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On coefficients of vector-valued Bloch functions

by

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Abstract. Let X be a complex Banach space and let $\mathrm{Bloch}(X)$ denote the space of X-valued analytic functions on the unit disc such that $\sup_{|z|<1}(1-|z|^2)\|f'(z)\| < \infty$. A sequence $(T_n)_n$ of bounded operators between two Banach spaces X and Y is said to be an operator-valued multiplier between $\mathrm{Bloch}(X)$ and $\ell_1(Y)$ if the map $\sum_{n=0}^{\infty} x_n z^n \to (T_n(x_n))_n$ defines a bounded linear operator from $\mathrm{Bloch}(X)$ into $\ell_1(Y)$. It is shown that if X is a Hilbert space then $(T_n)_n$ is a multiplier from $\mathrm{Bloch}(X)$ into $\ell_1(Y)$ if and only if $\sup_k \sum_{n=2^k}^{2^{k+1}} \|T_n\|^2 < \infty$. Several results about Taylor coefficients of vector-valued Bloch functions depending on properties on X, such as Rademacher and Fourier type p, are presented.

1. Introduction. Throughout the paper X stands for a complex Banach space and we write $\mathrm{Bloch}(X)$ for the space of X-valued analytic functions on the unit disc such that $\|f\|_{\mathrm{Bloch}(X)} = \|f(0)\| + \sup_{|z|<1}(1-|z|^2)\|f'(z)\| < \infty$. We write Bloch instead of $\mathrm{Bloch}(\mathbb{C})$.

Clearly, $f \in \mathrm{Bloch}(X)$ if and only if $x^*f(z) = \langle f(z), x^*\rangle \in \mathrm{Bloch}$ for all $x^* \in X^*$ and $\|f\|_{\mathrm{Bloch}(X)} \approx \sup_{\|x^*\|=1} \|x^*f\|_{\mathrm{Bloch}}$.

For $1 \le p, q \le \infty$ we denote by $\ell(p,q,X)$ the spaces of sequences $(x_n)_n$ in X such that $(\|(\|x_n\|)_{n\in I_k}\|_{\ell_p})_k \in \ell_q$, where $I_k = \{n \in \mathbb{N} : 2^{k-1} \le n < 2^k\}$ for $k \in \mathbb{N}$ and $I_0 = \{0\}$. We write $\ell_p(X)$ for $\ell(p,p,X)$.

For $1 \le p, q \le \infty$ we write $\|(x_n)\|_{p,q} = \|(\|(\|x_n\|)_{n\in I_k}\|_{\ell_p})_k\|_{\ell_q}$. As usual, when $X = \mathbb{C}$ we simply write $\ell(p,q)$. These classes were first introduced for the scalar-valued case by C. N. Kellogg in [20].

Let us recall the following well known fact on Taylor coefficients of Bloch functions. There exist $C_1, C_2 > 0$ such that

$$C_1\|(x_n)\|_{\infty} \le \|f\|_{\mathrm{Bloch}(X)} \le C_2\|(x_n)\|_{1,\infty} \tag{1}$$

for any $f(z) = \sum_{n=0}^{\infty} x_n z^n$ with $x_n \in X$.

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Indeed, for each n and $r \in (0,1)$,

$$x_n r^n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{i\theta}) e^{-in\theta}\, d\theta.$$

Hence $n\|x_n\| r^{n-1} \leq \sup_{|z|=r} \|f'(z)\|$ for all $n \in \mathbb{N}$ and $0 < r < 1$. Now selecting $r = 1 - 1/n$ we obtain $\|(x_n)\|_\infty \leq C\|f\|_{\mathrm{Bloch}(X)}$. For the other inequality, observe that

$$\|f'(z)\| \leq \sum_k \sum_{n\in I_k} n\|x_n\|\, |z|^{n-1} \leq \|(x_n)_n\|_{1,\infty} \sum_k 2^k |z|^{2^k-1} \leq C\, \frac{\|(x_n)_n\|_{1,\infty}}{1-|z|}.$$

The reader is referred to [1, 2, 6] for the general theory of Bloch functions.

Let $1 \leq p, q < \infty$. It is easy to see that $(\ell(p,q,X))^* = \ell(p',q',X^*)$ for $1/p + 1/p' = 1/q + 1/q' = 1$, under the natural pairing

$$\langle (x_n), (x_n^*) \rangle = \sum_n \langle x_n, x_n^* \rangle \tag{2}$$

(where we also use $\langle \cdot, \cdot \rangle$ for the dual pairing in X). Due to the fact that we would like to identify the analytic functions with the sequences corresponding to their Taylor coefficients, it is convenient to find a predual of $\mathrm{Bloch}(X^*)$ under the previous pairing.

We shall be denoting by $J_1(X)$ the space of X-valued analytic functions f on the disc $\mathbb{D}$ such that $\int_0^1 M_1(f', r)\, dr < \infty$, where

$$M_p(f,r) = \left(\int_0^{2\pi} \|f(e^{it})\|^p \,\frac{dt}{2\pi} \right)^{1/p} \quad \text{for } 1 \leq p \leq \infty.$$

Endowing the space with the norm $\|f\|_{J_1(X)} = \|f(0)\| + \int_0^1 M_1(f', r)\, dr$ one gets $(J_1(X))^* = \mathrm{Bloch}(X^*)$ under the pairing

$$\langle f, g \rangle = \sum_{n=0}^{\infty} \langle x_n^*, x_n \rangle \tag{3}$$

for any $g(z) = \sum_{n=0}^\infty x_n^* z^n \in \mathrm{Bloch}(X^*)$ and $f(z) = \sum_{n=0}^\infty x_n z^n \in J_1(X)$.

The reader is referred to [1] for this duality result in the scalar-valued case and to [7, 8] for its vector-valued extension. Another predual can be obtained in terms of Bergman spaces, namely $(A_1(X))^* = \mathrm{Bloch}(X^*)$ (see [27, 5]), where $A_1(X)$ denotes the space of X-valued analytic functions f on the disc $\mathbb{D}$ such that $\int_{\mathbb{D}} \|f(z)\|\, dA(z) < \infty$ and $dA(z)$ stands for the normalized area measure on $\mathbb{D}$, although in this duality the pairing is different from (2).

Hence from (1) and (3) we can conclude that there exist $C_1, C_2 > 0$ such that

$$C_1 \|(x_n)\|_{\infty,1} \leq \|f\|_{J_1(X)} \leq C_2 \|(x_n)\|_1 \tag{4}$$

for any $f \in J_1(X)$ with Taylor coefficients (x_n).

Vector-valued Bloch functions have been used in different papers and for different reasons (see [3, 4, 7–12]). We refer the reader to [5, 13] for new results on the subject.

In this paper we shall deal with the vector-valued analogues of the following result on multipliers due to J. M. Anderson and A. L. Shields (see [2]):

$$(\text{Bloch}, \ell_1) = \ell(2,1), \tag{5}$$

where (Bloch, ℓ_1) stands for the space of sequences $\lambda = (\lambda_n)$ such that the operator $T_\lambda(f) = (\lambda_n \alpha_n)_n$ for $f(z) = \sum_n \alpha_n z^n$ is bounded from Bloch into ℓ_1.

As a consequence of (5) one gets the following improvement of (1): There exists a constant $C > 0$ such that

$$\|(\alpha_n)_n\|_{2,\infty} \le C\|\phi\|_{\text{Bloch}} \tag{6}$$

for any $\phi(z) = \sum_{n=0}^{\infty} \alpha_n z^n$. We first observe that (6) does not hold in the vector-valued situation. Note that if e_n stands for the canonical basis of c_0 then $f(z) = \sum_{n=1}^{\infty} e_n z^n = (z^n)_n$ is a bounded c_0-valued analytic function. In particular $f \in \text{Bloch}(c_0)$, and $(e_n) \notin \ell(p, \infty, c_0)$ for any $p < \infty$. Hence (5) does not hold for general Banach spaces.

The aim of this paper is to understand whether (6) and (5) have natural extensions to vector-valued functions and how their vector-valued analogues depend on some geometrical properties of the Banach space X.

PROBLEM 1. For which Banach spaces X does the following hold:

$$f(z) = \sum_{n=0}^{\infty} x_n z^n \in \text{Bloch}(X) \ \Rightarrow\ (x_n)_n \in \ell(2, \infty, X)? \tag{7}$$

Let us give the following definition.

DEFINITION 1.1. Let X be a complex Banach space. Define $\Lambda_{\text{Bloch},\ell_1}(X)$ as the space of scalar-valued sequences $\lambda = (\lambda_n)_n$ such that the operator $T_\lambda(f) = (\lambda_n x_n)_n$ for $f(z) = \sum_{n=0}^{\infty} x_n z^n$ is bounded from $\text{Bloch}(X)$ into $\ell_1(X)$.

Obviously, taking $f(z) = x\phi(z)$ where $x \in X$ and $\phi \in$ Bloch one gets $\Lambda_{\text{Bloch},\ell_1}(X) \subseteq (\text{Bloch}, \ell_1) = \ell(2,1)$.

A dual argument shows that, for $1 < p \le 2$, the inequality

$$\|(x_n)_n\|_{p',\infty} \le C\Big\|\sum x_n z^n\Big\|_{\text{Bloch}(X)}$$

is equivalent to

$$\ell(p,1) \subseteq \Lambda_{\text{Bloch},\ell_1}(X).$$

Hence Problem 1 can be rephrased as follows: For which Banach spaces X, $\Lambda_{\text{Bloch},\ell_1}(X) = \ell(2,1)$?

The example given after (6) shows that $\ell(p,1)$ in not contained in $\Lambda_{\mathrm{Bloch},\ell_1}(c_0)$ for any $p>1$. This actually leads to a more general question.

PROBLEM 2. Find $\Lambda_{\mathrm{Bloch},\ell_1}(X)$ for a given Banach space X.

Similar problems and descriptions for vector-valued Hardy and Bergman spaces were considered in previous papers by the author (see [4, 14, 15]).

Another possible generalization of (5) is to consider sequences of bounded operators $(T_n)_n$ in $\mathcal{L}(X,Y)$ between two Banach spaces X and Y and to define operator-valued multipliers. This approach for different spaces of analytic functions and multipliers can be found in [3, 4, 9, 10, 12, 13].

DEFINITION 1.2. A sequence $(T_n)_n$ in $\mathcal{L}(X,Y)$ is said to be a *multiplier between* Bloch(X) *and* $\ell_1(Y)$, written $(T_n)\in(\mathrm{Bloch}(X),\ell_1(Y))$, if $(T_n(x_n))_n$ belongs to $\ell_1(Y)$ whenever $f(z)=\sum_{n=0}^\infty x_nz^n$ belongs to Bloch(X). This is equivalent to the existence of a constant $C>0$ such that

$$\sum_{n=0}^{N}\|T_n(x_n)\|\le C\sup_{|z|<1}(1-|z|^2)\Big\|\sum_{n=1}^{N}nx_nz^{n-1}\Big\| \tag{8}$$

for any $N\in\mathbb{N}$ and $x_0,x_1,\dots,x_N\in X$.

The infimum of the constants C satisfying (8) is the *multiplier norm*, which coincides with the norm of $\Phi_T(\sum x_nz^n)=(T_n(x_n))$ as the operator from Bloch(X) and $\ell_1(Y)$.

We shall address in the paper some partial answers to the more general problem of finding conditions on the Banach spaces X and Y to have

$$(\mathrm{Bloch}(X),\ell_1(Y))=\ell(2,1,\mathcal{L}(X,Y)). \tag{9}$$

Let us now collect several definitions of properties of Banach spaces to be used in what follows.

DEFINITION 1.3. Let $1\le p\le 2\le q<\infty$ and let X be a complex Banach space. X is said to have *Fourier type* p if there exists a constant C such that

$$\Big(\sum_{n=-\infty}^{\infty}\|\hat f(n)\|^{p'}\Big)^{1/p'}\le C\|f\|_{L^p(\mathbb{T},X)} \tag{10}$$

for all functions $f\in L^p(\mathbb{T},X)$.

X is said to have *Rademacher type* p, resp. *Rademacher cotype* q, if there exists a constant C such that

$$\int_0^1\Big\|\sum_{j=1}^{n}x_jr_j(t)\Big\|\,dt\le C\Big(\sum_{j=1}^{n}\|x_j\|^p\Big)^{1/p},$$

resp.

$$\Big(\sum_{j=1}^{n}\|x_j\|^q\Big)^{1/q} \le C\int_0^1\Big\|\sum_{j=1}^{n}x_jr_j(t)\Big\|\,dt,$$

for any finite family $x_1,\dots,x_n$ of vectors in X where r_j stand for the Rademacher functions on $[0,1]$.

The notion of Fourier type was first introduced by J. Peetre ([23]) and we refer the reader to the survey [18] for a complete study and references about this property. We just mention here that X has Fourier type p if and only if X^* does. In particular, if X has Fourier type p then

$$\|f\|_{L^{p'}(\mathbb{T},X)} \le C\Big(\sum_{n=-\infty}^{\infty}\|\widehat{f}(n)\|^p\Big)^{1/p}. \tag{11}$$

The notions of Rademacher type and cotype were introduced by B. Maurey and G. Pisier (see [16, 22, 26]) and were shown to be rather important in Banach space theory. Let us simply recall that Fourier type p implies Rademacher type p and that if X^* has type p then X has cotype p'.

The main examples of spaces of Fourier type p are $L^r(\mu)$ for any $p\le r \le p'$ or interpolation spaces $[X_0,X_1]_\theta$ between any Banach space X_0 and any Hilbert space X_1 where $1/p = 1-\theta/2$.

Recall also that $L^r(\mu)$ has Rademacher type $\min\{p,2\}$ and Rademacher cotype $\max\{p,2\}$.

2. Taylor coefficients. We start by mentioning a couple of examples of vector-valued Bloch functions to be used later on.

Example 2.1 (see [13, Example 3.1]). Let $1\le p\le\infty$ and define $f_p : \mathbb{D}\to\ell_p$ by $f_p(z)=\sum_{n=1}^{\infty} n^{-1/p}e_nz^n$, where e_n stands for the canonical basis. Then $f_p\in \mathrm{Bloch}(\ell_p)$.

Note that $f_p(z)=\sum_{n=1}^{\infty}x_nz^n$ with $\|x_n\| = n^{-1/p}$ and that $(x_n)\in \ell(2,\infty,\ell_p)$ if and only if $p\ge 2$.

Example 2.2 (see [13, Example 3.2]). Let $1\le p<\infty$ and define $F_p : \mathbb{D}\to L^p(\mathbb{T})$ by $F_p(z)(\xi)=(1-\overline{\xi}z)^{-1/p}$. Then $F_p\in\mathrm{Bloch}(L^p(\mathbb{T}))$.

Note that $F_p(z)=\sum_{n=1}^{\infty}x_n'z^n$ with $\|x_n'\|\approx n^{-1/p'}$ and that $(x_n)\in \ell(2,\infty,L^p(\mathbb{T}))$ if and only if $p\le 2$.

These examples show that

$$\Lambda_{\mathrm{Bloch},\ell_1}(\ell_p)\subsetneq \ell(2,1) \quad \text{for } p<2, \qquad \Lambda_{\mathrm{Bloch},\ell_1}(L^p(\mathbb{T}))\subsetneq \ell(2,1) \quad \text{for } p>2.$$

We now show that (7) holds for Hilbert spaces. The proof that we shall present is based upon Grothendieck's inequality.

THEOREM 2.1. *Let H be a Hilbert space. Then there exists a constant $C>0$ such that*

$$\|(x_n)_n\|_{2,\infty} \le C\|f\|_{\mathrm{Bloch}(H)}$$

for all $f(z)=\sum_{n=0}^{\infty} x_n z^n \in \mathrm{Bloch}(H)$. Hence $\Lambda_{\mathrm{Bloch},\ell_1}(H)=\ell(2,1)$.

Proof. Given $f\in \mathrm{Bloch}(H)$ we start by defining $T_f : A_1 \to H$ by the formula $T_f(u_n)=x_n$, where $u_n(z)=(n+1)z^n$, and extending the definition to all polynomials by linearity. That is,

$$T_f(\phi)=\sum_n \frac{x_n\alpha_n}{n+1}=\int_{\mathbb{D}} \phi(\overline{z})f(z)\,dA(z)$$

for $\phi(z)=\sum_{n=0}^{N}\alpha_n z^n$.

Using the fact that

$$\langle \phi,\psi\rangle=\sum_{n=0}^{\infty}\frac{\alpha_n\overline{\beta}_n}{n+1}=\int_{\mathbb{D}}\phi(z)\overline{\psi(z)}\,dA(z), \tag{12}$$

for any $\phi(z)=\sum_{n=0}^{N}\alpha_n z^n$ and $\psi(z)=\sum_{n=0}^{\infty}\beta_n z^n$, gives the duality $(A_1)^*=$ Bloch (see [27]), together with the facts that $\langle T_f(\phi),x^*\rangle=\langle x^*f,\phi\rangle$ and polynomials are dense in A_1, we can continuously extend T_f to A_1 as a bounded operator and $\|T_f\|\le C\|f\|_{\mathrm{Bloch}(H)}$.

On the other hand it is known (see [26] or [27]) that A_1 is isomorphic to ℓ_1. Hence by invoking the Grothendieck theorem (see [16]) we deduce that T_f is absolutely summing.

Let $\|(\lambda_n)\|_{2,1}\le 1$. It follows from (5) that

$$\sup_{\|g\|_{(A_1)^*}\le 1}\sum_n |\langle \lambda_n u_n, g\rangle| \le C.$$

This leads to

$$\sum_n |\lambda_n|\,\|T(u_n)\|\le C$$

for all $\|(\lambda_n)\|_{2,1}\le 1$. Or in other words $(x_n)\in \ell(2,\infty,X)$ and

$$\|(x_n)_n\|_{2,\infty}\le C\|T_f\|=C\|f\|_{\mathrm{Bloch}(H)}. \blacksquare$$

We shall try to see how some geometrical properties of the space X help to describe $\Lambda_{\mathrm{Bloch},\ell_1}(X)$.

We first improve the estimates in (4) under some assumptions on the Banach space X. To do that we use the following lemma.

LEMMA 2.2 (see [11] or [21]). *Let (α_n) be a sequence of nonnegative numbers and $0<q,\beta<\infty$. Then*

$$\int_0^1 (1-r)^{\beta q-1}\Big(\sum_{n=1}^{\infty}\alpha_n r^n\Big)^q dr \approx \sum_{k=1}^{\infty}\Big(\sum_{n\in I_k}\frac{\alpha_n}{n^\beta}\Big)^q. \tag{13}$$

THEOREM 2.3. *Let $1 \le p \le 2$ and X be a Banach space of Fourier type p.*

(i) *There exists a constant $C > 0$ such that*

$$\|f\|_{J_1(X)} \le C\|(x_n)\|_{p,1}$$

for all $(x_n) \in \ell(p,1,X)$ and $f(z) = \sum_{n=1}^{\infty} x_n z^n$.

(ii) *There exists a constant $C > 0$ such that*

$$\|(x_n)\|_{p',\infty} \le C\|f\|_{\mathrm{Bloch}(X)}$$

for all $f(z) = \sum_{n=1}^{\infty} x_n z^n \in \mathrm{Bloch}(X)$.

Proof. (i) Note that, by (11),

$$\|f\|_{J_1(X)} \le \|f(0)\| + \int_0^1 M_{p'}(f', r)\, dr \le C\Big(\|f(0)\| + \int_0^1 \Big(\sum_n n^p \|x_n\|^p r^{np}\Big)^{1/p} dr\Big).$$

Now apply Lemma 2.2 for $\beta = p$ and $q = 1/p$ to get $\|f\|_{J_1(X)} \le C\|(x_n)\|_{p,1}$.

(ii) Using the fact that $\mathrm{Bloch}(X)$ is isometrically included in $(J_1(X^*))^*$ together with (i) and the fact that X^* also has Fourier type p one gets, for $f(z) = \sum_{n=1}^{\infty} x_n z^n$,

$$\begin{aligned}\|(x_n)\|_{p',\infty} &= \sup\Big\{\sum_n \langle x_n, x_n^*\rangle : \|(x_n^*)\|_{p,1} = 1\Big\}\\ &\le C \sup\{\langle f, g\rangle : \|g\|_{J_1(X^*)} = 1\} \le C\|f\|_{\mathrm{Bloch}(X)}. \ \blacksquare\end{aligned}$$

THEOREM 2.4. *Let $1 < p < 2$ and let X be a Banach space.*

(i) *If $\ell(p,1) \subseteq \Lambda_{\mathrm{Bloch},\ell_1}(X)$ then X has cotype p'.*

(ii) *If $\ell(2,1) = \Lambda_{\mathrm{Bloch},\ell_1}(X)$ then X has Orlicz property, i.e. there exists $C > 0$ so that $(\sum_n \|x_n\|^2)^{1/2} \le C \sup_{\|x^*\|=1} \sum_n |\langle x_n, x^*\rangle|$.*

Proof. We shall see in both cases that $\ell(p,1) \subseteq \Lambda_{\mathrm{Bloch},\ell_1}(X)$ implies that if $\sup_{\|x^*\|=1} \sum_n |\langle x_n, x^*\rangle| < \infty$ then $\sum_n \|x_n\|^{p'} < \infty$. This, in the case $p < 2$, is equivalent to X having cotype p' (see [24, 25]).

Let $x_1, \ldots, x_N \in X$ be such that $\sup_{\|x^*\|=1} \sum_{n=1}^{N} |\langle x_n, x^*\rangle| = 1$. Take k such that $2^{k-1} \le N < 2^k$ and construct $f(z) = \sum_{n=2^k+1}^{2^k+N} x_{n-2^k} z^n$. Hence f belongs to $\mathrm{Bloch}(X)$ (because $x^* f \in \mathrm{Bloch}$ for all $x^* \in X^*$). Therefore $\sum_{k=1}^{N} \|\lambda_n x_n\| \le C$ for all (λ_n) such that $\|(\lambda_n)_{n\in I_k}\|_p = 1$. Hence $\sum_{n=1}^{N} \|x_n\|^{p'} \le C$. $\blacksquare$

COROLLARY 2.5. *Let X be a Banach space and $1 \le p \le 2$. Then*

X has Fourier type $p \ \Rightarrow\ \ell(p,1) \subseteq \Lambda_{\mathrm{Bloch},\ell_1}(X) \ \Rightarrow\ X$ has cotype p'.

3. Multipliers. Now we analyze the interplay between geometry of Banach spaces and questions (7) and (9).

Repeating the argument in Theorem 2.4 with $T_n = \lambda_n T$ for a fixed operator T we obtain the following result.

PROPOSITION 3.1. *Let $1 \le p \le 2$ and let X and Y be Banach spaces. If*

$$\ell(p,1,\mathcal{L}(X,Y)) \subseteq (\mathrm{Bloch}(X), \ell_1(Y))$$

then $\Pi_{p',1}(X,Y)=\mathcal{L}(X,Y)$, where $\Pi_{p',1}(X,Y)$ stands for the space of $(p',1)$-summing operators (see [16]).

PROPOSITION 3.2. *Let X and Y be Banach spaces and assume that X has Fourier type p. Then*

$$\ell(p,1,\mathcal{L}(X,Y)) \subseteq (\mathrm{Bloch}(X), \ell_1(Y)).$$

Proof. This follows easily from Theorem 2.3, since

$$\sum_{n=1}^{\infty} \|T_n(x_n)\| \le \|(T_n)\|_{p,1} \|(x_n)\|_{p',\infty} \le C\|f\|_{\mathrm{Bloch}(X)}$$

for $f(z) = \sum_{n=1}^{\infty} x_n z^n$. ∎

PROPOSITION 3.3. *Let X^* be a complex Banach space of Rademacher cotype p' and Y be any Banach space. Then*

$$(\mathrm{Bloch}(X), \ell_1(Y)) \subset \ell(p',1,\mathcal{L}(X,Y)).$$

Proof. Let (T_n) be a sequence of operators in $(\mathrm{Bloch}(X), \ell_1(Y))$. Using a simple duality argument we get

$$\Big\| \sum_{n=1}^{\infty} \varepsilon_n T_n^*(y_n^*) z^n \Big\|_{J_1(X^*)} \le C$$

for all $\varepsilon_n \in \{-1,1\}$ and $\|y_n^*\| = 1$.

Now writing $\varepsilon_n = r_n(t)$ for $t \in [0,1]$ and $f_t(z) = \sum_{n=1}^{\infty} r_n(t) T_n^*(y_n^*) z^n$, we have

$$\begin{aligned}\int_0^1 \|f_t\|_{J_1(X^*)}\, dt &= \int_0^1\int_0^1 |M_1(f_t', r)|\, dr\, dt \\ &= \int_0^1\int_0^{2\pi}\int_0^1 \Big| \sum_{n=1}^{\infty} n r_n(t) T_n^*(y_n^*) r^{n-1} e^{i(n-1)\theta} \Big|\, dt\, \frac{d\theta}{2\pi}\, dr \\ &\ge C \int_0^1 \Big(\sum_n n^{p'} \|T_n^*(y_n^*)\|^{p'} r^{np'} \Big)^{1/p'} dr.\end{aligned}$$

Applying Lemma 2.2 for $\beta = p'$ and $q = 1/p'$, we obtain $(T_n^*(y_n^*)) \in \ell(p',1,X^*)$ uniformly for $\|y_n^*\| = 1$. Hence $(T_n) \in \ell(p',1,\mathcal{L}(X,Y))$. ∎

Combining now Propositions 3.2 and 3.3 we get our final corollary:

COROLLARY 3.4. *Let H be a Hilbert space and let Y be a Banach space. Then*

$$(\mathrm{Bloch}(H), \ell_1(Y)) = \ell(2, 1, \mathcal{L}(X, Y)).$$

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