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Disjoint strict singularity of inclusions between rearrangement invariant spaces

by

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Abstract. It is studied when inclusions between rearrangement invariant function spaces on the interval $[0,\infty)$ are disjointly strictly singular operators. In particular suitable criteria, in terms of the fundamental function, for the inclusions $L^1\cap L^\infty\hookrightarrow E$ and $E\hookrightarrow L^1+L^\infty$ to be disjointly strictly singular are shown. Applications to the classes of Lorentz and Marcinkiewicz spaces are given.

1. Introduction. An operator between two Banach spaces is said to be *strictly singular* (or Kato) if it fails to be an isomorphism on any infinite-dimensional closed subspace. The class of strictly singular operators is a well known closed operator ideal. A weaker notion for Banach lattices is that of disjoint strict singularity: an operator T from a Banach lattice X to a Banach space Y is said to be *Disjointly Strictly Singular* (DSS for short) if there is no sequence $(x_n)_{n=1}^\infty$ of disjointly supported non-null vectors in X such that the restriction of T to the closed subspace spanned by $(x_n)_{n=1}^\infty$ is an isomorphism. This notion, introduced in [HR], is useful in the study of the lattice structure of function spaces (e.g. in constructing function spaces with singular ℓ_p-complemented copies).

The aim of this paper is to study the disjoint strict singularity of the inclusion operator between arbitrary rearrangement invariant spaces (r.i. spaces for short) on the interval $[0,\infty)$.

The analogous problem of DSS inclusions between r.i. spaces on the finite interval $[0,1]$ has been studied by Astashkin ([A]), Novikov ([N_1], [N_2]) and García del Amo, Ruiz and the present authors ([GHSS], [GHR]). If E is

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an r.i. space on $[0,1]$ different from $L^1[0,1]$ then the canonical inclusion $E \hookrightarrow L^1[0,1]$ is always DSS. This property in fact characterizes $L^1[0,1]$: an r.i. space F such that for any other different r.i. space E with $E \hookrightarrow F$ the inclusion $E \hookrightarrow F$ is DSS must be $L^1[0,1]$. A symmetric characterization also holds for $L^\infty[0,1]$ and both are deduced from a factorization result given in [GHSS]: for any inclusion $E \hookrightarrow F$ with $E \neq L^\infty[0,1]$ and $F \neq L^1[0,1]$ there exists an intermediate r.i. space G such that the inclusions $E \hookrightarrow G$ and $G \hookrightarrow F$ are not DSS. In the special context of Orlicz spaces, characterizations of when the inclusion operator $L^\varphi(\mu) \hookrightarrow L^\psi(\mu)$ is DSS have been given by Kalton [K_1] for sequence spaces with basis (where the notions of disjoint strict singularity and strict singularity coincide) and in [HR] and [GHR] for function spaces. In particular concrete criteria on the functions φ, ψ for the inclusions $L^p(\mu) \hookrightarrow L^\psi(\mu)$ and $L^\varphi(\mu) \hookrightarrow L^p(\mu)$ to be DSS were given. The classes of Lorentz function spaces $\Lambda(\phi)[0,1]$ and Marcinkiewicz function spaces $M(\phi)[0,1]$ have been studied in [A].

In general it is more delicate to determine the DSS behavior in the $[0,\infty)$ case than in the $[0,1]$ case. Thus, natural r.i. spaces with the same Boyd indices may have different behavior (e.g. the spaces $L^p \cap L^q$ and $L^p + L^q$ with respect to the inclusion in $L^1 + L^\infty$). First we analyse in Section 3 the inclusion $L^1 \cap L^\infty \hookrightarrow E$ characterizing the r.i. spaces E in terms of the associated fundamental function ϕ_E. Theorem 3.4 states that $L^1 \cap L^\infty \hookrightarrow E$ is DSS if and only if

$$\lim_{t\to 0} \phi_E(t) = \lim_{t\to\infty} \frac{\phi_E(t)}{t} = 0.$$

These conditions are also equivalent to $L^1 \cap L^\infty \hookrightarrow E$ being either strictly singular or weakly compact. In the proof of these statements we make use of the Dunford–Pettis property of $L^1 \cap L^\infty$. This was obtained by Kalton [K_2], during a visit to Madrid in the Spring of 1996, and it has also been proved by Kamińska and Mastyło [KM]. In this section we also determine when the canonical inclusion between a Lorentz space $\Lambda(\phi)$ and the Marcinkiewicz space with the same fundamental function $M(\widetilde{\phi})$ is weakly compact (Proposition 3.1). This extends earlier results for the $[0,1]$ case given by Kuzin-Aleksinskiĭ [K-A].

In Section 4 we study the disjoint strict singularity of the inclusion $E \hookrightarrow L^1 + L^\infty$ which is in general more complicated to determine; here the functions $t^{-1/p}\chi_{(0,\infty)}$ play a special role. It is proved that if $E \hookrightarrow L^1 + L^\infty$ is DSS then

$$\lim_{t\to 0} \frac{\phi_E(t)}{t} = \lim_{t\to\infty} \phi_E(t) = \infty$$

and

$$\sup_n \|t^{-1/p}\chi_{(1/n,n)}\|_E = \infty \quad \text{for any } 1 < p < \infty.$$

In particular the inclusion between the order continuous weak L^p-space $L_0^{p,\infty}$ and L^1+L^∞ is not DSS. One of the main results of this section (Theorem 4.5) gives a useful sufficient condition to be DSS: if an r.i. space E (different from L^1 and L^∞) has submultiplicative fundamental function, then $E \hookrightarrow L^1 + L^\infty$ is DSS, except when $E = L^{p,\infty}$ or $E = L_0^{p,\infty}$ for some $1 < p < \infty$. This result is obtained by carefully analyzing the inclusion of the associated Marcinkiewicz space $M(\widetilde{\phi}_E)$ in $L^1 + L^\infty$ (Lemma 4.4). As a consequence, we deduce some sufficient conditions on intermediate spaces F between E and $L^1 + L^\infty$ for the inclusion operator $E \hookrightarrow F$ to be DSS (Corollaries 4.6 and 4.7). We also give a criterion for inclusions between Lorentz spaces (Theorem 4.8): If $\lim_{t\to 0,\infty} \psi(t)/\phi(t) = 0$ and ϕ is submultiplicative then $\Lambda(\phi) \hookrightarrow \Lambda(\psi)$ is DSS.

2. Notations and previous results. Let us give some definitions and notations. We consider the interval $[0,\infty)$ and the Lebesgue measure λ. The *distribution function* λ_x associated with a measurable function x on $[0,\infty)$ is defined by $\lambda_x(s) = \lambda\{t \in [0,\infty) : |x(t)| > s\}$, and the *decreasing rearrangement function* x^* of x is

$$x^*(t) = \inf\{s \in [0,\infty) : \lambda_x(s) \le t\}.$$

A Banach space $E[0,\infty) \equiv E$ of measurable functions defined on $[0,\infty)$ is said to be a *rearrangement invariant space* (briefly r.i. space) if the following conditions are satisfied:

(a) if $y \in E$ and $|x(t)| \le |y(t)|$ λ-a.e. on $[0,\infty)$ then $x \in E$ and $\|x\|_E \le \|y\|_E$,

(b) if $y \in E$ and $\lambda_x = \lambda_y$ then $x \in E$ and $\|x\|_E = \|y\|_E$.

It is well known that every r.i. space E satisfies the condition $L^1 \cap L^\infty \hookrightarrow E \hookrightarrow L^1 + L^\infty$ where "$\hookrightarrow$" means continuous inclusion. Recall that the *fundamental function* ϕ_E of an r.i. space E is defined by $\phi_E(t) = \|\chi_{[0,t]}\|_E$ with $t \ge 0$. It is an increasing function and the associated function $\widetilde{\phi}_E$, defined by $\widetilde{\phi}_E(t) = t/\phi_E(t)$, is also increasing.

The *Köthe dual* E' of an r.i. space E is formed by the measurable functions x on $[0,\infty)$ such that

$$\|x\|_{E'} = \sup_{y \in B_E} \int_0^\infty x(t)y(t)\,dt < \infty$$

where B_E is the unit ball of E. The space E' is also an r.i. space and we denote $(E')'$ by E''. We shall consider r.i. spaces which are either *maximal* (i.e. $E = E''$) or *minimal* (i.e. E is the closed linear span of the simple integrable functions in E'').

We consider the Hardy–Littlewood–Pólya semi-order "$\prec$": If $x, y \in L^1 + L^\infty$, we say that $x \prec y$ if

$$\int_0^t x^*(s)\,ds \le \int_0^t y^*(s)\,ds \quad \text{for every } t \in [0,\infty).$$

If E is an r.i. space and $x \prec y$ with $y \in E$, then $x \in E$ and $\|x\|_E \le \|y\|_E$ (cf. [LT$_2$, p. 125]).

An r.i. space E has the *Fatou property* if for any increasing positive sequence $(x_n)_{n=1}^\infty$ in E with $\sup_n \|x_n\|_E < \infty$ we have $\sup_n x_n \in E$ and $\|\sup_n x_n\|_E = \sup_n \|x_n\|_E$. The Köthe dual E' has the Fatou property. Given r.i. spaces E_1 and E_2, we consider the sum space $E_1 + E_2$ with the norm $\|x\|_{E_1+E_2} = \inf\{\|x_1\|_{E_1} + \|x_2\|_{E_2} : x = x_1 + x_2,\ x_1 \in E_1,\ x_2 \in E_2\}$, and the intersection space $E_1 \cap E_2$ with the norm $\|x\|_{E_1\cap E_2} = \max(\|x\|_{E_1}, \|x\|_{E_2})$. Both are r.i. spaces with fundamental functions $\phi_{E_1+E_2} = \min(\phi_{E_1}, \phi_{E_2})$ and $\phi_{E_1\cap E_2} = \max(\phi_{E_1}, \phi_{E_2})$. If $E_1 = L^1$ and $E_2 = L^\infty$ then

$$\|x\|_{L^1+L^\infty} = \int_0^1 x^*(t)\,dt = \sup_{\lambda(E)=1} \int_E |x(t)|\,dt.$$

Important examples of r.i. spaces are the Orlicz, Lorentz and Marcinkiewicz spaces:

If φ is a positive convex function on $[0,\infty)$ with $\varphi(0) = 0$, the *Orlicz space* L^φ consists of all measurable functions x on $[0,\infty)$ for which

$$\|x\|_{L^\varphi} = \inf\left\{ s > 0 : \int_0^\infty \varphi\left(\frac{|x(t)|}{s}\right) dt \le 1 \right\} < \infty.$$

If $1 < p \le \infty$ and $1 \le q \le \infty$, the *classical Lorentz space* $L^{p,q}$ consists of all measurable functions x defined on $[0,\infty)$ such that

$$\|x\|_{p,q} = \left(\int_0^\infty (t^{1/p} x^*(t))^q \,\frac{dt}{t} \right)^{1/q} < \infty \quad \text{if } q < \infty,$$

and

$$\|x\|_{p,\infty} = \sup_{t>0}\{t^{1/p} x^*(t)\} < \infty.$$

We shall denote by $L_0^{p,\infty}$ the order continuous part of $L^{p,\infty}$.

Let Φ be the class of all increasing concave functions ϕ on $[0,\infty)$ with $\phi(0) = 0$. If $\phi \in \Phi$ the *Lorentz space* $\Lambda(\phi)$ consists of all measurable functions x defined on $[0,\infty)$ such that

$$\|x\|_{\Lambda(\phi)} = \int_0^\infty x^*(t)\,d\phi(t) < \infty.$$

The *Marcinkiewicz space* $M(\phi)$ consists of all measurable functions x defined on $[0,\infty)$ for which

$$\|x\|_{M(\phi)} = \sup_{t>0} \frac{\int_0^t x^*(s)\,ds}{\phi(t)} < \infty.$$

Given ϕ, the spaces $\Lambda(\phi)$ and $M(\widetilde{\phi})$ are respectively the smallest and the biggest r.i. space having the same fundamental function ϕ (cf. [KPS, Theorems 5.5 and 5.7]]):

THEOREM 2.1. *Every r.i. space E with fundamental function ϕ satisfies*

$$\Lambda(\phi) \hookrightarrow E \hookrightarrow M(\widetilde{\phi}).$$

Given $\phi \in \Phi$, we will consider the subspace $M_0(\phi)$ of $M(\phi)$ consisting of all functions $x \in M(\phi)$ such that

$$\lim_{t\to 0,\infty} \frac{1}{\phi(t)} \int_0^t x^*(s)\,ds = 0.$$

It is clear that if $M_0(\phi) \neq \{0\}$, then the function ϕ must satisfy

$$\lim_{t\to 0} \frac{\phi(t)}{t} = \lim_{t\to\infty} \phi(t) = \infty.$$

Conversely, it is well known that under both conditions and

$$\lim_{t\to 0} \phi(t) = 0$$

the space $M_0(\phi)$ is a separable closed subspace of $M(\phi)$ and $(M_0(\phi))^* = \Lambda(\phi)$ (cf. [KPS, Theorem II.5.4]). The function ϕ is the fundamental function of the space $M_0(\widetilde{\phi})$. Recall also that if $\lim_{t\to 0} \phi(t) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$ then the Lorentz space $\Lambda(\phi)$ is separable and $(\Lambda(\phi))^* = M(\phi)$. In general given an increasing function ϕ, since the function $\widetilde{\phi}$ is quasiconcave, there exists a concave function $\overline{\phi}$ such that $\widetilde{\phi} \le \overline{\phi} \le 2\widetilde{\phi}$ (cf. [BS], [KPS]).

For other properties of r.i. spaces we refer to [BS], [KPS], [LT_2].

3. The inclusion $L^1 \cap L^\infty \hookrightarrow E$. In this section we characterize when the inclusion $L^1 \cap L^\infty \hookrightarrow E$ is DSS. We will use the following

PROPOSITION 3.1. *Let $\phi \in \Phi$. The inclusion operator $\Lambda(\phi) \hookrightarrow M(\widetilde{\phi})$ is weakly compact if and only if the following conditions hold*:

$$\lim_{t\to 0} \frac{\phi(t)}{t} = \lim_{t\to\infty} \phi(t) = \infty \tag{1}$$

and

$$\lim_{t\to 0} \phi(t) = \lim_{t\to\infty} \frac{\phi(t)}{t} = 0. \tag{2}$$

Proof. First assume that (1) and (2) hold. If j denotes the inclusion operator, we have, by Theorem 2.1, $j : \Lambda(\phi) \hookrightarrow M_0(\widetilde{\phi}) \hookrightarrow M(\widetilde{\phi})$. Hence the adjoint operator j^* acts from $(M_0(\overline{\phi}))^* = \Lambda(\overline{\phi})$ into $(\Lambda(\phi))^*$ and then j^{**} transforms $(\Lambda(\phi))^{**}$ into $M(\overline{\phi})$. Thus, by a theorem of Gantmacher (see e.g. [PR, p. 250]) we conclude that j is weakly compact.

Conversely, if any of the conditions in (1) or (2) fails, then the inclusion operator $\Lambda(\phi) \hookrightarrow M(\widetilde{\phi})$ is not weakly compact since then it cannot be factorized through any reflexive space. Indeed, consider the space H of all functions $x(t) = \sum_k x_k \chi_{[k-1,k)}$ with $x_k \to 0$. It is easy to check that if $\lim_{t\to\infty} \phi(t) < \infty$ then

$$\|x\|_{\Lambda(\phi)} \approx \|x\|_{M(\widetilde{\phi})} \approx \sup_k |x_k|$$

and if $\lim_{t\to\infty} \phi(t)/t > 0$ then

$$\|x\|_{\Lambda(\phi)} \approx \|x\|_{M(\widetilde{\phi})} \approx \sum_k |x_k|$$

on the subspace H.

Consider now the space V of all functions $x \in L^\infty$ with $\mathrm{supp}(x) \subset [0,1]$. On V, if $\lim_{t\to 0} \phi(t) > 0$ then

$$\|x\|_{\Lambda(\phi)} \approx \|x\|_{M(\widetilde{\phi})} \approx \|x\|_{L^\infty},$$

and if $\lim_{t\to 0} \phi(t)/t < \infty$ then

$$\|x\|_{\Lambda(\phi)} \approx \|x\|_{M(\widetilde{\phi})} \approx \|x\|_{L^1}. \ \blacksquare$$

PROPOSITION 3.2. *Given $\phi \in \Phi$, there exists a reflexive r.i. space E with fundamental function $\phi_E = \phi$ if and only if the function ϕ satisfies conditions* (1) *and* (2).

Proof. The necessity part follows directly from Proposition 3.1 and Theorem 2.1. Conversely, under conditions (1) and (2), the inclusion operator $\Lambda(\phi) \hookrightarrow M(\widetilde{\phi})$ is weakly compact. Consider the real interpolation space $E = (\Lambda(\phi), M(\widetilde{\phi}))_{\theta,p}$ for $0 < \theta < 1$ and $1 < p < \infty$. Then, using [B, Proposition II.3.1] we deduce that E is a reflexive r.i. space, and it is clear that its fundamental function ϕ_E is equal to ϕ. ■

The above statements extend previous results for r.i. spaces on $[0,1]$ given by Kuzin-Aleksinskiĭ [K-A].

The following lemma will also be useful in order to characterize when the inclusion operator $L^1 \cap L^\infty \hookrightarrow E$ is DSS.

LEMMA 3.3. *If $\phi \in \Phi$ satisfies condition* (2) *then there exists a function $\psi \in \Phi$ such that ψ satisfies conditions* (1) *and* (2), *and*

$$\int_0^\infty \phi'(t)\psi'(t)\,dt < \infty.$$

Proof. This can be deduced from [P, Theorem 5]. We also give a self-contained alternative short proof provided by the referee. Using condition (2) we have

$$\lim_{t\to\infty}\phi'(t)\le \lim_{t\to\infty}\frac{1}{t}\int_0^t \phi'(s)\,ds=\lim_{t\to\infty}\frac{\phi(t)}{t}=0.$$

Choose a strictly increasing unbounded sequence $(a_n)_{n=1}^\infty$ which satisfies $a_1=1$,

$$a_{n+1}\ge 2a_n-a_{n-1}\quad\text{and}\quad \phi'(a_n)\le 1/n.$$

Also, choose a strictly decreasing sequence $(b_n)_{n=0}^\infty$ in $(0,1]$ such that

$$\phi(b_n)=2^{-n}\phi(1)$$

(so in particular $b_0=1$). Now we consider the function $f:(0,\infty)\to(0,\infty)$ defined by

$$f=\sum_{n=0}^\infty (n+1)\chi_{(b_{n+1},b_n]}+\sum_{n=1}^\infty \frac{1}{n(a_{n+1}-a_n)}\chi_{(a_n,a_{n+1}]}$$

and the function

$$\psi(t)=\int_0^t f(s)\,ds.$$

It is easy to verify that ψ has all the required properties. ∎

THEOREM 3.4. *Let E be an r.i. space. The following conditions are equivalent*:

(i) *The inclusion operator $L^1\cap L^\infty\hookrightarrow E$ is DSS.*
(ii) *The inclusion operator $L^1\cap L^\infty\hookrightarrow E$ is strictly singular.*
(iii) *The inclusion operator $L^1\cap L^\infty\hookrightarrow E$ is weakly compact.*
(iv) $\lim_{t\to 0}\phi_E(t)=\lim_{t\to\infty}\phi_E(t)/t=0$.

Proof. (i)⇒(iv). Suppose that $\lim_{t\to 0}\phi_E(t)=c>0$. Then

$$\|x\|_E\ge \lim_{t\to 0}\|x^*\chi_{[0,t]}\|_E\ge \lim_{t\to 0}x^*(t)\phi_E(t)=c\|x\|_\infty$$

for every $x\in E$. If V is the subspace of functions $x\in L^\infty$ with $\operatorname{supp}(x)\subset[0,1]$ then $\|x\|_1\le\|x\|_\infty$ for $x\in V$. Hence $\|x\|_{L^1\cap L^\infty}=\|x\|_\infty$. Thus $\|x\|_{L^1\cap L^\infty}\le c^{-1}\|x\|_E$ for $x\in V$. Since the converse inequality always holds, we deduce that the norms $\|\cdot\|_{L^1\cap L^\infty}$ and $\|\cdot\|_E$ are equivalent on V, so the inclusion $L^1\cap L^\infty\hookrightarrow E$ is not DSS.

Suppose now that $\lim_{t\to\infty}\phi_E(t)/t=b>0$. Then $\phi_E(t)\ge bt$ for every $t>0$. Hence $\widetilde{\phi}_E(t)\le 1/b$. Therefore, using Theorem 2.1, we have

$$\|x\|_E\ge\|x\|_{M(\widetilde{\phi}_E)}\ge b\|x\|_{L^1}$$

for every $x \in E$. Now, if H is the subspace of functions $x = \sum_k x_k \chi_{[k-1,k)}$, then

$$\|x\|_E \geq b \sum_k |x_k|$$

for every $x \in H$. Hence

$$\|x\|_{L^1 \cap L^\infty} = \sum_k |x_k| \leq \frac{1}{b} \|x\|_E$$

for every $x \in H$. As the converse inequality also holds, we conclude that the inclusion $L^1 \cap L^\infty \hookrightarrow E$ is not DSS.

(iv)⇒(iii). Let $\overline{\phi}_E$ be the least concave majorant function of ϕ_E. Then, by Lemma 3.3, there exists $\psi \in \Phi$ which satisfies (1) and (2) and

$$\int_0^\infty \overline{\phi}'_E(t)\psi'(t)\,dt < \infty. \tag{3}$$

From (3) we deduce the inclusion $M(\psi) \hookrightarrow \Lambda(\overline{\phi}_E)$. Now, if we denote by $\overline{\psi}$ the least concave majorant function of $\widetilde{\psi}$, then $\Lambda(\overline{\psi}) \hookrightarrow M(\psi)$. Thus, we have got the factorization

$$L^1 \cap L^\infty \hookrightarrow \Lambda(\overline{\psi}) \hookrightarrow M(\psi) \hookrightarrow \Lambda(\overline{\phi}_E) \hookrightarrow E.$$

Now, by Proposition 3.1, the inclusion $\Lambda(\overline{\psi}) \hookrightarrow M(\psi)$, and hence $L^1 \cap L^\infty \hookrightarrow E$, is weakly compact.

(iii)⇒(ii) follows from the Dunford–Pettis property of $L^1 \cap L^\infty$ ([K$_2$], [KM]), and (ii)⇒(i) is trivial. ■

Remark. A characterization of when the inclusion $L^1 \cap L^\infty \hookrightarrow E$ is strictly singular has also been obtained very recently by Cobos, Manzano, Martínez and Matos in [CMMM, Theorem 3.4] using a different technique based on qualitative interpolation methods.

Reasoning as in the above implication (iv)⇒(iii) we get the following

Corollary 3.5. *Given an r.i. space E, there exists a reflexive r.i. space F such that $F \hookrightarrow E$ if and only if the condition* (iv) *is satisfied.*

Using duality arguments and the equality $\phi_E(t)\phi_{E'}(t) = t$ we also deduce the following

Corollary 3.6. *Given an r.i. space E, there exists a reflexive r.i. space F such that $E \hookrightarrow F$ if and only if the condition* (1) *is satisfied.*

The above corollaries for r.i. spaces on $[0,1]$ were given in [K-A] and [N$_1$].

4. The inclusion $E \hookrightarrow L^1 + L^\infty$. In this section we study when the inclusion $E \hookrightarrow L^1 + L^\infty$ is DSS. First we give some necessary conditions.

THEOREM 4.1. *Let E be an r.i. space. If the inclusion operator $E \hookrightarrow L^1 + L^\infty$ is DSS, then*:

(i) $\lim_{t\to 0} \phi_E(t)/t = \lim_{t\to\infty} \phi_E(t) = \infty$.

(ii) $t^{-1/p}\chi_{(0,\infty)} \notin E$ *for any* $1 < p < \infty$.

Proof. (i) First suppose that $\lim_{t\to 0} \phi_E(t)/t < \infty$. If we consider the sequence $(x_n)_{n=1}^\infty = (2^n\chi_{(2^{-n},2^{-n+1}]})_{n=1}^\infty$, the norms of E and $L^1 + L^\infty$ are equivalent on the subspace $[(x_n)_{n=1}^\infty]$, which is isomorphic to ℓ_1. Indeed, since $\|x_n\|_E = 2^n\phi_E(2^{-n}) \le M$ for some $M < \infty$ and every $n \in \mathbb{N}$, we have

$$\sum_n |a_n| = \Big\|\sum_n a_n x_n\Big\|_{L^1+L^\infty} \le \Big\|\sum_n a_n x_n\Big\|_E \le M\sum_n |a_n|$$

for every scalar sequence $(a_n)_{n=1}^\infty$.

If $\lim_{t\to\infty} \phi_E(t) = A < \infty$, then the sequence $(\chi_{[n-1,n)})_{n=1}^\infty$ in E and $L^1 + L^\infty$ is equivalent to the canonical basis of c_0 since

$$\sup_n |a_n| = \Big\|\sum_n a_n\chi_{[n-1,n)}\Big\|_{L^1+L^\infty} \le \Big\|\sum_n a_n\chi_{[n-1,n)}\Big\|_E \le A\sup_n |a_n|$$

for every scalar sequence $(a_n)_{n=1}^\infty$.

(ii) Suppose now that there exists $1 < p < \infty$ such that $t^{-1/p}\chi_{(0,\infty)} \in E$. Let $(x_n)_{n=1}^\infty$ be a sequence of disjointly supported functions in E such that x_n and $t^{-1/p}\chi_{(0,\infty)}$ are equimeasurable for every $n \in \mathbb{N}$. Then the functions $\sum_n a_n x_n$ and $\|a\|_p t^{-1/p}\chi_{(0,\infty)}$ are equimeasurable for every scalar sequence $a = (a_n)_{n=1}^\infty$. Therefore

$$\begin{aligned}\|t^{-1/p}\chi_{(0,\infty)}\|_{L^1+L^\infty}\|a\|_p &= \Big\|\sum_n a_n x_n\Big\|_{L^1+L^\infty} \le \Big\|\sum_n a_n x_n\Big\|_E \\ &= \|t^{-1/p}\chi_{(0,\infty)}\|_E\|a\|_p\end{aligned}$$

for every scalar sequence $a = (a_n)_{n=1}^\infty$. So $(x_n)_{n=1}^\infty$ is equivalent in E and $L^1 + L^\infty$ to the canonical basis of ℓ_p. ■

From the above theorem we see that the inclusion $L^{p,\infty} \hookrightarrow L^1 + L^\infty$ is not DSS for any $1 < p < \infty$.

We turn to showing that the above conditions (i) and (ii) are not in general sufficient for $E \hookrightarrow L^1 + L^\infty$ to be DSS:

PROPOSITION 4.2. *Let $1 < p < \infty$ and $(x_n)_{n=1}^\infty$ be a disjointly supported sequence of functions in $L^{p,\infty}$ with $\|x_n\|_{p,\infty} \le 1$ for every $n \in \mathbb{N}$. Let $(\varepsilon_n)_{n=1}^\infty$ be a sequence in $[0,1]$ with $\sum_n \varepsilon_n < \infty$ and c be a positive constant such that*

$$\int_0^\tau x_n^*(t)\,dt \ge c\tau^{1-1/p} \tag{4}$$

for every $n \in \mathbb{N}$ and every $\tau \in [\varepsilon_n, 1]$. Then there exists a constant $M > 0$ such that

$$(5)\qquad \frac{1}{M}\|a\|_p \le \Big\|\sum_n a_n x_n\Big\|_{L^1+L^\infty} \le \Big\|\sum_n a_n x_n\Big\|_{p,\infty} \le M\|a\|_p$$

for every $a = (a_n)_{n=1}^\infty \in \ell_p$.

Proof. Since $\|x\|_{L^1+L^\infty} = \sup_{\lambda(E)=1} \int_E |x(t)|\,dt$, we have

$$\Big\|\sum_n a_n x_n\Big\|_{L^1+L^\infty} = \sup_{\sum_n \tau_n = 1} \sum_n |a_n| \int_0^{\tau_n} x_n^*(t)\,dt.$$

Choose $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0}^\infty \varepsilon_n \le 1/2$. Then, by (4), we have

$$\begin{aligned}\Big\|\sum_n a_n x_n\Big\|_{L^1+L^\infty} &\ge c \sup_{\substack{\tau_n \ge \varepsilon_n \\ \sum_{n=n_0}^\infty \tau_n = 1}} \sum_{n=n_0}^\infty |a_n| \tau_n^{1-1/p} \\ &\ge c \sup_{\substack{\tau_n \ge 0 \\ \sum_{n=n_0}^\infty \tau_n = 1/2}} \sum_{n=n_0}^\infty |a_n| \tau_n^{1-1/p} \\ &= 2^{1/p-1} c \sup_{\substack{\tau_n \ge 0 \\ \sum_{n=n_0}^\infty \tau_n = 1}} \sum_{n=n_0}^\infty |a_n| \tau_n^{1-1/p} \\ &= 2^{1/p-1} c \Big(\sum_{n=n_0}^\infty |a_n|^p\Big)^{1/p}.\end{aligned}$$

As $\|\sum_n a_n x_n\|_{L^1+L^\infty} \ge c \sup_n |a_n|$ we obtain the first inequality of (5).

The second inequality of (5) is obvious. Finally, since $L^{p,\infty}$ satisfies an upper p-estimate (cf. [CD$_1$]), we obtain the third inequality of (5). ■

Corollary 4.3. *The inclusion $L_0^{p,\infty} \hookrightarrow L^1 + L^\infty$ is not DSS for any $1 < p < \infty$.*

Proof. For a sequence $(\varepsilon_n)_{n=1}^\infty$ as in Proposition 4.2 we just consider a disjointly supported sequence of functions $(x_n)_{n=1}^\infty$ in $L_0^{p,\infty}$ such that

$$x_n^*(t) = \begin{cases} \varepsilon_n^{-1/p} & \text{if } 0 \le t \le \varepsilon_n, \\ t^{-1/p} & \text{if } \varepsilon_n < t < 1, \\ 0 & \text{if } t \ge 1. \end{cases}\ ■$$

Remark. The above corollary shows that in general the conditions (i) and (ii) of Theorem 4.1 are not sufficient for $E \hookrightarrow L^1 + L^\infty$ to be DSS. It also shows that the condition (ii) can be replaced, using the Fatou property of E'', by the sharper one:

$$\sup_n \|t^{-1/p}\chi_{(1/n,n)}\|_E = \infty \quad \text{for any } 1 < p < \infty.$$

We shall give some sufficient conditions for $E \hookrightarrow L^1 + L^\infty$ to be DSS. A very strong sufficient condition is that $1/\phi_E \in L^p$ for some $1 < p < \infty$ since in this case we have the inclusion $M(\widetilde{\phi_E}) \hookrightarrow L^p$. One of the main results of this section (Theorem 4.5) gives a milder sufficient condition for $E \hookrightarrow L^1 + L^\infty$ to be DSS. First we analyze the case of Marcinkiewicz spaces.

Let $\phi \in \Phi$ and assume that $M(\phi) \hookrightarrow L^1 + L^\infty$ is not DSS. Then there exist a disjointly supported sequence of functions $(x_n)_{n=1}^\infty \subset M(\phi)$ and a constant $D > 0$ such that

$$\Big\| \sum_n a_n x_n \Big\|_{M(\phi)} \le D \Big\| \sum_n a_n x_n \Big\|_{L^1+L^\infty} \tag{6}$$

for every scalar sequence $(a_n)_{n=1}^\infty$. Define

$$\varphi_n(\tau) = \int_0^\tau x_n^*(t)\,dt$$

for $n \in \mathbb{N}$ and $\tau > 0$. We may assume without loss of generality that $\varphi_n(1) = 1$ for every $n \in \mathbb{N}$. Under this hypothesis we have the following

Lemma 4.4. *There exists an increasing sequence $(n_k)_{k=1}^\infty$ of integers such that*

$$\varphi_n\Big(\frac{1}{k}\Big) \ge \frac{1}{D\phi(k)}$$

for every $k \in \mathbb{N}$ and every $n \ge n_k$, where D is the constant in (6).

Proof. The inequality (6) implies

$$\sup_{\tau_n \ge 0} \frac{\sum_n |a_n| \int_0^{\tau_n} x_n^*(t)dt}{\phi(\sum_n \tau_n)} \le D \sup_{\substack{\tau_n \ge 0 \\ \sum_n \tau_n = 1}} \sum_n |a_n| \int_0^{\tau_n} x_n^*(t)\,dt,$$

i.e.

$$\sup_{\tau_n \ge 0} \frac{\sum_n |a_n| \varphi_n(\tau_n)}{\phi(\sum_n \tau_n)} \le D \sup_{\substack{\tau_n \ge 0 \\ \sum_n \tau_n = 1}} \sum_n |a_n| \varphi_n(\tau_n)$$

or

$$\sup_{s_n \ge 0} \frac{\sum_n |a_n| s_n}{\phi(\sum_n \varphi_n^{-1}(s_n))} \le D \sup_{\substack{s_n \ge 0 \\ \sum_n \varphi_n^{-1}(s_n) = 1}} \sum_n |a_n| s_n. \tag{7}$$

Now, each φ_n is concave, therefore φ_n^{-1} is convex and the set

$$A = \Big\{ (t_n)_{n=1}^\infty \subset \mathbb{R} : \sum_n \varphi_n^{-1}(|t_n|) \le 1 \Big\}$$

is convex. Denote by ℓ_M the modular (or Musielak–Orlicz) sequence space generated by the sequence $(\varphi_n^{-1})_{n=1}^\infty$ (cf. [W], [LT_1]). Then A is the unit ball of ℓ_M. The inequality (7) implies that

$$\sup_{s_n\geq 0} \frac{\sum_n |a_n| s_n}{\phi(\sum_n \varphi_n^{-1}(s_n))} \leq D\|a\|_{\ell'_M}.$$

Therefore

$$\sup_{\|a\|_{\ell'_M}\leq 1} \sum_n |a_n| \frac{s_n}{\phi(\sum_m \varphi_m^{-1}(s_m))} \leq D$$

for each $(s_n)_{n=1}^\infty$ with $s_n \geq 0$ for every $n \in \mathbb{N}$. Since ℓ_M is maximal we obtain

$$\left\| \left(\frac{s_n}{\phi(\sum_m \varphi_m^{-1}(s_m))} \right)_{n=1}^\infty \right\|_{\ell_M} \leq D$$

and then

$$\left(\frac{s_n}{D\phi(\sum_m \varphi_m^{-1}(s_m))} \right)_{n=1}^\infty \in A.$$

Hence

$$\sum_n \varphi_n^{-1} \left(\frac{s_n}{D\phi(\sum_m \varphi_m^{-1}(s_m))} \right) \leq 1.$$

Given $k \in \mathbb{N}$ and a set $I \subset \mathbb{N}$ with $|I| = k$, we consider the sequence $(s_n)_{n=1}^\infty$ defined by

$$s_n = \begin{cases} 1 & \text{if } n \in I, \\ 0 & \text{if } n \notin I. \end{cases}$$

Using the assumption $\varphi_n(1) = 1$ for every $n \in \mathbb{N}$ we get

$$\sum_{n\in I} \varphi_n^{-1} \left(\frac{1}{D\phi(k)} \right) \leq 1.$$

This means that

$$\left| \left\{ n \in \mathbb{N} : \varphi_n^{-1} \left(\frac{1}{D\phi(k)} \right) > \frac{1}{k} \right\} \right| < k.$$

Consequently, there exists a sequence $(n_k)_{k=1}^\infty$ as in the statement. ∎

Now we are able to formulate a converse of Proposition 4.2.

THEOREM 4.5. *Let E be an r.i. space, different from L^1 and L^∞, with submultiplicative fundamental function. If the inclusion operator $E \hookrightarrow L^1 + L^\infty$ is not DSS, then $E = L^{p,\infty}$ or $E = L_0^{p,\infty}$ for some $1 < p < \infty$.*

Proof. There exist a sequence of disjointly supported functions $(x_n)_{n=1}^\infty \subset E$ and a constant $D > 0$ such that

$$\left\| \sum_n a_n x_n \right\|_E \leq D \left\| \sum_n a_n x_n \right\|_{L^1+L^\infty}$$

for every scalar sequence $(a_n)_{n=1}^{\infty}$. Now, by Theorem 2.1, we have

$$\Big\|\sum_n a_n x_n\Big\|_{M(\widetilde{\phi}_E)} \le D\Big\|\sum_n a_n x_n\Big\|_{L^1+L^\infty}.$$

We can assume that $(x_n)_{n=1}^{\infty}$ is normalized in L^1+L^∞ and $\widetilde{\phi}_E$ is concave. Using Lemma 4.4 and the concavity of $\widetilde{\phi}_E$ we get

$$\varphi_n(\tau) \ge \frac{1}{2D\widetilde{\phi}_E(1/\tau)}$$

for every $k\in\mathbb{N}$, $\tau\in[1/k,1]$ and $n\ge n_k$. Indeed, suppose that $\varphi_n(\tau) < 1/(2D\widetilde{\phi}_E(1/\tau))$ for some $k\in\mathbb{N}$, $\tau\in[1/k,1]$ and $n\ge n_k$. Then there exists $m\in\mathbb{N}$ such that $\tau\in(1/2^m,1/2^{m-1}]$. If $1/k\le 1/2^m$ then

$$\varphi_n(1/2^m) \le \frac{1}{2D\widetilde{\phi}_E(2^{m-1})} \le \frac{1}{D\widetilde{\phi}_E(2^m)},$$

which contradicts Lemma 4.4. The other case is analogous.

Now, by the submultiplicativity of ϕ_E there exists $C>0$ such that

$$\widetilde{\phi}_E(ts) \ge \frac{1}{C}\widetilde{\phi}_E(t)\widetilde{\phi}_E(s)$$

for every $t,s>0$. Hence $\widetilde{\phi}_E(1/\tau)\le C\widetilde{\phi}_E(1)/\widetilde{\phi}_E(\tau)$ for every $\tau>0$ and

$$\varphi_n(\tau) \ge \frac{1}{2CD\widetilde{\phi}_E(1)}\widetilde{\phi}_E(\tau) \tag{8}$$

for every $k\in\mathbb{N}$, $\tau\in[1/k,1]$ and $n\ge n_k$. Therefore, given $j\in\mathbb{N}$, we have

$$\begin{aligned}\sup_{|I|=j}\Big\|\sum_{n\in I} x_n\Big\|_E &\ge \sup_{|I|=j}\Big\|\sum_{n\in I} x_n\Big\|_{M(\widetilde{\phi}_E)} \ge \sup_{|I|=j}\sup_{0<\tau\le j}\frac{\sum_{n\in I}\varphi_n(\tau/j)}{\widetilde{\phi}_E(\tau)}\\ &= \sup_{0<\tau\le j}\sup_{|I|=j}\frac{\sum_{n\in I}\varphi_n(\tau/j)}{\widetilde{\phi}_E(\tau)} \ge \frac{1}{2CD\widetilde{\phi}_E(1)}\sup_{0<\tau\le j}\frac{j\widetilde{\phi}_E(\tau/j)}{\widetilde{\phi}_E(\tau)}.\end{aligned}$$

Now, since $\|x_n\|_{M(\widetilde{\phi}_E)}\le D$ for every $n\in\mathbb{N}$, we have $\varphi_n(\tau)\le D\widetilde{\phi}_E(\tau)$ for every $n\in\mathbb{N}$ and $\tau>0$. If $|I|=j$ then

$$\begin{aligned}\Big\|\sum_{n\in I} x_n\Big\|_{L^1+L^\infty} &= \sup_{\sum_{n\in I}\tau_n=1}\sum_{n\in I}\varphi_n(\tau_n)\\ &\le \sup_{\sum_{n\in I}\tau_n=1}\sum_{n\in I} D\widetilde{\phi}_E(\tau_n) = Dj\widetilde{\phi}_E\Big(\frac{1}{j}\Big).\end{aligned}$$

Comparing the last two inequalities we get

$$\frac{1}{2CD\widetilde{\phi}_E(1)}\sup_{0<\tau\le j}\frac{\widetilde{\phi}_E(\tau/j)}{\widetilde{\phi}_E(\tau)} \le D\widetilde{\phi}_E\Big(\frac{1}{j}\Big)$$

for every $j \in \mathbb{N}$. Consequently,

$$\widetilde{\phi}_E(ts) \leq C_1 \widetilde{\phi}_E(t) \widetilde{\phi}_E(s)$$

for every $t, s > 0$ such that $ts \leq 1$ where $C_1 = 4CD^2 \widetilde{\phi}_E(1)$.

Therefore

$$\frac{1}{C} \widetilde{\phi}_E(t) \widetilde{\phi_E}(s) \leq \widetilde{\phi}_E(ts) \leq C_1 \widetilde{\phi}_E(t) \widetilde{\phi}_E(s)$$

for every $0 < t, s \leq 1$. Now, using the well known fact that the only functions which satisfy such an equivalence are those which are equivalent to powers, and that $\widetilde{\phi}_E$ is an increasing concave function on $[0,1]$, we deduce that there exist $C_2 > 1$ and $\alpha \in [0,1]$ such that

$$\frac{1}{C_2} t^\alpha \leq \widetilde{\phi}_E(t) \leq C_2 t^\alpha$$

for every $t \in [0,1]$. If $t > 1$ and $0 < s \leq 1/t$, then

$$\frac{\widetilde{\phi}_E(ts)}{C_1 \widetilde{\phi}_E(s)} \leq \widetilde{\phi}_E(t) \leq C \frac{\widetilde{\phi}_E(ts)}{\widetilde{\phi}_E(s)}$$

and

$$\frac{1}{C_1 C_2^2}\, t^\alpha \leq \widetilde{\phi}_E(t) \leq C C_2^2\, t^\alpha$$

for every $t > 0$. If $\alpha = 0$ then $E = L^1$ and if $\alpha = 1$ then $E = L^\infty$, which are excluded. Now, there exist $1 < p < \infty$ and $C_3 > 1$ such that

$$\frac{1}{C_3} t^{1/p} \leq \phi_E(t) \leq C_3 t^{1/p}$$

for every $t > 0$. The right inequality and Theorem 2.1 imply that $E \hookrightarrow L^{p,\infty}$.

Let us now study the converse inclusion. By (8), there exist a constant $C_4 > 0$ and a sequence $(r_n)_{n=1}^\infty$ such that

$$\varphi_{r_n}(\tau) \geq C_4 \tau^{1-1/p}$$

for every $n \in \mathbb{N}$ and $\tau \in [1/2^n, 1]$. This means that

$$C_4(1 - 1/p) \min(t^{-1/p}, 2^{n/p}) \chi_{[0,1]} \prec x_{r_n}$$

for every $n \in \mathbb{N}$. Since the functions $(x_n)_{n=1}^\infty$ are disjointly supported, for every $\varepsilon > 0$ and $j \in \mathbb{N}$ there exist $I \subset \mathbb{N}$ with $|I| = j$ such that

$$C_4(1 - 1/p) \sum_{i=1}^{j} (t - i + 1)^{-1/p} \chi_{[i-1+\varepsilon, i]}(t) \prec \sum_{i \in I} x_i.$$

Applying now [LT_2, Proposition 2.a.8] and Proposition 4.2 we get

$$C_4(1 - 1/p) \Big\| \sum_{i=1}^{j} (t - i + 1)^{-1/p} \chi_{[i-1+\varepsilon, i]} \Big\|_E \leq \Big\| \sum_{i \in I} x_i \Big\|_E \leq C_5 j^{1/p}$$

for some constant $C_5 > 0$ which does not depend on ε. Since E'' has the Fatou property, letting ε tend to 0 we obtain

$$C_4(1-1/p)\Big\|\sum_{i=1}^{j}(t-i+1)^{-1/p}\chi_{(i-1,i]}\Big\|_{E''} \le C_5 j^{1/p}.$$

Now, since the functions

$$j^{-1/p}\sum_{i=1}^{j}(t-i+1)^{-1/p}\chi_{(i-1,i]} \quad \text{and} \quad t^{-1/p}\chi_{(0,j]}$$

are equimeasurable, we have

$$\|t^{-1/p}\chi_{(0,j]}\|_{E''} \le C_5[C_4(1-1/p)]^{-1}.$$

Using the Fatou property again, we get $t^{-1/p}\chi_{(0,\infty)} \in E''$. Hence $L^{p,\infty} \hookrightarrow E''$ and consequently $E'' = L^{p,\infty}$. Finally, if E is maximal then $E = L^{p,\infty}$, and if E is minimal then $E = L_0^{p,\infty}$. ∎

Examples. (i) The inclusion $L^{p,\infty} \cap L^{q,\infty} \hookrightarrow L^1 + L^\infty$ is DSS for $1 < p \ne q < \infty$.

(ii) If φ is a submultiplicative Orlicz function then the inclusion $L^\varphi \hookrightarrow L^1 + L^\infty$ is DSS except when $L^\varphi = L^1$.

Remark. The submultiplicativity of ϕ_E is essential in Theorem 4.5: the inclusion operator $L^p + L^q \hookrightarrow L^1 + L^\infty$ is not DSS for $1 \le p < q \le \infty$ (cf. [GHR]).

When considering interpolation spaces obtained by methods of genus s_ϱ (e.g. the real and complex interpolation methods) we have the following

Corollary 4.6. *Let E be an r.i. space with submultiplicative fundamental function and $E \ne L^1, L^\infty, L^{p,\infty}, L_0^{p,\infty}$ with $1 < p < \infty$. If F is an interpolation space between E and $L^1 + L^\infty$ obtained by an interpolation method of genus s_ϱ, then the inclusion operator $E \hookrightarrow F$ is DSS.*

Proof. This follows from Theorem 4.5 and [GHR, Theorem 3.4].

Corollary 4.7. *Let E be an r.i. space with submultiplicative fundamental function and $E \ne L^1, L^\infty, L^{p,\infty}, L_0^{p,\infty}$ with $1 < p < \infty$. If F is an intermediate Banach space between E and $L^1 + L^\infty$ such that*

$$\|x\|_F \le C\|x\|_E^\theta \|x\|_{L^1+L^\infty}^{1-\theta}$$

for some $0 < \theta < 1$ and $C > 0$ and for every $x \in E$, then the inclusion operator $E \hookrightarrow F$ is DSS.

Proof. This follows from Corollary 4.6 and [BL, Theorem 3.5.2(b)]. ∎

Finally we give some applications to Lorentz spaces $\Lambda(\phi)$. For classical Lorentz spaces $L^{p,q}$, the inclusion operator $L^{p,q} \hookrightarrow L^{p,q'}$ is DSS with $1 <$

$p < \infty$ and $1 \le q < q' \le \infty$. This follows easily from Proposition 2.3 and Lemma 2.1 of [CD_2].

THEOREM 4.8. *Let $\phi, \psi \in \Phi$ with $\psi \le C\phi$ for some constant $C > 0$. If*

$$\lim_{t\to 0,\infty} \frac{\psi(t)}{\phi(t)} = 0 \tag{9}$$

and ϕ is a submultiplicative function, then the inclusion operator $\Lambda(\phi) \hookrightarrow \Lambda(\psi)$ is DSS.

Proof. Suppose that $\Lambda(\phi) \hookrightarrow \Lambda(\psi)$ is not DSS. Then we can find a subspace $S \subset \Lambda(\phi)$ generated by a sequence of disjointly supported functions and $c > 0$ such that $\|x\|_{\Lambda(\psi)} \ge c$ for every $x \in S$ with $\|x\|_{\Lambda(\phi)} = 1$. By (9), there exists $0 < \delta < 1$ depending only on ϕ, ψ and c such that

$$\int\limits_{\lambda_x(s)\le\delta} \psi(\lambda_x(s))\,ds + \int\limits_{\lambda_x(s)\ge 1/\delta} \psi(\lambda_x(s))\,ds \le \frac{c}{2}.$$

Hence

$$\int\limits_{\delta<\lambda_x(s)<1/\delta} \psi(\lambda_x(s))\,ds \ge \frac{c}{2}$$

and therefore

$$\psi\left(\frac{1}{\delta}\right)\lambda\{s \in [0,\infty) : \delta < \lambda_x(s) < 1/\delta\} \ge \frac{c}{2}.$$

Consequently,

$$x^*(\delta) \ge \frac{c}{2\psi(1/\delta)}.$$

This implies that

$$\|x\|_{L^1+L^\infty} = \int_0^1 x^*(t)\,dt \ge \int_0^\delta x^*(t)\,dt \ge \delta x^*(\delta) \ge \frac{c\delta}{2\psi(1/\delta)}.$$

This means that the inclusion operator $\Lambda(\phi) \hookrightarrow L^1 + L^\infty$ is not DSS. Now, by Theorem 4.5, $\Lambda(\phi) = L^{p,\infty}$ or $\Lambda(\phi) = L_0^{p,\infty}$ for some $1 < p < \infty$, but this is a contradiction. ∎

REMARK. In general the submultiplicativity of ϕ in Theorem 4.8 cannot be removed as the following example shows:

Let $1 \le r < p < q < s < \infty$. If $\phi(t) = \min(t^{1/p}, t^{1/q})$ and $\psi(t) = \min(t^{1/r}, t^{1/s})$ then $\lim_{t\to 0,\infty} \psi(t)/\phi(t) = 0$ and the inclusion operator $\Lambda(\phi) \hookrightarrow \Lambda(\psi)$ is not DSS. Indeed, $t^{-1/l}\chi_{(0,\infty)} \in \Lambda(\phi)$ for $p < l < q$, so the inclusion $\Lambda(\phi) \hookrightarrow L^1 + L^\infty$, and hence the inclusion $\Lambda(\phi) \hookrightarrow \Lambda(\psi)$, is not DSS.

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