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Minimal projections with respect to various norms

by

ASUMAN GÜVEN AKSOY (Claremont, CA)
and GRZEGORZ LEWICKI (Kraków)

Abstract. A theorem of Rudin permits us to determine minimal projections not only with respect to the operator norm but with respect to various norms on operator ideals and with respect to numerical radius. We prove a general result about N-minimal projections where N is a convex and lower semicontinuous (with respect to the strong operator topology) function and give specific examples for the cases of norms or seminorms of p-summing, p-integral and p-nuclear operator ideals.

1. Introduction. Let X be a Banach space over $\mathbb{R}$ or $\mathbb{C}$. We write $B_X(r)$ for the closed ball with radius $r > 0$ and center at 0 (B_X if $r = 1$) and S_X for the unit sphere of X. The dual space is denoted by X^* and the Banach algebra of all continuous linear operators going from X into a Banach space Y is denoted by $B(X,Y)$ ($B(X)$ if $X = Y$).

DEFINITION 1.1. The *numerical range* of $T \in B(X)$ is defined by

$$W(T) = \{x^*(Tx) : x \in S_X,\ x^* \in S_{X^*},\ x^*(x) = 1\}\cdot$$

The *numerical radius* of T is then given by

$$\|T\|_w = \sup\{|\lambda| : \lambda \in W(T)\}\cdot$$

Clearly, $\|\cdot\|_w$ is a seminorm on $B(X)$ and $\|T\|_w \le \|T\|$ for all $T \in B(X)$. The *numerical index* of X is defined by

$$n(X) = \inf\{\|T\|_w : T \in S_{B(X)}\}\cdot$$

Equivalently, the numerical index $n(X)$ is the greatest constant $k \ge 0$ such that $k\|T\| \le \|T\|_w$ for every $T \in B(X)$. Note also that $0 \le n(X) \le 1$, and $n(X) > 0$ if and only if $\|\cdot\|_w$ and $\|\cdot\|$ are equivalent norms.

The concept of numerical index was introduced by Lumer [33] in 1968. Since then much attention has been paid to the constant of equivalence between the numerical radius and the usual norm in the Banach algebra of

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all bounded linear operators of a Banach space. Two classical books devoted to numerical range are [6] and [7]. For recent results we refer the reader to [1], [2], [18], [19], [21], [31], [34].

The following definition, given in [44], presents the general concept of an operator ideal.

Definition 1.2. We are given an *operator ideal* $\mathcal{U}$ if for each pair of Banach spaces X and Y we have a class of operators $\mathcal{U}(X,Y)$ ($\mathcal{U}(X)$ if $X=Y$) such that:

(a) $\mathcal{U}(X,Y)$ is a linear subspace (not necessarily closed) of $B(X,Y)$ containing all finite rank operators;
(b) if $T\in\mathcal{U}(X,Y)$, $A\in B(Z,X)$ and $B\in B(Y,V)$ then $BTA\in\mathcal{U}(Z,V)$ for all Banach spaces X,Y,Z,V and all operators A,B.

An operator ideal is a *Banach operator ideal* if on each $\mathcal{U}(X,Y)$ we have a norm N such that:

(a) $(\mathcal{U}(X,Y),N)$ is complete for each X,Y;
(b) $N(BTA)\le\|B\|N(T)\|A\|$, whenever the composition makes sense, where the symbol $\|\cdot\|$ denotes the operator norm;
(c) for every rank-one operator $T:X\to Y$ we have $N(T)=\|x^*\|\,\|y\|$, where $T(x)=x^*(x)y$.

The theory of Banach operator ideals was founded by A. Grothendieck and R. Schatten. Basic examples are the ideals of all continuous operators, compact operators, weakly compact operators, p-absolutely summing, p-integral and p-nuclear operators. For more details on operator ideals see [38] and [44].

If X is a Banach space and V is a linear, closed subspace of X, we denote by $\mathcal{P}(X,V)$ the set of all linear projections continuous with respect to the operator norm. Recall that an operator $P:X\to V$ is called a *projection* if $P|_V=\mathrm{id}_V$. A projection $P_0\in\mathcal{P}(X,V)$ is called *minimal* if

$$\|P_0\|=\inf\{\|P\|:P\in\mathcal{P}(X,V)\}=\lambda(V,X).$$

Minimal projections have been extensively studied by many authors in the context of functional analysis and approximation theory (see, e.g., [1], [5], [8]–[12], [14]–[17], [20], [22]–[32], [35]–[37], [39], [41]–[43]). Mainly the problems of existence and uniqueness of minimal projections, finding concrete formulas for minimal projections and estimates of the constant $\lambda(V,X)$ were considered. One of the main tools for finding minimal projections effectively is Rudin's Theorem (see [39] or [40]). This theorem plays a fundamental role in our discussion below.

Assume that X is a Banach space, $V\subset X$ is a closed subspace and $N:B(X,V)\to[0,+\infty]$ is a convex function. Let us define

$$\lambda_N(V,X) = \inf\{N(P) : P \in \mathcal{P}(X,V)\}.$$

We put $\lambda_N(V,X) = +\infty$ if $\mathcal{P}(X,V) = \emptyset$ or if $N(P) = +\infty$ for any $P \in \mathcal{P}(X,V)$. A projection $P_0 \in \mathcal{P}_N(X,V)$ is called *N-minimal* if

$$N(P_0) = \lambda_N(V,X).$$

In the following we will show that under some assumptions on N, Rudin's Theorem can be applied to obtain N-minimal projections effectively (see Theorem 2.2). Then we show that it is possible to apply Theorem 2.2 to many concrete cases. In particular we take for N the numerical radius $\|\cdot\|_w$ or a norm on a Banach operator ideal $(\mathcal{U}, N)$. Although our proofs follow from Rudin's result without much difficulty, our purpose is to give concrete applications, e.g. to Fourier projections. Applications presented in the last section justify our study of minimal projections in this context. We do not know of any paper (with the exception of [1] and [3]) concerning minimal projections with respect to norms different than the operator norm. In fact, in [1] a characterization of minimal numerical-radius extensions of operators from a normed linear space X onto its finite-dimensional subspaces and a comparison with minimal operator-norm extension are given. Furthermore, in the same paper, it is shown that the projection $P : l_3^p \to [v_1, v_2]$, where $v_1 = (1,1,1)$ and $v_2 = (-1,0,1)$, which is minimal with respect to the operator norm, is not minimal with respect to the numerical radius for $1 < p < \infty$ and $p \neq 2$.

Also, it is worth noticing that even if X is finite-dimensional (in this case all norms defined on $B(X)$ are equivalent) a minimal projection with respect to the operator norm is not automatically a minimal projection with respect to other norms in $B(X)$. Thus, the problems of existence and uniqueness of minimal projection are valid issues in this context too.

2. Main results. One of the key theorems on minimal projections is due to W. Rudin ([39] or [40, p. 127]). This theorem was motivated by an earlier result of Lozinskiĭ ([13, p. 216] or [32]) concerning the minimality of the classical nth Fourier projection in $\mathcal{P}(C(2\pi), \pi_n)$, where $C(2\pi)$ denotes the space of all 2π-periodic real-valued functions equipped with the supremum norm and π_n is the space of all trigonometric polynomials of degree less than or equal to n.

To explain the setting for this theorem and the main technique involved in its proof, let G be a compact topological group. We say that G *acts by isomorphisms* on a Banach space X if there exists a group homomorphism $g \mapsto \Phi_g$ into the group of (bounded and linear) isomorphisms of X, which is continuous in the strong operator topology on $B(X)$, that is, the map $(g,x) \mapsto \Phi_g(x)$ is continuous. A subset V of X is called *G-invariant* if $\Phi_g(V) \subset V$ for all $g \in G$, and a mapping $T : X \to X$ is said to *commute* with G if $T\Phi_g = \Phi_g T$ for all $g \in G$. In case $\|\Phi_g\| = 1$ for all $g \in G$ we say that g *acts on* G *by isometries*. To simplify the notation, in what follows we identify each $g \in G$ with $\Phi_g \in B(X)$, and simply view g as an isomorphism of X.

THEOREM 2.1 (Rudin, [39], [40]). *Let G be a compact topological group acting by isomorphisms on a Banach space X, and let V be a complemented G-invariant subspace of X. Then there exists a bounded linear projection Q of X onto V which commutes with G.*

The idea behind the proof of the above theorem is to obtain Q by averaging the operators $g^{-1}Pg$ with respect to the Haar measure on G, where P is any bounded projection of X onto V. Let us explain this averaging procedure, since it will be the key tool in our later results.

Given an operator $T \in B(X)$, the mapping $g \mapsto g^{-1}Tg$ is continuous and therefore integrable against the normalized Haar measure μ on G. This provides a bounded linear operator

$$T_G = \int_G (g^{-1}Tg)\, d\mu(g)$$

which belongs to the closed convex hull of the set $\{g^{-1}Tg : g \in G\}$. Here, the closure is taken in the strong operator topology ([40, Theorem 3.27]) and the above operator-valued integral should be understood in the Pettis sense. Thus, we have

$$\Lambda(T_G) = \int_G \Lambda(g^{-1}Tg)\, d\mu(g)$$

for every linear functional Λ on $B(X)$ which is continuous in the strong operator topology of $B(X)$ ([40, Definition 3.26]). Given $x \in X$ and $x^* \in X^*$ we may take $\Lambda(S) = x^*(Sx)$ for all $S \in B(X)$ to get

$$T_G x = \int_G (g^{-1}Tg)x\, d\mu(g).$$

It follows from the translation invariance of μ that the operator T_G commutes with G. Moreover, if P is a bounded linear projection of X onto a closed subspace V, then P_G is also a bounded linear projection onto V (see [40, Theorem 5.18]). This accounts for Rudin's proof of Theorem 2.1.

We point out for later use that the mapping $T \mapsto T_G$ is linear and that $T_G = T$ whenever T commutes with G. Therefore we have $(I_X - T)_G = I_X - T_G$, where I_X denotes the identity operator on X.

Now we are ready for our main results. Provided that G acts by isometries on X, we will show that the mapping $T \mapsto T_G$ is non-expansive, in a quite general sense.

MAIN THEOREM 2.2. *Let G be a compact topological group acting by isometries on a Banach space X, and let $N : B(X) \to [0, +\infty]$ be a convex function which is lower semicontinuous in the strong operator topology of $B(X)$. Assume furthermore that*

$$(2.1) \qquad N(g^{-1} \circ T \circ g) \leq N(T) \quad \text{for any } T \in B(X) \text{ and } g \in G.$$

Then

$$N(T_G) \leq N(T) \quad \text{for all } T \in B(X).$$

Therefore, if V is a closed subspace of X and there is a unique projection Q of X onto V which commutes with G, then Q is N-minimal and N-cominimal, that is, $N(Q) \leq N(P)$ and $N(I_X - Q) \leq N(I_X - P)$ for any other bounded linear projection P of X onto V.

Proof. Given $T \in B(X)$, by the convexity and lower semicontinuity of N, the set $E = \{S \in B(X) : N(S) \leq N(T)\}$ is convex and closed in the strong operator topology. It then follows from (2.1) that the closed convex hull of the set $\{g^{-1}Tg : g \in G\}$ is contained in E, so $T_G \in E$, as required.

For the second part of the theorem, if P is a bounded, linear projection onto V, we must have $P_G = Q$, so $N(Q) \leq N(P)$ and $N(I_X - Q) \leq N(I_X - P)$. ∎

Now we show that Theorem 2.2 can be applied in a wide collection of examples.

REMARK 2.3. If N is equal to the operator norm $\|\cdot\|$ on $B(X)$, then N is convex, lower semicontinuous in the strong operator topology and (2.1) is satisfied. Hence Theorem 2.2 can be applied. For the proof of Theorem 2.2 in the case of the operator norm see e.g. [16, Chapter 9].

REMARK 2.4. Assume that X is finite-dimensional and N is any norm on $B(X)$ satisfying (2.1). Then N is continuous with respect to the strong operator topology and hence satisfies the requirements of Theorem 2.2.

EXAMPLE 2.5. Let N denote the numerical radius on $B(X)$. It is clear that N is a convex function. To show that (2.1) is satisfied, fix $L \in B(X)$, $g \in G$ and $(x^*, x) \in S_{X^*} \times S_X$ satisfying $x^*(x) = 1$. Since g is a linear isometry, $(x^* \circ g^{-1}, g(x)) \in S_{X^*} \times S_X$ and $(x^* \circ g^{-1})(g(x)) = 1$. By the definition of numerical radius,

$$(x^* \circ g^{-1})L(g(x)) \leq N(L),$$

and consequently $N(g^{-1}LT_g) \leq N(L)$. It is also easy to see that N is lower semicontinuous with respect to the strong operator topology on $B(X)$.

Now assume that we have $W \subset S(X^*) \times S(X)$, $W \neq \emptyset$, such that for any $g \in G$, if $(x, x^*) \in W$ then $(x^*g^{-1}, gx) \in W$. Define for any $L \in \mathcal{L}(X)$ a seminorm $\|\cdot\|_W$ by

$$\|L\|_W = \sup\{|x^* Lx| : (x^*, x) \in W\}.$$

Observe that for

$$W = \{(x^*, x) \in S_{X^*} \times S_X : x^*(x) = 1\},$$

$\|L\|_W$ is equal to the numerical radius of L. Then the seminorm $\|\cdot\|_W$ also satisfies the requirements of Theorem 2.2.

Now let X be a Banach space and $(\mathcal{U}(X), N) \subset B(X)$ a Banach operator ideal (see Definition 1.2). Extending N to $B(X)$ by $N(L) = +\infty$ for $L \in B(X) \setminus \mathcal{U}(X)$ we see that N is convex and, by Definition 1.2, N satisfies (2.1). Hence to apply Theorem 2.2 in the case of operator ideals we only need to check the lower semicontinuity of N. This will be done in the next three examples.

Example 2.6. Let X be a Banach space and let $V \subset X$ be a closed subspace. Let $\mathcal{U}(X, V)$ denote the set of all p-summing operators for $1 \leq p < \infty$. Notice that by [44, p. 200], an operator $T \in B(X, V)$ is p-summing $(1 \leq p < \infty)$ if there exists $D > 0$ such that for any $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ we have

$$\Big(\sum_{j=1}^{n} \|Tx_j\|^p\Big)^{1/p} \leq D \sup\Big\{\Big(\sum_{j=1}^{n} |x^*(x_j)|^p\Big)^{1/p} : x^* \in X^*,\ \|x^*\| = 1\Big\}. \tag{2.2}$$

Consequently,

$$\begin{aligned} N(T) &= \inf\{D > 0 : (2.2) \text{ is satisfied with } D\} \\ &= \sup\Big\{\Big(\sum_{j=1}^{n} \|Tx_j\|^p\Big)^{1/p} : n \in \mathbb{N}, x_1, \ldots, x_n \in X, x^* \in S_{X^*}, \sum_{j=1}^{n} |x^*(x_j)|^p \leq 1\Big\}, \end{aligned}$$

and it follows that N is lower semicontinuous as the supremum of a family of continuous functions $T \mapsto (\sum_{j=1}^{n} \|Tx_j\|^p)^{1/p}$.

Example 2.7. Now we consider the case of p-integral operators. Recall that an operator $L \in B(X, V)$ is called *p-integral* for $1 \leq p \leq \infty$ if there exist a compact set K and a probability measure μ on K such that L admits a factorization

$$L = D \circ \mathrm{id} \circ A, \tag{2.3}$$

where $A \in B(X, C(K))$, $\mathrm{id} : C(K) \to L_p(K, \mu)$ and $D \in B(L_p(K, \mu), V)$. In this case,

$$N(L) = \inf\{\|A\|\,\|D\| : L \text{ has factorization } (2.3)\}.$$

Let X be a Banach space and let $V \subset X$ be a closed subspace. Assume furthermore that V is reflexive. Let $\mathcal{U}(X, V)$ denote the space of all p-integral operators for $1 \leq p < \infty$. To show that N is lower semicontinuous with respect to the strong operator topology, without loss of generality we can assume that a net $\{L_c\}_{c\in C} \subset B(X)$ converges to $L \in B(X)$ in the strong operator topology and $\lim_c N(L_c) = F < \infty$. Let i be an isometric embedding of X into $C(S)$, where $S = B_{X^*}$ with the weak* topology. By [44, p. 218, Th. 23] and the proof of the Pietsch factorization theorem (see [44, p. 203]), for any $\epsilon > 0$ there exists a probability measure μ on S (independent of $c \in C$) and $D_c \in B(L_p(S,\mu), V)$ such that

$$L_c = D_c \circ \mathrm{id} \circ i, \tag{2.4}$$

where $\mathrm{id} : C(S) \to L_p(S,\mu)$ and

$$\|D_c\| \le F + \epsilon. \tag{2.5}$$

Notice that by reflexivity of V for any $c \in C$,

$$D_c \in \mathcal{F}_\epsilon = \prod_{x\in L_p(S,\mu)} B_V(\|x\|(F+\epsilon)).$$

If for any $x \in L_p(S,\mu)$ we equip $B_V(\|x\|(N(P)+\epsilon))$ with the weak topology and $\mathcal{F}_\epsilon$ with the Tikhonov topology, then by the Eberlein Theorem and the Tikhonov Theorem, $\mathcal{F}_\epsilon$ is a compact set. Hence the net $\{D_c\}_{c\in C}$ has a cluster point $D \in \mathcal{F}_\epsilon$. It is easy to see that $D \in B(L_p(S,\mu), V)$ and $\|D\| \le F + \epsilon$. Since $L_c \to L$ in the strong operator topology,

$$L = D \circ \mathrm{id} \circ i,$$

and consequently $N(L) \le F + \epsilon$. Since $\epsilon > 0$ was arbitrary, $N(L) \le F$ as required.

Example 2.8. Now we consider the case of p-nuclear operators. Notice that by [44, p. 216], an operator $T \in B(X,V)$ is p-nuclear ($1 \le p < \infty$) if it can be written in the form

$$Tx = \sum_{j=1}^{\infty} x_j^*(x) v_j, \tag{2.6}$$

where $x_j^* \in X^*$, $v_j \in V$,

$$\Big(\sum_{j=1}^{\infty} \|x_j^*\|^p\Big)^{1/p} \sup\Big\{\Big(\sum_{j=1}^{\infty} |v^*(v_j)|^q\Big)^{1/q} : v^* \in V^*,\ \|v^*\| = 1\Big\} < \infty \tag{2.7}$$

and $1/p + 1/q = 1$. Then the p-nuclear norm $N(T)$ is understood as the infimum of the quantities (2.7) over all representations given by (2.6). Let X be a Banach space and let $V \subset X$ be a finite-dimensional subspace. Let $\mathcal{U}(X,V)$ denote the set of all p-nuclear operators for $1 \le p < \infty$. We show as in the previous example that if a net $\{L_c\}_{c\in C}$ converges to L in the strong operator topology and $\lim_c N(L_c) = D < \infty$ then $N(L) \le D$. To do this, fix

$\epsilon > 0$ and assume that for any $c \in C$,

$$L_c = \sum_{j=1}^{\infty} x_{j,c}^*(\cdot) v_{j,c}$$

and

$$\Big(\sum_{j=1}^{\infty} \|x_{j,c}^*\|^p\Big)^{1/p} \sup\Big\{\Big(\sum_{j=1}^{\infty} |v^*(v_{j,c})|^q\Big)^{1/q} : v^* \in V^*, \ \|v^*\| = 1\Big\} < D + \epsilon.$$

Without loss of generality we can assume that

$$\Big(\sum_{j=1}^{\infty} \|x_{j,c}^*\|^p\Big)^{1/p} \le 1$$

and

$$\sup\Big\{\Big(\sum_{j=1}^{\infty} |v^*(v_{j,c})|^q\Big)^{1/q} : v^* \in V^*, \ \|v^*\| = 1\Big\} < D + \epsilon.$$

Note that for any $c \in C$,

$$u_c = (x_{j,c}^*)_{j \in \mathbb{N}} \in \mathcal{F}_\epsilon = \prod_{n \in \mathbb{N}} B_{X^*}.$$

If we equip B_{X^*} with the weak* topology and $\mathcal{F}_\epsilon$ with the Tikhonov topology, by the Banach–Alaoglu Theorem and the Tikhonov Theorem, $\mathcal{F}_\epsilon$ is a compact set. Hence the net $\{u_c\}_{c \in C}$ has a cluster point $u = (x_1^*, x_2^*, \ldots) \in \mathcal{F}_\epsilon$. It is easy to see that $\|x_j^*\| \le 1$ for any $j \in \mathbb{N}$. Moreover, since for any $j \in \mathbb{N}$, $\|x_j^*\| \le \liminf_c \|x_{j,c}^*\|$, for any $n \in \mathbb{N}$ we have

$$\Big(\sum_{j=1}^{n} \|x_j^*\|^p\Big)^{1/p} \le \Big(\sum_{j=1}^{n} \Big(\liminf_c \|x_{j,c}^*\|\Big)^p\Big)^{1/p} \le \liminf_c \Big(\sum_{j=1}^{n} \|x_{j,c}^*\|^p\Big)^{1/p}$$
$$\le \liminf_c \Big(\sum_{j=1}^{\infty} \|x_{j,c}^*\|^p\Big)^{1/p} \le 1.$$

Hence

$$\Big(\sum_{j=1}^{\infty} \|x_j^*\|^p\Big)^{1/p} \le 1.$$

Since V is finite-dimensional, for any $c \in C$,

$$w_c = (v_{j,c})_{j \in \mathbb{N}} \in \mathcal{W}_\epsilon = \prod_{n \in \mathbb{N}} B_V(D + \epsilon).$$

If we equip $B_V(D+\epsilon)$ with the norm topology and $\mathcal{W}_\epsilon$ with the Tikhonov topology, reasoning as above we find that $\{w_c\}_{c \in C}$ has a cluster point

$(v_1, v_2, \ldots) \in \mathcal{W}_\epsilon$ satisfying

$$\sup\Big\{\Big(\sum_{j=1}^{\infty} |v^*(v_j)|^q\Big)^{1/q} : v^* \in V^*,\ \|v^*\| = 1\Big\} < D + \epsilon.$$

Hence passing to a convergent subnet if necessary, by the above estimates and the Hölder inequality we can assume that for any $x \in X$,

$$L_c x \to \sum_{j=1}^{\infty} x_j^*(x) v_j$$

weakly in X. Since L_c converges to L in the strong operator topology, we obtain

$$Lx = \sum_{j=1}^{\infty} x_j^*(x) v_j,$$

and $N(L) \le D + \epsilon$ for any $\epsilon > 0$, which completes our proof.

Remark 2.9. Notice that applications of Theorem 2.2 are interesting only under the condition that there exists $P \in \mathcal{P}(X, V)$ such that $N(P) < \infty$. Since every p-integral operator is weakly compact, $N(P) = \infty$ for any $P \in \mathcal{P}(X, V)$ if V is nonreflexive. Analogously, since any p-nuclear operator is compact, $N(P) = +\infty$ for any $P \in \mathcal{P}(X, V)$ if V is of infinite dimension. This explains our assumptions in Examples 2.7 and 2.8.

3. Applications. In this section we present some examples of X, V and N satisfying the requirements of Theorem 2.2 such that $N(P) < \infty$ for some $P \in \mathcal{P}(X, V)$. Notice that if N denotes the numerical radius and $\mathcal{P}(X, V) \neq \emptyset$, then $N(P) \le \|P\| < \infty$ for any $P \in \mathcal{P}(X, V)$, where $\|\cdot\|$ denotes the operator norm. If N denotes a norm in a Banach operator ideal and V is finite-dimensional then $\mathcal{P}(X, V) \neq \emptyset$ and by Definition 1.2 also $N(P) < \infty$ for any $P \in \mathcal{P}(X, V)$.

We start with a classical example which explains the origin of Rudin's Theorem.

Example 3.1. Let $C(2\pi)$ denote the set of all continuous 2π-periodic functions and π_n be the space of all trigonometric polynomials of order $\le n$ $(n \ge 1)$. The *Fourier projection* $F_n : C(2\pi) \to \pi_n$ is defined by

$$F_n(f) = \sum_{k=0}^{2n} \Big(\int_0^{2\pi} f(t) g_n(t)\, dt\Big) g_k$$

where $(g_k)_{k=0}^{2n}$ is an orthonormal basis in π_n with respect to the scalar product

$$\langle f, g\rangle = \int_{[0,2\pi]} f(t) g(t)\, dt.$$

In [32] it is shown that F_n is a minimal projection in $\mathcal{P}(C(2\pi), \pi_n)$. The method of proof is based on the Marcinkiewicz equality (see [13, p. 233]), which says that for any $P \in \mathcal{P}(C(2\pi), \pi_n)$, $f \in C(2\pi)$ and $t \in [0, 2\pi]$,

$$F_n f(t) = \frac{1}{2\pi} \int_{[0,2\pi]} (T_{g^{-1}} P T_g f)t \, d\mu(g).$$

Here μ is the Lebesgue measure and $(T_g f)t = f(t+g)$ for any $g \in \mathbb{R}$. Notice that F_n is the only projection which commutes with G, where $G = [0, 2\pi)$ with addition mod 2π. Hence in particular, F_n is an N-minimal projection for N as considered in Examples 2.5–2.8. Furthermore, it is known (see [13, p. 212]) that the operator norm of F_n satisfies

$$\frac{4}{\pi^2} \ln(n) \le \|F_n\| \le \ln(n) + 3.$$

In [1], it is shown that in cases of L_p, $p = 1, \infty$, numerical radius extensions and minimal norm extensions are equal. Since $C(2\pi) \subset L_\infty$, we also have

$$\frac{4}{\pi^2} \ln(n) \le \|F_n\|_w \le \ln(n) + 3.$$

The Marcinkiewicz equality holds true if we replace $C(2\pi)$ by $L_p[0, 2\pi]$ for $1 \le p \le \infty$ or by the Orlicz space $L^\phi[0, 2\pi]$ equipped with the Luxemburg or the Orlicz norm provided that ϕ satisfies the suitable Δ_2 condition. Hence, Theorem 2.2 can be applied to the numerical radius and norms in Banach operator ideals of p-summing, p-integral and p-nuclear operators generated by the L_p norm or the Luxemburg or the Orlicz norm.

Now we consider a more general situation.

Example 3.2. Let $m, n \in \mathbb{N}$, $n < m$. Let

$$V = \operatorname{span}[\sin(k_i \,\cdot), \cos(k_i \,\cdot) : i = 1, \dots, n],$$
$$X = \operatorname{span}[\sin(k_i \,\cdot), \cos(k_i \,\cdot) : i = 1, \dots, m],$$

where $k_i \in \mathbb{N}$ and $k_1 < \dots < k_m$. Assume that G is as in Example 3.1. It is easy to see that the only projection from X onto V which commutes with G is given by

$$Q(\sin(k_i \,\cdot)) = 0, \qquad Q(\cos(k_i \cdot)) = 0$$

for $i > n$. Assume that $\| \cdot \|_X$ is any norm on X such that the mapping

$$T_g : (X, \| \cdot \|_X) \to (X, \| \cdot \|_X)$$

is a linear isometry for any $g \in G$. Hence, in particular, Q is an N-minimal projection for N as considered in Examples 2.5–2.8. Typical examples of $\| \cdot \|_X$ are the L_p-norms, the Luxemburg and the Orlicz norms. The same situation holds true in the complex case with

$$X = \mathrm{span}[e^{ik_j t} : i = 1, \ldots, m],$$
$$V = \mathrm{span}[e^{ik_j t} : i = 1, \ldots, n] \quad \text{and} \quad G = \{e^{it} : t \in [0, 2\pi]\}.$$

Notice that it is possible to replace X by $L_p[0, 2\pi]$ for $1 \le p \le \infty$ or by an Orlicz space $L^\phi[0, 2\pi]$ equipped with the Luxemburg norm or the Orlicz norm provided that ϕ satisfies the suitable Δ_2 condition. Also we can apply Theorem 2.2 in multi-dimensional settings (see, e.g., [27]).

EXAMPLE 3.3. Let $X = L_p[0, 2\pi]$ and let $V = H^p[0, 2\pi]$ be the Hardy space for $1 < p < \infty$. By the M. Riesz Theorem (see [40, p. 152]), it follows that $\mathcal{P}(L_p[0, 2\pi], H^p[0, 2\pi]) \neq \emptyset$ and the projection Q given by

$$Q(e^{ik_j \cdot}) = 0$$

for $j < 0$ is the only projection which commutes with $G = \{e^{it} : t \in [0, 2\pi]\}$. Hence, in particular, Q is an N-minimal projection as considered in Example 2.5.

EXAMPLE 3.4. Let $M(n, m)$ be the space of all (real or complex) $n \times m$ matrices. Denote by $M(n, 1)$ ($M(1, m)$ respectively) the space of matrices from $M(n, m)$ with constant rows (constant columns respectively). Let S_n be the group of permutations of $\{1, \ldots, n\}$. Let $G = S_n \times S_m$. For any $g = \sigma \times \gamma \in G$ define a mapping $T_g : M(n, m) \to M(n, m)$ by

$$T_g(A)(i, j) = A(\sigma(i), \gamma(j))$$

for any $A \in M(n, m)$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Let

$$W = M(n, 1) + M(1, m).$$

It is easy to see that $T_g(W) \subset W$ for any $g \in G$. Now assume that

$$X = (M(n, m), \| \cdot \|)$$

where $\| \cdot \|$ is any norm such that the mappings T_g are isometries on G. Typical examples of such norms are L_p-norms and the Luxemburg and Orlicz norms. In [16, Chapter 9] it has been shown that there is a unique projection Q which commutes with G that is given by the formula

$$Qe_{rs}(i, j) = \begin{cases} \dfrac{n + m + 1}{nm}, & i = r,\ j = s, \\ \dfrac{m - 1}{nm}, & i \neq r,\ j = s, \\ \dfrac{n - 1}{nm}, & i = r,\ j \neq s, \\ \dfrac{-1}{nm}, & i \neq r,\ j \neq s, \end{cases}$$

where $e_{rs}(i, j) = \delta_{ri}\delta_{rj}$. Hence, in particular, Q is an N-minimal projection for N as considered in Examples 2.5–2.8.

Example 3.5. Let $[x]$ denote the integer part of $x \in \mathbb{R}$. The well-known Rademacher functions $r_0, r_1, \dots$ defined by $r_j(t) = (-1)^{[2^j t]}$ for $0 \le t \le 1$ play an important role in many areas of analysis. Let

$$\mathrm{Rad}_n = \mathrm{span}[r_0, \dots, r_n].$$

Fix $m, n \in \mathbb{N}$, $n < m$. Let us consider $X = \mathrm{Rad}_m$ as a subspace of $L_p[0,1]$, $1 \le p < \infty$, where $[0,1]$ is equipped with the Lebesgue measure. We will find an N-minimal projection from $X = \mathrm{Rad}_m$ onto $V = \mathrm{Rad}_n$ for N as considered in Examples 2.5–2.8. To do this, we need to define so-called dyadic addition on the interval $[0,1]$. Let Q denote the set of all dyadic rationals from the interval $[0,1)$, i.e.,

$$Q = \{x \in \mathbb{R} : x = p/2^n,\ p \in \mathbb{N},\ 0 \le p < 2^n\}.$$

Note that any $x \in [0,1]$ can be written in the form

$$x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)},$$

where each x_k is 0 or 1. For each $x \in [0,1] \setminus Q$ there is only one expression of this form. When $x \in Q$ there are two expressions, one which terminates in 0's and the other which terminates in 1's. By the dyadic expansion of $x \in Q$ we shall mean the one which terminates in 0's. Now we can define the dyadic sum of $x, y \in [0,1]$ by

$$x \oplus y = \sum_{k=0}^{\infty} |x_k - y_k| 2^{-(k+1)}.$$

Notice that $G = ([0,1], \oplus)$ is a group. Indeed, $x \oplus 0 = x$ and $x \oplus x = 0$. Also it is easy to see that for any $n \in \mathbb{N}$ and $x \in [0,1]$,

$$r_n(x \oplus y) = r_n(x) r_n(y) \tag{3.1}$$

provided that $x \oplus y \notin Q$. Moreover, for any $g \in [0,1]$ the operator $T_g : L_p[0,1] \to L_p[0,1]$ given by

$$(T_g f)(x) = f(x \oplus g)$$

is a linear surjective isometry.

Now we will show that if $f_n, f \in L_p[0,1]$, $\|f_n - f\|_p \to 0$ and $|g_n - g| \to 0$, then

$$\|T_{g_n}(f_n) - T_g(f)\|_p \to 0. \tag{3.2}$$

To do this note that

$$\|T_{g_n}(f_n) - T_g(f)\|_p \le \|T_{g_n}(f_n) - T_{g_n}(f)\|_p + \|T_{g_n}(f) - T_g(f)\|_p.$$

Observe that by changing variables from x to $x \oplus g_n$ we get

$$\begin{aligned}\|T_{g_n}(f_n) - T_g(f)\|_p^p &= \int\limits_{[0,1]} |f_n(x \oplus g_n) - f(x \oplus g_n)|^p \, d\mu(x) \\ &= \int\limits_{[0,1]} |f_n(x) - f(x)|^p \, d\mu(x) = \|f_n - f\|_p^p \to 0.\end{aligned}$$

Notice that, if f is a continuous function (and hence uniformly continuous on $[0,1]$), since $g_n \to g$, for any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for any $x \in [0,1]$ and $n \geq n_0$, $|f(x \oplus g_n) - f(x \oplus g)| \leq \epsilon$. Consequently, $\|T_{g_n} f - T_g f\|_p \to 0$ for any $f \in C[0,1]$. By the Banach–Steinhaus Theorem, since $1 \leq p < \infty$, it follows that $\|T_{g_n} f - T_g f\|_p \to 0$, which proves (3.2).

Note that, since Rad_n is a finite-dimensional subspace, $\mathcal{P}(\mathrm{Rad}_m, \mathrm{Rad}_n) \neq \emptyset$. By (3.1), $T_g(\mathrm{Rad}_n) \subset \mathrm{Rad}_n$ for any $n \in \mathbb{N}$. Consequently, applying the fact that $g^{-1} = g$ for any $g \in G$, for any $P \in \mathcal{P}(X, \mathrm{Rad}_n)$, the projection

$$Q_P f = \int\limits_{[0,1]} (T_g P T_g) f \, d\mu(g) \in \mathcal{P}(X, \mathrm{Rad}_n)$$

commutes with G.

Now we show that there is exactly one projection from X onto Rad_n which commutes with G. To do this, we show that for any $P \in \mathcal{P}(X, \mathrm{Rad}_n)$ $Q_P(r_k) = 0$ for $m \geq k > n$. So fix $x \in [0,1]$ and $g \in G$ with $x \oplus g \notin Q$. Note that

$$\begin{aligned}(T_g P T_g r_k)x = r_k(g)(T_g P T_g r_k)x &= r_k(g)\Big(T_g\Big(\sum_{j=0}^{n} a_j r_j\Big)\Big)x \\ &= r_k(g) \sum_{j=0}^{n} a_j r_j(x) r_j(g).\end{aligned}$$

Observe that $\int_{[0,1]} r_j(g) r_k(g) \, d\mu(g) = 0$ if $k \neq j$. Since for any $x \in [0,1]$,

$$\mu(\{g \in G : x \oplus g \in Q\}) = 0,$$

it follows that

$$(Q_P r_k)x = \int\limits_{[0,1]} r_k(g)\Big(\sum_{j=0}^{n} a_j r_j(x) r_j(g)\Big) \, d\mu(g) = 0,$$

which demonstrates our claim.

Consequently, for any $P \in \mathcal{P}(\mathrm{Rad}_m, \mathrm{Rad}_n)$ and $f \in \mathrm{Rad}_m$,

$$R_n f = Q_P f = \sum_{j=0}^{n} \Big(\int\limits_{[0,1]} r_j(t) f(t) \, d\mu(t) \Big) r_j$$

is an N-minimal projection for N as considered in Examples 2.5–2.8. For more information about the nth Rademacher projection R_n the reader is referred to [29].

Example 3.6. For all $n \in \mathbb{N}$, let $X_n = \mathcal{L}(\mathbb{R}^n)$. Set

$$Y_n = \{L \in X_n : L = L^T\}.$$

Let us equip X_n with the operator norm determined by any symmetric norm $\|\cdot\|$ on $\mathbb{R}^n$. (We say that $\|\cdot\|$ is *symmetric* if

$$\Big\|\sum_{j=1}^{n} a_j e_j\Big\| = \Big\|\sum_{j=1}^{n} \epsilon_j a_{\sigma(j)} e_j\Big\|$$

for any $a_1, \dots, a_n \in \mathbb{R}$, $\epsilon_j \in \{-1, 1\}$ and any permutation σ of $\{1, \dots, n\}$.) For $L \in X_n$ set

$$P(L) = (L + L^T)/2.$$

It is clear that $P \in \mathcal{P}(X_n, Y_n)$. Moreover, in [35] and [36] it was shown, applying Theorem 2.1, that P is a minimal projection in $\mathcal{P}(X_n, Y_n)$. Hence P is an N-minimal projection for N as considered in Examples 2.5–2.8.

Problem 3.7. Notice that in the above examples the N-minimal projections determined by Theorem 2.2 are, in general, not the unique N-minimal projection (see, e.g., [42] in the case of operator norm). However, it has been proven in [15] that the Fourier projection F_n is the unique minimal projection with respect to the operator norm determined by the supremum norm in $C_0(2\pi)$. Hence, we pose the following question:

Which N-minimal projections determined in the above examples are the only N-minimal projections?

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Asuman Güven Aksoy
Department of Mathematics
Claremont McKenna College
Claremont, CA 91711, U.S.A.
E-mail: aaksoy@cmc.edu

Grzegorz Lewicki
Department of Mathematics and Computer Science
Jagiellonian University
Łojasiewicza 6
30-348 Kraków, Poland
E-mail: Grzegorz.Lewicki@im.uj.edu.pl

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