converges uniformly in Pringsheim's sense.

COROLLARY 1. *Let $p(n)\uparrow\infty$ as $n\to\infty$ and $f\in C(I^N)\cap\bigcap_{i=1}^N BV_{i,\{p(n)\}}$. If there exists $i_0\in M$ such that*

$$\omega_{i_0}(1/2^k, f) = o\biggl(\biggl(\frac{1}{p(k)\log p(k)}\biggr)^N\biggr) \quad \text{as } k\to\infty$$

and

$$\omega_i(1/2^k, f) = O\biggl(\biggl(\frac{1}{p(k)\log p(k)}\biggr)^N\biggr) \quad \text{as } k\to\infty,\ 1\le i\le N,\ i\ne i_0,$$

then the N-dimensional Walsh–Fourier series of f converges uniformly in Pringsheim's sense.

COROLLARY 2. *Let $p(n)\uparrow\infty$ as $n\to\infty$ and $p(2m)\le cp(m)$ for all $m\ge1$, where $c>0$ is a constant, and let $f\in C(I^N)\cap\bigcap_{i=1}^N BV_{i,\{p(n)\}}$. If there exists $i_0\in M$ such that*

$$\omega_{i_0}(\delta, f) = o\biggl(\biggl(\frac{1}{p([\log(1/\delta)])\log p([\log(1/\delta)])}\biggr)^N\biggr) \quad \text{as } \delta\to0+$$

and

$$\omega_i(\delta, f) = O\biggl(\biggl(\frac{1}{p([\log(1/\delta)])\log p([\log(1/\delta)])}\biggr)^N\biggr)$$
$$\text{as } \delta\to0+,\ 1\le i\le N,\ i\ne i_0,$$

then the N-dimensional Walsh–Fourier series of f converges uniformly in Pringsheim's sense.

THEOREM 2. *Let $p(n)\uparrow\infty$ and $p(n)\log p(n)=o(n)$ as $n\to\infty$, and $p(2m)\le cp(m)$ for all $m\ge1$, where $c>0$ is a constant. Then for any $N\ge2$ there exists a function $f_0\in C(I^N)\cap\bigcap_{i=1}^N BV_{i,\{p(n)\}}$ such that*

$$\omega_i(\delta, f_0) = O\biggl(\biggl(\frac{1}{p([\log(1/\delta)])\log p([\log(1/\delta)])}\biggr)^N\biggr)$$
$$\text{as } \delta\to0+,\ i=1,\dots,N,$$

and the N-dimensional Walsh–Fourier cubic partial sums of f_0 diverge at some point.

4. Auxiliary propositions. We shall need the following.

Lemma 1. *Let $f \in C_W(I^N)$. Assume that for any nonempty $B \subset M$ we have*

$$V_{\mathbf{k}_B}(f,\mathbf{u}) = \sum_{\mathbf{q}_B=\widetilde{\mathbf{1}}_B}^{(\mathbf{2^k}-\widetilde{\mathbf{1}})_B} \left|\dot{\Delta}^B\left(f,\mathbf{u}\oplus\left(\frac{2\mathbf{q}}{\mathbf{2}^{\mathbf{k}+\widetilde{\mathbf{1}}}}\right)_B,\left(\frac{\widetilde{\mathbf{1}}}{\mathbf{2}^{\mathbf{k}+\widetilde{\mathbf{1}}}}\right)_B\right)\right| \prod_{j\in B}\frac{1}{q_j} \to 0$$

(as $k_i \to \infty$) *uniformly with respect to $u_i, i \in M$. Then the N-dimensional Walsh–Fourier series of f converges uniformly in Pringsheim's sense.*

For $N = 2$ the proof can be found in [8]. Using the method of [8], we can easily extend this criterion to N-dimensional Walsh–Fourier series.

Lemma 2. *Let $a_{i_1}, \dots, a_{i_N}$ and $b_{i_1,\dots,i_N}$ be real numbers. Then*

$$\sum_{\mathbf{i}_M=\widetilde{\mathbf{1}}_M}^{\mathbf{m}_M}\Big(\prod_{j\in M} a_{i_j}\Big)b_{i_1,\dots,i_N} = \sum_{B\subset M}\Big(\prod_{j\in B'} a_{m_j}\Big)\sum_{\mathbf{i}_B=\widetilde{\mathbf{1}}_B}^{\mathbf{m}_B-\widetilde{\mathbf{1}}_B}\prod_{j\in B}(a_{i_j}-a_{i_j+1})$$
$$\times\sum_{\mathbf{k}_B=\widetilde{\mathbf{1}}_B}^{\mathbf{i}_B}\sum_{\mathbf{k}_{B'}=\widetilde{\mathbf{1}}_{B'}}^{\mathbf{m}_{B'}} b_{k_1,\dots,k_N}.$$

For $N = 1$ this is the well known Abel transformation, and for $N = 2$ it is called the Hardy transformation. The validity of the above equality for any $N \geq 3$ can be easily verified by induction.

Lemma 3. *We have*

$$\int_{2^{i-2n-3}}^{2^{i-2n-2}} |D_{q_n}(t)|\,dt \geq c > 0, \qquad i = 1,\dots,2n+2,$$

where

$$q_n = 2^{2n+1} + 2^{2n-1} + \dots + 2^3 + 2^1 + 2^0, \qquad n = 1,2,\dots.$$

The proof can be found in [10].

5. Proofs of the main results

Proof of Theorem 1. By Lemma 1, it suffices to show that for all nonempty $B \subset M$,

$$\sum_{\mathbf{q}_B=\widetilde{\mathbf{1}}_B}^{(\mathbf{2}^k-\widetilde{\mathbf{1}})_B} \left|\dot{\Delta}^B\left(f,\mathbf{u}\oplus\left(\frac{2\mathbf{q}}{\mathbf{2}^{\mathbf{k}+\widetilde{\mathbf{1}}}}\right)_B,\left(\frac{\widetilde{\mathbf{1}}}{\mathbf{2}^{\mathbf{k}+\widetilde{\mathbf{1}}}}\right)_B\right)\right| \prod_{j\in B}\frac{1}{q_j} \to 0$$

uniformly with respect to u_i, $i \in M$, as $k_i \to \infty$, $i \in B$.

From Lemma 2, we write

$$\sum_{\mathbf{q}_B=\widetilde{\mathbf{1}}_B}^{(\mathbf{2}^k-\widetilde{\mathbf{1}})_B}\left|\dot{\Delta}^B\left(f,\mathbf{u}\oplus\left(\frac{2\mathbf{q}}{\mathbf{2^{k+\widetilde{1}}}}\right)_B,\left(\frac{\widetilde{\mathbf{1}}}{\mathbf{2^{k+\widetilde{1}}}}\right)_B\right)\right|\prod_{j\in B}\frac{1}{q_j} \tag{1}$$

$$= \sum_{B_1\subset B,\, B_1\neq\emptyset}\left(\prod_{i\in B\setminus B_1}\frac{1}{2^{k_i}-1}\right)\sum_{\mathbf{q}_{B_1}=\widetilde{\mathbf{1}}_{B_1}}^{(\mathbf{2}^k-\widetilde{\mathbf{2}})_{B_1}}\prod_{j\in B_1}\left(\frac{1}{q_j}-\frac{1}{q_j+1}\right)$$

$$\times\sum_{\mathbf{l}_{B_1}=\widetilde{\mathbf{1}}_{B_1}}^{\mathbf{q}_{B_1}}\sum_{\mathbf{l}_{B\setminus B_1}=\widetilde{\mathbf{1}}_{B\setminus B_1}}^{(\mathbf{2}^k-\widetilde{\mathbf{1}})_{B\setminus B_1}}\left|\dot{\Delta}^B\left(f,\mathbf{u}\oplus\left(\frac{2\mathbf{l}}{\mathbf{2^{k+\widetilde{1}}}}\right)_B,\left(\frac{\widetilde{\mathbf{1}}}{\mathbf{2^{k+\widetilde{1}}}}\right)_B\right)\right|$$

$$+\prod_{i\in B}\frac{1}{2^{k_i}-1}\sum_{\mathbf{l}_B=\widetilde{\mathbf{1}}_B}^{(\mathbf{2}^k-\widetilde{\mathbf{1}})_B}\left|\dot{\Delta}^B\left(f,\mathbf{u}\oplus\left(\frac{2\mathbf{l}}{\mathbf{2^{k+\widetilde{1}}}}\right)_B,\left(\frac{\widetilde{\mathbf{1}}}{\mathbf{2^{k+\widetilde{1}}}}\right)_B\right)\right|$$

$$= \sum_{B_1\subset B,\, B_1\neq\emptyset} I_B(f,B_1,\mathbf{u})+I_B(f,\emptyset,\mathbf{u}).$$

Since for all nonempty $B\subset M$,

$$\left|\dot{\Delta}^B\left(f,\mathbf{u}\oplus\left(\frac{2\mathbf{l}}{\mathbf{2^{k+\widetilde{1}}}}\right)_B,\left(\frac{\widetilde{\mathbf{1}}}{\mathbf{2^{k+\widetilde{1}}}}\right)_B\right)\right|\leq\dot{\omega}_B\left(\frac{1}{2^k},f\right), \tag{2}$$

we have

$$I_B(f,\emptyset,\mathbf{u})=O\left(\dot{\omega}_B\left(\frac{1}{2^k},f\right)\right). \tag{3}$$

It is evident that

$$I_B(f,B_1,\mathbf{u})=O\Bigg(\sum_{\mathbf{q}_{B_1}=\widetilde{\mathbf{1}}_{B_1}}^{(\mathbf{2}^k-\widetilde{\mathbf{2}})_{B_1}}\prod_{j\in B_1}\frac{1}{q_j^2}\sup_{u_i,i\in B_1'}\sum_{\mathbf{l}_{B_1}=\widetilde{\mathbf{1}}_{B_1}}^{\mathbf{q}_{B_1}}\Bigg|\dot{\Delta}^{B_1}\Bigg(f,\mathbf{u} \tag{4}$$

$$\oplus\left(\frac{2\mathbf{l}}{\mathbf{2^{k+\widetilde{1}}}}\right)_{B_1},\left(\frac{\widetilde{\mathbf{1}}}{\mathbf{2^{k+\widetilde{1}}}}\right)_{B_1}\Bigg)\Bigg|\Bigg).$$

Since for all nonempty $B_1\subset M$, and all $j\in B_1$,

$$\sup_{u_i,\, i\in B_1'}\sum_{\mathbf{l}_{B_1}=\widetilde{\mathbf{1}}_{B_1}}^{\mathbf{q}_{B_1}}\left|\dot{\Delta}^{B_1}\left(f,\mathbf{u}\oplus\left(\frac{2\mathbf{l}}{\mathbf{2^{k+\widetilde{1}}}}\right)_{B_1},\left(\frac{\widetilde{\mathbf{1}}}{\mathbf{2^{k+\widetilde{1}}}}\right)_{B_1}\right)\right| \tag{5}$$

$$=O\left(\prod_{i\in B_1\setminus\{j\}}q_i\sup_{u_i,\, i\in M\setminus\{j\}}\sum_{l_j=1}^{q_j}\left|\dot{\Delta}^{\{j\}}\left(f,\mathbf{u},\oplus\left(\frac{2\mathbf{l}}{\mathbf{2^{k+\widetilde{1}}}}\right)_{\{j\}},\left(\frac{\widetilde{\mathbf{1}}}{\mathbf{2^{k+\widetilde{1}}}}\right)_{\{j\}}\right)\right|\right),$$

we have

$$\sup_{u_i,\, i\in B_1'} \sum_{\mathbf{l}_{B_1}=\widetilde{\mathbf{1}}_{B_1}}^{\mathbf{q}_{B_1}} \left| \dot{\Delta}^{B_1}\left(f, \mathbf{u} \oplus \left(\frac{2\mathbf{l}}{\mathbf{2}^{\mathbf{k}+\widetilde{\mathbf{1}}}}\right)_{B_1}, \left(\frac{\widetilde{\mathbf{1}}}{\mathbf{2}^{\mathbf{k}+\widetilde{\mathbf{1}}}}\right)_{B_1}\right)\right| \tag{6}$$

$$= \left[\left(\sup_{u_i,\, i\in B_1'} \sum_{\mathbf{l}_{B_1}=\widetilde{\mathbf{1}}_{B_1}}^{\mathbf{q}_{B_1}} \left| \dot{\Delta}^{B_1}\left(f, \mathbf{u} \oplus \left(\frac{2\mathbf{l}}{\mathbf{2}^{\mathbf{k}+\widetilde{\mathbf{1}}}}\right)_{B_1}, \left(\frac{\widetilde{\mathbf{1}}}{\mathbf{2}^{\mathbf{k}+\widetilde{\mathbf{1}}}}\right)_{B_1}\right)\right|\right)^{|B_1|}\right]^{1/|B_1|}$$

$$= O\left(\prod_{j\in B_1} q_j^{1-1/|B_1|}\left[\sup_{u_i,\, i\in M\setminus\{j\}} \sum_{l_j=1}^{q_j} \left| \dot{\Delta}^{\{j\}}\left(f, \mathbf{u} \oplus \left(\frac{2\mathbf{l}}{\mathbf{2}^{\mathbf{k}+\widetilde{\mathbf{1}}}}\right)_{\{j\}}, \left(\frac{\widetilde{\mathbf{1}}}{\mathbf{2}^{\mathbf{k}+\widetilde{\mathbf{1}}}}\right)_{\{j\}}\right)\right|\right]^{1/|B_1|}\right)$$

By (4) and (6) we obtain

$$I_B(f, B_1, \mathbf{u}) = O\left(\prod_{j\in B_1} \sum_{q_j=1}^{2^{k_i}-2} \frac{1}{q_j^{1+1/|B_1|}}\left[\sup_{u_i,\, i\in M\setminus\{j\}} \sum_{l_j=1}^{q_j} \left| \dot{\Delta}^{\{j\}}\left(f, \mathbf{u} \oplus \left(\frac{2\mathbf{l}}{\mathbf{2}^{\mathbf{k}+\widetilde{\mathbf{1}}}}\right)_{\{j\}}, \left(\frac{\widetilde{\mathbf{1}}}{\mathbf{2}^{\mathbf{k}+\widetilde{\mathbf{1}}}}\right)_{\{j\}}\right)\right|\right]^{1/|B_1|}\right). \tag{7}$$

Define

$$\chi(k_j, B_1) = 4^{|B_1| p(k_j+1) \log_2 p(k_j+1)}.$$

If we apply Hölder's inequality, from (7) we get

$$I_B(f, B_1, \mathbf{u}) = O\left(\prod_{j\in B_1} \left\{ \sum_{q_j=1}^{\chi(k_j,B_1)} \frac{1}{q_j^{1+1/|B_1|}}\left[\sup_{u_i,\, i\in M\setminus\{j\}} \sum_{l_j=1}^{q_j} \left| \dot{\Delta}^{\{j\}}\left(f, \mathbf{u} \oplus \left(\frac{2\mathbf{l}}{\mathbf{2}^{\mathbf{k}+\widetilde{\mathbf{1}}}}\right)_{\{j\}}, \left(\frac{\widetilde{\mathbf{1}}}{\mathbf{2}^{\mathbf{k}+\widetilde{\mathbf{1}}}}\right)_{\{j\}}\right)\right|\right]^{1/|B_1|} \right.\right.$$
$$\left.\left. + \sum_{q_j=\chi(k_j,B_1)+1}^{2^{k_j}-2} \frac{1}{q_j^{1+1/|B_1|}}\left[\sup_{u_i,\, i\in M\setminus\{j\}} \sum_{l_j=1}^{q_j} \left| \dot{\Delta}^{\{j\}}\left(f, \mathbf{u} \oplus \left(\frac{2\mathbf{l}}{\mathbf{2}^{\mathbf{k}+\widetilde{\mathbf{1}}}}\right)_{\{j\}}, \left(\frac{\widetilde{\mathbf{1}}}{\mathbf{2}^{\mathbf{k}+\widetilde{\mathbf{1}}}}\right)_{\{j\}}\right)\right|\right]^{1/|B_1|} \right\}\right) \tag{8}$$

$$= O\biggl(\prod_{j\in B_1}\biggl\{\Bigl(\dot{\omega}_{\{j\}}\Bigl(\frac{1}{2^{k_j}},f\Bigr)\Bigr)^{1/|B_1|}\log\chi(k_j,B_1)$$

$$+\sum_{q_j=\chi(k_j,B_1)+1}^{2^{k_j}-2}\frac{1}{q_j^{1+1/|B_1|}}\biggl(\biggl[\sup_{u_i,\,i\in M\setminus\{j\}}\sum_{l_j=1}^{q_j}\biggl|\dot{\Delta}^{\{j\}}\biggl(f,\mathbf{u}$$

$$\oplus\Bigl(\frac{2\mathbf{l}}{\mathbf{2^{k+\tilde{1}}}}\Bigr)_{\{j\}},\Bigl(\frac{\tilde{\mathbf{1}}}{\mathbf{2^{k+\tilde{1}}}}\Bigr)_{\{j\}}\biggr)\biggr|^{p(k_j+1)}\biggr]^{1/p(k_j+1)}q_j^{1-1/p(k_j+1)}\biggr)^{1/|B_1|}\biggr\}\biggr).$$

By (8) and the assumption of the theorem we obtain

$$I_B(f,B_1,\mathbf{u}) = O\biggl(\prod_{j\in B_1}\biggl\{\Bigl(\dot{\omega}_{\{j\}}\Bigl(\frac{1}{2^{k_j}},f\Bigr)\Bigr)^{1/|B_1|}\log\chi(k_j,B_1)$$
$$+\sum_{q_j=\chi(k_j,B_1)+1}^{2^{k_j}-2}\frac{(V_{j,\{p(n)\}}(f))^{1/|B_1|}}{q_j^{1+1/(|B_1|p(k_j+1))}}\biggr\}\biggr)$$
$$= O\biggl(\prod_{j\in B_1}\biggl\{\Bigl(\dot{\omega}_{\{j\}}\Bigl(\frac{1}{2^{k_j}},f\Bigr)\Bigr)^{1/|B_1|}\log\chi(k_j,B_1)$$
$$+p(k_j+1)\Bigl(\frac{1}{\chi(k_j,B_1)}\Bigr)^{1/(|B_1|p(k_j+1))}\biggr\}\biggr)$$
$$= O\biggl(\prod_{j\in B_1}\biggl\{\Bigl(\dot{\omega}_{\{j\}}\Bigl(\frac{1}{2^{k_j}},f\Bigr)\Bigr)^{1/|B_1|}p(k_j+1)\log p(k_j+1)+\frac{1}{p(k_j+1)}\biggr\}\biggr). \tag{9}$$

Let $B_1 = B = M$. Then by (9) and the assumption of the theorem we get

$$I_M(f,M,\mathbf{u}) = O\biggl(\prod_{j=1}^{N}\biggl\{\Bigl(\dot{\omega}_{\{j\}}\Bigl(\frac{1}{2^{k_j}},f\Bigr)\Bigr)^{1/N}p(k_j+1)\log p(k_j+1)$$
$$+\frac{1}{p(k_j+1)}\biggr\}\biggr) = o(1) \quad \text{as } k_j\to\infty,\ j\in M. \tag{10}$$

Let $B_1\subset B\subset M$ and $|B_1|<N$. Then by (9) and the assumption of the theorem we get

$$I_B(f,B_1,\mathbf{u}) \tag{11}$$
$$= O\biggl(\prod_{j\in B_1}\biggl\{\Bigl(\dot{\omega}_{\{j\}}\Bigl(\frac{1}{2^{k_j}},f\Bigr)\Bigr)^{1/N}p(k_j+1)\log p(k_j+1)$$
$$\times\Bigl(\dot{\omega}_{\{j\}}\Bigl(\frac{1}{2^{k_j}},f\Bigr)\Bigr)^{1/|B_1|-1/N}+\frac{1}{p(k_j+1)}\biggr\}\biggr) = o(1) \quad \text{as } k_j\to\infty,\ j\in M.$$

Owing to (1), (2), (10) and (11) the proof of the theorem is complete.

Proof of Theorem 2. Let $1 < p(l_1)\log p(l_1) \le 2l_1+2$. Define the following closed intervals:

$$E_{1,j} = \left[\frac{j}{2^{2l_1+2}}, \frac{j+1}{2^{2l_1+2}}\right], \qquad j = 1, \dots, 2^{[p(l_1)\log p(l_1)]} - 1.$$

Denote by $\varphi_{1,j}$ the function equal to zero outside this interval, 1 at its center and linear on each half-interval. Let

$$\varphi_1(x) = \sum_{j=1}^{2^{[p(l_1)\log p(l_1)]}-1} \varphi_{1,j}(x),$$

$$f_1(x) = \varphi_1(x)\operatorname{sgn} D_{q_{l_1}}(x), \qquad f_1(x+l) = f_1(x), \qquad l \in \mathbb{Z},$$

where

$$q_{l_1} = 2^{2l_1+1} + 2^{2l_1-1} + \cdots + 2^3 + 2^1 + 2^0.$$

Suppose that the integers $l_1, \dots, l_{k-1}$ and 1-periodic functions $f_1, \dots$ $\dots, f_{k-1}$ are already defined. Then we define l_k to be an integer with the following properties:

$$l_k > l_{k-1},$$

$$\frac{2^{[p(l_k)\log p(l_k)]}}{2^{2l_k+2}} \le \frac{1}{2^{2l_{k-1}+2}},$$

$$\frac{p(l_k)\log p(l_k)}{l_k} \le 1, \tag{12}$$

$$\sum_{s=1}^{k-1}\left(\frac{1}{p(l_s)\log p(l_s)}\right)^N \prod_{i=1}^{N}\left|\int_{[1/2^{2l_{k-1}+2},1]} f_s(x_i) \times w_{q_{l_k}-q_{l_{k-1}}}(x_i) D_{q_{l_{k-1}}+1}(x_i)\,dx_i\right| \le \frac{1}{k}, \tag{13}$$

where

$$q_{l_k} = 2^{2l_k+1} + 2^{2l_k-1} + \cdots + 2^3 + 2^1 + 2^0.$$

Define

$$E_{k,j} = \left[\frac{j}{2^{2l_k+2}}, \frac{j+1}{2^{2l_k+2}}\right], \qquad j = 1, \dots, 2^{[p(l_k)\log p(l_k)]} - 1.$$

Denote by $\varphi_{k,j}$ the function equal to zero outside this interval, 1 at its center and linear on each half-interval. Let

$$\varphi_k(x) = \sum_{j=1}^{2^{[p(l_k)\log p(l_k)]}-1} \varphi_{k,j}(x),$$

$$f_k(x) = \varphi_k(x)\operatorname{sgn} D_{q_{l_k}}(x), \qquad f_k(x+l) = f_k(x), \qquad l \in \mathbb{Z}.$$

Define

$$f_0(\mathbf{x}) = \sum_{k=1}^{\infty} g_k(\mathbf{x}), \qquad f_0(\widetilde{\mathbf{0}}) = 0,$$

where

$$g_k(\mathbf{x}) = \left(\frac{1}{p(l_k)\log p(l_k)}\right)^N \prod_{i=1}^{N} f_k(x_i).$$

It is evident that $f_0 \in C(I^N)$. First we prove that $f_0 \in BV_{i,\{p(n)\}}$, $i = 1, \dots, N$. Let $\Pi^{(i)} : \cdots < t_{-1}^{(i)} < t_0^{(i)} < t_1^{(i)} < \cdots < t_{m_i}^{(i)} < \cdots$ be any partition with period 1 and $\varrho(\Pi^{(i)}) \geq 1/2^n$. For $n \geq 2l_1 + 2$, we can choose integers l_{k-1} and l_k for which $2^{2l_{k-1}+2} \leq 2^n < 2^{2l_k+2}$. Then

$$p(2l_{k-1} + 2) \leq p(n) \leq p(2l_k + 2).$$

Let $s > k$. Then it is evident that

$$\begin{aligned} (14)\quad \Big(\sum_{j=1}^{m_i} &|g_s(x_1, \dots, x_{i-1}, t_j, x_{i+1}, \dots, x_N) \\ &- g_s(x_1, \dots, x_{i-1}, t_{j-1}, x_{i+1}, \dots, x_N)|^{p(n)}\Big)^{1/p(n)} \\ &\leq \left(\frac{1}{p(l_k)\log p(l_k)}\right)^N. \end{aligned}$$

Let now $s < k$. Then from the construction of the function f_0 we obtain

$$\begin{aligned} (15)\quad \Big(\sum_{j=1}^{m_i} &|g_s(x_1, \dots, x_{i-1}, t_j^{(i)}, x_{i+1}, \dots, x_N) \\ &- g_s(x_1, \dots, x_{i-1}, t_{j-1}^{(i)}, x_{i+1}, \dots, x_N)|^{p(n)}\Big)^{1/p(n)} \\ &= \left(\frac{1}{p(l_s)\log p(l_s)}\right)^N \Big(\sum_{j=1}^{m_i} |f_s(t_j^{(i)}) - f_s(t_{j-1}^{(i)})|^{p(n)}\Big)^{1/p(n)} \prod_{q \neq i} |f_s(x_q)| \\ &\leq \left(\frac{1}{p(l_s)\log p(l_s)}\right)^N \exp_2\left\{\frac{p(l_s)\log p(l_s)}{p(l_{k-1})}\right\} \\ &\leq \left(\frac{1}{p(l_s)\log p(l_s)}\right)^N p(l_s) < \infty. \end{aligned}$$

It is evident that

$$\begin{aligned} (16)\quad \Big(\sum_{j=1}^{m_i} &|g_k(x_1, \dots, x_{i-1}, t_j^{(i)}, x_{i+1}, \dots, x_N) \\ &- g_k(x_1, \dots, x_{i-1}, t_{j-1}^{(i)}, x_{i+1}, \dots, x_N)|^{p(n)}\Big)^{1/p(n)} \end{aligned}$$

$$= \left(\frac{1}{p(l_k)\log p(l_k)}\right)^N \Big(\sum_{j=1}^{m_i} |f_k(t_j^{(i)}) - f_k(t_{j-1}^{(i)})|^{p(n)}\Big)^{1/p(n)} \prod_{q\neq i} |f_k(x_q)|$$

$$\leq c\left(\frac{1}{p(l_k)\log p(l_k)}\right)^N \left(\frac{2^n}{2^{2l_k}} \exp_2\{p(l_k)\log p(l_k)\}\right)^{1/p(n)}.$$

Let $2l_{k-1}+2 \leq n < l_k+1$. Then from (12) we get

$$(17)\quad \frac{2^n}{2^{2l_k}} \exp_2\{p(l_k)\log p(l_k)\} = \frac{\exp_2\{n + p(l_k)\log p(l_k)\}}{2^{2l_k}} \leq \frac{\exp_2\{l_k + 1 + l_k\}}{2^{2l_k}} = 2.$$

Let now $l_k + 1 \leq n < 2l_k + 2$. Then we get

$$(18)\quad \left(\frac{2^n}{2^{2l_k}} \exp_2\{p(l_k)\log p(l_k)\}\right)^{1/p(n)} \leq 4\exp_2\left\{\frac{p(l_k)\log p(l_k)}{p(l_k+1)}\right\} \leq 4p(l_k).$$

From (16)–(18) we have

$$(19)\quad V_{i,\{p(n)\}}(g_k) < \infty.$$

Owing to (14), (15) and (19) we obtain $f_0 \in BV_{i,\{p(n)\}}$.

Next we shall prove that

$$(20)\quad \omega_i(\delta, f) = O\left(\left\{\frac{1}{p([\log(1/\delta)])\log p([\log(1/\delta)])}\right\}^N\right) \quad \text{as } \delta \to 0+,$$

for $i = 1, \ldots, N$.

Let $1/2^{2l_k} \leq h < 1/2^{2l_{k-1}}$. Then it is evident that

$$p(2l_{k-1}) \leq p([\log_2(1/h)]) \leq p(2l_k) \leq cp(l_k).$$

Let $s \geq k$. Then we get

$$(21)\quad |g_s(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_N) - g_s(x)| \leq \left(\frac{1}{p(l_s)\log p(l_s)}\right)^N$$

$$\leq \left(\frac{1}{p(l_k)\log p(l_k)}\right)^N = O\left(\left\{\frac{1}{p([\log(1/h)])\log p([\log(1/h)])}\right\}^N\right).$$

Let now $s < k$. Then from the assumption on f_s we obtain

$$(22)\quad |g_s(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_N) - g_s(x_1, \ldots, x_N)|$$

$$= \left(\frac{1}{p(l_s)\log p(l_s)}\right)^N |f_s(x_i + h) - f_s(x_i)| \prod_{q\neq i} |f_s(x_q)|$$

$$\leq c\,\frac{h2^{2l_s}}{(p(l_s)\log p(l_s))^N} = O\left(\left\{\frac{1}{p([\log(1/h)])\log p([\log(1/h)])}\right\}^N\right).$$

From (21) and (22) we obtain (20).

Finally, we show that the N-dimensional Walsh–Fourier series of f_0 diverges at $\widetilde{\mathbf{0}} = (0,\dots,0)$. Indeed,

$$
\begin{aligned}
(23)\qquad S_{q_{l_k},\dots,q_{l_k}}(f_0,\widetilde{\mathbf{0}}) - f_0(\widetilde{\mathbf{0}}) &= \int\limits_{[0,1]^N} f_0(\mathbf{u}) \prod_{i=1}^{N} D_{q_{l_k}}(u_i)\,d\mathbf{u} \\
&= \int\limits_{[0,2^{-2l_k-2}]^N} f_0(\mathbf{u}) \prod_{i=1}^{N} D_{q_{l_k}}(u_i)\,d\mathbf{u} \\
&\quad + \int\limits_{[2^{-2l_k-2},2^{-2l_{k-1}-2}]^N} f_0(\mathbf{u}) \prod_{i=1}^{N} D_{q_{l_k}}(u_i)\,d\mathbf{u} \\
&\quad + \int\limits_{[2^{-2l_{k-1}-2},1]^N} f_0(\mathbf{u}) \prod_{i=1}^{N} D_{q_{l_k}}(u_i)\,d\mathbf{u} \\
&= I + II + III.
\end{aligned}
$$

From the construction of f_0 we obtain

$$(24)\qquad |I| = o(1) \quad \text{as } k \to \infty.$$

Since

$$D_{2^n}(x) = \begin{cases} 2^n & \text{if } x \in [0,2^{-n}), \\ 0 & \text{if } x \in [2^{-n},1), \end{cases}$$

for $x \in [2^{-2l_{k-1}-2},1)$ we obtain

$$D_{q_{l_k}}(x) = w_{q_{l_k}-q_{l_{k-1}}}(x) D_{q_{l_{k-1}}+1}(x).$$

Then by (13) we get

$$(25)\qquad III = o(1) \quad \text{as } k \to \infty.$$

From the construction of f_0 we have

$$
\begin{aligned}
(26)\qquad |II| &= \Bigg| \int\limits_{[2^{-2l_k-2},2^{-2l_{k-1}-2}]^N} f_0(\mathbf{u}) \prod_{i=1}^{N} D_{q_{l_k}}(u_i)\,d\mathbf{u} \Bigg| \\
&= \left(\frac{1}{p(l_k)\log p(l_k)} \right)^N \prod_{i=1}^{N} \Bigg| \int\limits_{2^{-2l_k-2}}^{2^{-2l_{k-1}-2}} f_k(u_i) D_{q_{l_k}}(u_i)\,du_i \Bigg| \\
&= \left(\frac{1}{p(l_k)\log p(l_k)} \right)^N \prod_{i=1}^{N} \int\limits_{2^{-2l_k-2}}^{2^{[p(l_k)\log p(l_k)]-2l_k-2}} \varphi_k(u_i) |D_{q_{l_k}}(u_i)|\,du_i \\
&\ge c \left(\frac{1}{p(l_k)\log p(l_k)} \right)^N \prod_{i=1}^{N} \int\limits_{2^{-2l_k-2}}^{2^{[p(l_k)\log p(l_k)]-2l_k-2}} |D_{q_{l_k}}(u_i)|\,du_i.
\end{aligned}
$$

From Lemma 3 we obtain

$$\int_{2^{-2l_k-2}}^{2^{[p(l_k)\log p(l_k)]-2l_k-2}} |D_{q_{l_k}}(u_i)|\,du_i = \sum_{i=1}^{[p(l_k)\log p(l_k)]} \int_{2^{i-2l_k-3}}^{2^{i-2l_k-2}} |D_{q_{l_k}}(u_i)|\,du_i \geq cp(l_k)\log p(l_k). \tag{27}$$

Combining (26) and (27) we have

$$|II| \geq c\left(\frac{1}{p(l_k)\log p(l_k)}\right)^N (p(l_k)\log p(l_k))^N \geq c > 0. \tag{28}$$

Owing to (23), (24), (25) and (28) we obtain

$$\overline{\lim_{k\to\infty}} |S_{q_{l_k},\dots,q_{l_k}}(f_0,\widetilde{\mathbf{0}}) - f_0(\widetilde{\mathbf{0}})| = c > 0.$$

The proof of Theorem 2 is complete.

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Received June 27, 2000
Revised version January 4, 2005 (4552)