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Characterization of Jordan derivations on $\mathcal{J}$-subspace lattice algebras

by

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Abstract. Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X and $\operatorname{Alg}\mathcal{L}$ the associated $\mathcal{J}$-subspace lattice algebra. Assume that $\delta : \operatorname{Alg}\mathcal{L} \to \operatorname{Alg}\mathcal{L}$ is an additive map. It is shown that δ satisfies $\delta(AB + BA) = \delta(A)B + A\delta(B) + \delta(B)A + B\delta(A)$ for any $A, B \in \operatorname{Alg}\mathcal{L}$ with $AB + BA = 0$ if and only if $\delta(A) = \tau(A) + \delta(I)A$ for all A, where τ is an additive derivation; if X is complex with $\dim X \geq 3$ and if δ is linear, then δ satisfies $\delta(AB+BA) = \delta(A)B+A\delta(B)+\delta(B)A+B\delta(A)$ for any $A, B \in \operatorname{Alg}\mathcal{L}$ with $AB+BA = I$ if and only if δ is a derivation.

1. Introduction. Let $\mathcal{A}$ be a ring (or an algebra) with unit I and $\delta : \mathcal{A} \to \mathcal{A}$ an additive (or a linear) map. Recall that δ is called a *derivation* if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathcal{A}$, and a *Jordan derivation* if $\delta(A^2) = \delta(A)A + A\delta(A)$ for all $A \in \mathcal{A}$, or equivalently, if

$$\delta(AB + BA) = \delta(A)B + A\delta(B) + \delta(B)A + B\delta(A)$$

for all $A, B \in \mathcal{A}$ in the case where $\mathcal{A}$ is not of characteristic 2. Derivations and Jordan derivations have been extensively studied (see for instance [B, C, H] and the references therein).

In recent years, more and more mathematicians are interested in conditions under which derivations (or Jordan derivations) can be completely determined by the action on some sets of elements. Recall that an additive (linear) map $\delta : \mathcal{A} \to \mathcal{A}$ is *derivable at a point* $Z \in \mathcal{A}$ if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathcal{A}$ with $AB = Z$, and *Jordan derivable at* $Z \in \mathcal{A}$ if $\delta(AB + BA) = \delta(A)B + A\delta(B) + \delta(B)A + B\delta(A)$ for all $A, B \in \mathcal{A}$ with $AB + BA = Z$. Lu [L] proved that every continuous linear map derivable at a left (or right) invertible element on a Banach algebra is a Jordan derivation.

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Let $\mathrm{Alg}\,\mathcal{N}$ be a nest algebra over a Banach space X. Qi and Hou [QH1] proved that every linear map derivable at the unit operator (or at an invertible operator, or at a nontrivial idempotent) on $\mathrm{Alg}\,\mathcal{N}$ is a derivation.

Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X with $\dim X > 2$ and $\mathrm{Alg}\,\mathcal{L}$ the associated $\mathcal{J}$-subspace lattice algebra. Hou and Qi [HQ] proved that every additive map δ derivable at zero on $\mathrm{Alg}\,\mathcal{L}$ is of the form $\delta(A) = \tau(A) + cA$ for all A, where τ is an additive derivation and c is a scalar; if X is complex, then every linear map on $\mathrm{Alg}\,\mathcal{L}$ derivable at the unit operator I is a derivation.

Let $\mathcal{D}$ be a strongly double triangle subspace lattice algebra on a nonzero complex reflexive Banach space X. Chen and Li [CL] proved that every linear map Jordan derivable at zero on $\mathrm{Alg}\,\mathcal{D}$ is a generalized derivation. For other results, see [JLL, QH2] and the references therein.

The purpose of this paper is to discuss additive or linear maps Jordan derivable at the zero or unit operator on another important family of algebras, namely, the family of $\mathcal{J}$-subspace lattice algebras.

Let X be a Banach space over the real or complex field $\mathbb{F}$. A family $\mathcal{L}$ of subspaces of X is called a *subspace lattice* on X if it contains $\{0\}$ and X, and is closed under the operations of closed linear span $\vee$ and intersection $\wedge$ in the sense that $\bigvee_{\gamma\in\Gamma} L_\gamma \in \mathcal{L}$ and $\bigwedge_{\gamma\in\Gamma} L_\gamma \in \mathcal{L}$ for every family $\{L_\gamma : \gamma \in \Gamma\}$ of elements in $\mathcal{L}$. For a subspace lattice $\mathcal{L}$ on X, the associated subspace lattice algebra $\mathrm{Alg}\,\mathcal{L}$ is the set of operators on X leaving every subspace in $\mathcal{L}$ invariant. Given a subspace lattice $\mathcal{L}$ on X, put

$$\mathcal{J}(\mathcal{L}) = \{K \in \mathcal{L} : K \neq \{0\} \text{ and } K_- \neq X\},$$

where $K_- = \bigvee\{L \in \mathcal{L} : K \nsubseteq L\}$. Call $\mathcal{L}$ a *$\mathcal{J}$-subspace lattice* (simply, JSL) on X if it satisfies the following conditions:

(1) $\bigvee\{K : K \in \mathcal{J}(\mathcal{L})\} = X$;
(2) $\bigwedge\{K_- : K \in \mathcal{J}(\mathcal{L})\} = \{0\}$;
(3) $K \vee K_- = X,\ \forall K \in \mathcal{J}(\mathcal{L})$;
(4) $K \wedge K_- = \{0\},\ \forall K \in \mathcal{J}(\mathcal{L})$.

If $\mathcal{L}$ is a JSL, the associated subspace lattice algebra $\mathrm{Alg}\,\mathcal{L}$ is called a *$\mathcal{J}$-subspace lattice algebra*, briefly, a *JSL-algebra* (see [LNP, LP, L1]). Note that JSL algebras are not prime. It should be mentioned that both atomic Boolean subspace lattices and pentagon subspace lattices are $\mathcal{J}$-subspace lattices [LP]. For $L \in \mathcal{L}$, denote $L_-^\perp = (L_-)^\perp$, where $L^\perp$ denotes the annihilator of L. Denote by $\langle \mathcal{J}(\mathcal{L})\rangle$ and $\langle \mathcal{J}(\mathcal{L})_-^\perp\rangle$ the (not necessarily closed) linear spans of $\bigcup\{K : K \in \mathcal{J}(\mathcal{L})\}$ and of $\bigcup\{K_-^\perp : K \in \mathcal{J}(\mathcal{L})\}$, respectively. For $x \in X$ and $f \in X^*$, $x \otimes f$ stands for the operator on X with rank not greater than one defined by $(x \otimes f)y = f(y)x$. Sometimes we use

$\langle x, f\rangle$ for $f(x)$. For $K \in \mathcal{J}(\mathcal{L})$, $\mathcal{F}_{\mathcal{L}}(K)$ stands for the subspace spanned by $\{x \otimes f : x \in K,\ f \in K_-^\perp\}$.

This paper is organized as follows. Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X, and $\delta : \operatorname{Alg}\mathcal{L} \to \operatorname{Alg}\mathcal{L}$ an additive map. In Section 2, we show that δ is Jordan derivable at zero (i.e. $\delta(AB+BA) = \delta(A)B + A\delta(B) + \delta(B)A + B\delta(A)$ whenever $AB+BA = 0$) if and only if there exists an additive derivation τ on $\operatorname{Alg}\mathcal{L}$ and a scalar λ such that $\delta(A) = \tau(A) + \lambda A$ for all $A \in \operatorname{Alg}\mathcal{L}$ (Theorem 2.1). In Section 3, we prove that if δ is linear and X is complex with $\dim X \geq 3$, then $\delta(AB+BA) = \delta(A)B + A\delta(B) + \delta(B)A + B\delta(A)$ for any $A, B \in \operatorname{Alg}\mathcal{L}$ with $AB + BA = I$ if and only if δ is a derivation (Theorem 3.1).

2. Additive maps Jordan derivable at zero. In this section, we consider the question of characterizing additive maps Jordan derivable at zero on $\mathcal{J}$-subspace lattice algebras.

THEOREM 2.1. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X over the real or complex field $\mathbb{F}$. Suppose that $\delta : \operatorname{Alg}\mathcal{L} \to \operatorname{Alg}\mathcal{L}$ is an additive map. Then δ satisfies $\delta(AB+BA) = \delta(A)B + A\delta(B) + \delta(B)A + B\delta(A)$ whenever $AB + BA = 0$ for any $A, B \in \operatorname{Alg}\mathcal{L}$ if and only if there exists an additive derivation $\tau : \operatorname{Alg}\mathcal{L} \to \operatorname{Alg}\mathcal{L}$ and a scalar $\lambda \in \mathbb{F}$ such that $\delta(A) = \tau(A) + \lambda A$ for all $A \in \operatorname{Alg}\mathcal{L}$.*

To prove Theorem 2.1, we need several lemmas.

LEMMA 2.2 ([L2]). *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X. Then $x \otimes f \in \operatorname{Alg}\mathcal{L}$ if and only if there exists a subspace $K \in \mathcal{J}(\mathcal{L})$ such that $x \in K$ and $f \in K_-^\perp$.*

LEMMA 2.3 ([LP]). *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X and let $K \in \mathcal{J}(\mathcal{L})$. Then, for any nonzero vector $x \in K$, there exists $f \in K_-^\perp$ such that $f(x) = 1$; dually, for any nonzero functional $f \in K_-^\perp$, there exists $x \in K$ such that $f(x) = 1$.*

LEMMA 2.4 ([HQ]). *Every rank one operator $x \otimes f \in \operatorname{Alg}\mathcal{L}$ is a linear combination of idempotents in* $\operatorname{Alg}\mathcal{L}$.

LEMMA 2.5. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X and $\delta : \operatorname{Alg}\mathcal{L} \to \operatorname{Alg}\mathcal{L}$ an additive map. If δ satisfies $\delta(AB+BA) = \delta(A)B + A\delta(B) + \delta(B)A + B\delta(A)$ whenever $AB + BA = 0$ for any $A, B \in \operatorname{Alg}\mathcal{L}$, then $\delta(I) = \lambda I$ for some scalar λ.*

Proof. For any idempotent $P \in \operatorname{Alg}\mathcal{L}$, it is obvious that $P(I-P) + (I-P)P = 0$. By the assumption on δ, we have

$$\begin{aligned} 0 &= \delta(P)(I-P) + P\delta(I-P) + \delta(I-P)P + (I-P)\delta(P) \\ &= 2\delta(P) - 2\delta(P)P + P\delta(I) - 2P\delta(P) + \delta(I)P, \end{aligned}$$

that is,

$$2\delta(P) + P\delta(I) + \delta(I)P = 2\delta(P)P + 2P\delta(P). \tag{2.1}$$

Multiplying (2.1) by P from the left and from the right, we get $P\delta(I) = \delta(I)P$. Taking any $K \in \operatorname{Alg}\mathcal{L}$, by Lemmas 2.2–2.3, for any $x \in K$, there exists $f \in K_-^\perp$ such that $f(x) = 1$ and $x \otimes f \in \operatorname{Alg}\mathcal{L}$. Thus, by Lemma 2.4, we get $x \otimes f\delta(I) = \delta(I)x \otimes f$, which implies that $\delta(I)x$ and x are linearly dependent. Let $\delta(I)x = \lambda_x x$ for some scalar λ_x. Since $\langle \mathcal{J}(\mathcal{L})\rangle$ is dense in X, we get $\delta(I) = \lambda I$ for some scalar λ. ∎

Lemma 2.6. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X and $\delta : \operatorname{Alg}\mathcal{L} \to \operatorname{Alg}\mathcal{L}$ an additive map satisfying $\delta(AB + BA) = \delta(A)B + A\delta(B) + \delta(B)A + B\delta(A)$ whenever $AB + BA = 0$ for any $A, B \in \operatorname{Alg}\mathcal{L}$. Then*

(i) *$\delta(P) = \delta(P)P + P\delta(P) - \delta(I)P$ for every idempotent $P \in \operatorname{Alg}\mathcal{L}$,*
(ii) *$\delta(N)N + N\delta(N) = 0$ for every $N \in \operatorname{Alg}\mathcal{L}$ with $N^2 = 0$.*

Proof. (i) By (2.1), this is obvious.

(ii) For every $N \in \operatorname{Alg}\mathcal{L}$ with $N^2 = 0$, we have

$$0 = \delta(N^2+N^2) = \delta(N)N+N\delta(N)+\delta(N)N+N\delta(N) = 2\delta(N)N+2N\delta(N).$$

It follows that $\delta(N)N + N\delta(N) = 0$. ∎

Lemma 2.7 ([LLL, Lemma 8.3.2]). *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X. Assume that $K \in \mathcal{J}(\mathcal{L})$ with $\dim K \geq 2$, $\phi : \mathcal{F}_{\mathcal{L}}(K) \to \mathcal{B}(X)$ is a ring homomorphism and $\psi : \mathcal{F}_{\mathcal{L}}(K) \to \mathcal{B}(X)$ is a ring anti-homomorphism. If $\phi(A) + \psi(A) = A$ for all $A \in \mathcal{F}_{\mathcal{L}}(K)$, then $\psi = 0$.*

Now we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1. Obviously, we only need to check the "only if" part.

Assume $\delta : \operatorname{Alg}\mathcal{L} \to \operatorname{Alg}\mathcal{L}$ is an additive map satisfying $\delta(AB + BA) = \delta(A)B + A\delta(B) + \delta(B)A + B\delta(A)$ whenever $AB + BA = 0$ for any $A, B \in \operatorname{Alg}\mathcal{L}$. Define $\tau : \operatorname{Alg}\mathcal{L} \to \operatorname{Alg}\mathcal{L}$ by $\tau(A) = \delta(A) - \delta(I)A = \delta(A) - \lambda A$ for all $A \in \operatorname{Alg}\mathcal{L}$. It is easy to check that τ is also an additive map satisfying $\tau(I) = 0$ and $\tau(AB + BA) = \tau(A)B + A\tau(B) + \tau(B)A + B\tau(A)$ whenever $AB + BA = 0$ for any $A, B \in \operatorname{Alg}\mathcal{L}$. Moreover, $\tau(P) = \tau(P)P + P\tau(P)$ for all idempotents $P \in \operatorname{Alg}\mathcal{L}$, and $\tau(N)N + N\tau(N) = 0$ for every $N \in \operatorname{Alg}\mathcal{L}$ with $N^2 = 0$. To complete the proof, we only need to show that τ is an additive derivation. We will prove it by checking several claims.

Claim 1. *There exists an additive map $h : \mathbb{F} \to \mathbb{F}$ such that $\tau(\alpha I) = h(\alpha)I$ and $\tau(\alpha P) = \tau(\alpha I)P + \alpha\tau(P) = h(\alpha)P + \alpha\tau(P)$ for every $\alpha \in \mathbb{F}$ and every idempotent $P \in \operatorname{Alg}\mathcal{L}$.*

Since $\alpha P(I-P)+(I-P)\alpha P=0$ and $\tau(I)=0$, we have

$$\tau(\alpha P)(I-P)+\alpha P\tau(I-P)+\tau(I-P)\alpha P+(I-P)\tau(\alpha P)=0,$$

that is,

$$2\tau(\alpha P)=\tau(\alpha P)P+\alpha P\tau(P)+\alpha\tau(P)P+P\tau(\alpha P). \tag{2.2}$$

Multiplying (2.2) by P from the left and the right, one gets, respectively,

$$P\tau(\alpha P)=P\tau(\alpha P)P+\alpha P\tau(P)+\alpha P\tau(P)P \tag{2.3}$$

and

$$\tau(\alpha P)P=\alpha P\tau(P)P+\alpha\tau(P)P+P\tau(\alpha P)P. \tag{2.4}$$

Comparing (2.3) and (2.4), we obtain

$$\tau(\alpha P)P+\alpha P\tau(P)=P\tau(\alpha P)+\alpha\tau(P)P. \tag{2.5}$$

Similarly, from the relation $\alpha(I-P)P+P\alpha(I-P)=0$, we get

$$\begin{aligned}2\alpha\tau(P)+P\tau(\alpha I)+\tau(\alpha I)P&\\ =\tau(\alpha P)P+P\tau(\alpha P)&+\alpha P\tau(P)+\alpha\tau(P)P.\end{aligned} \tag{2.6}$$

Multiplying (2.6) by P from the left and the right, and using (2.5), one can obtain, respectively,

$$P\tau(\alpha I)P+\tau(\alpha I)P=2P\tau(\alpha P)P \tag{2.7}$$

and

$$P\tau(\alpha I)P+P\tau(\alpha I)=2P\tau(\alpha P)P. \tag{2.8}$$

Combining (2.7) and (2.8), we get $\tau(\alpha I)P=P\tau(\alpha I)$ for all idempotents $P\in\operatorname{Alg}\mathcal{L}$. By a similar argument to that for Lemma 2.5, there exists a map $h:\mathbb{F}\to\mathbb{F}$ such that $\tau(\alpha I)=h(\alpha)I$. It is clear that h is additive. Combining this equation, (2.6) and (2.2), we get $\tau(\alpha P)=\tau(\alpha I)P+\alpha\tau(P)=h(\alpha)P+\alpha\tau(P)$.

Claim 2. *$\tau(P\alpha Q+\alpha QP)=\tau(P)\alpha Q+P\tau(\alpha Q)+\tau(\alpha Q)P+\alpha Q\tau(P)$ for every $\alpha\in\mathbb{F}$ and all idempotents $P,Q\in\operatorname{Alg}\mathcal{L}$.*

Take any $T,S\in\operatorname{Alg}\mathcal{L}$ with $ST=0$ and any idempotent $P\in\operatorname{Alg}\mathcal{L}$. Note that $TP(I-P)S+(I-P)STP=0$ and $T(I-P)PS+PST(I-P)=0$. We have

$$\tau(TP)(I-P)S+TP\tau((I-P)S)+\tau((I-P)S)TP+(I-P)S\tau(TP)=0$$

and

$$\tau(T(I-P))PS+T(I-P)\tau(PS)+PS\tau(T(I-P))+\tau(PS)T(I-P)=0.$$

That is,

$$\begin{aligned}&\tau(TP)S+TP\tau(S)+S\tau(TP)+\tau(S)TP\\&\qquad=\tau(TP)PS+TP\tau(PS)+PS\tau(TP)+\tau(PS)TP\end{aligned}$$

and

$$\begin{aligned}&\tau(T)PS+T\tau(PS)+PS\tau(T)+\tau(PS)T\\&\qquad=\tau(TP)PS+TP\tau(PS)+PS\tau(TP)+\tau(PS)TP.\end{aligned}$$

Comparing the above two equations, one obtains

$$\begin{aligned}&\tau(TP)S+TP\tau(S)+S\tau(TP)+\tau(S)TP\\&\qquad=\tau(T)PS+T\tau(PS)+PS\tau(T)+\tau(PS)T.\end{aligned}\tag{2.9}$$

Take any idempotent $Q\in\operatorname{Alg}\mathcal{L}$. Letting $T=Q$, $S=I-Q$, respectively $T=I-Q$, $S=Q$ in (2.9) and noting that $ST=0$, we have

$$\begin{aligned}&2\tau(QP)+\tau(Q)PQ+Q\tau(PQ)+PQ\tau(Q)+\tau(PQ)Q\\&\qquad=\tau(QP)Q+QP\tau(Q)+Q\tau(QP)+\tau(Q)QP\\&\qquad\quad+\tau(P)Q+P\tau(Q)+Q\tau(P)+\tau(Q)P\end{aligned}$$

and

$$\begin{aligned}&2\tau(PQ)+\tau(QP)Q+QP\tau(Q)+Q\tau(QP)+\tau(Q)QP\\&\qquad=\tau(Q)PQ+Q\tau(PQ)+PQ\tau(Q)+\tau(PQ)Q\\&\qquad\quad+\tau(P)Q+P\tau(Q)+Q\tau(P)+\tau(Q)P.\end{aligned}$$

The above two equations entail that

$$\tau(QP+PQ)=\tau(P)Q+P\tau(Q)+Q\tau(P)+\tau(Q)P\tag{2.10}$$

for all idempotents $P,Q\in\operatorname{Alg}\mathcal{L}$.

Now, take any $\alpha\in\mathbb{F}$. Letting $T=Q$, $S=\alpha(I-Q)$, respectively $T=\alpha(I-Q)$, $S=Q$ in (2.9) one gets

$$\begin{aligned}&2\alpha\tau(QP)+2\tau(\alpha I)QP+\tau(Q)\alpha PQ\\&\qquad+Q\tau(\alpha PQ)+\alpha PQ\tau(Q)+\tau(\alpha PQ)Q\\&\qquad=\tau(QP)\alpha Q+QP\tau(\alpha Q)+\alpha Q\tau(QP)+\tau(\alpha Q)QP\\&\qquad\quad+\tau(Q)\alpha P+Q\tau(\alpha P)+\alpha P\tau(Q)+\tau(\alpha P)Q\end{aligned}\tag{2.11}$$

and

$$\begin{aligned}&2\alpha\tau(PQ)+2\tau(\alpha I)PQ+\tau(\alpha QP)Q\\&\qquad+\alpha QP\tau(Q)+Q\tau(\alpha QP)+\tau(Q)\alpha QP\\&\qquad=\tau(\alpha P)Q+\alpha P\tau(Q)+Q\tau(\alpha P)+\tau(Q)\alpha P\\&\qquad\quad+\tau(\alpha Q)PQ\ +\alpha Q\tau(PQ)+PQ\tau(\alpha Q)+\tau(PQ)\alpha Q.\end{aligned}\tag{2.12}$$

Combining (2.10)–(2.12) and Claim 1, we get

$$\alpha\tau(QP+PQ)+\tau(\alpha I)(QP+PQ) = \tau(Q)\alpha P+Q\tau(\alpha P)+\alpha P\tau(Q)+\tau(\alpha P)Q.$$

This and (2.12) yield

$$\begin{aligned}\tau(QP+PQ)\alpha Q &+ (QP+PQ)\tau(\alpha Q)\\ &+ \alpha Q\tau(QP+PQ)+\tau(\alpha Q)(QP+PQ)\\ = \tau(Q)(\alpha PQ+\alpha QP) &+ Q\tau(\alpha PQ+\alpha QP)\\ &+ (\alpha PQ+\alpha QP)\tau(Q)+\tau(\alpha PQ+\alpha QP)Q.\end{aligned} \tag{2.13}$$

Finally, by taking $T = I - Q$, $S = \alpha Q$, respectively $T = \alpha Q$, $S = I - Q$ in (2.9), one obtains

$$\begin{aligned}\tau(P)\alpha Q &+ P\tau(\alpha Q)+\alpha Q\tau(P)+\tau(\alpha Q)P\\ &+ \tau(Q)\alpha PQ+Q\tau(\alpha PQ)+\alpha PQ\tau(Q)+\tau(\alpha PQ)Q\\ &= \tau(QP)\alpha Q+QP\tau(\alpha Q)+\alpha Q\tau(QP)+\tau(\alpha Q)QP+2\tau(\alpha PQ)\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}2\tau(\alpha QP) &+ \tau(\alpha Q)PQ+\alpha Q\tau(PQ)+PQ\tau(\alpha Q)+\tau(PQ)\alpha Q\\ &= \tau(\alpha QP)Q+\alpha QP\tau(Q)+Q\tau(\alpha QP)+\tau(Q)\alpha QP\\ &\quad + \tau(\alpha Q)P+\alpha Q\tau(P)+P\tau(\alpha Q)+\tau(P)\alpha Q.\end{aligned} \tag{2.15}$$

Comparing (2.14), (2.15) and (2.13), it follows that

$$\begin{aligned}&2[\tau(\alpha PQ+\alpha QP)-(\tau(P)\alpha Q+P\tau(\alpha Q)+\alpha Q\tau(P)+\tau(\alpha Q)P)]\\ &\quad = [\tau(Q)(\alpha PQ+\alpha QP)+Q\tau(\alpha PQ+\alpha QP)+(\alpha PQ+\alpha QP)\tau(Q)\\ &\qquad + \tau(\alpha PQ+\alpha QP)Q]-[\tau(QP+PQ)\alpha Q+(QP+PQ)\tau(\alpha Q)\\ &\qquad + \alpha Q\tau(QP+PQ)+\tau(\alpha Q)(QP+PQ)].\end{aligned}$$

This, together with (2.13), gives $\tau(P\alpha Q+\alpha QP) = \tau(\alpha Q)P+\alpha Q\tau(P)+P\tau(\alpha Q)+\tau(P)\alpha Q$ for all idempotents $P, Q \in \operatorname{Alg}\mathcal{L}$ and all $\alpha \in \mathbb{F}$, completing the proof of the claim.

Claim 3. *For any $A \in \operatorname{Alg}\mathcal{L}$ and any finite rank operator $F \in \operatorname{Alg}\mathcal{L}$, we have $\tau(AF+FA) = \tau(A)F+A\tau(F)+\tau(F)A+F\tau(A)$.*

Note that $\tau(QP+PQ) = \tau(Q)P+Q\tau(P)+\tau(P)Q+P\tau(Q)$ for all idempotents $P, Q \in \operatorname{Alg}\mathcal{L}$ (see (2.10)). By Lemma 2.4 and the fact that every finite rank operator of $\operatorname{Alg}\mathcal{L}$ is a sum of rank one operators in $\operatorname{Alg}\mathcal{L}$, for any finite rank operator $F \in \operatorname{Alg}\mathcal{L}$ we have

$$\tau(FP+PF) = \tau(F)P+F\tau(P)+\tau(P)F+P\tau(F). \tag{2.16}$$

Since $\tau(\alpha P) = \tau(\alpha I)P+\alpha\tau(P)$, by a similar argument we obtain $\tau(\alpha F) =$

$\tau(\alpha I)F + \alpha\tau(F)$ for all finite rank operators $F \in \operatorname{Alg}\mathcal{L}$. Hence

$$\begin{aligned}
&\tau(F(\alpha P) + (\alpha P)F)\\
&\qquad = \tau(\alpha F)P + \alpha F\tau(P) + P\tau(\alpha F) + \tau(P)\alpha F\\
&\qquad = \alpha\tau(F)P + \tau(\alpha I)FP + \alpha F\tau(P) + \alpha P\tau(F) + \tau(\alpha I)PF + \tau(P)\alpha F\\
&\qquad = \tau(F)\alpha P + \alpha P\tau(F) + F(\alpha\tau(P) + \tau(\alpha I)P) + (\alpha\tau(P) + \tau(\alpha I)P)F\\
&\qquad = \tau(F)\alpha P + F\tau(\alpha P) + \alpha P\tau(F) + \tau(\alpha P)F,
\end{aligned}$$

that is,

$$\tau(F(\alpha P) + (\alpha P)F) = \tau(F)\alpha P + F\tau(\alpha P) + \alpha P\tau(F) + \tau(\alpha P)F. \tag{2.17}$$

Now, for any $A \in \operatorname{Alg}\mathcal{L}$, let $T = (I - P)A(I - P)$. It is clear that $T(\alpha P) + (\alpha P)T = 0$. So

$$\tau(T)\alpha P + T\tau(\alpha P) + \tau(\alpha P)T + \alpha P\tau(T) = 0. \tag{2.18}$$

Note that $A - T$ is a finite rank operator. By (2.17)–(2.18), we have

$$\begin{aligned}
&\tau(A(\alpha P) + (\alpha P)A)\\
&\quad = \tau((A - T)\alpha P + \alpha P(A - T))\\
&\quad = \tau(A - T)\alpha P + (A - T)\tau(\alpha P) + \tau(\alpha P)(A - T) + \alpha P\tau(A - T)\\
&\quad = \tau(A)\alpha P - \tau(T)\alpha P + A\tau(\alpha P) - T\tau(\alpha P)\\
&\qquad + \tau(\alpha P)A - \tau(\alpha P)T + \alpha P\tau(A) - \alpha P\tau(T)\\
&\quad = \tau(A)\alpha P + A\tau(\alpha P) + \tau(\alpha P)A + \alpha P\tau(A).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\tau(Ax \otimes f + x \otimes fA)\\
&\qquad\qquad = \tau(A)x \otimes f + A\tau(x \otimes f) + \tau(x \otimes f)A + x \otimes f\tau(A)
\end{aligned}$$

for each $A \in \operatorname{Alg}\mathcal{L}$ and each rank one operator $x \otimes f \in \operatorname{Alg}\mathcal{L}$, which implies that the claim is true.

Claim 4. *For any $K \in \mathcal{J}(\mathcal{L})$, let τ_K denote the restriction to $\mathcal{F}_{\mathcal{L}}(K)$ of τ. Then τ_K is an additive derivation.*

We shall prove the claim by considering two cases.

Case 1: $\dim K = 1$. In this case, we have $\dim K_-^{\perp} = 1$, and so $\dim \mathcal{F}_{\mathcal{L}}(K) = 1$. Take $x_0 \in K$ and $f_0 \in K_-^{\perp}$ such that $\langle x_0, f_0\rangle = 1$. Write $P = x_0 \otimes f_0$. Then $\mathcal{F}_{\mathcal{L}}(K) = \{\lambda P : \lambda \in \mathbb{F}\}$. We claim that

$$h(\lambda\mu) = h(\lambda)\mu + \lambda h(\mu) \tag{2.19}$$

for all $\lambda, \mu \in \mathbb{F}$. Indeed, by Claims 1, 3 and the fact $\tau(P) = \tau(P)P + P\tau(P)$, we have

$$\begin{aligned}2\lambda\mu\tau(P)+2h(\lambda\mu)P &= 2\tau(\lambda\mu P)=\tau(2\lambda\mu P)=\tau(\lambda P\mu P+\mu P\lambda P)\\ &= \tau(\lambda P)\mu P+\lambda P\tau(\mu P)+\tau(\mu P)\lambda P+\mu P\tau(\lambda P)\\ &= 2h(\lambda)\mu P+2\lambda\mu\tau(P)P+2\lambda h(\mu)P+2\lambda\mu P\tau(P)\\ &= 2h(\lambda)\mu P+2\lambda h(\mu)P+2\lambda\mu\tau(P),\end{aligned}$$

which implies (2.19). Hence

$$\begin{aligned}\tau(\lambda P)\mu P+\lambda P\tau(\mu P) &= h(\lambda)\mu P+\lambda\mu\tau(P)P+\lambda h(\mu)P+\lambda\mu P\tau(P)\\ &= h(\lambda\mu)P+\lambda\mu\tau(P)=\tau(\lambda P\mu P).\end{aligned}$$

CASE 2: $\dim K>1$. Define a map $\Phi:\mathcal{F}_{\mathcal{L}}(K)\to\mathcal{B}(X\oplus X)$ by

$$\Phi(F)=\begin{pmatrix}F & \tau_K(F)\\ 0 & F\end{pmatrix}\qquad\text{for all } F\in\mathcal{F}_{\mathcal{L}}(K).$$

By Claim 3, it is easily checked that Φ is an additive Jordan homomorphism. Note that $\mathcal{F}_{\mathcal{L}}(K)$ is a locally matrix algebra. So, by [JR], Φ has the form

$$\Phi(F)=\phi(F)+\psi(F)=\begin{pmatrix}\phi_1(F) & \phi_2(F)\\ 0 & \phi_3(F)\end{pmatrix}+\begin{pmatrix}\psi_1(F) & \psi_2(F)\\ 0 & \psi_3(F)\end{pmatrix},$$

where $\phi:\mathcal{F}_{\mathcal{L}}(K)\to\mathcal{B}(X\oplus X)$ is a ring homomorphism, $\psi:\mathcal{F}_{\mathcal{L}}(K)\to\mathcal{B}(X\oplus X)$ is a ring anti-homomorphism, $\phi_1,\phi_3:\mathcal{F}_{\mathcal{L}}(K)\to\mathcal{B}(X)$ are ring homomorphisms and $\psi_1,\psi_3:\mathcal{F}_{\mathcal{L}}(K)\to\mathcal{B}(X)$ are ring anti-homomorphisms. Thus, for any $F\in\mathcal{F}_{\mathcal{L}}(K)$, we have

$$\phi_1(F)+\psi_1(F)=F\quad\text{and}\quad\phi_3(F)+\psi_3(F)=F.$$

It follows from Lemma 2.7 that $\psi_1=\psi_3=0$, and so $\phi_1(F)=\phi_3(F)=F$. Since $\phi(F_1F_2)=\phi(F_1)\phi(F_2)$ and $\psi(F_1F_2)=\psi(F_2)\psi(F_1)$ for all $F_1,F_2\in\mathcal{F}_{\mathcal{L}}(K)$, it is easy to check that ϕ_2 is an additive derivation and $\psi_2=0$. Hence $\tau_K=\phi_2$ is an additive derivation.

CLAIM 5. *τ is an additive derivation.*

Take any $K\in\mathcal{J}(\mathcal{L})$ and let τ_K be the restriction of τ to $\mathcal{F}_{\mathcal{L}}(K)$. Fix $f_K\in K_-^{\perp}$ and $x_K\in K$ such that $f_K(x_K)=1$. Then for any $x\in K$, we have $x\otimes f_K\in\mathcal{F}_{\mathcal{L}}(K)$. Define a map $T_K:K\to X$ as follows:

$$T_Kx=\tau_K(x\otimes f_K)x_K,\qquad\forall x\in K. \tag{2.20}$$

Then for any $F\in\mathcal{F}_{\mathcal{L}}(K)$, by Claim 4, we have

$$\tau_K(Fx\otimes f_K)=\tau_K(F)x\otimes f_K+F\tau_K(x\otimes f_K),\qquad\forall x\in K.$$

Multiplying the above equation by x_K from the right, one gets

$$\tau_K(F)x=T_KFx-FT_Kx,\qquad\forall x\in K. \tag{2.21}$$

Now take any $A \in \operatorname{Alg}\mathcal{L}$ and any $F \in \mathcal{F}_{\mathcal{L}}(K)$. Since $AF, FA \in \mathcal{F}_{\mathcal{L}}(K)$, for any $x \in K$, by (2.20)–(2.21), on the one hand, we have

$$\tau(AF+FA)x = \tau(AF)x + \tau(FA)x = T_K AFx - AFT_K x + T_K FAx - FAT_K x; \tag{2.22}$$

on the other hand, by Claim 3 and (2.21),

$$\begin{aligned}\tau(AF+FA)x &= \tau(A)Fx + A\tau(F)x + \tau(F)Ax + F\tau(A)x \\ &= \tau(A)Fx + AT_K Fx - AFT_K x \\ &\quad + T_K FAx - FT_K Ax + F\tau(A)x.\end{aligned} \tag{2.23}$$

Comparing (2.22) and (2.23), one gets

$$(\tau(A) + AT_K - T_K A)Fx = F(-AT_K + T_K A - \tau(A))x. \tag{2.24}$$

In particular, taking $x = x_K$, we have

$$(\tau(A) + AT_K - T_K A)Fx_K = F(-AT_K + T_K A - \tau(A))x_K \tag{2.25}$$

for all $A \in \operatorname{Alg}\mathcal{L}$ and $F \in \mathcal{F}_{\mathcal{L}}(K)$. Letting $F = x \otimes f_K$ in (2.25), one obtains $(\tau(A) + AT_K - T_K A)x = \lambda x$ for all $x \in K$, where $\lambda = -\langle(\tau(A) - T_K A + AT_K)x_K, f_K\rangle$. This and (2.24) imply that $\lambda = 0$. So $(\tau(A) + AT_K - T_K A)x = 0$, that is, $\tau(A)x = (T_K A - AT_K)x$ for all $x \in K$.

For any $A, B \in \operatorname{Alg}\mathcal{L}$ and any $x \in K$ with $K \in \mathcal{J}(\mathcal{L})$, we have

$$\begin{aligned}\tau(AB)x &= T_K ABx - ABT_K x \\ &= (T_K A - AT_K)Bx + A(T_K B - BT_K)x \\ &= \tau(A)Bx + A\tau(B)x.\end{aligned}$$

Since $\langle\mathcal{J}(\mathcal{L})\rangle$ is dense in X, it follows that $\tau(AB) = \tau(A)B + A\tau(B)$, that is, τ is a derivation. Note that $\delta(A) = \tau(A) + \delta(I)A = \tau(A) + \lambda A$ for all A. The proof of the theorem is complete. ∎

3. Linear maps Jordan derivable at the unit operator. In this section, we turn to a characterization of linear maps Jordan derivable at the unit operator on $\operatorname{Alg}\mathcal{L}$. The following is our main result.

THEOREM 3.1. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a complex Banach space X with $\dim X \geq 3$. Suppose that $\delta : \operatorname{Alg}\mathcal{L} \to \operatorname{Alg}\mathcal{L}$ is a linear map. Then δ satisfies $\delta(AB+BA) = \delta(A)B + A\delta(B) + \delta(B)A + B\delta(A)$ whenever $AB+BA = I$ for any $A, B \in \operatorname{Alg}\mathcal{L}$ if and only if δ is a derivation.*

We first prove two lemmas.

LEMMA 3.2. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X and $\delta : \operatorname{Alg}\mathcal{L} \to \operatorname{Alg}\mathcal{L}$ a linear map. If δ satisfies $\delta(AB+BA) = \delta(A)B + A\delta(B) + \delta(B)A + B\delta(A)$ whenever $AB+BA = I$ for any $A, B \in \operatorname{Alg}\mathcal{L}$, then*

(i) $\delta(P) = \delta(P)P + P\delta(P)$ *for every idempotent* $P \in \operatorname{Alg}\mathcal{L}$,

(ii) $\delta(N)N + N\delta(N) = 0$ *for every* $N \in \operatorname{Alg}\mathcal{L}$ *with* $N^2 = 0$.

Proof. It is obvious from $I = \frac{1}{2}I \cdot I + I \cdot \frac{1}{2}I$ that $\delta(I) = \delta(I)I + I\delta(I) = 2\delta(I)$. So $\delta(I) = 0$.

(i) Let $P \in \operatorname{Alg}\mathcal{L}$ be an idempotent operator. Since

$$I = (P - \tfrac{1}{2}I)(2P - I) + (2P - I)(P - \tfrac{1}{2}I),$$

we have

$$\begin{aligned} 0 = \delta(I) &= \delta(P - \tfrac{1}{2}I)(2P - I) + (P - \tfrac{1}{2}I)\delta(2P - I) \\ &\quad + \delta(2P - I)(P - \tfrac{1}{2}I) + (2P - I)\delta(P - \tfrac{1}{2}I) \\ &= 4\delta(P)P + 4P\delta(P) - 4\delta(P). \end{aligned}$$

That is, $\delta(P) = \delta(P)P + P\delta(P)$.

(ii) For every operator $N \in \operatorname{Alg}\mathcal{L}$ with $N^2 = 0$, since

$$(I + N)(\tfrac{1}{2}I - \tfrac{1}{2}N) + (\tfrac{1}{2}I - \tfrac{1}{2}N)(I + N) = I,$$

we have

$$\begin{aligned} 0 = \delta(I) &= \delta(I + N)(\tfrac{1}{2}I - \tfrac{1}{2}N) + (I + N)\delta(\tfrac{1}{2}I - \tfrac{1}{2}N) \\ &\quad + \delta(\tfrac{1}{2}I - \tfrac{1}{2}N)(I + N) + (\tfrac{1}{2}I - \tfrac{1}{2}N)\delta(I + N) \\ &= -\delta(N)N - N\delta(N). \end{aligned}$$

It follows that $\delta(N)N + N\delta(N) = 0$. ∎

LEMMA 3.3. *Let X be a real or complex linear space with* $\dim X = \infty$, *or a complex linear space with* $2 < \dim X < \infty$. *Assume that A, B are linear operators on X. If, for every $x \in X$, Bx is a linear combination of Ax and x, then B is a linear combination of A and I.*

Proof. Denote by $\mathcal{L}(X)$ the set of all linear operators from X into itself. We consider the two cases separately.

CASE 1: $\dim X = \infty$. In this case we will use a result due to Larson [Lar] (see also [Hou]). For a finite-dimensional subspace $\mathcal{S} \subseteq \mathcal{L}(X)$, define $\mathcal{S}_{\mathcal{G}} = \mathcal{S} \cap \mathcal{G}(X)$, where $\mathcal{G}(X)$ denotes the set of all finite rank operators in $\mathcal{L}(X)$, and define $\operatorname{ref}(\mathcal{S}) = \{A \in \mathcal{L}(X) \mid Ax \in \mathcal{S}x,\ \forall x \in X\}$. Larson [Lar] proved that $\operatorname{ref}(\mathcal{S}_{\mathcal{G}}) = \mathcal{S}_{\mathcal{G}}$ implies $\operatorname{ref}(\mathcal{S}) = \mathcal{S}$. Applying this result to $\mathcal{S} = \operatorname{span}\{I, A\}$, and noting that $\operatorname{ref}(\mathcal{S}_{\mathcal{G}}) = \mathcal{S}_{\mathcal{G}}$ as $\dim \mathcal{S}_{\mathcal{G}} \le 1$, we see that $\operatorname{ref}(\mathcal{S}) = \mathcal{S}$. Hence $B \in \operatorname{ref}(\mathcal{S}) = \mathcal{S}$, that is, $B = \alpha A + \beta I$ for some scalars α and β as desired.

CASE 2: $\dim X = n \ge 3$, $n \in \mathbb{N}$. Note that the hypotheses imply that $\operatorname{Lat} A \subseteq \operatorname{Lat} B$, where $\operatorname{Lat} T$ stands for the lattice of all invariant subspaces of T.

If A is of rank one, then there exist $x \in X$ and $f \in X^*$ such that $A = x \otimes f$. The inclussion $\operatorname{Lat} A \subseteq \operatorname{Lat} B$ implies that every subspace of $\ker A$ is invariant under B. It follows that B is a scalar multiple of the identity on $\ker f$. Hence, it has the form $B = y \otimes f + \beta I$. Now it is easily seen that $y = \alpha x$ for some scalar α and thus $B = \alpha A + \beta I$.

Thus we may assume that $A \notin \mathbb{F}I$ and the rank of A is greater than 1.

If A is diagonal, then $A = \bigoplus_{i=1}^k a_i I_i$ with respect to the space decomposition $X = \bigoplus_{i=1}^k X_i$, where $a_i \neq a_j$ whenever $i \neq j$. Then accordingly $B = \bigoplus_{i=1}^k b_i I_i$ as $\operatorname{Lat} A \subseteq \operatorname{Lat} B$. Let $x = (x_1, \ldots, x_n) \in X$ with all $x_i \neq 0$. Since $Bx \in \operatorname{span}\{Ax, x\}$, there are scalars β_x and γ_x such that $Bx = \beta_x Ax + \gamma_x x$. It follows that $b_i = \beta_x a_i + \gamma_x$, $i = 1, \ldots, k$. Thus

$$\beta_x = \frac{b_i - b_j}{a_i - a_j} = \alpha \quad \text{and} \quad \gamma_x = b_i - \frac{a_i(b_i - b_j)}{a_i - a_j} = \beta$$

are constants. It is easy to see that $Bx = \alpha Ax + \beta x$ for any $x \in X$ and hence $B = \alpha A + \beta I$.

Now assume that A is not diagonal. Let $J = \sum_{i=1}^{n-1} E_{i,i+1}$ where E_{ij} stands for the $n \times n$ matrix having the (i,j)th entry 1 and all other entries zero. If $A = J + \delta I$, then B has the form $(t_{ij})_{n\times n}$ with $t_{ij} = 0$ if $i > j$ since $\operatorname{Lat} A \subseteq \operatorname{Lat} B$. Furthermore, because $Bx \in \operatorname{span}\{Ax, x\}$ for every $x \in X$, we have $B = \alpha J + (\alpha\delta + \beta)I = \alpha A + \beta I$. Generally, since X is complex, A has the form $A = \bigoplus_{i=1}^k (\delta_i I_i + \epsilon_i J_i)$ with respect to some space decomposition $X = X_1 \oplus \cdots \oplus X_k$, where $\epsilon_i \in \{0, 1\}$. We may assume that $\epsilon_1 \neq 0$. Then the hypotheses imply that $B = \bigoplus_{i=1}^k B_i$, $\operatorname{Lat} B_i \supseteq \operatorname{Lat}(\delta_i I_i + \epsilon_i J_i)$, and $B_i x_i \in \operatorname{span}\{x_i, \epsilon_i J_i x_i\}$ for every $x_i \in X_i$. Thus, by what we have just proved, $B_i = \lambda_i I + \alpha_i \epsilon_i J_i$ for some scalars λ_i and α_i. So $B = \bigoplus_{i=1}^k (\lambda_i I + \alpha_i \epsilon_i J_i)$.

Now we will use the fact that $Bx \in \operatorname{span}\{Ax, x\}$ for all $x \in X$ to check that $B = \alpha A + \beta I$ for some scalars α and β. To do this, take $x = \bigoplus_{i=1}^k x_i$ with $J_1 x_1 \neq 0$. Then $Bx = \beta_x Ax + \gamma_x x$ for some scalars β_x and γ_x implies that $\beta_x = \alpha_1$ and $\gamma_x = \lambda_1 - \beta_x \delta_1 = \lambda_1 - \alpha_1 \delta_1$. For any $x \in X$ with $J_1 x_1 = 0$, by taking $x' = \bigoplus_{i=1}^k x_i' \in X$ so that $J_1 x_1' \neq 0$ and letting $y = x + x'$, we still get $\beta_x = \alpha_1$ and $\gamma_x = \lambda_1 - \alpha_1 \delta_1$. Hence, $B = \alpha A + \beta I$ with $\alpha = \alpha_1$ and $\beta = \lambda_1 - \alpha_1 \delta_1$. ■

LEMMA 3.4 ([LL, Proposition 1.1]). *Let E and F be nonzero subspaces of X and X^*, respectively. Let $\Phi : E \times F \to \mathcal{B}(X)$ be a bilinear map such that $\Phi(x, f) \ker(f) \subseteq \mathbb{F}x$ for all $x \in E$ and $f \in F$. Then there exist linear maps $T : E \to X$ and $S : F \to X^*$ such that $\Phi(x \otimes f) = Tx \otimes f + x \otimes Sf$ for all $x \in E$ and $f \in F$.*

Proof of Theorem 3.1. Again, we only need to prove the "only if" part. By Lemma 3.2, we have $\delta(P) = \delta(P)P + P\delta(P)$ for all idempotents P, and

$\delta(N)N + N\delta(N) = 0$ for all operators N with $N^2 = 0$. We complete the proof of the theorem by checking several claims.

CLAIM 1. *For any rank one operator $x \otimes f \in \operatorname{Alg}\mathcal{L}$, we have*

$$\delta(x \otimes f)\ker(x \otimes f) \subseteq \operatorname{span}\{x\}.$$

We will prove the claim by considering two cases.

CASE 1: $\langle x, f\rangle = \lambda \neq 0$. By the linearity of δ, we have

$$\delta(\lambda^{-1}x \otimes f) = \delta(\lambda^{-1}x \otimes f)(\lambda^{-1}x \otimes f) + (\lambda^{-1}x \otimes f)\delta(\lambda^{-1}x \otimes f),$$

that is, $\delta(x \otimes f) = \lambda^{-1}\delta(x \otimes f)(x \otimes f) + \lambda^{-1}(x \otimes f)\delta(x \otimes f)$, which implies that the claim is true.

CASE 2: $\langle x, f\rangle = 0$. By Lemma 2.2, there exists $K \in \mathcal{J}(\mathcal{L})$ such that $x \in K$ and $f \in K_-^\perp$. Then, by Lemma 2.3, there exists $z \in K$ such that $\langle z, f\rangle = 1$. Thus $(x + z) \otimes f, z \otimes f \in \operatorname{Alg}\mathcal{L}$ are both idempotents. So we have $\delta((x + z) \otimes f)\ker(f) \subseteq \operatorname{span}\{x + z\}$ and $\delta(z \otimes f)\ker(f) \subseteq \operatorname{span}\{z\}$. Note that $\delta(x \otimes f) = \delta((x + z) \otimes f) - \delta(z \otimes f)$. Hence $\delta(x \otimes f)\ker(f) \subseteq \operatorname{span}\{x + z\} - \operatorname{span}\{z\}$. Thus for any $y \in \ker(f)$, there exist $\alpha(y), \beta(y) \in \mathbb{C}$ such that

$$\delta(x \otimes f)y = \alpha(y)(x + z) - \beta(y)z = \alpha(y)x + (\alpha(y) - \beta(y))z. \tag{3.1}$$

Since $(x \otimes f)^2 = 0$, we get $\delta(x \otimes f)(x \otimes f) + (x \otimes f)\delta(x \otimes f) = 0$. It follows that $0 = (\delta(x \otimes f)(x \otimes f) + (x \otimes f)\delta(x \otimes f))y = \langle\delta(x \otimes f)y, f\rangle x$ for every $y \in \ker(f)$, that is, $\langle\delta(x \otimes f)y, f\rangle = 0$. Thus from (3.1) we get

$$\begin{aligned} 0 &= \langle\delta(x \otimes f)y, f\rangle = \langle\alpha(y)x + (\alpha(y) - \beta(y))z, f\rangle \\ &= (\alpha(y) - \beta(y))\langle z, f\rangle = \alpha(y) - \beta(y), \end{aligned}$$

that is, $\delta(x \otimes f)y = \alpha(y)x$ for every $y \in \ker(f)$, completing the proof of the claim.

CLAIM 2. *For each $K \in \mathcal{J}(\mathcal{L})$, there exist linear maps $S_K : K_-^\perp \to K_-^\perp$ and $T_K : K \to K$ such that*

$$\delta(x \otimes f) = x \otimes S_K f + T_K x \otimes f \quad \text{for all } x \in K \text{ and } f \in K_-^\perp.$$

By Lemma 3.4, the claim is obvious.

CLAIM 3. *There exists a linear operator $T : \langle\mathcal{J}(\mathcal{L})\rangle \to \langle\mathcal{J}(\mathcal{L})\rangle$ such that $\delta(x \otimes f) = T(x \otimes f) - (x \otimes f)T$ for every rank one operator $x \otimes f \in \operatorname{Alg}\mathcal{L}$.*

We first prove that $T_K : K \to K$ is bounded. In fact, for any $x \otimes f \in \operatorname{Alg}\mathcal{L}$ with $x \in K$, $f \in K_-^\perp$ and $\langle x, f\rangle = 1$, by Lemma 3.2, we can easily deduce that $(x \otimes f)\delta(x \otimes f)(x \otimes f) = 0$. So $(\langle x, S_K f\rangle + \langle T_K x, f\rangle)x \otimes f = 0$. It follows that

$$\langle x, S_K f\rangle + \langle T_K x, f\rangle = 0 \tag{3.2}$$

for all $x \in K$ and $f \in K_-^\perp$ with $\langle x, f\rangle = 1$. Now let $x \in K$ and $f \in K_-^\perp$ be arbitrary. If $\langle x, f\rangle \neq 0$, it is obvious that (3.2) holds. If $\langle x, f\rangle = 0$, there exists $f_1 \in K_-^\perp$ such that $\langle x, f_1\rangle = 1$. Let $f_2 = f_1 - f$. So $\langle x, f_2\rangle = 1$. Thus we have

$$\begin{aligned}\langle x, S_K f\rangle + \langle T_K x, f\rangle &= \langle x, S_K(f_1 - f_2)\rangle + \langle T_K x, (f_1 - f_2)\rangle \\ &= \langle x, S_K f_1\rangle + \langle T_K x, f_1\rangle - \langle x, S_K f_2\rangle - \langle T_K x, f_2\rangle = 0.\end{aligned}$$

Hence

$$\langle x, S_K f\rangle + \langle T_K x, f\rangle = 0 \quad \text{for all } x \in K,\ f \in K_-^\perp. \tag{3.3}$$

If $\{x_n\} \subseteq K$ with $x_n \to x_0$ and $T_K x_n \to y_0$ as $n \to \infty$, then

$$0 = \langle x_n, S_K f\rangle + \langle T_K x_n, f\rangle \mapsto \langle x_0, S_K f\rangle + \langle y_0, f\rangle = 0.$$

Combining (3.3) with the above equation, we have $\langle T_K x_0, f\rangle = \langle y_0, f\rangle$ for all $f \in K_-^\perp$. This entails that $T_K x_0 = y_0$, since, otherwise, there would be some $f \in K_-^\perp$ such that $\langle T_K x_0, f\rangle \neq \langle y_0, f\rangle$. It follows from the closed graph theorem that $T_K \in \mathcal{B}(K)$. Similarly, we can check that $S_K \in \mathcal{B}(K_-^\perp)$.

Note that $K \wedge K_- = \{0\}$ and $K \vee K_- = X$. We may regard that $K_-^\perp \subseteq K^*$ since, for any $f \in K_-^\perp$, $f|_K \in K^*$. It is clear that, for any $x \in K$, $\langle x, f\rangle = 0$ for all $f \in K_-^\perp$ entails that $x = 0$. By (3.3), we have

$$\langle x, (T_K^* + S_K)f\rangle = 0 \quad \text{for all } x \in K,\ f \in K_-^\perp.$$

It follows that $(T_K^* + S_K)f = 0$ since $(T_K^* + S_K)f \in K^*$. Thus we get $(T_K^* + S_K)|_{K_-^\perp} = 0$. Hence $S_K = -(T_K)^*|_{K_-^\perp}$, and so $\delta(x \otimes f) = T_K x \otimes f - x \otimes f T_K$ for all $x \in K$ and $f \in K_-^\perp$.

Now define a linear map $T : \langle \mathcal{J}(\mathcal{L})\rangle \to \langle \mathcal{J}(\mathcal{L})\rangle$ such that $T|_K = T_K$ for any $K \in \mathcal{J}(\mathcal{L})$. Since, by Lemma 2.3, $\mathcal{J}(\mathcal{L})$ is a collection of linearly independent subspaces K and $\langle \mathcal{J}(\mathcal{L})\rangle = \operatorname{span}\{K \mid K \in \mathcal{J}(\mathcal{L})\}$, T is well defined. Thus there exists a linear map $T : \langle \mathcal{J}(\mathcal{L})\rangle \to \langle \mathcal{J}(\mathcal{L})\rangle$ such that $\delta(x \otimes f) = Tx \otimes f - x \otimes fT$ for all $x \in K$ and $f \in K_-^\perp$.

Claim 4. *For every $A \in \operatorname{Alg}\mathcal{L}$, we have $\delta(A)|_{\langle \mathcal{J}(\mathcal{L})\rangle} = TA|_{\langle \mathcal{J}(\mathcal{L})\rangle} - AT$.*

Since X is complex, for any A and any $x \otimes f \in \operatorname{Alg}\mathcal{L}$, take $\lambda \in \mathbb{C}$ such that $|\lambda| > \|A\|$ and $\|(\lambda I - A)^{-1}x\|\,\|f\| < 1$. Then both $\lambda I - A$ and $\lambda I - A - x \otimes f = (\lambda I - A)(I - (\lambda I - A)^{-1}x \otimes f)$ are invertible and their inverses are still in $\operatorname{Alg}\mathcal{L}$. It is obvious that $(I - (\lambda I - A)^{-1}x \otimes f)^{-1} = I + (1-\alpha)^{-1}(\lambda I - A)^{-1}x \otimes f$, where $\alpha = \langle (\lambda I - A)^{-1}x, f\rangle$. Hence we have

$$\begin{aligned}0 = {}& \delta(\lambda I - A - x \otimes f)(I + (1-\alpha)^{-1}(\lambda I - A)^{-1}x \otimes f)(\lambda I - A)^{-1} \\ &+ (\lambda I - A - x \otimes f)\delta((\lambda I - A)^{-1} + (1-\alpha)^{-1}(\lambda I - A)^{-1}x \otimes f(\lambda I - A)^{-1}) \\ &+ \delta((\lambda I - A)^{-1} + (1-\alpha)^{-1}(\lambda I - A)^{-1}x \otimes f(\lambda I - A)^{-1})(\lambda I - A - x \otimes f) \\ &+ ((\lambda I - A)^{-1} + (1-\alpha)^{-1}(\lambda I - A)^{-1}x \otimes f(\lambda I - A)^{-1})\delta(\lambda I - A - x \otimes f)\end{aligned}$$

$$
\begin{aligned}
&= [\delta(\lambda I - A) - Tx \otimes f + x \otimes fT][(\lambda I - A)^{-1} \\
&\quad + (1-\alpha)^{-1}(\lambda I - A)^{-1}x \otimes f(\lambda I - A)^{-1}] + [\lambda I - A - x \otimes f][\delta((\lambda I - A)^{-1}) \\
&\quad + (1-\alpha)^{-1}T(\lambda I - A)^{-1}x \otimes f(\lambda I - A)^{-1} \\
&\quad - (1-\alpha)^{-1}(\lambda I - A)^{-1}x \otimes f(\lambda I - A)^{-1}T] \\
&\quad + [\delta((\lambda I - A)^{-1}) + (1-\alpha)^{-1}T(\lambda I - A)^{-1}x \otimes f(\lambda I - A)^{-1} \\
&\quad - (1-\alpha)^{-1}(\lambda I - A)^{-1}x \otimes f(\lambda I - A)^{-1}T][\lambda I - A - x \otimes f] \\
&\quad + [(\lambda I - A)^{-1} + (1-\alpha)^{-1}(\lambda I - A)^{-1}x \otimes f(\lambda I - A)^{-1}] \\
&\quad \cdot [\delta(\lambda I - A) - Tx \otimes f + x \otimes fT].
\end{aligned}
$$

Note that $\delta(B)B^{-1} + B\delta(B^{-1}) + \delta(B^{-1})B + B^{-1}\delta(B) = 0$ for each invertible operator $B \in \operatorname{Alg}\mathcal{L}$ by the assumption on δ and $\delta(I) = 0$. The above equation reduces to

$$
\begin{aligned}
0 &= (1-\alpha)^{-1}\delta(A)(\lambda I - A)^{-1}x \otimes f(\lambda I - A)^{-1} - x \otimes fT(\lambda I - A)^{-1} \\
&\quad + (1-\alpha)^{-1}Tx \otimes f(\lambda I - A)^{-1} \\
&\quad - (1-\alpha)^{-1}(\lambda I - A)T(\lambda I - A)^{-1}x \otimes f(\lambda I - A)^{-1} \\
&\quad + x \otimes f(\lambda I - A)^{-1}T + x \otimes f\delta((\lambda I - A)^{-1}) + \delta((\lambda I - A)^{-1})x \otimes f \\
&\quad - T(\lambda I - A)^{-1}x \otimes f + (1-\alpha)^{-1}(\lambda I - A)^{-1}x \otimes f(\lambda I - A)^{-1}T(\lambda I - A) \\
&\quad + (\lambda I - A)^{-1}Tx \otimes f + (1-\alpha)^{-1}(\lambda I - A)^{-1}x \otimes f(\lambda I - A)^{-1}\delta(A) \\
&\quad - (1-\alpha)^{-1}(\lambda I - A)^{-1}x \otimes fT.
\end{aligned}
$$

That is,

$$
\begin{aligned}
(3.4) \qquad 0 &= [\delta(A)(\lambda I - A)^{-1} + T - (\lambda I - A)T(\lambda I - A)^{-1}]x \otimes f(\lambda I - A)^{-1} \\
&\quad + x \otimes f(1-\alpha)[\delta((\lambda I - A)^{-1}) + (\lambda I - A)^{-1}T - T(\lambda I - A)^{-1}] \\
&\quad + (1-\alpha)[\delta((\lambda I - A)^{-1}) + (\lambda I - A)^{-1}T - T(\lambda I - A)^{-1}]x \otimes f \\
&\quad + (\lambda I - A)^{-1}x \otimes f[(\lambda I - A)^{-1}\delta(A) - T + (\lambda I - A)^{-1}T(\lambda I - A)].
\end{aligned}
$$

Now fix λ. Let $X_\lambda = \delta(A)(\lambda I - A)^{-1} + T - (\lambda I - A)T(\lambda I - A)^{-1}$ and $Y_\lambda = (\lambda I - A)^{-1}$. It is clear that f and $(\lambda I - A^*)^{-1}f$ are linearly independent. So there exists a vector $z \in X$ such that $\langle z, (\lambda I - A^*)^{-1}f\rangle = 1$ and $\langle z, f\rangle = 0$. Acting at z in (3.4), we find that, for every $x \in \langle \mathcal{J}(\mathcal{L})\rangle$, $X_\lambda x$ is a linear combination of $Y_\lambda x$ and x. By Lemma 3.3, X_λ is a linear combination of Y_λ and I. That is, there exist scalars α_λ and β_λ such that

$$\delta(A)(\lambda I - A)^{-1} + T - (\lambda I - A)T(\lambda I - A)^{-1} = \alpha_\lambda(\lambda I - A)^{-1} + \beta_\lambda I.$$

Multiplying the above equation by $(\lambda I - A)^{-1}$ from the right, we obtain

$$(3.5) \qquad \delta(A) - TA + AT = \alpha_\lambda I + \beta_\lambda(\lambda I - A)$$

on $\langle \mathcal{J}(\mathcal{L})\rangle$. By taking different λ in (3.5), we see that β_λ is independent of λ. Let $\beta = \beta_\lambda$. Thus $\delta(A) - TA + AT + \beta A = (\alpha_\lambda + \beta\lambda)I$ on $\langle \mathcal{J}(\mathcal{L})\rangle$, which implies that $\alpha = \alpha_\lambda + \beta\lambda$ is independent of λ. Hence

$$\delta(A) = TA - AT - \beta A + \alpha I$$

on $\langle \mathcal{J}(\mathcal{L}) \rangle$. Note that δ is Jordan derivable at I and $\delta(I) = 0$. One can easily check that $\alpha = \beta = 0$, and consequently $\delta(A)|_{\langle \mathcal{J}(\mathcal{L}) \rangle} = TA|_{\langle \mathcal{J}(\mathcal{L}) \rangle} - AT$. The claim holds.

Claim 5. *δ is a derivation.*

For any $A, B \in \operatorname{Alg} \mathcal{L}$, by Claim 5, we have

$$\delta(AB)|_{\langle \mathcal{J}(\mathcal{L}) \rangle} = TAB|_{\langle \mathcal{J}(\mathcal{L}) \rangle} - ABT = TA|_{\langle \mathcal{J}(\mathcal{L}) \rangle} B|_{\langle \mathcal{J}(\mathcal{L}) \rangle} - ABT$$

and

$$\begin{aligned}(\delta(A)B + A\delta(B)|_{\langle \mathcal{J}(\mathcal{L}) \rangle} &= (TA|_{\langle \mathcal{J}(\mathcal{L}) \rangle} - AT)B|_{\langle \mathcal{J}(\mathcal{L}) \rangle} + A(TB|_{\langle \mathcal{J}(\mathcal{L}) \rangle} - BT) \\ &= TA|_{\langle \mathcal{J}(\mathcal{L}) \rangle} B|_{\langle \mathcal{J}(\mathcal{L}) \rangle} - ABT.\end{aligned}$$

Comparing the above two equations, we get $\delta(AB)|_{\langle \mathcal{J}(\mathcal{L}) \rangle} = (\delta(A)B + A\delta(B)|_{\langle \mathcal{J}(\mathcal{L}) \rangle}$. Thus $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \operatorname{Alg} \mathcal{L}$ since $\langle \mathcal{J}(\mathcal{L}) \rangle$ is dense in X. ■

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