STUDIA MATHEMATICA 165 (2) (2004)

# A reaction-diffusion equation on a net-shaped thin domain

by

THOMAS ELSKEN (Rostock)

**Abstract.** Let $\Omega_\varepsilon \subset \mathbb{R}^{M+1}$, $0 < \varepsilon \le 1$, be a net-shaped Lipschitz domain which collapses to a one-dimensional net as $\varepsilon \downarrow 0$. On $\Omega_\varepsilon$ we consider the equation $u_t = \Delta u$ with von Neumann boundary conditions. We show under quite general conditions that the semiflows generated by this equation have a limit in a strong sense, the limit semiflow being generated by an abstract linear operator. Also, under an additional assumption, the eigenvalues and eigenfunctions of the corresponding operators converge. This allows us to apply the techniques in [14] to prove the convergence of the nonlinear semiflows generated by a reaction-diffusion equation on $\Omega_\varepsilon$ and the upper-semicontinuity of their attractors at $\varepsilon = 0$. Our technique also allows us to treat the case that $\Omega_\varepsilon$ is smooth and has holes which vanish of order at least $\varepsilon$ in all directions.

**1. Introduction.** Assume having a reaction-diffusion equation on a domain $\Omega_\varepsilon$ depending on a parameter $\varepsilon$. As $\varepsilon \to 0$, $\Omega_\varepsilon \subset \mathbb{R}^{N_x+N_y}$ collapses to a lower-dimensional set giving rise to a singular perturbation problem. Of particular interest is the behavior of the semiflows given by the reaction-diffusion equation in the limit, and given that these flows have attractors, how they behave in the limit.

Consider the reaction-diffusion equation

$$\begin{aligned} u_t(x,y) &= \Delta u(x,y) + f(u(x,y)), &\quad (x,y) \in \Omega_\varepsilon,\ t>0,\\ \partial_\nu u(x,y) &= 0, &\quad (x,y) \in \partial\Omega_\varepsilon,\ t>0, \end{aligned} \tag{1.1}$$

where $f$ is a nonlinearity with a suitable growth and dissipative condition, $\nu$ is the outer normal at $(x,y) \in \partial\Omega_\varepsilon$, and $\Omega_\varepsilon$ collapses to a lower-dimensional set, often a one-dimensional one.

One of the first to investigate this problem were Hale and Raugel [9]. Their domain $\Omega_\varepsilon$ is the ordinate set under a function, and they prove the existence of a limit flow and—in some sense—the upper-semicontinuity of their attractors.

M. Prizzi and K. P. Rybakowski generalized Hale and Raugel's result in [14] by squeezing a general Lipschitz domain $\Omega \subset \mathbb{R}^{N_x+N_y}$, which e.g. may

2000 *Mathematics Subject Classification*: 35B25, 35B41, 35K57, 35P99.

*Key words and phrases*: reaction-diffusion equations, thin net-shaped domain.

have holes or multiple branches. The corresponding limit equation is an abstract parabolic equation defined on a subspace $H^1_s(\Omega)$ of $H^1(\Omega)$. For a wide class of domains $\Omega \subset \mathbb{R}^2$ (so-called nicely decomposable domains) they described the limit problem explicitly. It is a system of second order differential equations on a graph, coupled by a compatibility condition and a Kirchhoff type balance condition. Under certain natural conditions on the nonlinearity $f$ they also proved for a general Lipschitz domain in $\mathbb{R}^{N_x+N_y}$ the existence of the limit semiflow in a strong sense, and the upper-semicontinuity of the family of attractors $\widetilde{\mathcal{A}}_\varepsilon$. In a second paper [15] they show these attractors to be contained in inertial manifolds of finite dimension.

There is a variety of generalizations in various ways. To mention a few: F. Antoci and M. Prizzi [2] consider an unbounded domain. For T. Elsken [5] the linear operator in (1.1) is not the Laplace operator but a general strongly elliptic one, which may have asymmetrical boundary conditions. In [13] M. Prizzi, M. Rinaldi and K. P. Rybakowski treat the case that $\Omega_\varepsilon$ contracts to a smooth curve. Hale and Raugel [10] consider an L-shaped domain, and Q. Fang [7] a thin tubular one. Kosugi [12] treats the corresponding elliptic equation on a net-shaped smooth domain. He also proves the existence of solutions for $\varepsilon > 0$ converging to a given solution of the limiting problem. Saito [17] characterizes the limit of the Laplacian for a domain which shrinks to a tree. Rubinstein and Schatzmann [16] show for a similar domain the convergence of the $n$th eigenvalue of the Laplacian to the $n$th eigenvalue of the limiting problem.

In this paper we extend some of the results of [13], [10], [17], [12] and [16] to more general domains. In particular we show that the L-shaped domains considered by Hale and Raugel [10] are net-shaped in our sense (see Example 2.1), but we also explicitly allow holes and multiple branches in them. On the other hand our convergence is slightly weaker than, say, in [10].

Under additional smoothness assumptions the domains can even have a finite number of holes which decrease in all directions of order $\varepsilon$ or less. To our knowledge this case has not been treated yet.

We assume $\Omega_\varepsilon \subset \mathbb{R}^{M+1}$ (i.e. $N_x = 1,\ N_y = M$) to be only Lipschitz, bounded and to consist of $K_E$ edges and $K_N$ nodes, $K_E, K_N \in \mathbb{N}$, all of which may have holes or multiple branches. The edges become smooth curves and the nodes points, as $\varepsilon \to 0$ (see Section 2 for the exact requirements on $\Omega_\varepsilon$). We prove that the semiflow generated by (1.1) (with $f \equiv 0$) converges in a strong sense to the semiflow generated by an abstract (linear) equation (see Theorem 1.1).

Given natural growth and dissipativity conditions on the nonlinearity $f$, the nonlinear semiflows exist (locally) and converge (see Theorem 1.3). In general one cannot expect the upper-semicontinuity of attractors as e.g.

in [14] (see Remark 2.1), but with an additional condition on $\Omega_\varepsilon$ one still gets the same result (see Theorem 1.4).

The conditions mentioned above are abstract ones. We also give some sufficient conditions which are easier to prove. One of these is that the edges connect nicely at the node, which is similar to the definition of nicely decomposed domains in [14]. This condition is also needed in Proposition 3.2 to describe explicitly the limit operator $A_0$ at a node. Just as for nicely decomposed domains it, is a continuity condition—roughly speaking the values at the end of edges which connect have to be equal—and a Kirchhoff type balance condition. We do not describe $A_0$ explicitly on the edges since this is almost like the description of $A_0$ for nicely decomposed domains in [14].

We also give an example of a domain which has holes which disappear of order $\varepsilon$ in all directions (see Example 3.2). This implies that under additional assumptions on the smoothness of $\Omega_\varepsilon$, the limiting problem is unperturbed upon changing $\Omega_\varepsilon$ by introducing finitely many small holes. Since even the domains considered in e.g. [14] can be viewed as net-shaped (one edge, no node), this generalizes the results of the afore-mentioned paper, allowing not only holes which contract in $y$-direction, but also in $x$-direction.

For notational simplicity, we restrict ourselves to an example having three edges meeting in one node, i.e. $K_E = 3$, $K_N = 1$. The case of $K_E$, $K_N$ arbitrary is a straightforward generalization of the case presented here.

Dividing $\Omega_\varepsilon$ into four parts: three edges $\Omega_{j,\varepsilon}$, $j = 1, 2, 3$, and the node $\Omega_{4,\varepsilon}$, we make a transformation from each edge onto a fixed domain $G_j$, $j = 1, 2, 3$, and expand the node to get $G_{4,\varepsilon}$. Thus $L^2(\Omega_\varepsilon)$ and $H^1(\Omega_\varepsilon)$ become $L^2_\varepsilon \subset L^2(G_1) \times L^2(G_2) \times L^2(G_3) \times L^2(G_{4,\varepsilon})$ and $H^1_\varepsilon \subset H^1(G_1) \times H^1(G_2) \times H^1(G_3) \times H^1(G_{4,\varepsilon})$ (see (2.3) and (2.4) for the definitions of $L^2_\varepsilon$ and $H^1_\varepsilon$). Note that we do not suppose that there is a transformation from the node to some fixed domain. In that sense the conditions on the node are very weak.

We write (1.1) as an abstract equation

$$[u_t] = -A_\varepsilon[u], \qquad t > 0, \tag{1.2}$$

where, as usual, $A_\varepsilon$ is defined via a bilinear form $a_\varepsilon$ (see (2.7)).

The "limiting equation" will be shown to be

$$[u_t] = -A_0[u], \qquad t > 0, \tag{1.3}$$

where the abstract linear operator $A_0 : D(A_0) \subset H^1_s \to L^2_s$ is defined by a bilinear form $a_0 : H^1_s \times H^1_s \to \mathbb{R}$ (see (2.8)). Here $L^2_s = L^2_s(G_1) \times L^2_s(G_2) \times L^2_s(G_3)$ and $H^1_s \subset H^1_s(G_1) \times H^1_s(G_2) \times H^1_s(G_3)$ with some boundary conditions depending on the node $G_{4,\varepsilon}$. As usual, $H^1_s(\Omega)$ denotes the space of functions on $\Omega \subset \mathbb{R} \times \mathbb{R}^M$ with derivative 0 in $y$-direction, and $L^2_s(\Omega)$ is the closure of $H^1_s(\Omega)$ in $L^2(\Omega)$ (see condition (C7) and Lemma 2.5).

Comparing the semiflows generated by (1.2) and (1.3) we have the difficulty that $e^{A_\varepsilon t}$, $\varepsilon \geq 0$, live on distinct spaces. So we embed $L^2_s$ and $H^1_s$ in $L^2_\varepsilon$ and $H^1_\varepsilon$ by continuous linear maps $\Phi^L_\varepsilon$ and $\Phi^H_\varepsilon$, respectively (see Lemma 2.6 and condition (C7); roughly speaking, both maps are the identity on each edge, $\Phi^L_\varepsilon$ is identically 0 and $\Phi^H_\varepsilon$ is small at the node).

For the convergence of the semigroups we need equivalent norms $\|\cdot\|_{\varepsilon,d}$, $0 \leq d \leq 1$, on $H^1_\varepsilon$ (see (2.2); roughly $\|\cdot\|_{\varepsilon,d}$ is the $H^1$-norm on the edges with the derivatives in $y$-direction being weighted by $\varepsilon^{-d}$).

We now state the central results of this article although the exact definitions will be presented in Section 2 as they are rather lengthy.

Theorem 1.1. *Assume $\Omega_\varepsilon$ satisfies conditions* (C1)–(C7) *of Section* 2. *Let $\varepsilon_n \downarrow 0$, $[u_n] \in L^2_{\varepsilon_n}$, $[u_0] \in L^2_s$ and assume $\|[u_n] - \Phi^L_{\varepsilon_n}[u_0]\|_{L^2_{\varepsilon_n}} \to 0$ as $n \to \infty$. Then $(\|e^{-A_{\varepsilon_n}t}[u_n] - \Phi^H_{\varepsilon_n}(e^{-A_0 t}[u_0])\|_{\varepsilon_n,1})_n$ is bounded uniformly on $[t_1, t_2]$ and for $0 \leq d < 1$,*

$$\|e^{-A_{\varepsilon_n}t}[u_n] - \Phi^H_{\varepsilon_n}(e^{-A_0 t}[u_0])\|_{\varepsilon_n,d} \to 0, \qquad n \to \infty,$$

*uniformly on $[t_1, t_2]$, for all $0 < t_1 < t_2 < \infty$.*

Theorem 1.2. *Assume $\Omega_\varepsilon$ satisfies conditions* (C1)–(C8) *of Section* 2. *Denote by $\lambda_{\varepsilon,l}$ and $\lambda_{0,l}$ the eigenvalues of $A_\varepsilon$ and $A_0$, respectively. Assume the eigenvalues to be ordered as $0 \leq \lambda_{\varepsilon,1} \leq \lambda_{\varepsilon,2} \leq \ldots$, $\varepsilon \geq 0$, and denote by $[u_{\varepsilon,l}] \in H^1_\varepsilon$ the corresponding eigenvectors which form a complete ONS of $L^2_\varepsilon$, $\varepsilon > 0$. If $\varepsilon_n \to 0$ then $\lambda_{\varepsilon_n,l} \to \lambda_{0,l}$ for all $l \in \mathbb{N}$. There is a subsequence, also called $\varepsilon_n$, and a complete ONS $([u_{0,l}])_l$ of $L^2_s$ consisting of eigenvectors belonging to $\lambda_{0,l}$ such that $\|[u_{\varepsilon_n,l}] - \Phi^H_{\varepsilon_n}[u_{0,l}]\|_{\varepsilon_n,d} \to 0$ as $n \to \infty$, for all $0 \leq d < 1$.*

The proofs of Theorems 1.1 and 1.2 will be given in Section 3. They show that these theorems also hold for any domain having $K_E$ edges and $K_N$ nodes, $K_E, K_N \in \mathbb{N}$ arbitrary.

The growth and dissipativeness conditions imposed on the nonlinearity $f$ are:

(H1) $f : \mathbb{R} \to \mathbb{R}$ is $C^1$, $|f'(s)| \leq C(|s|^\beta + 1)$ for all $s \in \mathbb{R}$, where $C, \beta \geq 0$ are some constants; if $M > 1$, then additionally $\beta \leq p^*/2 - 1$, where $p^* = 2(M+1)/(M-1) > 2$.

(H2) $\limsup_{|s|\to\infty} f(s)/s \leq -\xi$ for some $\xi > 0$.

Theorem 1.3. *Assume $\Omega_\varepsilon$ satisfies conditions* (C1)–(C6) *of Section* 2 *and* (C9), (C10) *of Section* 3. *Let $\varepsilon_n \downarrow 0$, $[u_n] \in H^1_{\varepsilon_n}$, $[u_0] \in H^1_s$, and $\|[u_n] - \Phi^L_{\varepsilon_n}([u_0])\|_{L^2_{\varepsilon_n}} \to 0$ as $n \to \infty$. Assume also that $f$ satisfies condition* (H1). *Then* (1.1) *generates a* (*local*) *semiflow, called $\pi_n$, on $H^1_{\varepsilon_n}$, and*

$$[u_t] = -A_0[u] + f([u]) \tag{1.4}$$

*generates a (local) semiflow, called* $\pi_0$, *on* $H^1_s$. *Assume that all these semiflows exist for* $0 \le t \le T$ *and some* $T > 0$, *and satisfy* $\|[u_n]\pi_{\varepsilon_n} t\|_{\varepsilon_n,1} \le C$, $0 \le t \le T$, $n \in \mathbb{N}$, *for some constant* $C > 0$.

*If* $t_0 \in \,]0, T[$, *then* $(\|[u_n]\pi_n t_0 - \Phi^H_{\varepsilon_n}([u_0]\pi_0 t_0)\|_{\varepsilon_n,1})_n$ *is bounded, and for* $0 \le d < 1$, $t_n \in \,]0, T[$, $t_n \to t_0$ *as* $n \to \infty$,

$$\|[u_n]\pi_n t_n - \Phi^H_{\varepsilon_n}([u_0]\pi_0 t_0)\|_{\varepsilon_n,d} \to 0, \qquad n \to \infty. \ \blacksquare$$

THEOREM 1.4. *Assume* $\Omega_\varepsilon$ *satisfies conditions* (C1)–(C6) *of Section* 2, (C9), (C10) *of Section* 3, *and f satisfies* (H1), (H2). *Then the semiflows generated by* (1.1) *and* (1.4) *are global ones, and they have attractors* $\mathcal{A}_\varepsilon \subset H^1_\varepsilon$, $\mathcal{A}_0 \subset H^1_s$ *consisting of all full bounded solutions on* $H^1_\varepsilon$ *and* $H^1_s$ *which attract every bounded set* $B \subset H^1_\varepsilon$ *and* $B \subset H^1_s$, *respectively. The family of attractors is upper-semicontinuous at* $\varepsilon = 0$, *i.e.*

$$\lim_{\varepsilon \downarrow 0} \sup_{[u] \in \mathcal{A}_\varepsilon} \inf_{[v] \in \mathcal{A}_0} \|[u] - \Phi^H_\varepsilon([v])\|_{\varepsilon,d} = 0$$

*for all* $0 \le d < 1$. $\blacksquare$

REMARK 1.1. In Theorems 1.1–1.4 above the convergence is always in $\|\cdot\|_{\varepsilon,d}$, that is, the derivatives in $y$-direction are weighted by $\varepsilon^{-d}$, $d < 1$. In other papers (e.g. [10], [14]) the convergence is in $\|\cdot\|_{\varepsilon,1}$, i.e. $\varepsilon^{-1}\|D_y \cdot\|_{L^2} \to 0$ as $\varepsilon \to 0$. This is not true here. The $y$-derivative divided by $\varepsilon$ is bounded in $L^2$, and may even converge in $L^2$, but the limit in general is not 0 (see e.g. Lemma 2.13 where the limit of a resolvent is given explicitly). $\blacksquare$

We shall not prove Theorems 1.3 and 1.4. The proofs are obvious adaptations of those in [14]. See also Remark 2.1 concerning Theorem 1.4.

The conditions posed in Section 2 do not allow loops because they cannot be mapped by a diffeomorphism onto a fixed domain. One can either change this condition, for example allowing the loop to be mapped by two diffeomorphisms onto two halves of a fixed domain, or by artificially introducing a node into the loop, thus creating a domain having one node and one edge more (see Example 3.1).

This article is organized as follows: in Section 2 we define the basic domains $\Omega_\varepsilon$ in an abstract way, present our notations and basic definitions and prove Theorems 1.1 and 1.2. In Section 3 we present sufficient simple conditions under which the abstract ones of the previous section hold. We characterize the abstract operator $A_0$ at the nodes and give examples of how introducing a new node one can cut for example a loop or allow very small holes.

**2. The general case.** In the rest of this paper $\varepsilon$ will always—unless stated otherwise—denote a number in $]0, 1]$.

$M \in \mathbb{N}$ is a fixed positive natural number. We will write $(x, y)$ for a generic point in $\mathbb{R} \times \mathbb{R}^M = \mathbb{R}^{M+1}$. Let $U \subset \mathbb{R}^{M+1}$. Then $\mathrm{proj}_x(U)$ and $\mathrm{proj}_y(U)$ are the projections onto the first coordinate and the last $M$ coordinates, respectively.

As in [14], [2], [5] and other papers, here also the set of functions on an open set $\Omega \subset \mathbb{R}^{M+1}$ which have derivative 0 in $y$-direction plays an important rôle. We define

$$H_s^1(\Omega) := \{u \in H^1(\Omega) \mid D_y u = 0\}, \qquad L_s^2(\Omega) := \overline{H_s^1}^{L^2(\Omega)}(\Omega).$$

Since $L_s^2(\Omega)$ is a closed subset of $L^2(\Omega)$, its orthogonal complement exists. Denote it by $L_\perp^2(\Omega)$.

For $n \in \mathbb{N}$ we denote by $E_n \in \mathbb{R}^{n \times n}$ the unit matrix and for a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes the Euclidian norm.

Let $V$ be a normed space, $z \in V$ and $\delta > 0$. Then $B_\delta(z) \subset V$ denotes the open ball around $z$ with radius $\delta$. Analogously define $B_\delta(U)$ for a set $U \subset V$.

If $U \subset \mathbb{R}^n$ then $|U|$ is the Lebesgue measure of $U$. The closure will be denoted by $\overline{U}$.

For a domain $\Omega \subset \mathbb{R}^{M+1}$ and $x \in \mathrm{proj}_x(\Omega)$, $(\Omega)_x := (\{x\} \times \mathbb{R}^M) \cap \Omega$ denotes the $x$-section. If $x \in \partial(\mathrm{proj}_x(\Omega))$, then $(\Omega)_x$ denotes the interior (as a set in $\mathbb{R}^M$) of $(\{x\} \times \mathbb{R}^M) \cap \overline{\Omega}$. The restriction $u|_{(\Omega)_x}$ of $u \in H^1(\Omega)$ is always understood in the sense of traces.

The letter $\chi$ will always denote a $C^\infty$ cut-off function with $\chi(x) \equiv 0$ for $x \le 1/2$ and $\chi(x) \equiv 1$ for $x \ge 1$.

In the proofs we shall often substitute an index $\varepsilon_n$ by the simpler $n$. For example $A_{\varepsilon_n}$, $H^1_{\varepsilon_n}$ and $\|\cdot\|_{\varepsilon_n,d}$ will be $A_n$, $H_n^1$ and $\|\cdot\|_{n,d}$.

We start by defining the domain $\Omega_\varepsilon$ which, as already mentioned, will be net-shaped and consist of one node and three edges. More precisely we assume $\Omega_\varepsilon \subset \mathbb{R}^{M+1}$ to be bounded, connected and Lipschitz. Set $\Omega_\varepsilon = \bigcup_{j=1}^3 \Omega_{j,\varepsilon} \cup \Omega_{4,\varepsilon}$, where the $\Omega_{j,\varepsilon}$ are mutually disjoint and satisfy the following:

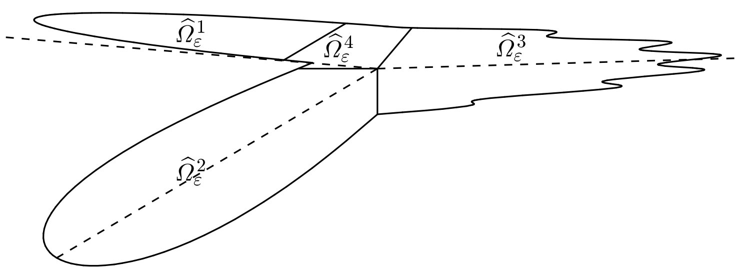


Fig. 1. An example of $\Omega_\varepsilon$. The dashed lines indicate the domain in the limit. Note that to $\Omega_{4,\varepsilon}$ belongs also the vertical line below the center.

The edges $\Omega_{j,\varepsilon}$, $j = 1, 2, 3$, have a description

$$\Omega_{j,\varepsilon} = \Psi_{\varepsilon,j}(G_j),$$

where $G_j \subset \mathbb{R} \times \mathbb{R}^M$ is open, bounded, connected and Lipschitz. To facilitate notation we assume $\mathrm{proj}_x(G_j) = ]0, 1[$.

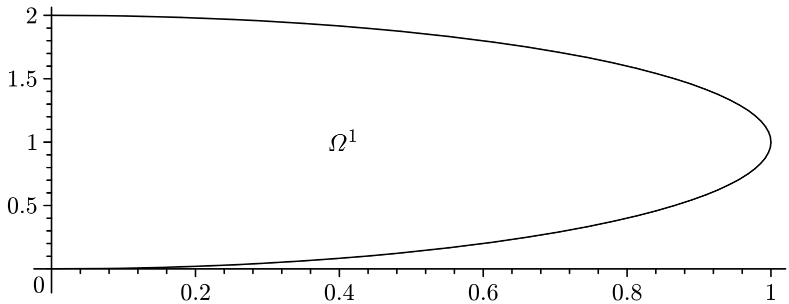


Fig. 2. $G_1$ for the domain in Fig. 1

The transformation $\Psi_{\varepsilon,j} : \overline{G}_j \to \Psi_{\varepsilon,j}(\overline{G}_j) \supset \Omega_{j,\varepsilon}$ is a $C^1$-diffeomorphism $T_{\varepsilon,j}$ which is close to the identity, followed by a contraction $S_\varepsilon$ in $y$-direction and a $C^1$-diffeomorphism $T_j$ which is independent of $\varepsilon$:

$$\Psi_{\varepsilon,j} = T_j \circ S_\varepsilon \circ T_{\varepsilon,j}.$$

Here $T_{\varepsilon,j} : Q_{1,j} \supset \overline{G}_j \to T_{\varepsilon,j}(\overline{G}_j) \subset Q_{2,j}$ is a $C^1$-diffeomorphism, $Q_{1,j}, Q_{2,j} \subset \mathbb{R}^{M+1}$ fixed, open, bounded sets; $S_\varepsilon(x, y) := (x, \varepsilon y)$; and $T_j : \widetilde{Q}_j \to T_j(\widetilde{Q}_j) \subset \mathbb{R}^{M+1}$ is again a $C^1$-diffeomorphism, $\widetilde{Q}_j \supset \overline{\bigcup_{0 \le \varepsilon \le 1} S_\varepsilon(T_{\varepsilon,j}(G_j))}$ open. Roughly speaking $T_{\varepsilon,j}$ is there to give some liberty in choosing the nodes, $S_\varepsilon$ is the normal squeezing, and $T_j$ moves an edge into the right position (i.e. to $[0, 1] \times \mathbb{R}^M$), possibly scaling and deforming it in a way independent of $\varepsilon$.

We want an edge to touch the node only at the side corresponding to $(\{0\} \times \mathbb{R}^M) \cap \overline{G}_j$, so we assume

$$\Psi_{\varepsilon,j}^{-1}(\Omega_{4,\varepsilon} \cap \overline{\Omega}_{j,\varepsilon}) \subset \{0\} \times \mathbb{R}^M$$

for all $j = 1, 2, 3$, i.e. $\Omega_{j,\varepsilon}$ begins at the node $\Omega_{4,\varepsilon}$.

The node $\Omega_{4,\varepsilon}$ converges to a one-point set, say $\Omega_{4,0} = \{z_{0,4}\} \in T_j(\widetilde{Q}_j) \subset \mathbb{R}^{M+1}$ for all $j = 1, 2, 3$, as $\varepsilon \to 0$. This means that with respect to say $L^2$-functions it disappears. Nevertheless it is important because it contains the information about which parts of the beginning of each edge are connected with each other (e.g. if $\Omega_{4,\varepsilon}$ is not connected, see Figure 5). What is of less importance is the exact shape of $\Omega_{4,\varepsilon}$ (see also remarks in [16], but keep in mind Figure 4 and Remark 2.1 where the shape does destroy convergence of eigenvalues).

We assume the node $\Omega_{4,\varepsilon}$ has a description $\Omega_{4,\varepsilon} = \Psi_{\varepsilon,4}(G_{4,\varepsilon})$, where $\Psi_{\varepsilon,4}(z) = \varepsilon z + z_{\varepsilon,4},\ z_{\varepsilon,4} \to z_{0,4}$ as $\varepsilon \to 0$. Note that since $\Omega_{j,\varepsilon},\ j = 1,2,3$, are open, $\Omega_{4,\varepsilon}$ is closed in $\Omega_\varepsilon$. It may even have empty interior.

Throughout this article we consider the following additional conditions (C1)–(C7) on $G_j$, $T_{\varepsilon,j}$, $T_j$ and $G_{4,\varepsilon}$, where always $j = 1,2,3$. The technical condition (C2) will only be used in Proposition 2.1. It is an open question if one could do without it. Condition (C8) will only be used for Theorem 1.2. (C7) and (C8) are abstract conditions: there will be more explicit sufficient ones in Section 3.

For $j = 1,2,3$ we suppose

(C1) $\overline{G}_j \cap (\{0\} \times \mathbb{R}^M)$ has finitely many connected components with positive $M$-dimensional measure.

(C2) There are at most countably many open, connected, pairwise disjoint $U^{j,l} \subset G_j$, $l \in I_\Omega$, such that each $U^{j,l}$ has connected $x$-sections and $E := \{x \in \mathbb{R} \mid \exists y \in \mathbb{R}^M\ (x,y) \in G_j \setminus \bigcup_{l \in I_\Omega} U^{j,l}\}$ has at most finitely many accumulation points.

(C3) $T_{\varepsilon,j}(x,y) \to (x,y),\ \varepsilon \to 0$, pointwise for all $(x,y) \in \overline{G}_j$, and if $(T_{\varepsilon,j})_x$ denotes the $x$-component of $T_{\varepsilon,j}$, then $(T_{\varepsilon,j})_x \to \mathrm{proj}_x|_{\overline{G}_j}$ uniformly on $\overline{G}_j$.

(C4) There is a $C > 0$ such that for all $\varepsilon \le 1$, $v \in \mathbb{R}^{M+1}$, $\|v\| = 1$,

$$\sup_{(x,y)\in\overline{G}_j} \|DT_{\varepsilon,j}(x,y)v\|, \qquad \sup_{(x,y)\in T_{\varepsilon,j}(\overline{G}_j)} \|DT_{\varepsilon,j}^{-1}(x,y)v\| < C.$$

(C5) Define $\mathcal{T}_{\varepsilon,j},\ \mathcal{T}^*_{\varepsilon,j}$ by $DT_{\varepsilon,j}(x,y) = E_{M+1} - \mathcal{T}_{\varepsilon,j}(x,y)$, $(DT_{\varepsilon,j}(x,y))^{-1} = E_{M+1} + \mathcal{T}^*_{\varepsilon,j}(x,y)$. Denote the elements of these matrix functions by $\mathcal{T}_{\varepsilon,j,l,k}$ and $\mathcal{T}^*_{\varepsilon,j,l,k}$, $l,k = 0,\dots,M$. We assume

$$\sup_{0<\varepsilon\le 1,\,(x,y)\in G_j} \left(\frac{1}{\varepsilon}\,|\mathcal{T}_{\varepsilon,j,0,l}(x,y)|, \frac{1}{\varepsilon}\,|\mathcal{T}^*_{\varepsilon,j,0,l}(x,y)|\right) < \infty,$$
$$\mathcal{T}_{\varepsilon,j}(x,y), \mathcal{T}^*_{\varepsilon,j}(x,y) \to 0, \qquad \varepsilon \to 0,$$

and there are maps $\mathcal{T}_j = (\mathcal{T}_{j,1},\dots,\mathcal{T}_{j,M}) : \overline{G}_j \to \mathbb{R}^M$ such that

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\,\mathcal{T}_{\varepsilon,j,0,l}(x,y) = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\,\mathcal{T}^*_{\varepsilon,j,0,l}(x,y) = \mathcal{T}_{j,l}(x,y)$$

for all $(x,y) \in G_j,\ l = 1,\dots,M$.

(C6) $G_{4,\varepsilon}$ is bounded independently of $\varepsilon$, i.e. there is a positive $R_\Omega$ such that $G_{4,\varepsilon} \subset B_{R_\Omega}(0)$ for all $0 < \varepsilon \le 1$.

(C7) Define $H^1_s$ as the set of all $[u] = [u_1,u_2,u_3] \in H^1_s(G_1) \times H^1_s(G_2) \times H^1_s(G_3)$ such that there are a constant $\beta > 0$, a sequence $\varepsilon_n \downarrow 0$ (both dependent on $[u]$), and $\widehat{u}_n \in H^1(\Omega_{\varepsilon_n})$ such that $\widehat{u}_n \circ \Psi_{\varepsilon_n,j} \rightharpoonup u_j$ weakly in $H^1(G_j), j = 1,2,3$, and

$$\sum_{j=1}^{3} \varepsilon_n^{-1} \|D_y(\widehat{u}_n \circ \Psi_{\varepsilon_n,j})\|_{L^2(G_j)} + \varepsilon_n \|\widehat{u}_n \circ \Psi_{\varepsilon_n,4}\|^2_{L^2(G_{4,\varepsilon_n})} + \frac{1}{\varepsilon_n} \|D(\widehat{u}_n \circ \Psi_{\varepsilon_n,4})\|^2_{L^2(G_{4,\varepsilon_n})} < \beta.$$

We assume $H^1_s$ is a closed subspace of $H^1_s(G_1) \times H^1_s(G_2) \times H^1_s(G_3)$ and for every $\varepsilon > 0$ there is a linear map $\Phi^H_\varepsilon : H^1_s \to H^1(G_1) \times H^1(G_2) \times H^1(G_3) \times H^1(G_{4,\varepsilon})$ and a constant $C > 0$, independent of $\varepsilon$, such that $(\Phi^H_\varepsilon[u])_j = (\Phi^H_\varepsilon[u_1, u_2, u_3])_j = u_j$ for all $[u] = [u_1, u_2, u_3]$, $j = 1, 2, 3$,

$$C \sum_{j=1}^{3} \|u_j\|^2_{H^1(G_j)} \ge \varepsilon \|(\Phi^H_\varepsilon[u])_4\|^2_{L^2(G_{4,\varepsilon})} + \frac{1}{\varepsilon} \|D(\Phi^H_\varepsilon[u])_4\|^2_{L^2(G_{4,\varepsilon})} \to 0 \tag{2.1}$$

as $\varepsilon \to 0$, and $\widehat{u}_\varepsilon := (\Phi^H_\varepsilon[u_1, u_2, u_3])_j \circ \Psi^{-1}_{\varepsilon,j}$ on $\Omega_{j,\varepsilon}$, $j = 1, \ldots, 4$, is a function in $H^1(\Omega_\varepsilon)$ (i.e. $\Phi^H_\varepsilon[u]$ comes from the $H^1$-function $\widehat{u}_\varepsilon$ via the transformations $\Psi_{\varepsilon,j}$).

(C8) If $C > 0$, $\varepsilon_n \to 0$, $[u_n] \in H^1_{\varepsilon_n}$, $\|[u_n]\|_{\varepsilon_n,1} \le C$ and $\|[u_n]\|_{L^2_{\varepsilon_n}} = 1$ for all $n$, then $\varepsilon_n \|u_{n,4}\|^2_{L^2(G_{4,\varepsilon_n})} \to 0$ as $n \to \infty$.

In the limit $\Omega_\varepsilon$ collapses to the one-dimensional net

$$\Omega_0 = \bigcup_{j=1}^{3} \Omega_{j,0} \cup \{z_{0,4}\}.$$

If we set $\Psi_{0,j}(x) := T_j(x,0) : [0,1] \to \mathbb{R}^{M+1}$, then $\Omega_{j,0} = \Psi_{0,j}(]0,1[)$ (see Lemma 2.2 below), $j = 1, 2, 3$.

Example 2.1 (L-shaped domains with holes). Let $g_j \in C^2([0,1], ]0,\infty[)$, $j = 1, 2$, and set

$$\Omega_\varepsilon := \{(x,y) \in \mathbb{R}^2 \mid 0 < y < \varepsilon g_1(x),\, 0 < x < 1\} \cup \{(x,y) \in \mathbb{R}^2 \mid 0 < x < \varepsilon g_2(y),\, 0 < y < 1\}.$$

$\Omega_\varepsilon$ is the L-shaped domain considered in [10]. We will show that $\Omega_\varepsilon$ is net-shaped in our sense, with $K_E = 2$ and $K_N = 1$.

We have to divide $\Omega_\varepsilon$. To do that let $C_g > \|g_j\|_\infty$, $j = 1, 2$, and $0 < \varepsilon_0 < 1/3C_g$. Divide $\Omega_\varepsilon$ into two edges $\Omega_{j,\varepsilon} \subset \Omega_\varepsilon$, $j = 1, 2$, and $\Omega_{3,\varepsilon} := \Omega_\varepsilon \setminus (\Omega_{1,\varepsilon} \cup \Omega_{2,\varepsilon})$, $0 < \varepsilon \le \varepsilon_0$, by setting

$$\Omega_{1,\varepsilon} := \{(x,y) \in \mathbb{R}^2 \mid 0 < y < \varepsilon g_1(x),\, \varepsilon C_g < x < 1\},$$
$$\Omega_{2,\varepsilon} := \{(x,y) \in \mathbb{R}^2 \mid 0 < x < \varepsilon g_2(x),\, \varepsilon C_g < y < 1\}.$$

Then $\Omega_\varepsilon = \bigcup_{j=1}^{3} \Omega_{j,\varepsilon}$ is bounded, connected, and since $x \mapsto (x, \varepsilon g_1(x))$ and $y \mapsto (\varepsilon g_2(y), y)$ do not intersect tangentially for $0 < \varepsilon \le \varepsilon_0$ (possibly upon decreasing $\varepsilon_0$ a little), $\Omega_\varepsilon$ is also Lipschitz. The sets $\Omega_{1,\varepsilon}$, $\Omega_{2,\varepsilon}$, $\Omega_{3,\varepsilon}$ are mutually disjoint.

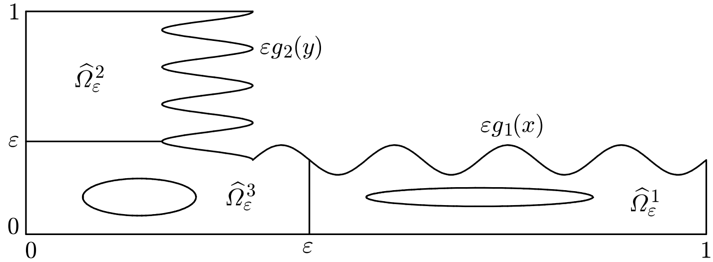


Fig. 3. An example of an L-shaped domain $\Omega_\varepsilon$ with holes

Now we define the diffeomorphisms $T_{\varepsilon,j}$, $j = 1,2$. Recall that $\chi$ is a cut-off function, $\chi' \geq 0$, $\chi(x) = 0$ for $x \leq 1/2$ and $\chi(x) = 1$ for $x \geq 1$. Extend $g_1, g_2$ to $C^2$-functions on $\mathbb{R}$ in such a way that

$$0 < \inf_{x\in\mathbb{R}} |g_j(x)| \leq \sup_{x\in\mathbb{R}} |g_j(x)| < C_g, \qquad j = 1,2.$$

Set

$$\begin{aligned}
T_1(x,y) &:= (x, g_1(x)y), \qquad T_2(x,y) := (g_2(x)y, x), \qquad S_\varepsilon(x,y) := \varepsilon(x,y),\\
T_{\varepsilon,1}(x,y) &:= T_{\varepsilon,2}(x,y)\\
&:= \left(x\chi\left(\frac{x}{\varepsilon C_g} - 2\right) + \left(\frac{1}{2}x + \varepsilon C_g\right)\left(1 - \chi\left(\frac{x}{\varepsilon C_g} - 2\right)\right), y\right).
\end{aligned}$$

Then $T_{\varepsilon,j}(x,y) = \left(\frac{1}{2}x + \varepsilon C_g, y\right)$ for $x < 2\varepsilon C_g$, $T_{\varepsilon,j} = \mathrm{id}$ for $x > 3\varepsilon C_g$, and $1/2 \leq \partial_x T_{\varepsilon,j}(x,y) \leq 1 + \|\chi'\|_\infty$ for $2\varepsilon C_g \leq x \leq 3\varepsilon C_g$, $j = 1,2$.

Hence $T_{\varepsilon,j}, T_j, S_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ are $C^1$-diffeomorphisms, as are $\Psi_{\varepsilon,j} = T_j \circ S_\varepsilon \circ T_{\varepsilon,j}$, $j = 1,2$.

Set $G_1 := G_2 := \,]0,1[\, \times \,]0,1[$. These are obviously bounded, open, connected, Lipschitz sets and $\mathrm{proj}_x(G_j) = \,]0,1[$. It is easily seen that $\Psi_{\varepsilon,j}|_{\overline{G}_j} : \overline{G}_j \to \overline{\Omega}_{j,\varepsilon}$ is bijective.

Set $G_{3,\varepsilon} := \{(x,y) \in \mathbb{R}^2 \mid 0 < y < g_1(\varepsilon x),\ 0 < x \leq C_g\} \cup \{(x,y) \in \mathbb{R}^2 \mid 0 < x < g_2(\varepsilon y),\ 0 < y \leq C_g\}$ and $\Psi_{\varepsilon,3}(z) := \varepsilon z$. Then $\Psi_{\varepsilon,z}|_{G_{3,\varepsilon}} : G_{3,\varepsilon} \to \Omega_{3,\varepsilon}$ is bijective.

Since $\Psi_{\varepsilon,1}(0,y) = \varepsilon(C_g, g_1(\varepsilon C_g)y)$ and $\Psi_{\varepsilon,2}(x,y) = \varepsilon(g_2(\varepsilon C_g)y, C_g)$, it follows that $\Psi_{\varepsilon,j}^{-1}(\Omega_{3,\varepsilon} \cap \overline{\Omega}_{j,\varepsilon}) \subset \{0\} \times \mathbb{R}$, $j = 1,2$.

Conditions (C1)–(C4) and (C6) are obviously satisfied. We show that (C9) and (C10) of Section 3 hold; then by Proposition 3.1 below, (C7) and (C8) also hold.

Note that $\Omega_{3,\varepsilon}$ is connected. Set $\omega_{j,x} := \,]0,1[$. Then $G_j = \bigcup_{0<x<1} \omega_{j,x}$. That is, in Definition 3.1 we have $L_j = 1$, $\delta = 1$, $G_{j,1} = G_j$, $j = 1,2$, and $S_\Omega = \{(1,1),(2,1)\}$.

For all $0 < \varepsilon \leq \varepsilon_0$ we have $(0,1/2) \in \partial G_1 \cap \partial G_2$, $\Psi_{\varepsilon,1}(0,1/2) = \varepsilon\left(C_g, \frac{1}{2}g_1(\varepsilon C_g)\right)$, and $\Psi_{\varepsilon,2}(0,1/2) = \varepsilon\left(\frac{1}{2}g_2(\varepsilon C_g), C_g\right) \in \Omega_{3,\varepsilon}$. Choose $0 <$

$\delta_1 < \frac{1}{3}\inf_{x\in\mathbb{R}}(g_1(x), g_2(x))$ and set

$$U := (]\delta_1, C_g + \delta_1[\times]\delta_1, 2\delta_1[) \cup (]\delta_1, 2\delta_1[\times]\delta_1, C_g + \delta_1[).$$

This is the $U_{1,1,2,1}$ of Definition 3.1. If $x \in ]\delta_1, C_g + \delta_1[$ and $y \in ]\delta_1, 2\delta_1[$, then by the choice of $\varepsilon_0$ and $\delta_1$ we have $0 < \varepsilon x < 2/3$ and $0 < y < \varepsilon g_1(x)$, yielding $\Psi_{\varepsilon,3}(x,y) \in \Omega_\varepsilon$. If additionally $x > C_g$, then $\Psi_{\varepsilon,3}(x,y) \in \Omega_{1,\varepsilon}$ and $\mathrm{proj}_x(\Psi_{\varepsilon,1}^{-1} \circ \Psi_{\varepsilon,3}(x,y)) = 2\varepsilon(x - C_g) \in \left]0, \frac{1}{2}\varepsilon C_g\right[$. Analogous statements hold for the second set in the definition of $U$.

Hence, if in Definition 3.1 we choose $r := \delta_1/3$, $z_{\varepsilon,1,1} := \left(C_g + \frac{1}{2}\delta_1, \frac{3}{2}\delta_1\right)$ and $z_{\varepsilon,2,1} := \left(\frac{3}{2}\delta_1, C_g + \frac{1}{2}\delta_1\right)$, all conditions of the definition are satisfied, i.e. $G_1$ and $G_2$ connect nicely and $\Omega_\varepsilon$ is nicely connected, that is, (C9) holds.

By Proposition 3.1, (C7) holds and

$$H_s^1 = \{[u_1, u_2] \mid u_j \in H_s^1(]0,1[),\ j = 1,2,\ u_1(0,\cdot)|_{]0,1[} = u_2(0,\cdot)|_{]0,1[} \text{ as traces}\}.$$

To prove (C10) we divide $G_{3,\varepsilon}$ into three parts: a rectangle in the middle with corners at 0 and the intersection of $\varepsilon g_1(x)$ with $\varepsilon g_2(y)$, and each of the remaining parts connecting it to the edges.

By the Implicit Function Theorem (possibly decreasing $\varepsilon_0$ slightly) there is a neighborhood $W \subset \mathbb{R}^2$ of 0 and a function $x : ]0, \varepsilon_0] \to \mathbb{R}$ such that $y = \varepsilon g_1(x)$, $x = \varepsilon g_2(y)$ for $(x,y) \in W$ if and only if $x = x(\varepsilon)$, $y = y(\varepsilon) := \varepsilon g_1(x(\varepsilon))$. The functions $x(\varepsilon), y(\varepsilon)$ are $C^2$ and $x(\varepsilon)/\varepsilon$, $y(\varepsilon)/\varepsilon$ are bounded away from $\infty$ and 0 as $\varepsilon \downarrow 0$.

Set $G_{3,j} := ]0,1[\times]0,1[$, $j = 1,2,3$, and define $C^1$-diffeomorphisms $\Psi_{\varepsilon,3,j} : \mathbb{R}^2 \to \mathbb{R}^2$, $j = 1,2,3$, by

$$\begin{aligned}
&\Psi_{\varepsilon,3,1}(x,y) := \varepsilon^{-1}(y(\varepsilon)x, x(\varepsilon)y),\\
&\Psi_{\varepsilon,3,2}(x,y) := ((C_g - x(\varepsilon))x + x(\varepsilon), g_1(\varepsilon((C_g - x(\varepsilon))x + x(\varepsilon)))y),\\
&\Psi_{\varepsilon,3,3}(x,y) := (g_2(\varepsilon((C_g - y(\varepsilon))y + y(\varepsilon)))x, (C_g - y(\varepsilon))y + y(\varepsilon)).
\end{aligned}$$

We get

$$G_3 \setminus \bigcup_{j=1}^{3} \Psi_{\varepsilon,3,j}(G_{3,j}) = (\{x(\varepsilon)\} \times ]0, y(\varepsilon)[) \cup (]0, x(\varepsilon)[\times \{y(\varepsilon)\}).$$

For $U_k$ and $U_{\varepsilon,k,j,l}$ in (C10) we can simply choose the same sets as before, that is, $U$ and $B_{\delta_1/3}(z_{\varepsilon,1,1})$ (choose $(j,l) = (1,1)$). Then all conditions of (C10) are satisfied.

We have just shown that the L-shaped domains Hale and Raugel consider in [10] are net-shaped in our sense. Their convergence is stronger than ours (see Remark 1.1), but we can handle L-shaped domains with holes.

For example, if we keep everything as before, only changing $G_1$ and $G_{3,1}$ by setting $G_1 := G_{3,1} := ]0,1[^2 \setminus \overline{B_{1/4}(1/2,1/2)}$, then $\Omega_{1,\varepsilon}$ and $\Omega_{3,\varepsilon}$ each have a hole which contracts, for the former, only in $y$-direction, and for the latter in all directions of order $\varepsilon$. (C1)–(C6) obviously remain true. For (C9) and (C10) we may have to change $U$ by making $\delta_1$ smaller. For example choosing

$0 < \delta_1 < \frac{1}{12} \inf_{x\in\mathbb{R}}(g_1(x), g_2(x))$ will be sufficient. Hence $\Omega_\varepsilon$ connects nicely, $H^1_s$ remains unchanged, and (C9) and (C10) are satisfied.

Solving the limit problem $A_0[u_1, u_2] = [w_1, w_2]$ means solving the following problems (see [10] for the domain without holes, [14] and Proposition 3.2 if there are holes): there are $u_1, u_2, w_1, w_2 : [0,1] \to \mathbb{R}$ such that

$$\begin{aligned}
&(u_j'(x)g_j(x))' = -w_j(x)g_j(x), \qquad x \in \,]0,1[,\ j = 1,2,\\
&u_1(0) = u_2(0),\\
&u_1'(1) = u_2'(1) = 0, \qquad u_1'(0)g_1(0) + u_2'(0)g_2(0) = 0,
\end{aligned}$$

for the domain without holes. With the holes we have to divide $G_1$ into four subsets with connected $x$-sections (see the definition of nicely decomposed in [14]). Set $h : [1/4, 3/4] \to \mathbb{R}$, $h(x) := \sqrt{1/16 - (x-1/2)^2}$. Then these subsets are: $]0,1/4[\, \times \,]0,1[$, $\{(x,y) \in \mathbb{R}^2 \mid 1/4 \le x \le 3/4,\ 0 < y < 1/2 - h(x)\}$, $\{(x,y) \in \mathbb{R}^2 \mid 1/4 \le x \le 3/4,\ 1/2 + h(x) < y < 1\}$, $]3/4,1[\, \times \,]0,1[$. On each of these sets $u_1$ and $w_1$ are functions of $x$ only, hence we have to find $u_2, w_2 : [0,1] \to \mathbb{R}$, $u_{1,1}, w_{1,1} : [0,1/4] \to \mathbb{R}$, $u_{1,2}, u_{1,3}, w_{1,2}, w_{1,3} : [1/4,3/4] \to \mathbb{R}$, and $u_{1,4}, w_{1,4} : [3/4,1] \to \mathbb{R}$ such that

$$\begin{aligned}
&(u_2'(x)g_2(x))' = -w_2(x)g_2(x), \qquad x \in \,]0,1[,\\
&(u_{1,1}'(x)g_1(x))' = -w_{1,1}(x)g_1(x), \qquad x \in \,]0,1/4[,\\
&(u_{1,j}'(x)g_1(x)(1/2 - h(x)))' = -w_{1,j}(x)g_1(x)(1/2 - h(x)),\\
&\hspace{20em} x \in \,]1/4,3/4[,\ j = 2,3,\\
&(u_{1,4}'(x)g_1(x))' = -w_{1,4}(x)g_1(x), \qquad x \in \,]3/4,1[,\\
&u_{1,1}(0) = u_2(0), \qquad u_{1,1}(1/4) = u_{1,2}(1/4) = u_{1,3}(1/4),\\
&u_{1,2}(3/4) = u_{1,3}(3/4) = u_{1,4}(3/4),\\
&u_{1,4}'(1) = u_2'(1) = 0, \qquad u_{1,1}'(0)g_1(0) + u_2'(0)g_2(0) = 0,\\
&2u_{1,1}'(1/4) = u_{1,2}'(1/4) + u_{1,3}'(1/4), \qquad 2u_{1,4}'(3/4) = u_{1,2}'(3/4) + u_{1,3}'(3/4).
\end{aligned}$$

Note that the hole in $\Omega_{3,\varepsilon}$ has no influence at all on the limit problem. ∎

Define $\mathcal{A}_{\varepsilon,j} : \overline{G}_j \to \mathbb{R}^{(M+1)\times(M+1)}$, $j = 1,2,3$, by

$$\mathcal{A}_{\varepsilon,j}(x,y) := \begin{pmatrix} 1 & & & \\ & \varepsilon & & \\ & & \ddots & \\ & & & \varepsilon \end{pmatrix} (DT_{\varepsilon,j}(x,y))^{-1} \begin{pmatrix} 1 & & & \\ & 1/\varepsilon & & \\ & & \ddots & \\ & & & 1/\varepsilon \end{pmatrix}$$
$$\times\, (DT_j(S_\varepsilon \circ T_{\varepsilon,j}(x,y)))^{-1}.$$

In [14] the convergence of the semigroups is in the norm $\|u\|_{L^2} + \|D_x u\|_{L^2} + \varepsilon^{-1}\|D_y u\|_{L^2}$. Here we will have convergence with respect to the equivalent norm $\|\cdot\|_{\varepsilon,d}$, $0 \le d < 1$, on $H^1_\varepsilon$ defined by (for $0 \le d \le 1$)

$$\|[u]\|_{\varepsilon,d}^2 := \sum_{j=1}^{3} \int\limits_{G_j} \left( u_j^2 + (D_x u_j)^2 + \frac{1}{\varepsilon^{2d}} |D_y u_j|^2 \right) dx\, dy + \int\limits_{G_{4,\varepsilon}} \left( \varepsilon u_4^2 + \frac{1}{\varepsilon} |Du_4|^2 \right) dz. \tag{2.2}$$

We divide $\Omega_\varepsilon$ into the above-mentioned three edges $\Omega_{j,\varepsilon}$, $j = 1, 2, 3$, and the node $\Omega_{4,\varepsilon}$, which in turn get transformed by $\Psi_{\varepsilon,j}$ into $G_j$, $j = 1, 2, 3$, and $G_{4,\varepsilon}$. Thus we can identify $L^2(\Omega_\varepsilon)$, $H^1(\Omega_\varepsilon)$ with

$$L_\varepsilon^2 := \{[u] = [u_1, \ldots, u_4] \mid u_j \in L^2(G_j),\ j = 1, 2, 3,\ u_4 \in L^2(G_{4,\varepsilon})\}, \tag{2.3}$$

$$([u], [v])_{L_\varepsilon^2} := \sum_{j=1}^{3} \int\limits_{G_j} u_j v_j \, d\lambda_{\varepsilon,j} + \varepsilon \int\limits_{G_{4,\varepsilon}} u_4 v_4 \, dz,$$

$$H_\varepsilon^1 := \{[u] \in L_\varepsilon^2 \mid u_j \in H^1(G_j),\ j = 1, 2, 3,\ u_4 \in H^1(G_{4,\varepsilon}),\ \exists \widehat{u} \in H^1(\Omega_\varepsilon)\ \widehat{u} \circ \Psi_{\varepsilon,j} = u_j,\ j = 1, \ldots, 4\}, \tag{2.4}$$

$$([u], [v])_{H_\varepsilon^1} := ([u], [v])_{L_\varepsilon^2} + \frac{1}{\varepsilon} \int\limits_{G_{4,\varepsilon}} Du_j Dv_j \, dz + \sum_{j=1}^{3} \int\limits_{G_j} \left( D_x u_j, \frac{1}{\varepsilon} D_y u_j \right) \mathcal{A}_{\varepsilon,j}(x,y) \mathcal{A}_{\varepsilon,j}^T(x,y) \left( D_x v_j, \frac{1}{\varepsilon} D_y v_j \right)^T d\lambda_{\varepsilon,j},$$

with norms $\|\cdot\|_{L_\varepsilon^2}$, $\|\cdot\|_{H_\varepsilon^1}$, respectively. Here we used measures on $\mathbb{R}^{M+1}$ defined by

$$\lambda_{\varepsilon,j}(A) := \int\limits_A |\det DT_{\varepsilon,j}(x,y)|\, |\det DT_j(S_\varepsilon \circ T_{\varepsilon,j}(x,y))| \, dx\, dy,$$

$$\lambda_j(A) := \int\limits_A |\det DT_j(x,0)| \, dx\, dy$$

for all Lebesgue measurable sets $A \subset \overline{G}_j$, $j = 1, 2, 3$.

Two functions $u_j \in L^2(G_j)$, $\widehat{u}_j \in L^2(\Omega_{j,\varepsilon})$ ($u_j \in L^2(G_{4,\varepsilon})$ if $j = 4$) will always—unless stated otherwise—be related by $u_j = \widehat{u}_j \circ \Psi_{\varepsilon,j}$, $j = 1, \ldots, 4$.

Given $u_j$, $j = 1, 2, 3$ or $j = 1, \ldots, 4$, we write $[u]$ for $[u_1, u_2, u_3]$ and $[u_1, \ldots, u_4]$, respectively. It will be clear from the context which case is meant.

The definition of $L_\varepsilon^2$ and $H_\varepsilon^1$ with the respective scalar products in (2.3), (2.4) is just a change of variables on each subset $\Omega_{j,\varepsilon}$, $j = 1, \ldots, 4$, the measures $\lambda_{\varepsilon,j}$ being the Jacobians of the respective transformations dropping the common factor $\varepsilon^M$. Thus $\widehat{u} \in L^2(\Omega_\varepsilon)$ iff $[u] \in L_\varepsilon^2$ and $\|\widehat{u}\|_{L^2(\Omega_\varepsilon)}^2 = \varepsilon^M \|[u]\|_{L_\varepsilon^2}^2$; $\widehat{u} \in H^1(\Omega_\varepsilon)$ iff $[u] \in H_\varepsilon^1$ and $\|\widehat{u}\|_{H^1(\Omega_\varepsilon)}^2 = \varepsilon^M \|[u]\|_{H_\varepsilon^1}^2$. Also, if $[u_\varepsilon] \in H_\varepsilon^1$ is such that $(\|[u_\varepsilon]\|_{\varepsilon,1})_{\varepsilon>0}$ is bounded, then $(\varepsilon^{-M} \|\widehat{u}_\varepsilon\|_{H^1(\Omega_\varepsilon)})_{\varepsilon>0}$ is bounded as well (see Lemma 2.7(iii) below).

We have already introduced the space $H_s^1$ in (C7); let $L_s^2$ be the closure of $H_s^1$ in $L^2(G_1)\times L^2(G_2)\times L^2(G_3)$. We introduce inner products on them by

$$([u],[v])_{L_s^2} := \sum_{j=1}^{3}\int_{G_j} u_j v_j\,d\lambda_j, \tag{2.5}$$

$$([u],[v])_{H_s^1} := ([u],[v])_{L_s^2} + \sum_{j=1}^{3}\int_{G_j} D_x u_j D_x v_j\,d\lambda_j. \tag{2.6}$$

Denote the respective norms by $\|\cdot\|_{L_s^2}$ and $\|\cdot\|_{H_s^1}$.

We write equation (1.1) and the limit equation as abstract equations on $L_\varepsilon^2$ and $L_s^2$, respectively. As usual, the abstract linear operators involved, namely $A_\varepsilon$ and $A_0$, are generated by bilinear forms $a_\varepsilon : H_\varepsilon^1\times H_\varepsilon^1\to\mathbb{R}$ and $a_0 : H_s^1\times H_s^1\to\mathbb{R}$, respectively. These bilinear forms are defined as follows:

$$\begin{aligned} a_\varepsilon([u],[v]) := {} & \sum_{j=1}^{3}\int_{G_j}\Big(D_x u_j, \frac{1}{\varepsilon}D_y u_j\Big)\mathcal{A}_{\varepsilon,j}\mathcal{A}_{\varepsilon,j}^T\Big(D_x v_j, \frac{1}{\varepsilon}D_y v_j\Big)^T d\lambda_{\varepsilon,j} \\ & + \frac{1}{\varepsilon}\int_{G_{4,\varepsilon}} Du_4 Dv_4\,dz, \end{aligned} \tag{2.7}$$

$$\begin{aligned} a_0([u],[v]) := {} & \sum_{j=1}^{3}\int_{G_j} D_x u_j D_x v_j |\Psi_{0,j}'(x)|^{-2}|\det\ DT_j(x,0)|\,dx\,dy \\ = {} & \sum_{j=1}^{3}\int_{G_j} D_x u_j D_x v_j |(1,0)DT_j^T(x,0)|^{-2}\,d\lambda_j. \end{aligned} \tag{2.8}$$

It is well known (see e.g. [14, Proposition 2.2]) that if $V, H$ are two infinite-dimensional Hilbert spaces, $V\subset H$ densely and compactly, $\|\cdot\|_V,\ \|\cdot\|_H$ denote the norms on $V$ and $H$, respectively, $\langle\cdot,\cdot\rangle_H$ the inner product on $H$, and $a : V\times V\to\mathbb{R}$ is a symmetric bilinear form satisfying

$$|a(u,v)|\le C_1\|u\|_V\|v\|_V, \qquad a(u,u)\ge C_2\|u\|_V^2 - C_3\|u\|_H^2,$$

$C_1, C_2, C_3$ constants, then $a$ defines, via $a(u,v)=\langle w,v\rangle_H$ for all $v\in V$, a linear selfadjoint operator $A : D(A)\subset V\to H$ with compact resolvent. Moreover, $D(A)\subset V$, $D(A)\subset H$ densely and there is a complete orthonormal system (ONS) of $H$ consisting only of eigenvalues of $A$.

By Lemmas 2.1 and 2.7 below we can apply this to the cases above. Call the resulting operators $A_\varepsilon$ and $A_0$. They are sectorial, and the linear semigroups $e^{-A_\varepsilon t}$, $\varepsilon\ge 0$, as well as the fractional power spaces exist. There are complete ONS of $L_\varepsilon^2$ and $L_s^2$ consisting of eigenvectors of $A_\varepsilon$ and $A_0$, respectively.

Equation (1.1) then becomes

$$[u_t] = -A_\varepsilon[u], \qquad t > 0, \tag{2.9}$$

and the limit equation will be $[u_t] = -A_0[u]$, $t > 0$.

It is clear that it suffices to investigate the behavior of the semiflow generated by equation (2.9) because a simple transformation changes it into the semiflow generated by (1.1).

Henceforth we shall only treat equation (2.9).

We start with a few lemmas which are easy consequences of conditions (C1)–(C7).

LEMMA 2.1. *There is a constant $C > 0$ such that for all $j = 1, 2, 3$, $0 < \varepsilon \le 1$ and $v \in \mathbb{R}^{M+1}$, $\|v\| = 1$,*

$$\frac{1}{C} \le \|DT_j(x,y)v\|, |\det DT_j(x,y)|, \|DT_{\varepsilon,j}(x,y)v\|, |\det DT_{\varepsilon,j}(x,y)| \le C,$$

*for all possible $(x,y)$.* ∎

Note that by Lemma 2.1 similar estimates hold for $DT_j^{-1}$, $DT_{\varepsilon,j}^{-1}$, $DT_j^T$, and $DT_{\varepsilon,j}^T$. The lemma is easily proved by using the fact that $T_j$ is a diffeomorphism on a compact set and condition (C4). ∎

LEMMA 2.2. *As $\varepsilon \to 0$,*

$$\Psi_{\varepsilon,j}(x,y) \to \Psi_{0,j}(x), \qquad D\Psi_{\varepsilon,j}(x,y) \to (\Psi'_{0,j}(x), 0, \ldots, 0)$$

*pointwise for all $(x,y) \in G_j$, $j = 1,2,3$. Moreover, there is a constant $C$ such that $\|\Psi_{\varepsilon,j}(x,y)\|, \|D\Psi_{\varepsilon,j}(x,y)v\| < C$ for all $v \in \mathbb{R}^{M+1}$, $\|v\| = 1$, $0 < \varepsilon \le 1$, $(x,y) \in G_j$, and $\Psi'_{0,j}(x) \ne 0$ for all $x$ and $j$.* ∎

*Proof.* By (C3) and (C5), $S_\varepsilon \circ T_{\varepsilon,j}(x,y) \to (x,0)$ and $D(S_\varepsilon \circ T_{\varepsilon,j})(x,y) \to (e_1, 0, \ldots, 0)$. Together with Lemma 2.1 this proves the result. ∎

LEMMA 2.3. *For all $j = 1,2,3$ the following hold*:

(i) *There is a constant $C > 0$ such that for all $0 < \varepsilon \le 1$, $(x,y) \in \overline{G}_j$, $v \in \mathbb{R}^{M+1}$, $\|v\| = 1$,*

$$\frac{1}{C} \le |\det \mathcal{A}_{\varepsilon,j}(x,y)|, \|\mathcal{A}_{\varepsilon,j}(x,y)v\| \le C.$$

(ii) *We have*

$$\mathcal{A}_{\varepsilon,j}(x,y) \to \begin{pmatrix} 1 & \mathcal{T}_j(x,y) \\ 0 & E_M \end{pmatrix} DT_j^{-1}(x,0)$$

*pointwise on $G_j$, as $\varepsilon \downarrow 0$, where $\mathcal{T}_j$ is the function of condition* (C5).

*Proof.* This is straightforward, using conditions (C3)–(C5) and Lemma 2.1. ∎

For completeness we now bring in a technical lemma we shall need.

Lemma 2.4. *Let $\Omega \subset ]0,\infty[ \times \mathbb{R}^M$ be open, bounded, Lipschitz, $0 \in \mathrm{proj}_x(\overline{\Omega})$ and assume $(\Omega)_0 \neq \emptyset$ (recall that $(\Omega)_0$ is the interior, as an $M$-dimensional set, of the intersection $(\{0\} \times \mathbb{R}^M) \cap \overline{\Omega})$. Let $\varepsilon_n \to 0$ and $w \in H^1(\Omega)$ with $w|_{(\Omega)_0} = 0$ (as a trace). Then*

$$\frac{1}{\varepsilon_n}\|w\|^2_{L^2(\{(x,y)\in\Omega:0<x\le\varepsilon_n\})} \to 0, \qquad n \to \infty.$$

*Proof.* Extend $w$ to $\widetilde{w} \in H^1(\mathbb{R}^{M+1})$ and set $\{0\}\times\omega = \overline{\Omega} \cap (\{0\}\times\mathbb{R}^M)$. Approximate $\widetilde{w}$ in $H^1(\mathbb{R}^{M+1})$ by $C^\infty$-functions $\widetilde{w}_n$. Then the $\widetilde{w}_n|_\Omega$ approximate $w$ in $H^1(\Omega)$ and $\widetilde{w}_n|_{(\Omega)_0} \to w|_{(\Omega)_0} = 0$ in $L^2((\Omega)_0)$. We get for $\delta > 0$, as $n \to \infty$,

$$\begin{aligned}\|\widetilde{w}\|^2_{L^2(\{0\}\times B_\delta(\omega))} \leftarrow \|\widetilde{w}_n\|^2_{L^2(\{0\}\times B_\delta(\omega))} &= \int\limits_{B_\delta(\omega)} \widetilde{w}_n^2(0,y)\,dy \\ &= \int\limits_{B_\delta(\omega)\setminus\omega} \widetilde{w}_n^2(0,y)\,dy + \int\limits_{\omega} \widetilde{w}_n^2(0,y)\,dy \to \int\limits_{B_\delta\omega\setminus\omega} \widetilde{w}^2(0,y)\,dy\end{aligned}$$

and as $\delta \downarrow 0$,

$$\|\widetilde{w}\|^2_{L^2(\{0\}\times B_\delta(\omega))} \to 0. \tag{2.10}$$

There is a sequence of positive numbers $\delta_n \to 0$ such that

$$\mathrm{proj}_y(\{(x,y) \in \Omega \mid 0 < x \le \varepsilon_n\}) \subset B_{\delta_n}(\omega). \tag{2.11}$$

We get

$$\begin{aligned}&\frac{1}{\varepsilon_n} \int\limits_{\{(x,y)\in\Omega\,|\,0<x\le\varepsilon_n\}} w^2\,dx\,dy \\ &\le \frac{2}{\varepsilon_n} \int\limits_{\{(x,y)\in\Omega\,|\,0<x\le\varepsilon_n\}} |w(x,y) - \widetilde{w}(0,y)|^2\,dx\,dy \\ &\quad + \frac{2}{\varepsilon_n} \int\limits_{\{(x,y)\in\Omega\,|\,0<x\le\varepsilon_n\}} \widetilde{w}(0,y)^2\,dx\,dy \\ &\le \underbrace{\frac{2}{\varepsilon_n}\int_0^{\varepsilon_n} \|\widetilde{w}(x,y)-\widetilde{w}(0,y)\|^2_{L^2(\mathbb{R}^M)}\,dx}_{\le (C/\varepsilon_n)\int_0^{\varepsilon_n} x\|\widetilde{w}\|^2_{H^1(\mathbb{R}^{M+1})}dx} + 2\|\widetilde{w}\|^2_{L^2(\{0\}\times\mathrm{proj}_y\{(x,y)\in\Omega\,|\,0<x\le\varepsilon_n\})} \to 0,\end{aligned}$$

where $C$ is a constant and we have used Theorem 6.2.29 of [8], (2.10) and (2.11). ■

The next lemma characterizes $H^1_s(\Omega)$ and $L^2_s(\Omega)$.

Lemma 2.5. (i) *Let $u_j \in H^1_s(G_j)$, $j = 1,2,3$. Then $[u] \in H^1_s$ iff there is a $[v] \in H^1_s$ and $u_j|_{(G_j)_0} = v_j|_{(G_j)_0}$, $j = 1,2,3$ (as traces). Note that if $(G_j)_0 = \{0\} \times \omega_{j,0}$, then $\omega_{j,0}$ is open and nonempty, for $j = 1,2,3$.*

(ii) *$H^1_s \subset L^2_s$ densely and compactly and $L^2_s = L^2_s(G_1) \times L^2_s(G_2) \times L^2_s(G_3)$. Moreover, setting $L^2_\perp = L^2_\perp(G_1) \times L^2_\perp(G_2) \times L^2_\perp(G_3)$ we have $L^2(G_1) \times L^2(G_2) \times L^2(G_3) = L^2_s \oplus L^2_\perp$.*

*Proof.* $L^2_s = \prod_{j=1}^3 L^2_s(G_j)$ follows because every $u_j \in L^2_s(G_j)$ can be approximated by $u_{n,j} \in H^1_s(G_j)$ with $u_{n,j} = 0$ for $x$ near 0. By part (i), $[u_n] \in H^1_s$. The rest of part (ii) is trivial.

To proof (i), notice that if $j, k \in \{1,2,3\}$ and $j \neq k$, then $\Omega_\varepsilon$ being open and connected implies there is a path $\gamma$ in $\Omega_\varepsilon$ starting in $\Omega_{j,\varepsilon}$ and ending in $\Omega_{k,\varepsilon}$. Moreover, $\Omega_{j,\varepsilon} \cap \Omega_{k,\varepsilon} = \emptyset$, and both sets are open, so $\gamma$ has to pass through $\overline{\Omega}_{j,\varepsilon} \cap \Omega_{4,\varepsilon}$. But then the assumption $\Psi^{-1}_{\varepsilon,j}(\overline{\Omega}_{j,\varepsilon} \cap \Omega_{4,\varepsilon}) \subset \{0\} \times \mathbb{R}^M$ implies $(G_j)_0$ has nonempty interior, i.e. $\omega_{j,0}$ is open and nonempty.

Let $[v] \in H^1_s$ and $u_j \in H^1_s(G_j)$ be as in (i). By condition (C7) there are $\widehat{v}_\varepsilon \in H^1(\Omega_\varepsilon)$ and $C > 0$ such that $\widehat{v}_\varepsilon \circ \Psi_{\varepsilon,j} = v_j$. Set

$$u_{\varepsilon,j}(x,y) := v_j(x,y) + \chi(\varepsilon^{-1}x)(u_j(x,y) - v_j(x,y)) \in H^1_s(G_j),$$

$$\widehat{u}_\varepsilon(z) := \begin{cases} \widehat{v}_\varepsilon(z), & z \in \Omega_{4,\varepsilon}, \\ u_{\varepsilon,j} \circ \Psi^{-1}_{\varepsilon,j}(z), & z \in \Omega_{j,\varepsilon},\ j = 1,2,3. \end{cases}$$

Then $\widehat{u}_\varepsilon \in H^1(\Omega_\varepsilon)$ and

$$\begin{aligned} \|u_{\varepsilon,j} - u_j\|^2_{L^2(G_j)} &\le \int\limits_{\{(x,y)\in G_j \,|\, 0<x\le\varepsilon\}} |v_j - u_j|\, dx\, dy \to 0, \\ \|D_y u_{\varepsilon,j} - D_y u_j\|_{L^2(G_j)} &= 0, \\ \|D_x u_{\varepsilon,j}\|_{L^2(G_j)} &\le 2\|D_x v_j\|_{L^2(G_j)} + \|D_x u_j\|_{L^2(G_j)} \\ &\quad + \frac{1}{\varepsilon}\|\chi'\|_\infty \int\limits_{\{(x,y)\in G_j \,|\, 0<x\le\varepsilon\}} |u_j - v_j|\, dx\, dy. \end{aligned}$$

Let $\varepsilon_n \to 0$. By Lemma 2.4 the last term above tends to zero. Since $(\|u_{\varepsilon_n,j}\|_{H^1(G_j)})_n$ is bounded, taking a subsequence, also called $\varepsilon_n$, we can assume $(u_{\varepsilon_n,j})_n$ to converge weakly in $H^1(G_j)$, the weak limit being $u_j$. By (C7), $[u] \in H^1_s$ follows. ∎

PROPOSITION 2.1. *Fix $j \in \{1,2,3\}$. Denote by $\widetilde{L^2_s}$ the set of all $L^2$-functions which are locally functions of $x$ only, i.e.*

$$\begin{aligned} \widetilde{L^2_s} := \{u \in L^2(G_j) \mid{} & \exists S \subset G_j,\ |S| = 0\ \forall (x_0,y_0) \in G_j\ \exists \widetilde{V} = \widetilde{V}(x_0,y_0) \subset G_j \\ & \textit{open}, (x_0,y_0) \in \widetilde{V},\ \widetilde{u} = \widetilde{u}(x_0,y_0) \in L^2(\mathrm{proj}_x(\widetilde{V})) \\ & \textit{such that } u(x,y) = \widetilde{u}(x)\ \forall (x,y) \in \widetilde{V} \setminus S\}. \end{aligned}$$

*Then $L^2_s(G_j) = \widetilde{L^2_s}$ and $L^\infty(G_j)$ is dense in $L^2_s(G_j)$. Moreover, if $u \in H^1_s(G_j)$, then $\partial_x u \in L^2_s(G_j)$.* ∎

*Proof.* In this proof we drop the index $j$, that is, $G_j$ becomes $\Omega$.

We only suppose of $\Omega$ that it is open, bounded, connected and there is a subdivision of $\Omega$ as in condition (C2), i.e. there are open, connected, pairwise disjoint $U_l \subset \Omega$, each $U_l$ has connected $x$-sections and $E := \{x \in \mathbb{R} \mid \exists y \in \mathbb{R}^M\ (x,y) \in \Omega \setminus \bigcup_l U_l\}$ has at most finitely many accumulation points.

In this proof we write $L^2$ for $L^2(\Omega)$. If the underlying space is not $\Omega$ it will be mentioned explicitly. Other function spaces will be treated likewise.

Recall that for an open set $U \subset \mathbb{R}^{M+1}$ and $x \in \mathbb{R}$ the set $(U)_x := U \cap (\{x\} \times \mathbb{R}^M)$ is the $x$-section of $U$. For $(x,y) \in U$ we denote the connected component of $(U)_x$ which contains $(x,y)$ by $U_x(y)$.

The proof will be given through a series of claims.

Claim 1. *There are at most countably many $\widetilde{V}_j$ such that $\Omega = \bigcup_j \widetilde{V}_j$ and*

$$\begin{aligned} \widetilde{L}^2_s = \{u \in L^2 \mid \exists S \subset \Omega,\ |S| = 0\ \forall j\ \exists \widetilde{u}_j \in L^2(\mathrm{proj}_x(\widetilde{V}_j)), \\ u(x,y) = \widetilde{u}_j(x) \text{ for all } (x,y) \in \widetilde{V}_j \setminus S\}. \end{aligned}$$

*If $u \in \widetilde{L}^2_s$, then $u$ has a representative $\widetilde{u}$ and there are a nullset $\widetilde{S} \subset \Omega$ and $\widetilde{u}_j \in L^2(\widetilde{V}_j)$ such that $|\mathrm{proj}_x(\widetilde{S})| = 0$ and $\widetilde{u}(x,y) = \widetilde{u}_j(x)$ for $(x,y) \in \widetilde{V}_j \setminus \widetilde{S}$ and all $j$.*

Claim 2. *$\overline{E}$ is at most countable and we can assume that $\Omega \setminus \bigcup_l U_l = \bigcup_{x \in \overline{E}} \{x\} \times (\Omega)_x$.*

Claim 3. *For all $l \neq \widetilde{l}$, $x \in \mathrm{proj}_x(U_l) \cap \mathrm{proj}_x(U_{\widetilde{l}})$ we have $(U_l)_x \cap (U_{\widetilde{l}})_x = \emptyset$. Also, if $(x,y) \in U_l$ for some $l$, then $(U_l)_x = \Omega_x(y)$.*

Claim 4. *Let $U$ be an open, connected, bounded set with connected $x$-sections and $u \in L^2(U)$ be a function depending locally on $x$ only. Then for every $\delta > 0$ there is a function $w \in H^1_s(U)$ with $\|u - w\|_{L^2(U)} < \delta$. Moreover $w$ is a function of $x$ only and $w \in C_0^\infty(\mathrm{proj}_x(U))$.*

Claim 5. *Given $u \in \widetilde{L}^2_s$ there is a sequence $w_n \in H^1_s \cap C^\infty$ such that $\|u - w_n\|_{L^2} \to 0$, $n \to \infty$.*

Claim 5 immediately implies $\widetilde{L}^2_s \subset L^2_s$.

Assuming the claims above, we now show that $L^2_s \subset \widetilde{L}^2_s$. This then immediately implies $\partial_x u \in L^2_s$ for each $u \in H^1_s$, and $C^\infty(\Omega)$ is dense in $L^2_s$.

Let $u \in H^1_s$. Theorem 2.5 of [14] implies there is a nullset $S \subset \Omega$ such that for all $(x_0, y_0) \in \Omega$ there is an open neighborhood $\widetilde{V}$ of $(x_0, y_0)$ and a function $\widetilde{u}(x)$ defined on $\mathrm{proj}_x(\widetilde{V})$ with $u(x,y) = \widetilde{u}(x)$ for all $(x,y) \in \widetilde{V} \setminus S$. Thus $H^1_s \subset \widetilde{L}^2_s$. By the first claim $\widetilde{L}^2_s$ is closed in $L^2$, and $L^2_s \subset \widetilde{L}^2_s$ follows.

We now prove the claims.

*Proof of Claim 1.* If open $\widetilde{V}_j$ are such that $\Omega = \bigcup_j \widetilde{V}_j$, then obviously $\widetilde{L}^2_s$ contains the set in Claim 1. We have to show the existence of $\widetilde{V}_j$ such that the other inclusion is true as well.

For $(x,y) \in \Omega$ there is an $r = r(x,y) > 0$ such that $]x-r, x+r[ \times \prod_{j=1}^{M} ]y_j - r, y_j + r[ \subset \Omega$. We can cover $\Omega$ by countably many of these sets; denote them by $\widetilde{V}_j$, $j \in \mathbb{N}$.

Let $u \in \widetilde{L}^2_s$. We have to show there is a nullset $\widetilde{S} \subset \Omega$ and for each $j$ a function $\widetilde{u}_j \in L^2(\mathrm{proj}_x(\widetilde{V}_j))$ with $u(x,y) = \widetilde{u}_j(x)$ for all $(x,y) \in \widetilde{V}_j \setminus \widetilde{S}$.

The sets $\widetilde{V}(x_0, y_0)$ in the definition of $L^2_s$ give an open covering of $\Omega$ (depending on $u$). Choose a countable subcovering, denoted by $\widetilde{U}_l$, and let $\widetilde{u}_l \in L^2(\mathrm{proj}_x(\widetilde{U}_l))$ be the corresponding functions.

We change $u$ on a nullset: if $(x,y) \in S$ and there is no $r > 0$ with $\{x\} \times B_r(y) \subset S$, then for $l \neq \widetilde{l}$ and $(x,y) \in \widetilde{U}_l \cap \widetilde{U}_{\widetilde{l}}$ we have $\widetilde{U}_l \cap \widetilde{U}_{\widetilde{l}} \cap (\{x\} \times \mathbb{R}^M) \neq \emptyset$, and $\widetilde{u}_l(x) = u(x, y_1) = \widetilde{u}_{\widetilde{l}}(x)$ for suitable $y_1 \in \mathbb{R}^M$. That is, we can define $u(x,y) := \widetilde{u}_l(x)$ if $(x,y) \in \widetilde{U}_l \cap S$.

Redefining $u$ in this way we can assume that for all $(x,y) \in S$ there is an $r = r(x,y) > 0$ with $\{x\} \times B_r(y) \subset S$. But then $\mathrm{proj}_x(S)$ is a one-dimensional nullset.

Set $\widetilde{S} = \bigcup_{x \in \mathrm{proj}_x(S)} \{x\} \times (\Omega)_x$. Then $|\widetilde{S}| = 0$ and for $(x,y) \in \widetilde{V}_j \setminus \widetilde{S}$ we can define $\widetilde{u}_j(x) := u(x,y)$. Now, $u \in L^2$ and $|(\widetilde{V}_j)_x| > r_j$ for some $r_j > 0$ imply $\widetilde{u}_j \in L^2(\mathrm{proj}_x(\widetilde{V}_j))$.

*Proof of Claim 2.* Since $E$ has at most finitely many accumulation points, $\overline{E}$ is countable. For each $l$ the set $U_l \setminus \bigcup_{x \in \overline{E}} \{x\} \times (\Omega)_x$ consists of at most countably many sets which are open, connected and have connected $x$-sections. Using these sets instead of $U_l$ we can without loss of generality assume $\Omega \setminus \bigcup_l U_l = \bigcup_{x \in \overline{E}} \{x\} \times (\Omega)_x$.

*Proof of Claim 3.* We have $(U_l)_x \cap (U_{\widetilde{l}})_x = \emptyset$ since $U_l \cap U_{\widetilde{l}} = \emptyset$. Now let $(x,y) \in U_l$. Obviously $(U_l)_x \subset \Omega_x(y)$. Let $(x, y_1) \in \Omega_x(y) \setminus (U_l)_x$. Then $x \notin \overline{E}$ by Claim 1 and there is an $\widetilde{l} \neq l$ such that $(x, y_1) \in (U_{\widetilde{l}})_x$. Hence the connectable open set $\Omega_x(y)$ (viewed as a set in $\mathbb{R}^M$) is the union of open (in $\mathbb{R}^M$), pairwise disjoint sets $(U_k)_x$, with $k$ varying in an at most countable set containing $l$ and $\widetilde{l}$. This cannot be, thus $\Omega_x(y) \setminus (U_l)_x = \emptyset$.

*Proof of Claim 4.* Let $\delta > 0$ and $U$, $u$ be as stated in the claim. Note that $U$ satisfies the conditions imposed on $\Omega$ at the beginning of this proof, the division into the $(U_l)_l$ having just the one element $U$ itself.

We can apply Claim 1: without loss of generality there are $S, \widetilde{V}_j \subset U$ such that $\widetilde{u}_j \in L^2(\mathrm{proj}_x(\widetilde{V}_j))$, $|\mathrm{proj}_x(S)| = 0$, $U = \bigcup_j \widetilde{V}_j$, and $u(x,y) = u_j(x)$ for $(x,y) \in \widetilde{V}_j \setminus S$.

We can redefine $u$ and $\widetilde{u}$ by setting $u(x,y) := \widetilde{u}(x) := 0$ if $x \in \mathrm{proj}_x(S)$. This allows us to define $\widetilde{u} : \mathrm{proj}_x(U) \to \mathbb{R}$ by $\widetilde{u}(x) := u(x,y)$ if $(x,y) \in (U)_x$.

Let $\mathrm{proj}_x(U) = \,]a,b[$. Since $U$ is open and connected, it follows that $\widetilde{u} \in L^2(]a+\delta_1, b-\delta_1[)$ for all $\delta_1 > 0$. Let $\delta_1 > 0$ be such that

$$\int_U u^2\,dx\,dy - \int_{\{(x,y)\in U \mid x \in ]a+\delta_1,b-\delta_1[\}} u^2\,dx\,dy < \delta.$$

There is a $w \in C_0^\infty(]a+\delta_1, b-\delta_1[)$ with $\|w-\widetilde{u}\|_{L^2(]a+\delta_1,b-\delta_1[)} < \delta^{1/2}$, and extending $w$ trivially we get

$$\|u-w\|^2_{L^2(u)} \le \delta + \int_{a+\delta_1}^{b-\delta_1} (\widetilde{u}-w)^2 \int_{(U)_x} dy\,dx < C\delta,$$

where $C = 1 + \sup_{x\in]a,b[} \int_{(U)_x} dy$ depends on $U$ only.

*Proof of Claim 5.* We have $\Omega = \bigcup_l U_l \cup \bigcup_{x\in\overline{E}} \{x\} \times (\Omega)_x$, where $\overline{E}$ is countable and has at most finitely many accumulation points, the set $(U_l)_l$ and $\bigcup_{x\in\overline{E}}\{x\}\times(\Omega)_x$ are pairwise disjoint, and each $U_l$ is open, connected and has connected $x$-sections. Setting $U_l := \emptyset$ for those $l \in \mathbb{N}$ for which $U_l$ is not defined, we can assume $l \in \mathbb{N}$ and $|U_l| \to 0$ as $l \to \infty$.

By Claim 4 for each $l,n \in \mathbb{N}$ there is a $u_{l,n} \in C_0^\infty(\mathrm{proj}_x(U_l))$ with $\|u-u_{l,n}\|_{L^2(U_l)} < 2^{-l}/n$.

Set $v_n(x,y) := u_{l,n}(x)$ if $(x,y) \in U_l$, $l = 1,\dots,n$, and $v(x,y) := 0$ elsewhere. Then $v_n$ is well defined for all $n \in \mathbb{N}$, and

$$\|u-v_n\|^2_{L^2} = \underbrace{\sum_{l=1}^n \|u-u_{l,n}\|^2_{L^2(U_l)}}_{\le 2/n} + \sum_{l>n} \|u\|^2_{L^2(U_l)} \to 0, \qquad n \to \infty.$$

If $(x_1,y_1) \in U_l$ for some $l \in \mathbb{N}$, then for $n$ large $v_n(x,y) = u_{l,n}(x)$ close to $(x_1,y_1)$, and $v_n$ is $C^\infty$ around this point. If $(x_1,y_1) \in \Omega \setminus \bigcup_l U_l$, then $x_1 \in \overline{E}$. If $x_1$ is an isolated point of $\overline{E}$, then there are $l \ne \widetilde{l}$ such that $x_1 \in \overline{\mathrm{proj}_x(U_l)} \cap \overline{\mathrm{proj}_x(U_{\widetilde{l}})}$ and for $n$ large $v_n(x,y) = 0$ on a neighborhood of $(x_1,y_1)$.

But if $x_1$ is not an isolated point of $\overline{E}$, then it could happen that $v_n$ is not $C^\infty$ at the point $(x_1,y_1)$. So we choose open neighborhoods $E_n$ of all (finitely many) accumulation points of $\overline{E}$ such that $\|v_n\|_{L^2(\Omega\cap(E_n\times\mathbb{R}^M))} < 1/n$, and cut-off functions $\chi_n$ such that $\chi_n(x) = 0$ near all accumulation points of $\overline{E}$, and $\chi_n(x) = 1$ on $\mathbb{R} \setminus E_n$.

Set $w_n(x,y) := \chi_n(x)v_n(x,y)$. Then $w_n \in C^\infty(\Omega)$, $\partial_y w_n = 0$, and as $n \to \infty$,

$$\|u-w_n\|_{L^2} \le \|u-v_n\|_{L^2} + \|v_n-w_n\|_{L^2} \le \|u-v_n\|_{L^2} + \|v_n\|_{L^2(\Omega\cap(E_n\times\mathbb{R}^M))} \to 0,$$

$$\int_{\Omega} (\partial_x w_n)^2 \, dx \, dy \le 2 \sum_{l=1}^{n} \|\chi_n'(x) u_{l,n}(x)\|^2_{L^2(U_l)} + \|\partial_x u_{l,n}(x)\|^2_{L^2(U_l)} < \infty.$$

Hence $w_n \in H^1_s$ and the last claim has been proven. ∎

The next lemma presents some tools for comparing $H^1_\varepsilon$ and $H^1_s$.

LEMMA 2.6. (i) *If $\varepsilon_n \to 0$ and $u_n \in L^p(G_j) \to u_0$ in $L^p(G_j)$, then*
$u_n|\det DT_{\varepsilon_n,j}(x,y)|\,|\det DT_j(S_{\varepsilon_n} \circ T_{\varepsilon_n,j}(x,y))| \to u_0|\det DT_j(x,0)|$
*in* $L^p(G_j)$, $p \ge 1$, $j = 1,2,3$.

(ii) *For $0 < \varepsilon \le 1$ define $\Phi^L_\varepsilon : L^2_s \to L^2_\varepsilon$ by $(\Phi^L_\varepsilon[u])_j = u_j$, $j = 1,2,3$, $(\Phi^L_\varepsilon[u])_4 = 0$. Then $\Phi^L_\varepsilon$ is continuous.*

(iii) *The linear operator $\Phi^H_\varepsilon : H^1_s \to H^1_\varepsilon$ of* (C7) *satisfies the following*:
   (1) $(\Phi^H_\varepsilon[u])_j = u_j$, $j = 1,2,3$, *and* (2.1) *holds*, *that is*, *as* $\varepsilon \to 0$,

$$C \sum_{j=1}^{3} \|u_j\|^2_{H^1(G_j)} \ge \varepsilon \|(\Phi^H_\varepsilon[u])_4\|^2_{L^2(G_{4,\varepsilon})} + \frac{1}{\varepsilon} \|D(\Phi^H_\varepsilon[u])_4\|^2_{L^2(G_{4,\varepsilon})} \to 0.$$

   (2) $\|\Phi^H_\varepsilon[u]\|_{H^1_\varepsilon} \le C_1 \|[u]\|_{H^1_s}$.
   (3) *If $[v_\varepsilon] \in L^2_\varepsilon$, $[v_0] \in L^2_s$, $[u] \in H^1_s$ are such that $\|[v_\varepsilon]\|_{L^2_\varepsilon} \le C_2 < \infty$, $v_{\varepsilon,j} \to v_{0,j}$ in $L^2(G_j)$, $j = 1,2,3$, then $([v_\varepsilon], \Phi^H_\varepsilon[u])_{L^2_\varepsilon} \to ([v_0],[u])_{L^2_s}$ as $\varepsilon \to 0$.*
   (4) $(\Phi^H_\varepsilon[u], \Phi^H_\varepsilon[v])_{L^2_\varepsilon} \to ([u],[v])_{L^2_s}$ *as* $\varepsilon \to 0$ *for all* $[u],[v] \in H^1_s$.
   (5) $\|\Phi^H_\varepsilon[u] - \Phi^L_\varepsilon[u]\|_{L^2_\varepsilon} \to 0$ *as* $\varepsilon \to 0$ *for all* $[u] \in H^1_s$.

*The constants $C_1, C_2$, are independent of $\varepsilon$.*

*Proof.* (C5) and Lemma 2.1 prove (i) and (ii).

Condition (C7) implies (iii)(1); and (iii)(2) follows from this and Lemmas 2.1 and 2.3.

Now let $[v_\varepsilon]$ be as in (3). Then by parts (iii)(1) and (i),

$$([v_\varepsilon], \Phi^H_\varepsilon[u])_{L^2_\varepsilon} = \sum_{j=1}^{3} \int_{G_j} v_{\varepsilon,j} u_j \, d\lambda_{\varepsilon,j} + \varepsilon \underbrace{\int_{G_{4,\varepsilon}} v_{\varepsilon,4} (\Phi^H_\varepsilon[u])_4 \, dz}_{\le (c_2/\sqrt{\varepsilon}) \|(\Phi^H_\varepsilon[u])_4\|_{L^2(G_{4,\varepsilon})}} .$$

This proves (iii)(3) and together with (2) also (4). Part (5) follows from (1). ∎

We collect some facts about $\|\cdot\|_{\varepsilon,d}$.

LEMMA 2.7. *Let $0 \le d \le 1$.*

(i) *$\|\cdot\|_{\varepsilon,d}$ is equivalent to $\|\cdot\|_{H^1_\varepsilon}$.*

(ii) *There is a constant $C > 0$, independent of $d$ and $\varepsilon$, such that $\|\cdot\|_{\varepsilon,d} \le C\|\cdot\|_{H^1_\varepsilon}$ and $C^{-1}\|\cdot\|_{H^1_\varepsilon} \le \|\cdot\|_{\varepsilon,1}$.*

(iii) *If* $\widehat{u}_\varepsilon \in H^1(\Omega_\varepsilon)$ *corresponds to* $[u_\varepsilon] \in H^1_\varepsilon$, *then* $\|[u_\varepsilon]\|_{\varepsilon,1}$ *bounded as* $\varepsilon \to 0$ *implies* $\varepsilon^{-M}\|\widehat{u}_\varepsilon\|_{H^1(\Omega_\varepsilon)}$ *is also bounded.*

(iv) $\|[u]\|^2_{H^1_\varepsilon} = \|[u]\|^2_{L^2_\varepsilon} + a_\varepsilon([u],[u])$.

(v) *There is a* $C > 0$, *independent of* $\varepsilon$, *such that for all* $[u] \in H^1_s$,

$$Ca_0([u],[u]) \geq a_\varepsilon(\Phi^H_\varepsilon[u], \Phi^H_\varepsilon[u]),$$

$$\frac{1}{C}\|[u]\|^2_{H^1_s} \leq \|[u]\|^2_{L^2_s} + a_0([u],[u]) \leq C\|[u]\|^2_{H^1_s}.$$

*Proof.* Lemmas 2.1 and 2.3(i) prove (i) and (ii). (iii) is a consequence of (ii) and $\|\widehat{u}\|_{H^1(\Omega_\varepsilon)} = \varepsilon^M\|[u]\|_{H^1_\varepsilon}$. Part (iv) is obvious. The second inequality in (v) follows from Lemma 2.1, which also together with Lemmas 2.3(i) and (2.1) implies the first inequality. ■

The next two lemmas provide some rules which are helpful for working in $L^2_s$ and $L^2_\perp$.

Lemma 2.8. *Fix* $j \in \{1,2,3\}$. *If* $w \in L^\infty(G_j)$, *then* $w = w_s + w_\perp$, *where* $w_s \in L^\infty(G_j) \cap L^2_s(G_j)$, $w_\perp \in L^\infty(G_j) \cap L^2_\perp(G_j)$ *and* $\|w_s\|_\infty \leq \|w\|_\infty$, $\|w_\perp\|_\infty \leq 2\|w\|_\infty$.

*Proof.* Since $w \in L^2(G_j)$, we can decompose it as $w = w_s + w_\perp$ with $w_s \in L^2_s(G_j)$, $w_\perp \in L^2_\perp(G_j)$.

For $c, k > 0$ define $U_{c,k} := \{(x,y) \in G_j \mid c < u(x,y) < c+k\}$. It could be that for given $c, k$ we have $|U_{c,k}| = 0$, but if $U_{c,k}$ is not a nullset, then setting $\widetilde{w}_s(x,y) := w_s(x,y)$ if $c < w_s(x,y) < c+k$ and $\widetilde{w}_s(x,y) := 0$ elsewhere, we find by Proposition 2.1 that $\widetilde{w}_s \in L^2_s$ and thus

$$|U_{c,k}|c^2 \leq \int_{G_j} w_s\widetilde{w}_s\,dx\,dy = \int_{G_j} w\widetilde{w}_s\,dx\,dy \leq \|w\|_\infty(c+k)|U_{c,k}|.$$

This in turn would imply $c \leq \|w\|_\infty$. Thus $\{(x,y) \in G_j \mid w_s(x,y) > \|w\|_\infty\}$ is a nullset.

Considering $-w_s$ we see $\{(x,y) \in G_j \mid w_s(x,y) < -\|w\|_\infty\}$ is a nullset as well, i.e. $\|w_s\|_\infty \leq \|w\|_\infty$. Finally, $\|w_\perp\|_\infty \leq 2\|w\|_\infty$ follows immediately. ■

Lemma 2.9. *Fix* $j \in \{1,2,3\}$.

(i) *If* $v \in L^2_s(G_j)$ *and* $w \in L^\infty(]0,1[)$, *then* $(x,y) \mapsto v(x,y)w(x) \in L^2_s(G_j)$.

(ii) *If* $u \in L^2(G_j)$, *then*

$$\int_{G_j} uv\,dx\,dy = 0\ \forall v \in L^2_s(G_j) \Leftrightarrow \int_{G_j} uv\,d\lambda_j = 0\ \forall v \in L^2_s(G_j).$$

(iii) *If* $u, v \in L^2_s(G_j)$ *and* $w \in L^\infty(G_j) \cap L^2_\perp(G_j)$, *then* $\int_{G_j} uvw\,d\lambda_j(x,y) = 0$.

(iv) *If* $u \in L^2_s(G_j)$, $w \in L^\infty(G_j) \cap L^2_s(G_j)$, *then* $uw \in L^2_s(G_j)$.

*Proof.* (i) There are $w_n \in C_0^\infty(\mathbb{R})$ and $C > 0$ such that $\|w_n\|_{L^\infty(]0,1[)} \le C\|w\|_{L^\infty(]0,1[)}$, and $\|w_n - w\|_{L^2(]0,1[)} \to 0$ as $n \to \infty$.

There are also $v_n \in H_s^1(G_j)$ such that $\|v_n - v\|_{L^2(G_j)} \to 0$ as $n \to \infty$. But then $v_n w_n \in H_s^1(G_j)$ and

$$\|v_n w_n - vw\|_{L^2(G_j)} \le C\|w\|_{L^\infty(]0,1[)}\|v_n - v\|_{L^2(G_j)} + \|v(w_n - w)\|_{L^2(G_j)}.$$

The first term tends to 0 as $n \to \infty$. Since $v(w_n - w) \to 0$ almost everywhere on $G_j$ and $|v(w_n - w)(x,y)| \le |v(x,y)|(1+C)\|w\|_{L^\infty(]0,1[)}$, by the Dominated Convergence Theorem we also get $\|v(w_n - w)\|_{L^2(G_j)} \to 0$. This proves $vw \in L_s^2(G_j)$.

By (i) and Lemma 2.1, $v \in L_s^2(G_j)$ iff $v|\det DT_j(x,0)| \in L_s^2(G_j)$, hence (ii) holds. Using (ii) it is sufficient to prove $\int_{G_j} uvw\,dx\,dy = 0$ to prove (iii). So let $u, v, w$ be as in (iii). For $m \in \mathbb{N}$ let $u_m$ be the truncated function $u_m(x,y) = u(x,y)$ if $|u(x,y)| < m$ and $u_m(x,y) = 0$ elsewhere. Analogously define $v_m$. These functions are locally functions of $x$ only, hence by Proposition 2.1, $u_m v_m \in L_s^2(G_j)$ and (iii) follows from

$$0 = \int_{G_j} u_m v_m w\,dx\,dy \to \int_{G_j} uvw\,dx\,dy, \qquad m \to \infty.$$

Now let $u, w$ be as in (iv). By Proposition 2.1, $u, w$ are locally functions of $x$ only, hence so is $vw \in L^2(G_j)$ as well, and $vw \in L_s^2(G_j)$ follows. ■

We shall need the following estimates on the linear semigroups $e^{-A_\varepsilon t}$, $e^{-A_0 t}$.

LEMMA 2.10. *The following hold*:

(2.12) (i) $\|e^{-A_\varepsilon t}[u]\|_{L_\varepsilon^2} \le \|[u]\|_{L_\varepsilon^2}$, $\forall [u] \in L_\varepsilon^2,\ t \ge 0$,

(2.13) (ii) $\|e^{-A_0 t}[u]\|_{L_s^2} \le \|[u]\|_{L_s^2}$, $\forall [u] \in L_s^2,\ t \ge 0$,

(2.14) (iii) $\|e^{-A_\varepsilon t}[u]\|_{H_\varepsilon^1} \le \sqrt{1 + \frac{1}{2t}}\,\|[u]\|_{L_\varepsilon^2}$, $\forall [u] \in L_\varepsilon^2,\ t > 0$,

(2.15) (iv) $\|e^{-A_0 t}[u]\|_{H_s^1} \le C\sqrt{1 + \frac{1}{2t}}\,\|[u]\|_{L_s^2}$, $\forall [u] \in L_s^2,\ t > 0$,

*where the constant $C > 0$ in* (iv) *is independent of* $t$, $[u]$.

*Proof.* The inequalities can be proven easily by expressing each vector with respect to the ONS of eigenvectors, and using Lemma 2.7(iv) together with the fact that all eigenvalues are positive and tend to infinity. Note that the constant $C$ in (iv) exists by Lemma 2.7(v). ■

We shall prove the convergence of the resolvents by following the ideas of [2] and [5]. To be able to do this, we need two more technical lemmas.

LEMMA 2.11. *Let $\varepsilon_n \downarrow 0$, $C > 0$, $[u_n] \in D(A_{\varepsilon_n})$ and $[w_n] \in H_{\varepsilon_n}^1$ be such that* $\|[u_n]\|_{H_{\varepsilon_n}^1}, \|A_{\varepsilon_n}[u_n]\|_{L_{\varepsilon_n}^2} \le C$, $\|w_n\|_{L_{\varepsilon_n}^2} \to 0$, $\|D_x w_{n,j}\|_{L^2(G_j)} \to 0$,

*$\varepsilon_n^{-1} D_y w_{n,j} \to w_{0,j} = (w_{0,j,1}, \dots, w_{0,j,M})$ in $L^2(G_j)$, $j = 1,2,3$, and $\|Dw_{n,4}\|_{L^2(G_{4,\varepsilon_n})} \le \varepsilon_n$. Assume $u_{n,j} \rightharpoonup u_{0,j} \in H^1(G_j)$ weakly in $H^1(G_j)$ and $\varepsilon_n^{-1} D_y u_{n,j} \rightharpoonup \widetilde{u}_{0,j}$ weakly in $L^2(G_j)$, $j = 1,2,3$. Then*

$$\sum_{j=1}^{3} \int_{G_j} (D_x u_{0,j}(1, \mathcal{T}_j(x,y)) + (0, \widetilde{u}_{0,j})) \times DT_j^{-1}(x,0)(DT_j^{-1}(x,0))^T (0, w_{0,j})^T \, d\lambda_j = 0.$$

*Proof.* We have

$$\begin{aligned}
0 \leftarrow (A_n[u_n],[w_n])_{L^2_n} &= a_n([u_n],[w_n]) \\
&= \sum_{j=1}^{3} \Bigg( \int_{G_j} D_x u_{n,j} \underbrace{(1,0)\mathcal{A}_{n,j}\mathcal{A}_{n,j}^T (1,0)^T D_x w_{n,j}}_{\to 0 \text{ in } L^2} \, d\lambda_{n,j} \\
&\quad + \int_{G_j} \Bigg( D_x u_{n,j} \quad \underbrace{(1,0)\mathcal{A}_{n,j}\mathcal{A}_{n,j}^T \left(0, \frac{1}{\varepsilon_n} D_y w_{n,j}\right)^T}_{\to (1,\mathcal{T}_j) DT_j^{-1}(x,0)(DT_j^{-1}(x,0))^T (0,w_{0,j})^T \text{ in } L^2} \\
&\quad + \left(0, \frac{1}{\varepsilon_n} D_y u_{n,j}\right) \underbrace{\mathcal{A}_{n,j}\mathcal{A}_{n,j}^T (1,0)^T D_x w_{n,j}}_{\to 0 \text{ in } L^2} \Bigg) \, d\lambda_{n,j} \\
&\quad + \int_{G_j} \left(0, \frac{1}{\varepsilon_n} D_y u_{n,j}\right) \quad \underbrace{\mathcal{A}_{n,j}\mathcal{A}_{n,j}^T \left(0, \frac{1}{\varepsilon_n} D_y w_{n,j}\right)^T}_{\to \left(\begin{smallmatrix} 1 & \mathcal{T}_j \\ 0 & E_M \end{smallmatrix}\right) DT_j^{-1}(x,0)(DT_j^{-1}(x,0))^T (0,w_{0,j})^T} \quad d\lambda_{n,j} \Bigg) \\
&\quad + \underbrace{\frac{1}{\varepsilon_n} \int_{G_{4,n}} Du_{n,4} Dw_{n,4} \, dz}_{\le \sqrt{\varepsilon_n} C_1} \\
&\to \sum_{j=1}^{3} \int_{G_j} (D_x u_{0,j}(1,\mathcal{T}_j) DT_j^{-1}(x,0)(DT_j^{-1}(x,0))^T (0,w_{0,j})^T \\
&\quad + (0,\widetilde{u}_{0,j}) DT_j^{-1}(x,0)(DT_j^{-1}(x,0))^T (0,w_{0,j})^T) \, d\lambda_j,
\end{aligned}$$

where $C_1 > 0$ is a constant, and we have applied Lemmas 2.3 and 2.6. ∎

LEMMA 2.12. *Let $\varepsilon_n \to 0$, $C > 0$, and $[u_n] \in D(A_{\varepsilon_n})$ be a sequence such that $\|[u_n]\|_{H^1_{\varepsilon_n}}, \|A_{\varepsilon_n}[u_n]\|_{L^2_{\varepsilon_n}} \le C$. Then there are a subsequence, called $\varepsilon_n$ again, and $[u_0] \in H^1_s$ such that $u_{n,j} \rightharpoonup u_{0,j}$ weakly in $H^1(G_j)$ and strongly in $L^2(G_j)$, $j = 1,2,3$, as $n \to \infty$. Moreover, $\varepsilon_n^{-1} D_y u_{n,j} \rightharpoonup \widetilde{u}_{0,j} \in (L^2(G_j))^M$ as $n \to \infty$, weakly in $L^2$. Decompose $L^2(G_1) \times L^2(G_2) \times L^2(G_3) = L^2_s \oplus L^2_\perp$*

*to get* $\widetilde{u}_{0,j,l} = \widetilde{u}_{0,j,l,s} + \widetilde{u}_{0,j,l,\perp}$, $l = 1,\dots,M$, $\mathcal{T}_j = \mathcal{T}_{j,s} + \mathcal{T}_{j,\perp}$; *then for* $j = 1,2,3$,

$\widetilde{u}_{0,j,s} = D_x u_{0,j}(|(1,0)DT_j^T(x,0)|^{-2}(1,0)DT_j^T(x,0)DT_j(x,0)(0,E_M)^T - \mathcal{T}_{j,s})$. *Also*

$$a_{\varepsilon_n}([u_n], \Phi^H_{\varepsilon_n}[v]) \to a_0([u_0],[v]) \quad \forall [v] \in H^1_s,\ n \to \infty.$$

*Proof.* Since $\|[u_n]\|_{H^1_n}$ is bounded, so is $\|u_{n,j}\|_{H^1(G_j)}$, and a suitable subsequence satisfies $u_{n,j} \rightharpoonup u_{0,j} \in H^1(G_j)$ weakly. By Lemma 2.3, the sequence $\varepsilon_n^{-1}\|D_y u_{n,j}\|_{L^2(G_j)}$ is bounded, i.e. $\|D_y u_{n,j}\|_{L^2(G_j)} \to 0$ and $u_{0,j} \in H^1_s(G_j)$, $j = 1,2,3$.

Also—taking again a subsequence—there are $\widetilde{u}_{0,j} \in (L^2(G_j))^M$ such that $\varepsilon_n^{-1} D_{y_l} u_{n,j} \rightharpoonup \widetilde{u}_{0,j,l}$ in $L^2(G_j)$, $l = 1,\dots,M$, $j = 1,2,3$.

The boundedness of $[u_n]$ in $H^1_n$ shows that $([u_n])_n$ is admissible in the definition of $H^1_s$ and thus $[u_0] \in H^1_s$. Now let $[v] \in H^1_s$. Then

$$\begin{aligned}
&a_n([u_n], \Phi^H_n[v]) \\
&\quad = \sum_{j=1}^{3} \int_{G_j} \Big(D_x u_{n,j}, \frac{1}{\varepsilon_n} D_y u_{n,j}\Big) \mathcal{A}_{n,j}\mathcal{A}^T_{n,j}(1,0)^T D_x v_j \, d\lambda_{n,j} \\
&\qquad \underbrace{+ \frac{1}{\varepsilon_n} \int_{G_{4,n}} D u_{n,4} D(\Phi^H_n[v])_4 \, dx\, dy}_{:= c(n) \to 0} \\
&\quad = \sum_{j=1}^{3} \int_{G_j} \Big( D_x u_{n,j} \underbrace{|(1,0)\mathcal{A}_{n,j}(x,y)|^2 D_x v_j}_{\to |(1,\mathcal{T}_j) DT_j^{-1}|^2 D_x v_j \text{ in } L^2} \\
&\qquad + \Big(0, \frac{1}{\varepsilon_n} D_y u_{n,j}\Big) \underbrace{\mathcal{A}_{n,j}(x,y)\mathcal{A}^T_{n,j}(x,y)(1,0)^T D_x v_j}_{\to \left(\begin{smallmatrix} 1 & \mathcal{T}_j \\ 0 & E_M \end{smallmatrix}\right) DT_j^{-1}((1,\mathcal{T}_j)DT_j^{-1})^T D_x v_j \text{ in } L^2} \Big) d\lambda_{n,j} + c(n)
\end{aligned}$$

$$\tag{2.16} \begin{aligned} \to \sum_{j=1}^{3} \int_{G_j} &(D_x u_{0,j} |(1,\mathcal{T}_j(x,y)) DT_j^{-1}(x,0)|^2 \\ &+ (0,\widetilde{u}_{0,j}) DT_j^{-1}(x,0)(DT_j^{-1}(x,0))^T (1,\mathcal{T}_j(x,y))^T) D_x v_j \, d\lambda_j, \end{aligned}$$

where we made use of Lemmas 2.3 and 2.6. Unfortunately we do not know $\widetilde{u}_{0,j}$. But using certain test functions $w_n \in H^1_n$ we can eliminate it in (2.16).

First we use test functions related to $\mathcal{T}_j$. Denote by $(T_{n,j})_x : \mathbb{R} \times \mathbb{R}^M \to \mathbb{R}$ the $x$-component of $T_{n,j}$. We now fix $j \in \{1,2,3\}$, $u_j \in H^1_s(G_j)$, and $c_n \le \|\mathrm{proj}_x - (T_{n,j})_x\|^{-1/2}_{L^\infty(G_j)}$, with $c_n \to \infty$ (see (C3)), and set

$$w_{n,j}(x,y) := -(x - (T_{n,j})_x(x,y))\chi(c_n x) u_j(x,y).$$

Then $w_{n,j} \in H^1(G_j)$, $w_{n,j}(x,y) \to 0$ in $L^2(G_j)$ and $w_{n,j}(x,y) \equiv 0$ for $x$ close enough to 0.

If we set $w_{n,l} := 0$ for $j \neq l = 1,\dots,4$, then $[w_n] = [w_{n,1},\dots,w_{n,4}] \in H_n^1$ and $\|[w]\|_{L_n^2} \to 0$. Using (C3)–(C5) one can show that $\|D_x w_{n,j}\|_{L^2(G_j)}$ and $\|\varepsilon_n^{-1} D_y w_{n,j} - \mathcal{T}_j(x,y)u_j\|_{L^2(G_j)}$ tend to 0 as $n \to \infty$. We can apply Lemma 2.11 to get

$$\int_{G_j} (D_x u_{0,j}(1,\mathcal{T}_j) + (0,\widetilde{u}_{0,j}))DT_j^{-1}(x,0)(DT_j^{-1}(x,0))^T(0,\mathcal{T}_j)^T u_j \, d\lambda_j = 0. \tag{2.17}$$

By density (2.17) holds for all $u_j \in L_s^2(G_j)$.

For the second set of test functions fix again for a moment $j \in \{1,2,3\}$, $m \in \mathbb{N}$, $l \in \{1,\dots,M\}$ and $u_j \in H_s^1(G_j)$. Set

$$w_{n,l}(x,y) := \begin{cases} u_j(x,y)\chi(mx)y_l, & (x,y) \in G_j,\\ 0, & (x,y) \neq G_j. \end{cases}$$

Then $[w_{n,l}] \in H_n^1$ and $[\varepsilon_n w_{n,l}]$ satisfies all conditions as a test function in Lemma 2.11, with

$$\frac{1}{\varepsilon_n} D_{y_{k_1}} w_{n,l,k_2} = \begin{cases} 0, & l \neq k_1 \text{ or } j \neq k_2,\\ u_j(x,y)\chi(mx), & l = k_1,\ j = k_2. \end{cases}$$

We get

$$\int_{G_j} (D_x u_{0,j}(1,\mathcal{T}_j) + (0,\widetilde{u}_{0,j}))DT_j^{-1}(x,0)(DT_j^{-1}(x,0))^T(0,e_l)^T u_j \chi(mx) \, d\lambda_j = 0,$$

where $e_l$ is the $l$th unit vector. Letting $m \to \infty$ yields

$$0 = \int_{G_j} (D_x u_{0,j}(1,\mathcal{T}_j(x,y)) + (0,\widetilde{u}_{0,j}))DT_j^{-1}(x,0)(DT_j^{-1}(x,0))^T(0,e_l)^T u_j \, d\lambda_j \tag{2.18}$$

for all possible $j$ and $l$, and by density also for all $u_j \in L_s^2(G_j)$.

Write $\widetilde{u}_{0,j} = \widetilde{u}_{0,j,s} + \widetilde{u}_{0,j,\perp}$, $\mathcal{T}_j = \mathcal{T}_{j,s} + \mathcal{T}_{j,\perp}$, with $\widetilde{u}_{0,j,s}, \mathcal{T}_{j,s} \in L_s^2(G_j)$, $\widetilde{u}_{0,j,\perp}, \mathcal{T}_{j,\perp} \in L_\perp^2(G_j)$, $j = 1,2,3$. By (C5) and Lemma 2.8, $\mathcal{T}_{j,s}, \mathcal{T}_{j,\perp} \in L^\infty(G_j)$.

Now equation (2.18) and Lemma 2.9 imply that for all $j = 1,2,3$ there is a $\widetilde{v}_j \in L_s^2(G_j)$ such that

$$\widetilde{v}_j(x,y)(1,0) = (D_x u_{0,j}(1,\mathcal{T}_{j,s}(x,y)) + (0,\widetilde{u}_{0,j,s}))DT_j^{-1}(x,0)(DT_j^{-1}(x,0))^T.$$

Thus

$$D_x u_{0,j} = \widetilde{v}_j(1,0)DT_j^T(x,0)DT_j(x,0)(1,0)^T = \widetilde{v}_j|(1,0)DT_j^T(x,0)|^2,$$

$$(0,\widetilde{u}_{0,j,s}) = D_x u_{0,j}(|(1,0)DT_j^T(x,0)|^{-2}(1,0)DT_j^T(x,0)DT_j(x,0) - (1,\mathcal{T}_{j,s})).$$

Insert this and (2.17) into (2.16), and then use Proposition 2.1 and Lemma 2.9 to obtain

$$\begin{aligned} a_n([u_n], \Phi_n^H[v]) &\\ \to \sum_{j=1}^{3} \int_{G_j} & D_x v_j (D_x u_{0,j}(1, \mathcal{T}_j) \\ &+ (0, \widetilde{u}_{0,j})) DT_j^{-1}(x,0)(DT_j^{-1}(x,0))^T (1,0)^T \, d\lambda_j \\ = \sum_{j=1}^{3} \int_{G_j} & D_x u_{0,j} D_x v_j |(1,0)) DT_j^T(x,0)|^{-2} \, d\lambda_j = a_0([u_0],[v]). \ \blacksquare \end{aligned}$$

Lemma 2.12 allows us to prove the convergence of the resolvents.

LEMMA 2.13. *Let* $\varepsilon_n \downarrow 0$, $\lambda \in \mathbb{C}$, *and* $|\arg(\lambda - 1/2)| < \pi - \delta$ *for some small* $\delta > 0$. *Assume* $[w_n] \in L^2_{\varepsilon_n}$ *and* $[w_0] \in L^2_s$ *are such that* $\|[w_n] - \Phi^L_{\varepsilon_n}[w_0]\|_{L^2_{\varepsilon_n}} \to 0$ *as* $n \to \infty$. *Set* $[u_n] := (A_{\varepsilon_n} + \lambda I)^{-1}[w_n]$ *and* $[u_0] := (A_0 + \lambda I)^{-1}[w_0]$. *Then there is a constant* $C > 0$, *independent of* $\lambda$, *such that* $\|[u_n]\|_{\varepsilon_n,1} < C$ *for all* $n$, *and as* $n \to \infty$, *for* $j = 1, 2, 3$: $u_{n,j} \to u_{0,j}$ *in* $H^1(G_j)$, $\varepsilon_n^{-1} D_y u_{n,j} \to D_x u_{0,j}(|(1,0)DT_j^T(x,0)|^{-2}(1,0)DT_j^T(x,0)DT_j(x,0)(0,E_M)^T - \mathcal{T}_j)$ *in* $L^2(G_j)$, *and* $\varepsilon_n \|u_{n,4}\|^2_{L^2(G_{4,\varepsilon_n})} + \varepsilon_n^{-1}\|Du_{n,4}\|^2_{L^2(G_{4,\varepsilon_n})} \to 0$. *In particular we have* $\|[u_n] - \Phi^H_{\varepsilon_n}[u_0]\|_{\varepsilon_n,d} \to 0$ *as* $n \to \infty$, *for all* $0 \le d < 1$.

*Proof.* By Lemma 2.6(ii), $\|\Phi^L_n[w_0]\|_{L^2_n} \le C_1$. Using the numerical range it is not difficult to show

$$\|[u_n]\|_{L^2_n} \le C_2 \frac{\|[w]\|_{L^2_n}}{|\lambda| - C_3},$$

where $C_3 > 0$ can be chosen arbitrarily small and $C_2 = C_2(c_3) > 0$ is independent of $n$ and $\lambda$ (see e.g. Exercise 6, Chapter 1.3 of [11]). Together with Lemma 2.7(iv) this gives

$$\begin{aligned} \|[u_n]\|^2_{H^1_n} &= \|[u_n]\|^2_{L^2_n} + a_n([u_n],[u_n]) \\ &= ((A_n + \lambda I)[u_n],[u_n])_{L^2_n} - (\lambda - 1)\|[u_n]\|^2_{L^2_n} \\ &\le (\|[w_n]\|_{L^2_n} + (|\lambda| + 1)\|[u_n]\|_{L^2_n})\|[u_n]\|_{L^2_n} \le C_4 \|[w_n]\|_{L^2_n}\|[u_n]\|_{L^2_n}, \end{aligned}$$

where $C_4$ is a constant independent of $n$ and $\lambda$. This in turn implies the boundedness of $\|[u_n]\|_{H^1_n}$ and hence of $\|[u_n]\|_{n,1}$ as well (see Lemma 2.7).

We can apply Lemma 2.12: there is a subsequence, called $\varepsilon_n$ again, and $[\widetilde{u}_0] \in H^1_s$ such that $u_{n,j} \rightharpoonup \widetilde{u}_{0,j}$ in $H^1(G_j)$, $j = 1, 2, 3$, and $a_n([u_n], \Phi^H_n[v]) \to a_0([\widetilde{u}_0],[v])$ for all $[v] \in H^1_s$. Since also

$$\begin{aligned} a_n([u_n], \Phi^H_n[v]) &= (A_n[u_n], \Phi^H_n[v])_{L^2_n} = ([w_n], \Phi^H_n[v])_{L^2_n} - \lambda([u_n], \Phi^H_n[v])_{L^2_n} \\ &\to ([w_0],[v])_{L^2_s} - \lambda([\widetilde{u}_0],[v])_{L^2_s} \end{aligned}$$

by Lemma 2.6(iii), we get $[\widetilde{u}_0] \in D(A_0)$ and $[\widetilde{u}_0] = (A_0 + \lambda I)^{-1}[w_0] = [u_0]$.

Lemma 2.12 also shows $\varepsilon_n^{-1} D_y u_{n,j} \rightharpoonup \widetilde{v}_{0,j}$ and

$$\widetilde{v}_{0,j,s} = D_x u_{0,j}(|(1,0)DT_j^T(x,0)|^{-2} \times (1,0)DT_j^T(x,0)DT_j(x,0)(0,E_M)^T - \mathcal{T}_{j,s}). \tag{2.19}$$

Now

$$\begin{aligned}
0 &\leftarrow ([w_n],[u_n])_{L^2_n} - ([w_0],[u_0])_{L^2_s} \\
&= a_n([u_n],[u_n]) - a_0([u_0],[u_0]) + \lambda(([u_n],[u_n])_{L^2_n} - ([u_0],[u_0])_{L^2_s}) \\
&= \sum_{j=1}^{3} \Bigg( \int_{G_j} \Bigg( \bigg| (D_x u_{0,j}, \widetilde{v}_{0,j}) \begin{pmatrix} 1 & \mathcal{T}_j(x,y) \\ 0 & E_M \end{pmatrix} DT_j^{-1}(x,0) \bigg|^2 \\
&\quad - (D_x u_{0,j})^2 |(1,0)DT_j^T(x,0)|^{-2} \Bigg) \, d\lambda_j \\
&\quad + \int_{G_j} \Bigg[ \bigg| \underbrace{\left(D_x u_{n,j}, \frac{1}{\varepsilon_n} D_y u_{n,j}\right) \mathcal{A}_{n,j}(x,y) \sqrt{|\det DT_{n,j}|\,|\det DT_j(S_n \circ T_{n,j})|}}_{\rightharpoonup (D_x u_{0,j}, \widetilde{v}_{0,j}) \left(\begin{smallmatrix} 1 & \mathcal{T}_j \\ 0 & E_M \end{smallmatrix}\right) DT_j^{-1} \sqrt{|\det DT_j|}} \bigg|^2 \\
&\quad \underbrace{- \bigg| (D_x u_{0,j}, \widetilde{v}_{0,j}) \begin{pmatrix} 1 & \mathcal{T}_j(x,y) \\ 0 & E_M \end{pmatrix} DT_j^{-1}(x,0) \bigg|^2 |\det DT_j(x,0)|}_{=:E_{1,n,j}} \Bigg] \, dx\, dy \\
&\quad + \underbrace{\lambda \int_{G_j} (u_{n,j}^2 |\det DT_{n,j}|\,|\det DT_j(S_n \circ T_{n,j})| - u_{0,j}^2 |\det DT_j(x,0)|) \, dx\, dy}_{=:E_{2,n,j} \to 0} \Bigg) \\
&\quad + \underbrace{\int_{G_{4,n}} \left( \frac{1}{\varepsilon_n} |Du_{n,4}|^2 + \varepsilon_n \lambda (u_{n,4})^2 \right) dz}_{=:E_{3,n}} \\
&= \sum_{j=1}^{3} \Bigg( \int_{G_j} (| \underbrace{(D_x u_{0,j}, D_x u_{0,j} \mathcal{T}_{j,s} + \widetilde{v}_{0,j,s}) DT_j^{-1}}_{=(D_x u_{0,j})^2 |(1,0)DT_j^T|^{-2}} |^2 \\
&\quad - (D_x u_{0,j})^2 |(1,0)DT_j^T|^{-2}) \, d\lambda_j \\
&\quad + \underbrace{\int_{G_j} |(0, D_x u_{0,j} \mathcal{T}_{j,\perp} + \widetilde{v}_{0,j,\perp}) DT_j^{-1}|^2 \, d\lambda_j}_{=:E_{4,j} \geq 0} + E_{1,n,j} + E_{2,n,j} \Bigg) + E_{3,n} \\
&= \sum_{j=1}^{3} (E_{1,n,j} + E_{2,n,j} + E_{4,j}) + E_{3,n},
\end{aligned}$$

where we have used equation (2.19), decomposed $\mathcal{T}_j$ and $\widetilde{v}_{0,j}$ as in that equation, and used Proposition 2.1 and Lemmas 2.3, 2.6 and 2.9.

Now $\liminf_{n\to\infty} E_{1,n,j} \geq 0$ and either $\lambda > 0$ or $\operatorname{Im}\lambda \neq 0$. In both cases necessarily $E_{1,n,j}, E_{3,n} \to 0$ as $n \to \infty$, $j = 1,2,3$, which in turn implies $D_x u_{0,j}\mathcal{T}_{j,\perp} = -\widetilde{v}_{0,j,\perp}$, $D_x u_{n,j} \to D_x u_{0,j}$ and $\varepsilon_n^{-1} D_y u_{n,j} \to \widetilde{v}_{0,j}$ in $L^2(G_j)$. Hence with (2.19),

$$\widetilde{v}_{0,j} = D_x u_{0,j}(|(1,0)DT_j^T(x,0)|^{-2}(1,0)DT_j^T(x,0)DT_j(x,0)(0,E_M)^T - \mathcal{T}_j)$$

for $j = 1,2,3$, and

$$\lim_{n\to\infty} \frac{1}{\varepsilon_n} \|Du_{n,4}\|^2_{L^2(G_{4,n})} = \lim_{n\to\infty} \varepsilon_n \|u_{n,4}\|^2_{L^2(G_{4,n})} = 0.$$

The remaining convergence $\|[u_n] - \Phi_n^H[u_0]\|_{n,d} \to 0$ as $n \to \infty$ follows easily from the definition of $\|\cdot\|_{\varepsilon,d}$, $d < 1$. ∎

We have now prepared everything to prove the main results of this paper.

*Proof of Theorem 1.1.* Let $0 < t_1 \leq t$. Then by Lemmas 2.10 and 2.7,

$$\begin{aligned}
&\|e^{-A_n t}[u_n] - \Phi_n^H(e^{-A_0 t}[u_0])\|_{n,d} \\
&\quad \leq \|e^{-A_n t}[u_n] - e^{-A_n t}\Phi_n^L[u_0]\|_{n,d} + \|e^{-A_n t}\Phi_n^L[u_0] - \Phi_n^H(e^{-A_0 t}[u_0])\|_{n,d} \\
&\quad \leq C_1\|[u_n] - \Phi_n^L[u_0]\|_{L^2_n} + \|e^{-A_n t}\Phi_n^L[u_0] - \Phi_n^H(e^{-A_0 t}[u_0])\|_{n,d},
\end{aligned}$$

where $C_1 = C_1(t_1)$ is independent of $n$ and $t$. Hence it is sufficient to show

$$\|e^{-A_n t}\Phi_n^L[u_0] - \Phi_n^H(e^{-A_0 t}[u_0])\|_{n,d} \to 0$$

and that the $\|\cdot\|_{\varepsilon,1}$-norm of the above expression is bounded, both uniformly on $[t_1, t_2]$.

If $\gamma : \,]-\infty,\infty[\, \to \{\lambda \in \mathbb{C} \mid |\arg(\lambda - 1/2)| < \pi - \delta\}$, $\delta > 0$ small, is as in the definition of linear semigroups, then for $t_1 \leq t \leq t_2$ and $[u_0] \in L^2_s$,

$$\begin{aligned}
&\|e^{-A_n t}\Phi_n^L[u_0] - \Phi_n^H(e^{-A_0 t}[u_0])\|_{n,d} \\
&\quad = \frac{1}{2\pi}\Big\| \int_\gamma ((\lambda I + A_n)^{-1} e^{\lambda t}\Phi_n^L[u_0] - \Phi_n^H((\lambda I + A_n)^{-1} e^{\lambda t}[u_0]))\, d\lambda \Big\|_{n,d} \\
&\quad \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \|(\gamma(s)I + A_n)^{-1}\Phi_n^L[u_0] - \Phi_n^H((\gamma(s)I + A_0)^{-1}[u_0])\|_{n,d} \\
&\qquad \times |e^{\gamma(s)t}|\,|\gamma'(s)|\, dx.
\end{aligned}$$

By Lemma 2.13 the integrand tends pointwise to 0 if $d < 1$, and is bounded for all $d$ by an integrable function independent of $t \in [t_1, t_2]$. This concludes the proof. ∎

*Proof of Theorem 1.2.* Given the situation of the theorem, it is well known that

$$\lambda_{n,l} = \min_{E \in \Theta_l} \max_{[u] \in E\setminus\{0\}} \frac{a_n([u],[u])}{\|[u]\|^2_{L^2_n}},$$

where $\Theta_l$ is the set of all $l$-dimensional linear subspaces of $H^1_n$ (see e.g. Proposition 2.2 in [14]).

Fix $l_0 \in \mathbb{N}$. If $[v_{0,l}],\ 1 \le l \le l_0$, are $l_0$ independent eigenvectors of $A_0$, then $\Phi^H_n[v_{0,l}],\ 1 \le l \le l_0$, are $l_0$ independent vectors in $H^1_n$. Hence

$$\lambda_{n,l_0} \le \max_{(\alpha_1,\dots,\alpha_{l_0})\in\mathbb{R}^{l_0}\setminus\{0\}} \frac{a_n(\Phi^H_n(\sum_{l=1}^{l_0}\alpha_l[v_{0,l}]), \Phi^H_n(\sum_{l=1}^{l_0}\alpha_l[v_{0,l}]))}{\|\Phi^H_n(\sum_{l=1}^{l_0}\alpha_l[v_{0,l}])\|^2_{L^2_n}}$$

Using Lemmas 2.7(iv), (v) and 2.6(iii) it is straightforward to show that $\lambda_{n,l_0} \le C_2$ for some constant $C_2$ independent of $n$.

Take a subsequence $\lambda_{n,l_0} \to \mu_0 \in \mathbb{R}$. Then $\lambda_{n,l_0} = a_n([u_{n,l_0}],[u_{n,l_0}])$ shows that $(\|[u_{n,l_0}]\|_{H^1_n})_n$, and hence $(\|u_{n,l_0}\|_{n,1})_n$, is bounded. We can apply Lemma 2.12: taking again a subsequence, there is a $[\widetilde{u}_{l_0}] \in H^1_s$ such that $u_{n,l_0,j} \to \widetilde{u}_{l_0,j}$ in $L^2(G_j)$, $j = 1,2,3$, which by (C8) implies $\|[\widetilde{u}_{l_0}]\|_{L^2_s} = 1$. This together with Lemmas 2.12, 2.6(iii) shows $(\mu_{l_0}, [\widetilde{u}_{l_0}])$ is an eigenvalue-vector pair for $A_0$ and by Theorem 1.1 and Lemma 2.6(ii) we conclude that $\|[u_{n,l_0}] - \Phi^H_n[\widetilde{u}_{l_0}]\|_{n,d} \to 0$ for $d < 1$.

Since $l_0$ was arbitrary, we can use the Cantor diagonal procedure to find a subsequence, called $\varepsilon_n$ again, such that for all $l \in \mathbb{N}$, $(\mu_l, [u_{0,l}])$ is an eigenvalue-vector pair for $A_0$, $\|[u_{0,l}]\|_{L^2_s} = 1$, $\lambda_{n,l} \to \mu_l$ and $\|[u_{n,l}] - \Phi^H_n[u_{0,l}]\|_{n,d} \to 0$ for $d < 1$.

It is easy to show $([u_{0,l}])_l$ is an ONS. The only thing we still have to show is the completeness, since then necessarily $\mu_l = \lambda_{0,l}$ for all $l \in \mathbb{N}$.

Let $[v] \in L^2_s$ be such that $([u_{0,l}],[v])_{L^2_s} = 0$ for all $l$ and $\|[v]\|_{L^2_s} = 1$. Since the set of all eigenvectors of $A_0$ forms a complete ONS of $L^2_s$, we can assume $[v]$ to be an eigenvector as well, i.e. $[v] \in H^1_s$.

Fix $l_1 \in \mathbb{N}$. Then by Lemma 2.6(iii) for any $\delta > 0$ there exists an $n_1 = n_1(\delta)$ such that for $n \ge n_1$ and $1 \le l \le l_1$,

$$|([u_{n,l}], \Phi^H_n[v])_{L^2_n}| < \delta.$$

Moreover, $\Phi^H_n[v] = \sum_{l\ge1}\alpha_{n,l}[u_{n,l}]$, where $\alpha_{n,l} = ([u_{n,l}],\Phi^H_n[v])_{L^2_n}$, and by the inequality above $|\alpha_{n,l}| \le \delta$ for $l = 1,\dots,l_1$ and $n$ large enough. It follows again by Lemma 2.6(iii) that

$$1 = ([v],[v])_{L^2_s} \leftarrow (\Phi^H_n[v], \Phi^H_n[v])_{L^2_n} = \sum_{l\ge1}\alpha^2_{n,l} \le l_1\delta^2 + \sum_{l>l_1}\alpha^2_{n,l}.$$

By Lemma 2.7(v) there exists a constant $C_3 > 0$, independent of $n$ and $l_1$, such that

$$\begin{aligned} C_3 a_0([v],[v]) &\ge a_n(\Phi^H_n[v], \Phi^H_n[v]) = \sum_{l\ge1}\alpha^2_{n,l}\lambda_{n,l} \\ &\ge \lambda_{l_1+1}\sum_{l>l_1}\alpha^2_{n,l} \ge \lambda_{l_1+1}(1 - l_1\delta^2). \end{aligned}$$

Since $\lambda_l \to \infty$ as $l \to \infty$, this is a contradiction. This means there is no such $[v]$, and $([u_{0,l}])_l$ is indeed a complete ONS of $L^2_s$. ∎

REMARK 2.1. (i) We do not have in general the convergence of the $n$th eigenvalue of $A_\varepsilon$ to the $n$th one of $A_0$. A very simple example with a domain consisting of two edges and one node is shown in Figure 4.

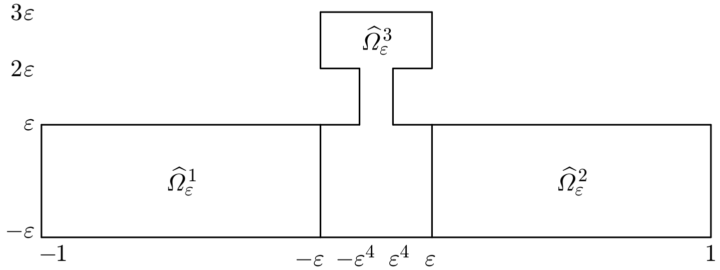


Fig. 4. An example of domains $\Omega_\varepsilon$ for which the $n$th eigenvector of $A_\varepsilon$ does not converge to the $n$th eigenvector of $A_0$.

(ii) In [14] (and other papers) the boundedness of a sequence $u_n \in H^1(\Omega)$ with respect to the $\varepsilon$-norm implies, by taking a subsequence, $u_n \to u \in H^1_s(\Omega)$ in $\|\cdot\|_{L^2}$. Thus uniformly bounded full solutions of the nonlinear $\varepsilon$-problems induce a full solution of the limit problem. This is then used to prove the upper-semicontinuity of the attractors at $\varepsilon = 0$ for equation (1.1).

In our case without condition (C8) we can have eigenvectors $[u_\varepsilon]$ for which the corresponding eigenvalues $\lambda_\varepsilon \to \lambda_0$, i.e. $[u_\varepsilon]$ is bounded in the $\varepsilon$-norm $\|.\|_{\varepsilon,d}$, but there is no $[u_0] \in H^1_s$ such that $\|[u_\varepsilon] - \Phi^L_n[u_0]\|_{L^2_\varepsilon} \to 0$ (see Figure 4 for an example). Hence in general we cannot expect the continuity of attractors as mentioned above.

If (C8) is satisfied however, then if $\varepsilon_n \to 0$, $[u_n] \in H^1_{\varepsilon_n}$ and $(\|[u_n]\|_{\varepsilon_n,1})_n$ is bounded, we can find a subsequence such that $u_{n,j} \to u_{0,j} \in H^1_s(G_j)$ in $L^2$, $j = 1,2,3$, $\varepsilon_n \|u_{n,4}\|^2_{L^2(G_{4,\varepsilon_n})} \to 0$, i.e. $[u_0] \in H^1_s$ and $\|[u_n] - \Phi^L_{\varepsilon_n}[u_0]\|_{L^2_{\varepsilon_n}} \to 0$ as $n \to \infty$.

Thus we can apply the method of [14], getting the upper-semicontinuity of attractors.

**3. Special cases.** In this section we will present concrete sufficient conditions which guarantee that conditions (C7) and (C8) of the previous section are satisfied and Theorems 1.3 and 1.4 hold (see hypotheses (C9), (C10) below). We will give an explicit description of the operator $A_0$ at the nodes under quite general assumptions (see Proposition 3.2) and present some examples. The first one shows (under weak additional assumptions) that one can cut an edge introducing a new node. In this way one can treat

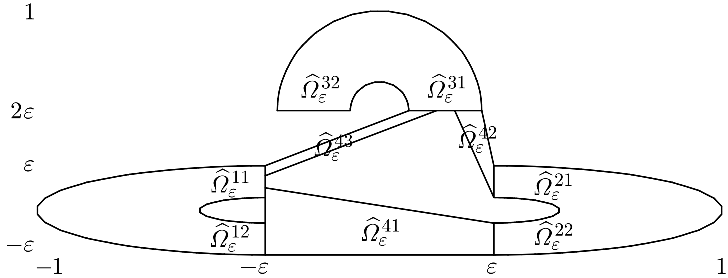


Fig. 5. An example of a nicely connected domain. Here $S_1 = \{(1,1),(1,2),(2,1),(2,2),(3,1)\}$, $S_2 = \{(3,2)\}$.

net-shaped domains having loops, for which the requirements of Section 2 are not satisfied. An alternative way of treating loops is to relax the requirement that $T_j$ is bijective, for example only supposing bijectivity on each of the two halves of $G_j$. The second example shows (under additional smoothness assumptions) that the domains $\Omega_\varepsilon$ can have holes which decrease at least proportionally with $\varepsilon$ (in all directions). This generalizes the domains treated e.g. in [14], which can have holes which scale proportionally to $\varepsilon$ but only in $y$-direction.

As before—unless stated otherwise—we will again treat the example of a net-shaped domain having three edges and one node which satisfies the conditions of Section 2, with the possible exception of (C7) and (C8).

Of crucial importance for condition (C7) is how the edges meet at a node. In [14] the authors define nicely decomposed domains; we will use a similar idea to define when edges connect nicely at a node (see Figure 5 for an example).

Definition 3.1. We say the edges $(G_j,\ j=1,2,3)$ *connect nicely* at the node $(G_{4,\varepsilon})$, or simply the domain $\Omega_\varepsilon$ is *nicely connected*, if the following is satisfied: There are $\delta, C > 0$, open, connected, Lipschitz, pairwise disjoint $G_{j,l} \subset G_j$, connected $\omega_{j,l,x} \subset \mathbb{R}^M$, $|\omega_{j,l,x}| \geq \delta$ for all $0 < x < \delta$, such that

$$G_{j,l} = \bigcup_{0<x<\delta} \{x\} \times \omega_{j,l,x}, \qquad G_j \cap (]0,\delta[ \times \mathbb{R}^M) = \bigcup_{l=1}^{L_j} G_{j,l}$$

for $l = 1,\dots,L_j,\ j = 1,2,3$. Set $S_\Omega := \{(j,l) \mid l = 1,\dots,L_j,\, j = 1,2,3\}$ and $\Omega_{j,l,\varepsilon} := \Psi_{\varepsilon,j}(G_{j,l}),\ (j,l) \in S_\Omega$.

If there are an $\varepsilon_1 > 0$ and $(j_i,l_i) \in S_\Omega$, $(0,y_i) \in \partial G_{j_i,l_i}$, $i = 1,2$, such that $\Psi_{\varepsilon_1,j_1}(0,y_1)$ and $\Psi_{\varepsilon_1,j_2}(0,y_2)$ belong to the same connected component of $\Omega_{4,\varepsilon_1}$, then there are an open, connected, bounded, Lipschitz $U_{j_1,l_1,j_2,l_2} \subset \Psi_{\varepsilon,4}^{-1}(\Omega_\varepsilon)$ and $r > 0$, both independent of $\varepsilon$, and open $U_{\varepsilon,j_i,l_i} = B_r(z_{\varepsilon,j_i,l_i}) \subset U_{j_1,l_1,j_2,l_2} \cap \Psi_{\varepsilon,4}^{-1}(\Omega_{j_i,l_i,\varepsilon})$, $\Psi_{\varepsilon,j_i}^{-1} \circ \Psi_{\varepsilon,4}(U_{\varepsilon,j_i,l_i}) \subset\ ]0,\varepsilon C[ \times \mathbb{R}^M$, for all $\varepsilon$ and

$i = 1, 2$. In this case we say $G_{j_1,l_1}$ and $G_{j_2,l_2}$ *join each other* (at the node $G_{4,\varepsilon}$). Note that $j_1 = j_2,\ l_1 \neq l_2$ could happen (see Figure 5 for an example).

If $\Omega_\varepsilon$ is nicely connected define an equivalence relation on $S_\Omega$ by: $(j,l)$ is equivalent to $(\widetilde{j},\widetilde{l})$ iff $(j,l) = (\widetilde{j},\widetilde{l})$ or there are $(j,l) = (j_1,l_1),\dots,(j_m,l_m) = (\widetilde{j},\widetilde{l})$ such that $G_{j_i,l_i}$ and $G_{j_{i+1},l_{i+1}}$, $i = 1,\dots,m-1$, join each other. Denote the equivalence classes by $S_k,\ k = 1,\dots,N_\Omega$. Note that the equivalence classes $S_k$ are independent of $\varepsilon$. Thus for a nicely connected domain there is a partition $S_1,\dots,S_{N_\Omega}$ of $S_\Omega$, independent of $\varepsilon$.

Consider the following hypotheses:

(C9) The domain $\Omega_\varepsilon$ is nicely connected.

(C10) One of the following holds:

(i) $G_{4,\varepsilon}$ has empty interior for all $\varepsilon > 0$.

(ii) There are $G_{4,1},\dots,G_{4,N_4} \subset \mathbb{R}^{M+1}$ open, bounded, connected, Lipschitz, $C > 0$, $\overline{G}_{4,j} \subset Q_j \subset \mathbb{R}^{M+1}$ open, and $C^1$-diffeomorphisms $\Psi_{\varepsilon,4,j} : Q_j \to \Psi_{\varepsilon,4,j}(Q_j) \subset \mathbb{R}^{M+1}$ such that

$$\frac{1}{C} \le |\det D\Psi_{\varepsilon,4,j}(z)|, \|D\Psi_{\varepsilon,4,j}(z)v\| \le C,$$

$$\Big|G_{4\varepsilon} \setminus \bigcup_{j=1}^{N_4} \Psi_{\varepsilon,4,j}(G_{4,j})\Big| = 0,$$

for all possible $z, j, \varepsilon$ and $v \in \mathbb{R}^{M+1},\ \|v\| = 1$. For all $k \in \{1,\dots,N_4\}$ there exist open, bounded, connected, Lipschitz $U_k \subset \Psi_{\varepsilon,4}^{-1}(\Omega_\varepsilon)$, $(j,l) \in S_\Omega$ and $r > 0$, all independent of $\varepsilon$, and open $U_{\varepsilon,k,j,l} = B_r(z_{\varepsilon,k,j,l}) \subset U_k \cap \Psi_{\varepsilon,4}^{-1}(\Omega_{j,l,\varepsilon})$, such that $|\Psi_{\varepsilon,4,k}^{-1}(U_k \cap \Psi_{\varepsilon,4,k}(G_{4,k}))| \ge 1/C$ and $\Psi_{\varepsilon,j}^{-1} \circ \Psi_{\varepsilon,4}(U_{\varepsilon,k,j,l}) \subset \,]0,\varepsilon C[\, \times \mathbb{R}^M$ for all $\varepsilon$.

Proposition 3.1. *Assume $\Omega_\varepsilon$ satisfies the requirements of Section 2, and conditions* (C1)–(C6), (C9) *hold. Then* (C7) *holds with*

$$H_s^1 = \{[u] \mid u_j \in H_s^1(G_j),\ j = 1,2,3, \text{ and } u_j|_{(\{0\}\times\mathbb{R}^M)\cap\partial G_{j,l}} = u_{\widetilde{j}}|_{(\{0\}\times\mathbb{R}^M)\cap\partial G_{\widetilde{j},\widetilde{l}}} \text{ if } (j,l),(\widetilde{j},\widetilde{l}) \in S_k \text{ for some } k \in \{1,\dots,N_\Omega\}\}. \tag{3.1}$$

*Note that* $(\{0\}\times\mathbb{R}^M)\cap\partial G_{j,l}$ *has positive measure and is connected, hence* $u_j|_{(\{0\}\times\mathbb{R}^M)\cap\partial G_{j,l}} \equiv \text{const}$ *for all possible fixed* $j, l$.

*If additionally* (C10) *holds, then so does* (C8).

*Proof.* In this proof $C_1, C_2,\dots$ denote positive constants which are independent of $\varepsilon$, respectively $n$ if $\varepsilon_n \to 0$, unless stated otherwise.

The set $(\{0\}\times\mathbb{R}^M)\cap\partial G_{j,l}$ has positive measure and is connected because $G_{j,l}$ is bounded, Lipschitz, $|\omega_{j,l,x}|\geq\delta$, and $\omega_{j,l,x}$ is connected, for all possible $j,l,x$.

Denote by $\widetilde{H}^1_s$ the set in (3.1). Note that it is closed in $H^1_s(G_1)\times H^1_s(G_2)\times H^1_s(G_3)$. To prove (C7) let $\varepsilon_n\to 0$, $[u_n]\in H^1_n$, $u_{n,j}\rightharpoonup u_{0,j}\in H^1(G_j)$, $(\varepsilon_n^{-1}\|D_y u_{n,j}\|_{L^2(G_j)})_n$ bounded, $j=1,2,3$, and $\sup_n(\varepsilon_n\|u_{n,4}\|^2_{L^2(G_{4,n})}+\varepsilon_n^{-1}[\varepsilon_n]\|Du_{n,4}\|^2_{L^2(G_{4,n})})<\infty$. Then $u_{0,j}\in H^1_s(G_j)$ and $u_{0,j}|_{(\{0\}\times\mathbb{R}^M)\cap\partial G_{j,l}}=c_{j,l}$ is constant, for all $(j,l)\in S_\Omega$.

We claim that $c_{j,l}=c_{\tilde{j},\tilde{l}}=c_k$ if $(j,l),(\widetilde{j},\widetilde{l})\in S_k$ for some $k\in\{1,\dots,N_\Omega\}$.

Assume for a moment the claim is true. Then $H^1_s\subset\widetilde{H}^1_s$. We can define $\Phi^H_\varepsilon:\widetilde{H}^1_s\to H^1_\varepsilon$ by $(\Phi^H_\varepsilon[u])_j:=u_j$, $j=1,2,3$, $(\Phi^H_\varepsilon[u])_4:=c_k$ on the connected component of $G_{4,\varepsilon}$ which has nonempty intersection with $\Psi^{-1}_{\varepsilon,4}\circ\Psi_{\varepsilon,j}(\partial G_{j,l})$ for some $(j,l)\in S_k$. Since $\Omega_\varepsilon$ is connected, $(\Phi^H_\varepsilon[u])_4$ is defined on all $G_{4,\varepsilon}$, it is well defined and $u_j|_{(\{0\}\times\mathbb{R}^M)\cap\partial G_{j,l}}=(\Phi^H_\varepsilon[u])_4|_{\Psi^{-1}_{\varepsilon,4}(\Omega_{4,\varepsilon}\cap\Psi_{\varepsilon,j}(\overline{G}_{j,l,\varepsilon}))}$, hence indeed $\Phi^H_\varepsilon[u]\in H^1_\varepsilon$. So $\Phi^H_\varepsilon$ satisfies all conditions of (C7). Also, if $[u]\in\widetilde{H}^1_s$, then $\Phi^H_{\varepsilon_n}[u]$ satisfies the condition for the sequence in the definition of $H^1_s$, hence $[u]\in H^1_s$, i.e. $\widetilde{H}^1_s\subset H^1_s$ and (C7) holds with $H^1_s=\widetilde{H}^1_s$.

Now we prove the claim. It is sufficient to prove $c_{j,l}=c_{\tilde{j},\tilde{l}}$ if $G_{j,l}$ and $G_{\tilde{j},\tilde{l}}$ join each other. For this it is sufficient to prove the following condition:

(∗) If $C,r>0$, $U_1\subset\Psi^{-1}_{\varepsilon,4}(\Omega_\varepsilon)$ is open, bounded, connected, Lipschitz, independent of $\varepsilon$, $U_{\varepsilon,0}=B_r(z_\varepsilon)\subset U_1\cap\Psi^{-1}_{\varepsilon,4}(\Omega_{j,l,\varepsilon})$ is open, $(j,l)\in S_\Omega$ is independent of $\varepsilon$, $\Psi^{-1}_{\varepsilon,j}\circ\Psi_{\varepsilon,4}(U_{\varepsilon,0})\subset\,]0,\varepsilon C[\,\times\mathbb{R}^M$ for all $\varepsilon$, and $u_{U,n}(z):=u_{n,k}\circ\Psi^{-1}_{n,k}\circ\Psi_{n,4}(z)$ if $z\in U_1\cap\Psi^{-1}_{n,4}(\Omega_{k,n})$, $k=1,\dots,4$, then there is a subsequence, called $\varepsilon_n$ as well, such that $\|u_{U,n}-c_{j,l}\|_{H^1(U_1)}\to 0$ as $n\to\infty$. (Recall that $c_{j,l}=u_{0,j}(\partial G_{j,l}\cap(\{0\}\times\mathbb{R}^M))$.)

We can extend $u_{n,j}$ to $\widetilde{u}_{n,j}\in H^1(\mathbb{R}^{M+1})$, $n\geq 0$. Then without loss of generality $\widetilde{u}_{n,j}\rightharpoonup\widetilde{u}_{0,j}$ in $H^1(\mathbb{R}^{M+1})$.

Using Lemmas 2.1, 2.3 and 2.4 we get $\|Du_{U,n}\|^2_{L^2(U_1)}\to 0$ as $n\to\infty$, and

$$\begin{aligned}\|u_{U,n}\|^2_{L^2(U_{n,0})}&\leq\int\limits_{\{(x,y)\in G_{j,l}\,|\,0<x<\varepsilon_n C\}}u^2_{n,j}|\det D\Psi_{n,j}|\varepsilon_n^{-M-1}\,dx\,dy\\&\leq\frac{C_4}{\varepsilon_n}\int\limits_{\{(x,y)\in G_{j,l}\,|\,0<x<\varepsilon_n C\}}((u_{n,j}(x,y)-\widetilde{u}_{n,j}(0,y))^2\\&\qquad+\widetilde{u}^2_{n,j}(0,y))\,dx\,dy\leq C_5.\end{aligned}$$

Taking a subsequence we can without loss of generality assume $z_n\to z_0$ and $B_r(z_0)\subset\overline{U}_1$. Thus, for $n$ large enough, $U_{n,0}\supset B_{r/2}(z_0)$.

Define $S_U := \{u \in H^1(U_1) \mid \|u\|^2_{L^2(B_{r/2}(z_0))} \le C_5\}$. Then $S_U$ is closed, convex, $0 \in S_U$ and the constant function $\widetilde{C}$ is in $S_U$ only if $\widetilde{C}^2 \le C_6$. The conditions of the general Poincaré inequality are satisfied (see e.g. 5.15 in [1]), hence there is a $C_7$ such that $\|u\|_{L^2(U_1)} \le C_7(\|Du\|_{L^2(U_1)} + 1)$ for all $u \in S_U$.

Since $u_{U,n} \in S_U$, the sequence $(\|u_{U,n}\|_{H^1(U_1)})_n$ is bounded, and a subsequence satisfies $u_{U,n} \to u \in S_U$ weakly in $H^1(U_1)$. But then $Du = 0$, $u \equiv c$ is a constant and $u_{U,n} \to c$ in $H^1(U_1)$. We get

$$\begin{aligned}
&\|u_{n,j} - c_{j,l}\|^2_{L^2(\Psi^{-1}_{n,j}\Psi_{n,4}(U_{n,0}))} \\
&\qquad \le C_8 \int\limits_{\{(x,y)\in G_{j,l}\,|\,0<x<\varepsilon_n C\}} ((\widetilde{u}_{n,j}(x,y) - \widetilde{u}_{n,j}(0,y))^2 \\
&\qquad\qquad + (\widetilde{u}_{n,j}(0,y) - \widetilde{u}_{0,j}(0,y))^2 + (\widetilde{u}_{0,j}(0,y) - c_{j,l})^2)\,dx\,dy \\
&\qquad \le \varepsilon_n C(C_9(n) + \|\widetilde{u}_{n,j} - \widetilde{u}_{0,j}\|^2_{L^2(\mathrm{proj}_y((]0,\varepsilon_n C[\times\mathbb{R}^M)\cap G_{j,l}))} \\
&\qquad\qquad + \|\widetilde{u}_{0,j} - c_{j,l}\|^2_{L^2(\mathrm{proj}_y((]0,\varepsilon_n C[\times\mathbb{R}^M)\cap G_{j,l}))}).
\end{aligned}$$

By Lemma 2.4, $C_9(n) \to 0$. Now, $\widetilde{u}_{n,j} \rightharpoonup \widetilde{u}_{0,j}$ in $H^1$ implies $\widetilde{u}_{n,j} \to \widetilde{u}_{0,j}$ as traces on $L^2(B_\delta(\partial G_{j,l} \cap (\{0\} \times \mathbb{R}^M)))$ for each $\delta > 0$ fixed, hence by (2.11) the second term above also tends to 0. Again (2.11) and $\widetilde{u}_{0,j} = c_{l,j}$ on $\partial G_{j,l} \cap (\{0\} \times \mathbb{R}^M)$ imply the last term above also tends to 0. We get

$$\frac{1}{\varepsilon_n}\|u_{n,j} - c_{j,l}\|_{L^2(\Psi^{-1}_{n,j}\Psi_{n,4}(U_{n,0}))} \to 0, \qquad n \to \infty.$$

On the other hand

$$\frac{1}{\varepsilon_n}\|u_{n,j} - c_{j,l}\|_{L^2(\Psi^{-1}_{n,j}\Psi_{n,4}(U_{n,0}))} \ge C_{11} \int\limits_{U_{n,0}} (u_{U,n} - c_{j,l})^2\,dz$$

and $c_{j,l} = c$ follows. This proves the claim.

We now prove (C8). If $G_{4,\varepsilon}$ has empty interior, (C8) holds trivially.

Assume now the situation in (10)(ii) and let $\varepsilon_n \to 0$, $[u_n] \in H^1_n$, with $(\|[u_n]\|_{n,1})_n$ bounded, and $\|[u_n]\|_{L^2_n} = 1$. Note that, taking a subsequence, $[u_n]$ satisfies all conditions we imposed in the proof of (C7).

Fix $k \in \{1, \dots, N_4\}$. We can apply $(*)$ above: for suitable $(j,l) \in S_\Omega$ and $c_{k,j,l} := u_{0,j}(\partial G_{j,l} \cap (\{0\} \times \mathbb{R}^M))$, we have $\|u_{U,n} - c_{k,j,l}\|_{H^1(U_k)} \to 0$ as $n \to \infty$.

For $u_4 \in L^2(\Psi_{\varepsilon,4,k}(G_{4,k}))$ set $u_{4,k} := u_4 \circ \Psi_{\varepsilon,4,k} : G_{4,k} \to \mathbb{R}$. Then $u_4 \in L^2(\Psi_{\varepsilon,4,k}(G_{4,k}))$ (resp. $u_4 \in H^1(\Psi_{\varepsilon,4,k}(G_{4,k}))$) iff $u_{4,k} \in L^2(G_{4,k})$ (resp. $u_{4,k} \in H^1(G_{4,k})$) with equivalent norms.

Let $u_{n,4,k} := u_{n,4} \circ \Psi_{n,4,k}$. Note that

$$\|Du_{n,4,k}\|_{L^2(G_{4,k})}, \|u_{n,4,k} - c_{k,j,l}\|_{H^1(\Psi^{-1}_{n,4,k}(U_k\cap\Psi_{n,4,k}(G_{4,k}))} \to 0, \qquad n \to \infty.$$

The set $S_{4,k} := \{u \in H^1(G_{4,k}) \mid (u,1)_{L^2(G_{4,k})} = 0\}$ is closed, convex, $0 \in S_{4,k}$ and the only constant function in it is $u \equiv 0$. Hence $S_{4,k}$ satisfies the conditions of the generalized Poincaré inequality, and there is a constant $C_{13}$ such that $\|u\|_{L^2(G_{4,k})} \le C_{13}(\|Du\|_{L^2(G_{4,k})} + 1)$ for all $u \in S_{4,k}$.

This implies that setting $\alpha_{n,k} := |G_{4,k}|^{-1} \int_{G_{4,k}} u_{n,4,k}\, dz$ we have $u_{n,4,k} - \alpha_{n,k} \in S_{4,k}$, and, taking a subsequence, $u_{n,4,k} - \alpha_{n,k}$ converges in $H^1(G_{4,k})$ to a constant, which necessarily has to be 0. That is, for the original sequence $u_{n,4,k} - \alpha_{n,k} \to 0$ in $H^1(G_{4,k})$, implying $\alpha_{n,k} \to c_{k,j,l}$. This in turn proves $\|u_{n,4,k}\|_{L^2(G_{4,k})}$ is bounded and thus so is $\|u_{n,4}\|_{L^2(G_{4,n})}$, i.e. (C8) holds. ■

We now give an explicit description of the limit operator $A_0$ at the node. The problem of giving such a description for the edges is essentially the same as that of describing the limit operator in the case of squeezing a Lipschitz domain. This has been done for the case of so-called nicely decomposed domains in [14], so we will not treat it here.

We need the following notation. Let the edges $G_1, G_2, G_3$ connect nicely at the node $G_{4,\varepsilon}$ and $[u] \in L^2_s$. With the notations of Definition 3.1 set $p_{j,l}(x) := |G_{j,l} \cap (\{0\} \times \mathbb{R}^M)| = |\omega_{j,l,x}|$ for $0 < x < \delta$. By Proposition 6.1 of [14] we can without loss of generality assume $u_j|_{G_{j,l}}(x,y) = u_{j,l}(x)$ and $p_{j,l}^{1/2} u_{j,l} \in L^2(0,\delta)$ and, if $[u] \in H^1_s$, additionally $\partial_x u_j|_{G_{j,l}}(x,y) = u'_{j,l}(x)$, $p_{j,l}^{1/2} u'_{j,l} \in L^2(0,\delta)$ and $u_{j,l}$ is absolutely continuous.

Proposition 3.2. *Let $\Omega_\varepsilon$ be as in Section 2 and assume that the edges $G_1, G_2, G_3$ connect nicely at the node $G_{4,\varepsilon}$ (condition (C8) is not required). Assume the notations introduced above and let $[u] \in H^1_s$, $[w] \in L^2_s$. Then $[u] \in D(A_0)$, $A_0[u] = [w]$ iff the distributional derivative*

$$(u'_{j,l}(x) p_{j,l}(x) |(1,0) DT_j^T(x,0)|^{-2} |\det DT_j(x,0)|)'$$

*exists and is equal to $-w_{j,l}(x) p_{j,l})(x) |\det DT_j(x,0)|$ for all $0 < x < \delta$, $l = 1, \ldots, L_j$, $j = 1,2,3$,*

$$\int_{G_j} D_x u_j D_x v_j |(1,0) DT_j^T(x,0)|^{-2}\, d\lambda_j = \int_{G_j} w_j v_j\, d\lambda_j \tag{3.2}$$

*for all $v_j \in H^1_s(G_j)$, $v_j(0,\cdot) \equiv 0$ (as a trace), $j = 1,2,3$, and*

$$\sum_{(j,l) \in S_k} u'_{j,l}(0) p_{j,l}(0) |(1,0) DT_j^T(0)|^{-2} |\det DT_j(0)| = 0 \tag{3.3}$$

*for all $k = 1, \ldots, N_\Omega$.*

*Proof.* The proof is very similar to the one of Proposition 6.1 in [14], so we only outline it. We use the notations introduced above, that is, $u(x,y)|_{G_{j,l}}(x,y) = u_{j,l}(x)$ if $[u] \in L^2_s$. Note that we can apply Proposition 3.1 and $H^1_s$ is as in (3.1).

Case $\Rightarrow$. Let $\delta > 0$ be as in Definition 3.1 (or slightly smaller) and $0 < \delta_1 < \frac{1}{2}\delta$. Fix $k \in \{1, \dots, N_\Omega\}$ and $(j,l) \in S_k$. Set $\alpha_j(x) := |(1,0)DT_j^T(x,0)|^{-2} \times |\det DT_j(x,0)|$. Let $\widetilde{v} \in H_0^1(\delta_1, \delta-\delta_1)$ and set $v_i(x,y) := \widetilde{v}(x)$ if $i = j$, $(x,y) \in G_{j,l}$, $\delta_1 < x < \delta - \delta_1$, and $v_i(x,y) := 0$ elsewhere. Then $[v] \in H_s^1$ and

$$a_0([u],[v]) = \int_0^\delta u'_{j,l}\widetilde{v}'\alpha_j p_{j,l}\,dx, \qquad ([w],[v])_{L_s^2} = \int_0^\delta w_{j,l}\widetilde{v}p_{j,l}|\det DT_j(x,0)|\,dx.$$

Letting $\delta_1 \to 0$, we see that $A_0[u] = [w]$ implies

$$(u'_{j,l}\alpha_j p_{j,l})'(x) = -w_{j,l}(x)p_{j,l}(x)|\det DT_j(x,0)| \in L^2(0,\delta).$$

In particular $\lim_{x\downarrow 0} u'_{j,l}(x)\alpha_j(x)p_{j,l}(x)$ exists.

Now set $v_i(x,y) := 1 - \chi(x/\delta_1)$ if $(x,y) \in G_{\tilde{j},\tilde{l}}$ for all $(\widetilde{j},\widetilde{l}) \in S_k$, and $v_i(x,y) := 0$ elsewhere. Then $[v] \in H_s^1$ and as $\delta_1 \to 0$,

$$\begin{aligned}
0 \leftarrow ([w],[v])_{L_s^2} &= a_0([u],[v]) \\
&= \sum_{(\tilde{j},\tilde{l})\in S_k} \int_{G_{\tilde{j},\tilde{l}}} u'_{\tilde{j},\tilde{l}} v'_{\tilde{j}}\alpha_{\tilde{j}}\,dx\,dy = \sum_{(\tilde{j},\tilde{l})\in S_k} \int_0^\delta u'_{\tilde{j},\tilde{l}}\alpha_{\tilde{j}}p_{\tilde{j},\tilde{l}}v'_{\tilde{j}}\,dx \\
&= -\sum_{(\tilde{j},\tilde{l})\in S_k} \int_0^\delta (u'_{\tilde{j},\tilde{l}}\alpha_{\tilde{j}}p_{\tilde{j},\tilde{l}})'(x)v_{\tilde{j}}\,dx + (u'_{\tilde{j},\tilde{l}}\alpha_{\tilde{j}}p_{\tilde{j},\tilde{l}}v_{\tilde{j}})(0) \\
&\to -\sum_{(\tilde{j},\tilde{l})\in S_k} (u'_{\tilde{j},\tilde{l}}\alpha_{\tilde{j}}p_{\tilde{j},\tilde{l}})(0).
\end{aligned}$$

Thus equation (3.3) holds. In the same way using test functions $v_i(x,y) := \widetilde{v}(x,y)$ if $(x,y) \in G_j$, $v_i(x,y) := 0$ elsewhere, $\widetilde{v} \in H_s^1(G_j)$ arbitrary with trace $\widetilde{v}(0,.) = 0$, one can prove equation (3.2).

Case $\Leftarrow$. Assume the distributional derivative exists and equations (3.2) and (3.3) hold. Let $[v] \in H_s^1$ and $\delta_1 \downarrow 0$. Then

$$\begin{aligned}
&a_0([u],[v]) \\
&\quad = \sum_{j=1}^3 \int_{G_j} D_x u_j D_x\left(v_j\chi\left(\frac{x}{\delta_1}\right)\right)\alpha_j\,dx\,dy \\
&\qquad + \sum_{j=1}^3 \int_{G_j} D_x u_j D_x\left(v_j\left(1-\chi\left(\frac{x}{\delta_1}\right)\right)\right)\alpha_j\,dx\,dy \\
&\quad = \sum_{j=1}^3 \int_{G_j} w_j v_j \chi\left(\frac{x}{\delta_1}\right)d\lambda_j + \sum_{j=1}^3\sum_{l=1}^{L_j} -\int_0^\delta (u'_{j,l}p_{j,l}\alpha_j)'v_j\left(1-\chi\left(\frac{x}{\delta_1}\right)\right)dx \\
&\qquad - u'_{j,l}(0)p_{j,l}(0)\alpha_j(0)v_j(0) \to ([w],[v])_{L_s^2}. \ \blacksquare
\end{aligned}$$

Example 3.1. Let $\Omega_\varepsilon$ be as in Section 2. Assume $0 < x_0 < 1$ is such that $G_a := G_1 \cap (]0, x_0[ \times \mathbb{R}^M)$ and $G_b := G_1 \cap (]x_0, 1[ \times \mathbb{R}^M)$ are both connected, Lipschitz and have finitely many connected components. Set $T_{\varepsilon,a} := T_{\varepsilon,1}|_{\overline{G}_a}$, $T_{\varepsilon,b} := T_{\varepsilon,1}|_{\overline{G}_b}$, $T_a := T_b := T_1$, $\Psi_{\varepsilon,a} := T_a \circ S_\varepsilon \circ T_{\varepsilon,a} = \Psi_{\varepsilon,1}|_{\overline{G}_a}$, $\Psi_{\varepsilon,b} := T_b \circ S_\varepsilon \circ T_{\varepsilon,b} = \Psi_{\varepsilon,1}|_{\overline{G}_b}$, $\Omega_{a,\varepsilon} := \Psi_{\varepsilon,a}(G_a)$, $\Omega_{b,\varepsilon} := \Psi_{\varepsilon,b}(G_b)$ and $\Omega_{c,\varepsilon} := \Psi_{\varepsilon,1}(G_1 \cap (\{x_0\} \times \mathbb{R}^M))$, $\Psi_{\varepsilon,c}(z) := \Psi_{\varepsilon,1}(x_0, y_0) + \varepsilon z$, $G_{c,\varepsilon} := \Psi_{\varepsilon,c}^{-1}(\Omega_{c,\varepsilon})$. Then $\Omega_{a,\varepsilon} \cup \Omega_{b,\varepsilon} \cup \Omega_{c,\varepsilon} = \Omega_{1,\varepsilon}$, but instead of three edges and one node, the domain $\Omega_\varepsilon$ consists now of four edges $G_a, G_b, G_2, G_3$ and two nodes $G_{c,\varepsilon}, G_{4,\varepsilon}$.

Note that a simple linear transformation would yield $\mathrm{proj}_x(G_a) = \mathrm{proj}_x(G_b) = ]0, 1[$ and $\Psi_{\varepsilon,a}^{-1}(\Omega_{c,\varepsilon}), \Psi_{\varepsilon,b}^{-1}(\Omega_{c,\varepsilon}) \subset (\{0\} \times \mathbb{R}^M)$, as assumed in Section 2.

It is easy to show that for this "new" domain conditions (C1)–(C6), (C8) of Section 2 hold. (C7) holds because $[u_a, u_b, u_c, u_2, u_3, u_4] \in H_\varepsilon^1$ iff $u_a|_{(\{0\}\times\mathbb{R}^M)\cap G_1} = u_b|_{(\{0\}\times\mathbb{R}^M)\cap G_1}$ (as traces).

We can apply the methods of Section 2 and get a limiting operator $A_0$(new) induced by $a_0$(new), where it turns out that $a_0$(new) $= a_0$(old).

In other words: if $x_0 \in ]0, 1[$ is as stated above, the we can divide an edge putting in a new node. The resulting net-shaped domain satisfies all conditions stated in Section 2 which hold for the original domain. ∎

Example 3.2. Let $M = 1$ and $\Omega_\varepsilon \subset \mathbb{R}^2$ satisfy conditions (C1)–(C7) of Section 2. Assume $\|\mathcal{T}_{\varepsilon,1,0,0}\|_\infty < 1$ if $\varepsilon \le \varepsilon_0$ for some $0 < \varepsilon_0 \le 1$.

In the following we shall always assume $\varepsilon \le \varepsilon_0$, possibly decreasing $\varepsilon_0 > 0$ slightly. Assume additionally that there are $x_0 \in ]0, 1[$, $\delta > 0$, $\varrho_l \in C^1([x_0 - 2\delta, x_0 + 2\delta], \mathbb{R})$, $l = 1, 2$, such that $\varrho_1 < \varrho_2$ and

$$G_1 \cap ([x_0 - 2\delta, x_0 + 2\delta] \times \mathbb{R}) = \bigcup_{x\in[x_0-2\delta,x_0+2\delta]} \{x\} \times ]\varrho_1(x), \varrho_2(x)[.$$

Fix $y_0 \in ]\varrho_1(x_0), \varrho_2(x_0)[$ and set $z_{\varepsilon,4} := \Psi_{\varepsilon,1}(x_0, y_0)$, $\Psi_{\varepsilon,c}(z) := z_{\varepsilon,4} + \varepsilon z$. We assume $\varrho_1(x) < y_0 < \varrho_2(x)$ for all $x_0 - 2\delta < x < x_0 + 2\delta$.

By Lemma 2.1, (C4) and (C5) there is a constant $C_1 > 0$ such that

$$\begin{aligned}
&|\mathrm{proj}_x(\Psi_{\varepsilon,1}^{-1}(z_1) - \Psi_{\varepsilon,1}^{-1}(z_2))| \le C_1\|z_1 - z_2\|,\\
&\|\mathrm{proj}_y(\Psi_{\varepsilon,1}^{-1}(z_1) - \Psi_{\varepsilon,1}^{-1}(z_2))\| \le \frac{C_1}{\varepsilon}\,\|z_1 - z_2\|,\\
&\|\Psi_{\varepsilon,1}(x_1, y_1) - \Psi_{\varepsilon,1}(x_2, y_2)\| \le C_1(|x_1 - x_2| + \varepsilon|y_1 - y_2|),\\
&\|\Psi_{\varepsilon,1}(x_3, y_0) - \Psi_{\varepsilon,1}(x_0, y_0)\| \ge \frac{1}{C_1}\,(|x_3 - x_0|
\end{aligned}\tag{3.4}$$

for all $z_1, z_2, x_1, y_1, x_2, y_2, x_3$ such that $\lambda z_1 + (1-\lambda) z_2 \in \Omega_{1,\varepsilon}$ and $\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) \in G_1$, for all $\lambda \in [0, 1]$, and $|x_3 - x_0| < 2\delta$, possibly decreasing $\varepsilon_0$ to get the last inequality. The displayed inequalites imply there are $r_0, \widetilde{r}_0 > 0$ such that $\Psi_{\varepsilon,1}^{-1} \circ \Psi_{\varepsilon,c}(B_{r_0}(0)) \subset B_{\widetilde{r}_0}(x_0, y_0) \subset G_1$. Possibly decreasing $\varepsilon_0$ again and defining $z_{4,\varepsilon,\pm} := \Psi_{\varepsilon,c}^{-1} \circ \Psi_{\varepsilon,1}(x_0 \pm \varepsilon r_0/4C_1, y_0) \in B_{r_0}(0)$, we

find that there is an $r_1 > 0$ such that $B_{r_1}(z_{\varepsilon,4,\pm}) \subset B_{r_0}(0)$ and
$\Psi_{\varepsilon,1}^{-1}\Psi_{\varepsilon,c}(B_{r_1}(z_{\varepsilon,4,\pm}))$

$$\subset \left(\left(\left[x_0 - \varepsilon\,\frac{r_0}{2C_1}, x_0 - \varepsilon\,\frac{r_0}{8C_1}\right] \cup \left[x_0 + \varepsilon\,\frac{r_0}{8C_1}, x_0 + \varepsilon\,\frac{r_0}{2C_1}\right]\right) \times \mathbb{R}\right) \cap G_1.$$

Set $\alpha := r_0/10C_1$ and define $G_a := (]0, x_0[\times \mathbb{R}) \cap G_1$, $G_b := (]x_0, 1[\times \mathbb{R}) \cap G_1$, $\Omega_{c,\varepsilon} := \Psi_{\varepsilon,1}(([x_0 - \alpha\varepsilon, x_0 + \alpha\varepsilon] \times \mathbb{R}) \cap G_1)$, $G_{c,\varepsilon} := \Psi_{\varepsilon,c}^{-1}(\Omega_{c,\varepsilon})$, and

$$X_\varepsilon(x) := x + \left(\frac{1}{2}\,(x_0 - x) - \alpha\varepsilon\right)\chi\left(\frac{1}{\alpha\varepsilon}\,(x - x_0 + 3\alpha\varepsilon)\right),$$

$$Y_\varepsilon(x,y) := \frac{\varrho_2(X_\varepsilon(x)) - \varrho_1(X_\varepsilon(x))}{\varrho_2(x) - \varrho_1(x)}\left(y - \frac{1}{2}\,(\varrho_1(x) + \varrho_2(x))\right) + \frac{1}{2}\,(\varrho_1(X_\varepsilon(x)) + \varrho_2(X_\varepsilon(x))), \qquad (x,y) \in G_a,$$

where we have extended $Y_\varepsilon(x,y) = y$ if $x \le x_0 - 3\alpha\varepsilon$. Then

$$\overline{G}_a \ni (x,y) \mapsto (X_\varepsilon(x), Y_\varepsilon(x,y)) =: XY_\varepsilon(x,y) \in ([0, x_0 - \alpha\varepsilon] \times \mathbb{R}) \cap \overline{G}_1$$

is bijective for $\varepsilon \le \varepsilon_0$. Thus

$$T_{\varepsilon,a}(x,y) := T_{\varepsilon,1} \circ XY_\varepsilon : \overline{G}_a \to T_{\varepsilon,a}(\overline{G}_a) = T_{\varepsilon,1}(([0, x_0 - \alpha\varepsilon] \times \mathbb{R}) \cap G_1)$$

is a $C^1$-diffeomorphism. If we set $T_a := T_1$ and analogously define $T_{\varepsilon,b}$, $T_b$, it is straightforward to show that the requirements of Section 2 which do not involve the node are satisfied (for $\varepsilon \le \varepsilon_0$).

Note that $\mathcal{T}_a = \mathcal{T}_b = \mathcal{T}_1$ in (C5). Moreover, (3.4) shows $G_{c,\varepsilon}$ is bounded, i.e. (C6) holds. The balls $B_{r_0}(0)$ and $B_{r_1}(z_{\varepsilon,4,\pm})$ satisfy the conditions in Definition 3.1, hence $G_a$ and $G_b$ connect nicely at $G_{c,\varepsilon}$ and (C7) holds as well.

Define $T_\varepsilon : [-\alpha, \alpha]^2 \to ([x_0 - \alpha\varepsilon, x_0 + \alpha\varepsilon] \times \mathbb{R}) \cap \overline{G}_1 = \bigcup_{|x - x_0| \le \alpha\varepsilon} \{x\} \times [\varrho_1(x), \varrho_2(x)]$ by

$$T_\varepsilon(x,y) = \left(x_0 + \varepsilon x, \frac{1}{2}\left(\varrho_1(x_0 + \varepsilon x) + \varrho_2(x_0 + \varepsilon x) + \frac{y}{\alpha}\,(\varrho_2(x_0 + \varepsilon x) - \varrho_1(x_0 + \varepsilon x))\right)\right).$$

Then $T_\varepsilon$ and $\Psi_{\varepsilon,4,1} : [-\alpha, \alpha]^2 \to \overline{G}_{c,\varepsilon}$, $\Psi_{\varepsilon,4,1} := \Psi_{\varepsilon,c}^{-1} \circ \Psi_{\varepsilon,1} \circ T_\varepsilon$ are $C^1$-diffeomorphisms. It is straightforward to show that $|\det D\Psi_{\varepsilon,4,1}|$ is bounded away from 0 and infinity. (C5) implies that $\|D\Psi_{\varepsilon,4,1}\|$, $\|D\Psi_{\varepsilon,4,1}^{-1}\|$ are bounded uniformly in $(x,y)$ and $\varepsilon$. Next, $B_{r_0}(0)$ and $B_{r_1}(z_{\varepsilon,4,\pm})$ satisfy the requirements for $U_1$ and $U_{\varepsilon,1,j,l}$ in (C10). That is, (C10) holds and by Proposition 3.1 so does (C8) at $G_{c,\varepsilon}$.

We have $\Omega_\varepsilon = \Omega_{a,\varepsilon} \cap \Omega_{b,\varepsilon} \cap \Omega_{c,\varepsilon} \cap \Omega_{2,\varepsilon} \cap \Omega_{3,\varepsilon} \cap \Omega_{4,\varepsilon}$, i.e. as in Example 3.1 we look at $\Omega_\varepsilon$ as a net having four edges and two nodes, but unlike in 3.1 now $G_{c,\varepsilon}$ has nonempty interior. As in Example 3.1 we get the same limiting operator $A_0$ as for the original net-shaped domain with three edges and one node.

Now perturb $\Omega_\varepsilon$ slightly by changing $\Omega_{c,\varepsilon}$: let $U$ be a (finite number of) small Lipschitz domain(s), $\overline{U} \subset ]-\alpha,\alpha[^2$, and set $G_{c,\varepsilon} := \Psi_{\varepsilon,4,1}([-\alpha,\alpha]^2 \setminus \overline{U})$. Then $\Omega_\varepsilon$ has a (finite number of) hole(s) which decrease(s) proportionally to $\varepsilon$ in all directions, $G_a$ and $G_b$ still connect nicely at $G_{c,\varepsilon}$, (C10) holds, and $H^1_s$ and $a_0$ do not change.

In other words: under the additional assumptions on the smoothness above, the domains $\Omega_\varepsilon$ can have a finite number of holes in any edge if these holes decrease of order $\varepsilon$ in all directions. The limiting problem does not change under this perturbation.

In our example, $M = 1$ and $(G_1)_{x_0}$ has only one connected component, but the technique could be extended to domains for which $M > 1$ and the $x$-sections $(G_1)_{x_0}$ have a finite number of connected components.

If a hole in an edge disappears faster than of order $\varepsilon$, we can no longer find a diffeomorphism of the node (containing this hole) onto a fixed Lipschitz domain satisfying the requirements of condition (C10). In this case one would have to divide the node and then apply this proposition.

In particular one can apply this technique to domains $\Omega_\varepsilon = S_\varepsilon(\Omega)$ with $\Omega \subset \mathbb{R}^2$ Lipschitz and bounded, viewing $\Omega_\varepsilon$ as a net-shaped domain having only one edge and no node. Thus—under the additional smoothness assumptions mentioned above and taking into account the weaker convergence in our theorems—the results of [14] also hold if the relevant domains have finitely many smooth holes of order $\varepsilon$ or less. ∎

## References


[1] H. W. Alt, *Lineare Funktionalanalysis*, Springer, Berlin, 1992.

[2] F. Antoci and M. Prizzi, *Reaction-diffusion equations on unbounded thin domains*, Topol. Methods Nonlinear Anal. 18 (2001), 283–302.

[3] M. C. Carbinatto and K. P. Rybakowski, *Conley index continuation and thin domain problems*, ibid. 16 (2000), 201–252.

[4] T. Elsken, *Limiting behavior of attractors for systems on thin domains*, Hiroshima Math. J. 32 (2002), 389–415.

[5] —, *Attractors for reaction-diffusion equations on thin domains whose linear part is nonselfadjoint*, submitted.

[6] T. Elsken and M. Prizzi, *Characterization of the limit of some higher dimensional thin domain problems*, Topol. Methods Nonlinear Anal. 20 (2002), 151–178.

[7] Q. Fang, *Inertial manifold theory for a class of reaction-diffusion equations on thin tubular domains*, Hiroshima Math. J. 23 (1993), 459–508.

[8] W. Hackbusch, *Elliptic Differential Equations: Theory and Numerical Treatment*, Springer Ser. Comput. Math. 18, Springer, Berlin, 1992.

[9] J. K. Hale and G. Raugel, *Reaction-diffusion equation on thin domains*, J. Math. Pures Appl. (9) 71 (1992), 33–95.

[10] —, —, *A reaction-diffusion equation on a thin L-shaped domain*, Proc. Roy. Soc. Edinburgh Sect. A 125 (1995), 283–327.

[11] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Math. 840, Springer, New York, 1981.
[12] S. Kosugi, *Semilinear elliptic equations on thin network-shaped domains with variable thickness*, J. Differential Equations 183 (2002), 165–188.
[13] M. Prizzi, M. Rinaldi and K. P. Rybakowski, *Curved thin domains and parabolic equations*, Studia Math. 151 (2002), 109–140.
[14] M. Prizzi and K. P. Rybakowski, *The effect of domain squeezing upon the dynamics of reaction-diffusion equations*, J. Differential Equations 173 (2001), 271–320.
[15] —, —, *Inertial manifolds on squeezed domains*, J. Dynam. Differential Equations 15 (2003), 1–48.
[16] J. Rubinstein and M. Schatzman, *Variational problems on multiply connected thin strips I: basic estimates and convergence of the Laplacian spectrum*, Arch. Rat. Mech. Anal. 160 (2001), 271–308.
[17] Y. Saito, *The limiting equation for Neumann Laplacians on shrinking domains*, Electron. J. Differential Equations 2000, No. 31, 25 pp.

Fachbereich Mathematik
Universität Rostock
D-18051 Rostock, Germany
E-mail: thomas.elsken@mathematik.uni-rostock.de

*Received November 17, 2003*
*Revised version June 7, 2004* (5299)