STUDIA MATHEMATICA 189 (1) (2008)

Non-separable Banach spaces with non-meager Hamel basis

by

TARAS BANAKH (Kielce and Lviv), MIRNA DŽAMONJA (Norwich) and
LORENZ HALBEISEN (Bern)

Abstract. We show that an infinite-dimensional complete linear space X has:

- a dense hereditarily Baire Hamel basis if $|X| \leq \mathfrak{c}^+$;
- a dense non-meager Hamel basis if $|X| = \kappa^\omega = 2^\kappa$ for some cardinal κ.

According to Corollary 3.4 of [BDHMP] each infinite-dimensional separable Banach space X has a non-meager Hamel basis. This is a special case of Theorem 3.3 of [BDHMP], asserting that an infinite-dimensional Banach space X has a non-meager Hamel basis provided $2^{d(X)} = d(X)^\omega$, where $d(X)$ is the density of X. Having in mind those results the authors of [BDHMP] asked if each infinite-dimensional Banach space has a non-meager Hamel basis. In this paper we shall give two partial answers to this question generalizing the above-mentioned Corollary 3.4 and Theorem 3.3 of [BDHMP] in two directions.

THEOREM 1. *Each infinite-dimensional linear complete metric space X of size $|X| \leq \mathfrak{c}^+$ has a dense hereditarily Baire Hamel basis.*

We recall that a topological space X is *hereditarily Baire* if each closed subspace F of X is *Baire* (in the sense that the intersection of a countable family of open dense subsets of F is dense in F).

Our next result treats Banach spaces of even larger size. We define a subset A of a topological space X to be *κ-perfect* for some cardinal κ if each non-empty open set U of A has size $|U| \geq \kappa$. Note that a Hausdorff space X is ω-perfect if and only if it has no isolated points (so is perfect in the standard sense).

It is well-known (see [BDHMP, 2.8]) that each Banach space X has size $|X| = d(X)^\omega$. Our second principal result generalizes Theorem 3.3 of [BDHMP].

2000 *Mathematics Subject Classification*: 46B15, 03E75.

Key words and phrases: Hamel basis, Banach space, Baire space.

Mirna Džamonja thanks EPSRC for support through an Advanced Research Fellowship.

Theorem 2. *If an infinite-dimensional linear complete linear space X has size $|X| = \kappa^\omega = 2^\kappa$ for some cardinal κ, then X has a non-meager Hamel basis $H \subset X$ such that for any closed $|X|$-perfect subset $C \subset X$ the space $C \cap H$ is Baire.*

Let us observe that there are many cardinals κ with $\kappa^\omega = 2^\kappa$.

Proposition 1. *For any sequence of cardinals $(\kappa_i)_{i\in\omega}$ with $\kappa_{i+1} \geq 2^{\kappa_i}$, $i \in \omega$, the cardinal $\kappa = \sup_{i\in\omega} \kappa_i$ has the property $2^\kappa = \kappa^\omega$.*

Proof. Since $\kappa^\omega \leq 2^\kappa$ always holds, it suffices to prove that $\kappa^\omega \geq 2^\kappa$. For this take a sequence $(X_i)_{i\in\omega}$ of pairwise disjoint sets of size $|X_i| = \kappa_i$ and let $X = \bigcup_{i\in\omega} X_i$. It is clear that $|X| = \kappa$ and the power set $\mathcal{P}(X)$ of X has size $|\mathcal{P}(X)| = 2^\kappa$. Since each subset $A = \bigcup_{i\in\omega} A \cap X_i$ of X can be uniquely identified with the sequence $(A \cap X_i)_{i\in\omega}$, we get

$$2^\kappa = |\mathcal{P}(X)| = \Big|\prod_{i\in\omega} \mathcal{P}(X_i)\Big| = \prod_{i\in\omega} 2^{\kappa_i} \leq \prod_{i\in\omega} \kappa_{i+1} \leq \kappa^\omega. \ \blacksquare$$

In fact, one can make an easy observation about κ^ω which is helpful in calculating this value, and in particular implies Proposition 1. We use $\mathrm{cof}(\kappa)$ to denote the cofinality of κ.

Proposition 2. *Suppose that $\mathrm{cof}(\kappa)=\aleph_0$. Then $2^\kappa=(\sup\{2^\lambda : \lambda<\kappa\})^\omega$. If $\mathrm{cof}(\kappa) > \aleph_0$ then $\kappa^\omega = \kappa \cdot \sup\{\lambda^\omega : \lambda < \kappa\}$.*

Proof. If $\kappa = \aleph_0$ then the proposed equality easily holds. Suppose that $\kappa > \aleph_0$. Then clearly $2^\kappa = (2^\kappa)^\omega \geq (\sup\{2^\lambda : \lambda < \kappa\})^\omega$. Let $(\lambda_i)_{i\in\omega}$ be a sequence of regular cardinals increasing to κ, with $\lambda_0 = 0$, and let $\theta = \sup\{2^\lambda : \lambda < \kappa\}$. Every subset A of κ can be identified with the sequence $(A \cap [\lambda_{i+1} \setminus \lambda_i))_{i\in\omega}$, therefore $2^\kappa \leq |{}^\omega\theta| = \theta^\omega$.

For the second equality, observe first that the left side of the equality is always no smaller than the right side. If $\mathrm{cof}(\kappa) > \aleph_0$ and κ is a limit cardinal, then notice that every countable subset of κ is already a subset of some $\lambda < \kappa$, so $\kappa^\omega \leq \sup\{\lambda^\omega : \lambda < \kappa\}$, which does not exceed the quantity on the right side of the equation. Finally, if $\kappa = \lambda^+$ for some λ then $\kappa^\omega = \bigcup_{\alpha\in[\lambda,\kappa)} \alpha^\omega$, and the latter set has size $\leq \kappa \cdot \lambda^\omega \leq 2^\lambda$, which is exactly the quantity on the right side of the equation. $\blacksquare$

Corollary 1. *Suppose that a complete metric space X satisfies $d(X) \in [\kappa, 2^\kappa]$ for some κ with $\kappa^\omega = 2^\kappa$. Then X contains a non-meager Hamel basis.*

Under the Generalized Continuum Hypothesis GCH, each cardinal κ of countable cofinality satisfies $\kappa^\omega = \kappa^+$. Consequently, each complete metric space X with density $d(X) \in \{\kappa, \kappa^+\}$ contains a non-meager Hamel basis.

Proof. Suppose that $d(X) = \lambda \in [\kappa, \kappa^\omega]$. Then $|X| = \lambda^\omega = \kappa^\omega = 2^\kappa$, so X contains a non-meager Hamel basis by Theorem 2. For the conclusion under GCH notice that by König's lemma we have $\kappa^\omega > \kappa$, and since $\kappa^\omega \le 2^\kappa$ we may conclude that $\kappa^\omega = \kappa^+$. ∎

We comment that Corolllary 1 shows that our Theorem 2 is more general then Theorem 3.3 of [BDHMP], since by assuming GCH and taking for example X to be an infinite-dimensional Banach space of density $\lambda = \aleph_{\omega+1}$ (such as $l_\infty(\aleph_\omega)$), we find that X has a non-meager Hamel basis by Corollary 1, while $\lambda^\omega = \lambda < 2^\lambda$ so Theorem 3.3 of [BDHMP] does not apply.

1. Proof of Theorem 1. The proof of Theorem 1 is divided into three lemmas. The first of them supplies us with many linearly independent Cantor sets.

A topological space X is called a *Cantor set* if it is homeomorphic to the Cantor cube $\{0,1\}^\omega$. This happens if and only if X is compact, metrizable, zero-dimensional and has no isolated points (see [Ke, 7.4]).

By the *algebraic dimension* of a subset A of a linear space L we understand the algebraic dimension (= the cardinality of a Hamel basis) of the linear hull $\mathrm{Lin}(A)$ of A in L.

Lemma 1. *Let L be a linear metric space and L_∞ a linear subspace which can be written as the countable union $L_\infty = \bigcup_{n\in\omega} L_n$ of a non-decreasing sequence $(L_n)_{n\in\omega}$ of closed linear subspaces of L. Denote by $\pi : L \to L/L_\infty$ the quotient operator. Let $X \subset L$ be a completely metrizable subspace of L such that for every non-empty open set $U \subset X$ the projection $\pi(U)$ has infinite algebraic dimension in L/L_∞. Then X contains a Cantor set $C \subset X$ whose projection is linearly independent in L/L_∞ and has size $\mathfrak{c}$.*

Proof. Fix a complete metric ρ on X. Let $2 = \{0,1\}$ and let $2^{<\omega} = \bigcup_{n\in\omega} 2^n$ denote the set of finite binary sequences. For a binary sequence $s = (s_1,\dots,s_l) \in 2^{<\omega}$ and $i \in \{0,1\}$, we denote by $s\hat{}i = (s_1,\dots,s_n,i)$ the concatenation of s and i.

By induction, to each sequence $s \in 2^{<\omega}$ we shall assign a non-empty open set $U_s \subset X$ so that the following conditions are satisfied for every $n \in \omega$ and $s \in 2^n$:

(1) $\mathrm{diam}(U_s) \le 2^{-n}$;
(2) $U_{s\hat{}0} \cup U_{s\hat{}1} \subset U_s$;
(3) $\overline{U}_{s\hat{}0} \cap \overline{U}_{s\hat{}1} = \emptyset$;
(4) for any points $x_t \in U_t$, $t \in 2^n$, and real numbers λ_t, $t \in 2^n$, the inclusion $\sum_{t\in 2^n} \lambda_t x_t \in L_n$ is possible only if all $\lambda_t = 0$.

We put $U_\emptyset = X \setminus L_0$. Assume that for some n the sets U_s, $s \in 2^n$, have been constructed. The projection $\pi(U)$ of each open set $U \subset X$ has infinite algebraic dimension. Consequently, for every finite-dimensional linear subspace F of L the intersection $(F + L_n) \cap U$ is nowhere dense in U. Using this fact, by finite induction of length 2^{n+1} in each set U_s, $s \in 2^n$, we can select two distinct points $x_{s\hat{}0}, x_{s\hat{}1} \in U_s$ so that the indexed set $\{x_t + L_n : t \in 2^{n+1}\}$ is linearly independent in L/L_n. Next we can select open neighborhoods U_t of the points x_t to satisfy the conditions (1)–(4). This finishes the inductive construction.

Now it is easy to see that the intersection $C = \bigcap_{n\in\omega} \bigcup_{s\in 2^n} \overline{U}_s$ is a Cantor set in X. It follows from (4) that the image $\pi(C)$ in L/L_∞ is linearly indepenedent and has size $\mathfrak{c}$. ∎

LEMMA 2. *Let L be a complete linear metric space of size $|L| \le \mathfrak{c}$ and $(L_n)_{n\in\omega}$ be a non-decreasing sequence of closed linear subspaces of L with infinite-dimensional quotient space L/L_∞ where $L_\infty = \bigcup_{n\in\omega} L_n$. Let H_∞ be a Hamel basis for L_∞ such that for every $n \in \omega$ the intersection $H_\infty \cap L_n$ is a hereditarily Baire Hamel basis in L_n. Then H_∞ can be enlarged to a dense hereditarily Baire Hamel basis H for L.*

Proof. Let $\pi : L \to L/L_\infty$ denote the quotient homomorphism and let $\mathcal{C}$ be the family of Cantor sets $C \subset L$ whose projection $\pi(C)$ on L/L_∞ has algebraic dimension $\mathfrak{c}$. The family $\mathcal{C}$ has size $|\mathcal{C}| \le |L|^\omega \le \mathfrak{c}$ because each Cantor set $C \in \mathcal{C}$ is a continuous image of the Cantor cube 2^ω and each continuous map $f : 2^\omega \to L$ is uniquely determined by values of f on a countable dense subset of 2^ω. Let $\mathcal{C} = \{C_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of the family $\mathcal{C}$ by ordinals $< \mathfrak{c}$.

By transfinite induction we can construct a transfinite sequence of points $\{x_\alpha : \alpha < \mathfrak{c}\} \subset L$ so that $x_\alpha \in C_\alpha \setminus (L_\infty + \mathrm{Lin}\{x_\beta : \beta < \alpha\})$. At each step α the choice of the point x_α is possible because each set $\pi(C_\alpha)$ has algebraic dimension $\mathfrak{c}$.

After completing the inductive construction we will get a set $E = \{x_\alpha : \alpha < \mathfrak{c}\}$ whose projection onto L/L_∞ is injective and has linearly independent image in L/L_∞. Then the union $H_\infty \cup E$ is a linearly independent subset of L and can be enlarged to a Hamel basis H for L. Since H_∞ is a Hamel basis for L_∞, we have $H \cap L_\infty = H_\infty$. We claim that the space H is hereditarily Baire and dense in L.

To prove the density of H, take any non-empty open subset $U \subset L$. By Lemma 1, the set U contains a Cantor set $C \subset U$ belonging to the family $\mathcal{C}$. By the inductive construction, $E \cap C \ne \emptyset$ and hence $H \cap U \ne \emptyset$ too.

Next we show that H is hereditarily Baire. Assuming the converse and applying [De], we can find a closed countable subset $C \subset H$ without isolated points. Then the closure $\overline{C}$ of C in X is a Polish space without isolated points and so is the complement $\overline{C} \setminus C$. We claim that for each open set $U \subset \overline{C}$ the set $W = U \setminus H$ has an infinite-dimensional image $\pi(W)$ in L/L_∞. The density of $\overline{C} \setminus H$ in $\overline{C}$ implies that $U \subset \overline{W}$. Assuming that $\pi(W)$ is finite-dimensional, we would find that $W \subset U \subset \overline{W} \subset L_\infty + F = \bigcup_{n\in\omega}(L_n + F)$ for some finite-dimensional linear subspace $F \subset L$ with $F \cap L_\infty = \{0\}$. The Baire theorem guarantees that some non-empty open subset of U lies in $L_n + F$. Replacing U by this open set we can assume that $U \subset L_n + F$. Since $H_n = H \cap L_n = H_\infty \cap L_n$ is a Hamel basis for L_n, we have $H \cap (L_n + F) = H_n \cup B$ for some finite set B disjoint from L_∞. Then $U \cap H = U \cap (L_n + F) \cap H = U \cap (H_n \cup B) \subset U \cap L_n \cup (U \cap B)$. We claim that $U \cap B = \emptyset$. Assuming the converse, we would infer that $U \cap B$ is a non-empty closed subset of $U \cap H$, which is not possible because $U \cap H = U \cap C$ has no isolated points. Thus $U \cap H = U \cap H_n \subset L_n$ is a countable set without isolated points in H_n, which contradicts the fact that $H_n = H \cap L_n$ is a hereditarily Baire Hamel basis for L_n. ∎

Applying Lemma 2 to the sequence (L_n) of trivial linear spaces $L_n = \{0\}$ we obtain a part of Theorem 1.

LEMMA 3. *Each infinite-dimensional linear complete metric space X with $|X| \le \mathfrak{c}$ contains a dense hereditarily Baire Hamel basis.*

The remaining part of Theorem 1 is proved in

LEMMA 4. *Each complete metric linear space X of size $|X| = \mathfrak{c}^+$ contains a dense hereditarily Baire Hamel basis.*

Proof. Given a complete linear metric space X of size $|X| = \mathfrak{c}^+$, write X as the union $X = \bigcup_{\alpha<\mathfrak{c}^+} X_\alpha$ of an increasing transfinite sequence $(X_\alpha)_{\alpha<\mathfrak{c}^+}$ of closed linear subspaces of size $|X_\alpha| = \mathfrak{c}$ such that, for every $\alpha < \mathfrak{c}^+$,

- the quotient $X_{\alpha+1}/X_\alpha$ is infinite-dimensional;
- $X_\alpha = X_{<\alpha} = \bigcup_{\beta<\alpha} X_\beta$ if α has uncountable cofinality;
- $X_\alpha/\overline{X}_{<\alpha}$ is infinite-dimensional if α has countable infinite cofinality.

It is convenient to assume that $X_{-1} = \{0\}$. By transfinite induction, for every $\alpha < \mathfrak{c}^+$ we shall construct a dense hereditarily Baire Hamel basis H_α in X_α so that $H_\alpha \supset \bigcup_{\beta<\alpha} H_\beta$. To start the inductive construction let $H_0 = \emptyset$.

Assume that for some ordinal α, dense hereditarily Baire Hamel bases H_β have been constructed in each space X_β for $\beta < \alpha$. Now consider three cases:

1) $\alpha = \beta + 1$ is a successor ordinal. In this case apply Lemma 2 with $L = X_\alpha$ and $L_n = X_\beta$, $n \in \omega$, to enlarge the Hamel basis H_β to a dense hereditarily Baire Hamel basis H_α for the space X_α.

2) α is a limit ordinal with countable cofinality. In this case we can find an increasing sequence of ordinals $(\alpha_n)_{n\in\omega}$ with $\alpha = \sup_n \alpha_n$ and apply Lemma 2 with $L = X_\alpha$, $L_n = X_{\alpha_n}$ and $H_\infty = \bigcup_{n\in\omega} H_{\alpha_n}$ to enlarge the Hamel basis H_∞ to a dense hereditarily Baire Hamel basis H_α for X_α.

3) α is of uncountable cofinality. In this case $X_{<\alpha} = \bigcup_\beta X_\beta$ and we can put $H_\alpha = \bigcup_{\beta<\alpha} H_\beta$. The density of the Hamel bases in X_β implies the den-

sity of H_α in X_α. Let us show that the Hamel basis H_α is hereditarily Baire. Assuming the converse, and applying [De], we can find a closed countable subset $C \subset H_\alpha$ without isolated points. Since α has uncountable cofinality, $C \subset H_\beta$ for some $\beta < \alpha$. Then H_β contains a closed meager subspace C and thus is not hereditarily Baire, which is a contradiction. ∎

2. Proof of Theorem 2. Given an infinite cardinal κ, we denote by $\sqrt[\omega]{\kappa}$ the smallest infinite cardinal λ with $\lambda^\omega \geq \kappa$. The proof of Theorem 2 is similar to that of Theorem 1 and relies on

LEMMA 5. *For every κ-perfect complete metric space X and a comeager subspace $G \subset X$ there is a subspace $\Pi \subset G$ homeomorphic to the countable product λ^ω, where the cardinal $\lambda = \sqrt[\omega]{\kappa}$ is endowed with the discrete topology.*

Proof. The complement $X \setminus G$, being meager in X, lies in the countable union $\bigcup_{n\in\omega} Z_n$ of closed nowhere dense subsets Z_n in X. Since X is κ-perfect, each non-empty open subset $U \subset X$ has size $|U| \geq \kappa$ and density $d(U) \geq \sqrt[\omega]{\kappa} = \lambda$. By the Erdős–Tarski theorem [ET] (see also [En, 4.1.H]), the metrizable space $X \setminus Z_0$ contains a family $\mathcal{U}_0$ consisting of λ many open subsets of $X \setminus Z_0$ of diameter $< 1/2^0$ such that the family $\overline{\mathcal{U}}_0 = \{\overline{U} : U \in \mathcal{U}_0\}$ is disjoint. Repeating this argument, inductively construct a sequence $(\mathcal{U}_n)_{n\in\omega}$ of families of non-empty open sets of $X \setminus Z_n$ having diameter $< 1/2^n$ so that $\overline{\mathcal{U}} = \{\overline{U} : U \in \mathcal{U}_n\}$ is disjoint, $\bigcup \overline{\mathcal{U}}_{n+1} \subset \bigcup \mathcal{U}_n$ and for every $U \in \mathcal{U}_n$ the family $\mathcal{U}_{n+1}(U) = \{W \in \mathcal{U}_{n+1} : \overline{W} \subset U\}$ has size λ. It is easy to see that the space $F = \bigcap_{n\in\omega} \bigcup \overline{\mathcal{U}}_n \subset X \setminus \bigcup_{n\in\omega} Z_n \subset G$ is homeomorphic to the product $\prod_{n\in\omega} \mathcal{U}_n$ where each $\mathcal{U}_n$ is endowed with the discrete topology, and the latter product is homeomorphic to λ^ω. ∎

With Lemma 5 in hand, we are now able to present

Proof of Theorem 2. Let X be an infinite-dimensional linear complete metric space of size $|X| = 2^\kappa = \kappa^\omega$ for some cardinal κ. Without loss of generality, κ is the smallest infinite cardinal with that property. If $|X| \leq \mathfrak{c}$, then X has a hereditarily Baire Hamel basis by Theorem 1 and we are done. So asume that $|X| > \mathfrak{c}$ and hence $\kappa > \omega$.

Let $\mathcal{K}$ denote the family of all subspaces $K \subset X$ that are homeomorphic to the countable product κ^ω where κ is endowed with the discrete topology. Observe that each embedding $f : \kappa^\omega \to X$ is uniquely determined by the values of f on a dense subset of κ^ω. Since κ^ω has density κ, the family $\mathcal{K}$ has size $|\mathcal{K}| \leq |X|^\kappa = (2^\kappa)^\kappa = 2^\kappa = |X|$ and hence can be enumerated as $\mathcal{K} = \{K_\alpha : \alpha < |X|\}$. Observe that each space $K \in \mathcal{K}$ has size $|K| = \kappa^\omega > \mathfrak{c}$ and algebraic dimension κ^ω.

By transfinite induction we can construct a transfinite sequence of points $\{x_\alpha : \alpha < \mathfrak{c}\} \subset X$ so that $x_\alpha \in K_\alpha \setminus \mathrm{Lin}\{x_\beta : \beta < \alpha\}$. At each step α the choice of the point x_α is possible because each set K_α has algebraic dimension $\kappa^\omega > \alpha$.

After completing the inductive construction we will get a linearly independent set $E = \{x_\alpha : \alpha < \mathfrak{c}\}$ that meets each set $K \in \mathcal{K}$. Complete E to a Hamel basis $H \supset E$.

We claim that for each closed $|X|$-perfect subset $F \subset X$ the intersection $F \cap H$ is non-meager. Assuming the converse, we can apply Lemma 5 to find a topological copy $K \subset F \setminus H$ of κ^ω. It follows from the construction of H that $K \cap H \neq \emptyset$, which contradicts the inclusion $K \subset F \setminus H$.

3. Some remarks and open problems. Our Theorem 2 generalizes Corollary 3.4 of [BDHMP] supplying a non-meager Hamel basis in each Banach space X whose density $d(X)$ satisfies the equality $2^{d(X)} = d(X)^\omega$. In its turn, this corollary was derived from Theorem 3.3 of [BDHMP] guaranteeing the existence of a non-meager Hamel basis in each Banach space X satisfying $\mathrm{cof}(\mathcal{M}_X) \le |X|$, where $\mathrm{cof}(\mathcal{M}_X)$ stands for the cofinality of the ideal of meager sets in X. Having this result in mind, the authors of [BDHMP] asked in [BDHMP, Question 2] if the inequality $\mathrm{cof}(\mathcal{M}_X) > |X|$ holds for a suitable Banach space X. This is indeed so if $d(X) = |X|$. We shall prove a somewhat more general result giving lower and upper bounds for the cardinal $\mathrm{cof}(\mathcal{M}_X)$ via the weight $w(X)$ and the cellularity $c(X)$ of a linear topological space X.

PROPOSITION 3. *Let X be a Baire topological space without isolated points. Then*

(1) $\mathrm{cof}(\mathcal{M}_X) \le w(X)^{c(X)}$;
(2) $\mathrm{cof}(\mathcal{M}_X) > |\mathcal{U}|$ *for any disjoint family $\mathcal{U}$ of open sets in X.*

Proof. (1) Fix a base $\mathcal{B}$ of the topology of X of size $|\mathcal{B}| = w(X)$. Let $\mathcal{N} = \{X \setminus \bigcup\mathcal{U} : \mathcal{U} \subset \mathcal{B},\ |\mathcal{U}| \le c(X)\}$. It is clear that $|\mathcal{N}| \le w(X)^{c(X)}$. We claim that each nowhere dense subset $Z \subset X$ lies in some set $N \in \mathcal{N}$. Indeed, take a maximal disjoint subfamily $\mathcal{U} \subset \mathcal{B}$ with $\bigcup\mathcal{U} \subset X \setminus Z$ and note that $|\mathcal{U}| \le c(X)$. Then $Z \subset X \setminus \bigcup\mathcal{U} \in \mathcal{N}$. It follows that the family $\mathcal{N}_\infty = \{\bigcup\mathcal{C} : \mathcal{C}$ is a countable subfamily of $\mathcal{N}\}$ is cofinal in $\mathcal{M}_X$ and has size $|\mathcal{N}_\infty| \le |\mathcal{N}|^\omega \le (w(X)^{c(X)})^\omega = w(X)^{c(X)}$. Then $\mathrm{cof}(\mathcal{M}_X) \le |\mathcal{N}_\infty| \le w(X)^{c(X)}$.

(2) Assume conversely that $\mathrm{cof}(\mathcal{M}_X) \le |\mathcal{U}|$ for some disjoint family $\mathcal{U}$ of non-empty open sets in X. Pick a cofinal family $\mathcal{M}$ in $\mathcal{M}_X$ of size $|\mathcal{M}| \le |\mathcal{U}|$ and enumerate $\mathcal{M} = \{M_U : U \in \mathcal{U}\}$ by elements of the family $\mathcal{U}$. Each open set $U \in \mathcal{U}$ is not meager because X is Baire. Consequently, $U \not\subset M_U$ and we can pick a point $x_U \in U \setminus M_U$. Then the set $A = \{x_U : U \in \mathcal{U}\}$, being discrete, is nowhere dense in X. On the other hand, A lies in no set $M \in \mathcal{M}$, which means that $\mathcal{M}$ is not cofinal in the ideal $\mathcal{M}_X$. ∎

Since each metrizable space X contains a disjont family $\mathcal{U}$ of open sets of size $|\mathcal{U}| = d(X)$ (see [ET] or [En, 4.1.H]), Proposition 3 implies the following corollary answering Question 2 of [BDHMP].

Corollary 2. *For any metrizable Baire space X without isolated points we get $d(X) < \mathrm{cof}(\mathcal{M}_X) \le 2^{d(X)}$.*

A typical linear topological space with countable cellularity is the Tikhonov product $\mathbb{R}^\kappa$ of κ many lines. Then repeating the argument of the proof of Theorem 3.3 [BDHMP] we can prove

Proposition 4. *For any infinite cardinal κ the linear topological space $X = \mathbb{R}^\kappa$ has a non-meager Hamel basis and satisfies $\mathrm{cof}(\mathcal{M}_X) \le \kappa^\omega \le 2^\kappa = |X|$.*

In spite of (partial) results proven in this paper we still do not know the complete answer to the basic

Problem 1. *Let X be an infinite-dimensional Banach space.*

(1) *Does X have a non-meager Hamel basis?*
(2) *Does X have a non-meager Hamel basis if $|X| = \mathfrak{c}^{++}$?*
(3) *Does X have a Hamel basis containing no uncountable compact subset?*

References

[BDHMP] T. Bartoszyński, M. Džamonja, L. Halbeisen, E. Murtinová and A. Plichko, *On bases in Banach spaces*, Studia Math. 170 (2005), 147–171.

[De] G. Debs, *Espaces héréditairement de Baire*, Fund. Math. 129 (1988), 199–206.

[En] R. Engelking, *General Topology*, PWN, Warszawa, 1977.

[ET] P. Erdős and A. Tarski, *On families of mutually exclusive sets*, Ann. of Math. 44 (1943), 315–329.

[Ke] A. Kechris, *Classical Descriptive Set Theory*, Springer, New York, 1995.

Instytut Matematyki
Uniwersytet Humanistyczno-Przyrodniczy
im. Jana Kochanowskiego w Kielcach
Świętokrzyska 15
25-406 Kielce, Poland
and
Department of Mathematics
Ivan Franko National University of Lviv
1 Universytetska St.
79000 Lviv, Ukraine
E-mail: tbanakh@yahoo.com

Department of Mathematics
University of East Anglia
Norwich, NR4 7TJ, United Kingdom
E-mail: M.Dzamonja@uea.ac.uk

Theoretische Informatik und Logik
Universität Bern
Neubrückstr. 10
3012 Bern, Switzerland
E-mail: halbeis@iam.unibe.ch

Received July 14, 2006
Revised version June 29, 2008 (5965)