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On some properties of generalized Marcinkiewicz spaces

by

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Abstract. We give a full solution of the following problems concerning the spaces $M_\varphi(\vec{X})$: (i) to what extent two functions φ and ψ should be different in order to ensure that $M_\varphi(\vec{X}) \neq M_\psi(\vec{X})$ for any nontrivial Banach couple $\vec{X}$; (ii) when an embedding $M_\varphi(\vec{X}) \subsetneq M_\psi(\vec{X})$ can (or cannot) be dense; (iii) which Banach space can be regarded as an $M_\varphi(\vec{X})$-space for some (unknown beforehand) Banach couple $\vec{X}$.

Introduction. The generalized Marcinkiewicz spaces $M_\varphi(\vec{X})$ (sometimes denoted also by $\vec{X}_{\varphi,\infty}$) apparently are the simplest intermediate (and interpolation) spaces for every Banach couple $\vec{X} = (X_0, X_1)$. The only condition for $x \in \Sigma(\vec{X}) = X_0 + X_1$ to belong to the unit ball of M_φ is the inequality $K(t, x, \vec{X}) = \|x\|_{X_0+tX_1} \le \varphi(t)$ for all $t > 0$. Such simplicity makes these spaces a good "touchstone" for various hypotheses on global properties of intermediate (or interpolation) spaces for abstract Banach couples. Very often they turn out to be extreme spaces having a required property and can be used for description of other spaces.

The spaces $M_\varphi(\vec{X})$ appeared for the first time in [D] just in such a role. Together with their dual spaces $\Lambda_\varphi(\vec{X})$ they were used for description of spaces which are interpolation with respect to one-dimensional operators. Further they were systematically studied in [P1], where their extreme position was stated for many other problems. It was also proved there that any real interpolation space can be represented as the sum of some collection of $M_\varphi(\vec{X})$-spaces. At the same time the paper [DKO] associated these spaces with certain extreme interpolation functors. Notice that, for some special couples, these spaces were known and intensively studied previously,

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e.g. as usual (rearrangement invariant) Marcinkiewicz spaces for the couple (L_1, L_∞) and as generalized Hölder spaces for the couple (C, C^1).

A full bibliography concerning generalized Marcinkiewicz spaces is likely to be very long. They were studied in [M] as A_φ-spaces, in [Mi] as extrapolation ones. Even some monographs (e.g., [O], [BK]) devoted significant parts to them. These investigations concentrated on the interpolation properties of these spaces. At the same time their inner structure has been studied rather insufficiently. The main problem here is to compare the spaces $M_\varphi(\vec{X})$ and $M_\psi(\vec{X})$ for different functions φ and ψ. A partial solution of this problem was given in [P2] and [P4]. It turned out that it is closely connected with other difficult problems concerning K-functional properties, such as K-divisibility and K-abundance of a given Banach couple. For example, the usual Marcinkiewicz spaces M_φ as well as the Hölder spaces H_φ are different for any two nonequivalent parameter functions.

Another problem concerns embeddings of such spaces. In the above mentioned particular cases every embedding $M_\varphi \subsetneq M_\psi \neq L_1$ (on $[0,1]$) and $H_\varphi \subsetneq H_\psi \neq C$ is known to be nondense. For the general case, an analogous property was established in [P2] under the condition that the couple $\vec{X}$ is K-abundant. A question concerning the embedding $\Delta(\vec{X}) = X_0 \cap X_1 \subsetneq M_\varphi(\vec{X})$ was considered in [CM], and nondensity of this embedding was proved for any nontrivial Banach couple. Some ideas from [CM] will be used in the present paper.

The last problem we mention is to characterize those Banach spaces which could be regarded as generalized Marcinkiewicz spaces for some Banach couple. Theorem 4.6.28 on p. 663 of [BK] claims to give necessary conditions, but the proof given there is incorrect. In fact, this proof gives some necessary conditions for a Banach space X to be the closure of $\Delta(\vec{X})$ in some generalized Marcinkiewicz space $M_\varphi(\vec{X})$ (the usual notation for this closure is $M^\circ_\varphi(\vec{X})$). As mentioned in the previous paragraph, such a closure never equals the space $M_\varphi(\vec{X})$ itself. However, some ideas and partial assertions from the above mentioned proof can (and will) be used for a corrected proof below.

In the present paper we give full answers to all the above mentioned problems. As a main tool, we use a special characteristic of intermediate spaces, which will be named K-*envelope* here (in fact, this characteristic in close forms was used formerly by some authors, see Remark in Section 1 below). In short, it is the exact upper bound of K-functionals on the unit ball of the space considered. Various properties of K-envelopes will be stated in the next section. In particular, we prove that for any space with K-monotone norm (all $M_\varphi(\vec{X})$-spaces are of this kind), its K-envelope can be calculated using elements from $\Delta(\vec{X})$ only.

Section 2 is devoted to the comparison of spaces M_φ with nonequivalent parameter functions. The core of all proofs is Lemma 2 which states that the K-envelope of $M_\varphi(\vec{X})$ always coincides with φ on some sequence $\{t_n\}$ which tends to 0 or to ∞. This enables us to prove that $M_\varphi(\vec{X}) \neq M_\psi(\vec{X})$ *for any nontrivial Banach couple* $\vec{X}$ *if and only if* $\lim \varphi(t)/\psi(t) = 0$ (*or* ∞) *as* $t \to 0$ *in the case when* $X_0^\circ \not\subset X_1^\circ$ *or as* $t \to \infty$ *in the case when* $X_1^\circ \not\subset X_0^\circ$. Notice that otherwise we have some monotone sequence $t_n \to 0$ (or $t_n \to \infty$) on which the functions φ and ψ are equivalent (i.e. $A\varphi(t_n) \le \psi(t_n) \le B\varphi(t_n)$ for some $A, B > 0$ and all $n = 1, 2, \ldots$). In that case, we show that $M_\varphi(\vec{X}) = M_\psi(\vec{X})$ for some couple $\vec{X}$ which is constructed explicitly.

In the last section we solve the remaining problems. We give a full characterization of Banach spaces which can be regarded as generalized Marcinkiewicz spaces. Namely, *let* $\varphi(t)$, $0 < t < \infty$, *be a quasiconcave function which is not equivalent to any of the functions* $1, t, \min(1,t), \max(1,t)$. *Then a Banach space* X *is equal to* $M_\varphi(\vec{X})$ *for some nontrivial Banach couple* $\vec{X}$ *if and only if* X *contains a subspace isomorphic to* l_∞. The excluded functions are not problematic either, because they correspond to the extreme spaces $X_0, X_1, \Delta(\vec{X})$ and $\Sigma(\vec{X})$ respectively, and each X may be taken as one of such spaces in some couple $\vec{X} = (X_0, X_1)$ without any condition.

The same technique is used for solving the problem on embedding density. We prove the following statement: *let* $M_\psi(\vec{X}) \subsetneq M_\varphi(\vec{X})$ *and suppose that* $M_\varphi(\vec{X})$ *is not larger than either of the spaces* X_0, X_1. *Then* $M_\psi(\vec{X})$ *is not dense in* $M_\varphi(\vec{X})$. A counterexample shows that the condition $X_i \not\subset M_\varphi(\vec{X})$ is essential when $\Delta(\vec{X})$ is dense in the corresponding X_i ($i = 0, 1$), but otherwise it may be omitted. All such possibilities are described in the paper.

As a consequence of our proofs, we obtain a new result relating the problem of K-abundance of Banach couples. Recall that a Banach couple $\vec{X}$ is termed *K-abundant* if, for any given quasiconcave function φ, there exists $f \in \Sigma(\vec{X})$ such that $K(t, f, \vec{X})$ is equivalent to $\varphi(t)$. As shown in [BK], for K-abundance of a couple $\vec{X}$, it is sufficient that such an f exists for the single function $\varphi(t) = \sqrt{t}$. A large variety of nonabundant couples is given in [P4]. The new assertion which we have got in the present paper is: *for any nontrivial Banach couple* $\vec{X}$ *and any quasiconcave function* $\varphi(t)$ (*not equivalent to* $\max(1,t)$), *there exists an element* $f \in \Sigma(\vec{X})$ *such that* $K(t_n, f, \vec{X})$ *is equivalent to* $\varphi(t_n)$ *for some sequence* $\{t_n\}$ *which tends to* 0 *or to* ∞.

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1. K-envelope of an intermediate Banach space. Let us recall some basic notions of interpolation theory. Two Banach spaces X_0, X_1 are said to form a *Banach couple* $\vec{X}$ if they are continuously embedded into some Hausdorff topological vector space. Their intersection $\Delta(\vec{X}) = X_0 \cap X_1$ and sum $\Sigma(\vec{X}) = X_0 + X_1$ are also Banach spaces endowed with the norms

$$\|f\|_{\Delta(\vec{X})} = \max(\|f\|_{X_0}, \|f\|_{X_1}), \quad \|f\|_{\Sigma(\vec{X})} = \inf_{f=f_0+f_1}(\|f_0\|_{X_0} + \|f_1\|_{X_1}).$$

A space X is called *intermediate* in the couple $\vec{X}$ if $\Delta(\vec{X}) \subset X \subset \Sigma(\vec{X})$; notice that any embedding of intermediate spaces into each other is automatically continuous (see, e.g., [KPS], p. 13). A Banach couple $\vec{X}$ is called *ordered* if $X_0 \subset X_1$; this implies that $\Delta(\vec{X}) = X_0$, $\Sigma(\vec{X}) = X_1$ (here and below any equality $X = Y$ means that the spaces have the same elements and equivalent norms). The Banach couple $\vec{X}^T = (X_1, X_0)$ is called *transposed* with respect to $\vec{X} = (X_0, X_1)$; both couples have the same collection of intermediate spaces.

We use the following two operations defined on all intermediate spaces. By X° we denote the closure of $\Delta(\vec{X})$ in X (which may coincide with X or be a closed subset of X). By X^c we denote the *Gagliardo completion* of X; its unit ball is defined as the closure of the unit ball of X in the space $\Sigma(\vec{X})$. Obviously $\Delta^\circ(\vec{X}) = \Delta(\vec{X})$ and $\Sigma^c(\vec{X}) = \Sigma(\vec{X})$. If $X_i^\circ = X_i$, $i = 0, 1$, the couple is called *regular*.

An important role in this paper will be played by Peetre's K-*functional*

$$K(t, f, \vec{X}) = \|f\|_{X_0+tX_1} = \inf_{f=f_0+f_1}(\|f_0\|_{X_0} + t\|f_1\|_{X_1}), \quad t > 0,\ f \in \Sigma(\vec{X}).$$

For any fixed t it is equivalent to the norm on $\Sigma(\vec{X})$. An intermediate space X is called a K-*space* if it has K-monotone norm, i.e. the conditions $f \in \Sigma(\vec{X})$, $g \in X$, $K(t, f, \vec{X}) \le K(t, g, \vec{X})$ imply that $f \in X$, $\|f\|_X \le \|g\|_X$. A fundamental study of K-spaces can be found in [BK]; there are many Banach couples where all interpolation spaces are only of that type.

Definition. The K-*envelope* of an intermediate space X is the function

$$\mu(t, X, \vec{X}) = \sup_{\|f\|_X \le 1} K(t, f, \vec{X}), \quad t > 0.$$

Remark. The function $\mu(t, X, \vec{X})$ is closely connected to the fundamental co-function $\psi_X(t, \vec{X})$, defined in [P1] for an arbitrary Banach couple (for regular couples it was already considered in [D] without the use of the K-functional). In a later version of definition, given in [P3], the functions are exactly the same, but the term "K-envelope" seems now to be more suitable. Some other related functions were considered in [DKO], [M] and [MM]; in the recent paper [CCM], the function $\mu(t, X, \vec{X})$ plays an important role in interpolation of operator ideals. Note also the embedding functions from [P5] which generalize all such characteristics, based on K and J-functionals.

Let us indicate some general properties of K-envelopes, mostly established in the papers mentioned in the last remark. If $X \subset Y$ with embedding constant 1 then $\mu(t, X, \vec{X}) \le \mu(t, Y, \vec{X})$ and thus

$$\begin{aligned} &\mu(t, X \cap Y, \vec{X}) \le \min(\mu(t, X, \vec{X}), \mu(t, Y, \vec{X})), \\ &\mu(t, X + Y, \vec{X}) \ge \max(\mu(t, X, \vec{X}), \mu(t, Y, \vec{X})). \end{aligned} \tag{1}$$

Since $K(t, f, \vec{X}^T) = tK(1/t, f, \vec{X})$, an analogous relation holds for the K-envelopes:

$$\mu(t, X, \vec{X}^T) = t\mu(1/t, X, \vec{X}).$$

It allows us (if necessary) to study the behaviour of K-envelopes only for $t \to 0$ or only for $t \to \infty$. If $f \in X_0^\circ + X_1^\circ$ then $K(t, f, \vec{X}) = K(t, f, \vec{X}^\circ)$, and hence for spaces in which $\Delta(\vec{X})$ is dense, the K-envelope does not change if we replace $\vec{X}$ by $\vec{X}^\circ$. So, for such spaces, we may regard the couple $\vec{X}$ as regular from the beginning.

As shown in [KPS], p. 12,

$$X_0 \not\subset X_1 \ \Rightarrow \sup_{\|f\|_{X_0} \le 1} \|f\|_{\Sigma(\vec{X})} = 1.$$

In the case when neither of X_0, X_1 is embedded in the other, this implies immediately that $\mu(t, X_0, \vec{X}) = 1$, $\mu(t, X_1, \vec{X}) = t$ for all $t > 0$. By (1) then $\mu(t, \Delta(\vec{X}), \vec{X}) \le \min(1, t)$ and $\mu(t, \Sigma(\vec{X}), \vec{X}) \ge \max(1, t)$. We can obtain even more if we take into account that any K-functional is concave as a function of t, and thus any K-envelope is quasiconcave, i.e. $\mu(t)\uparrow$, $\mu(t)/t\downarrow$. This implies that $\mu(1)\min(1, t) \le \mu(t) \le \mu(1)\max(1, t)$. Consequently, $\mu(t, \Delta(\vec{X}), \vec{X}) \sim \min(1, t)$ ($\sim$ means equivalence), while $\mu(t, \Sigma(\vec{X}), \vec{X}) = \max(1, t)$ because $\mu(1, \Sigma(\vec{X}), \vec{X}) = 1$ for any couple.

If $X_0 \subset X_1$ then $\mu(t, X, \vec{X}) \sim t$ on the interval $(0, 1)$ for any intermediate space X. If $X_1 \subset X_0$ then always $\mu(t, X, \vec{X}) \sim 1$ on $(1, \infty)$. Such behaviour of K-envelopes allows us not to consider the corresponding intervals in the case of ordered couples. A similar behaviour of K-envelopes can also be observed in nonordered couples, if the space X considered is larger or smaller than one of the spaces X_0, X_1. Indeed, $X \supset X_0$ implies that $\mu(t, X, \vec{X}) \sim 1$ on $(0, 1)$ and $X \subset X_0$ implies the same on $(1, \infty)$. Similarly $\mu(t, X, \vec{X}) \sim t$ on $(0, 1)$ if $X \subset X_1$ and on $(1, \infty)$ if $X \supset X_1$.

If $f_n \to f$ in $\Sigma(\vec{X})$ then $K(t, f_n, \vec{X}) \to K(t, f, \vec{X})$ for any fixed $t > 0$. Hence $\mu(t, X^c, \vec{X}) = \mu(t, X, \vec{X})$ for any intermediate space X. A problem arises to compare analogously the K-envelopes $\mu(t, X, \vec{X})$ and $\mu(t, X^\circ, \vec{X})$. It turns out that they may be inequivalent even when X is one of the basic spaces X_0, X_1 or $\Sigma(\vec{X})$ (at least this can happen on one of the intervals $(0,1)$ or $(1,\infty)$). Such a situation occurs for Banach couples with a particular structure, which we now describe in order to be able to exclude these couples from further considerations.

Let us say that the space X_0 is an *improper component* of the Banach couple $\vec{X}$ if $X_0^\circ \subset X_1$ but $X_0 \not\subset X_1$. In this case $\Delta(\vec{X}) = X_0^\circ \neq X_0$, so $\Delta(\vec{X})$ is a closed subspace of X_0. Similarly we define X_1 to be an improper component if $X_1^\circ \subset X_0$ but $X_1 \not\subset X_0$. In this case $\Delta(\vec{X}) = X_1^\circ$ and it is a closed subspace of X_1. If both situations occur simultaneously, namely, $\Delta(\vec{X})$ is closed both in X_0 and in X_1, the couple $\vec{X}$ is called *trivial*. All these "pathological" cases were studied in the fundamental paper [AG]. In particular, it was proved there that any trivial Banach couple has only 4 interpolation spaces: $X_0, X_1, \Delta(\vec{X})$ and $\Sigma(\vec{X})$.

Summing up the above, $\mu(t, X_i^\circ, \vec{X})$ is not equivalent to $\mu(t, X_i, \vec{X})$ (for the same $i = 0,1$) if and only if X_i is an improper component of the couple $\vec{X}$. Clearly in such cases $\mu(t, \Sigma^\circ(\vec{X}), \vec{X})$ is not equivalent to $\mu(t, \Sigma(\vec{X}), \vec{X})$. The remaining cases can be divided into three categories: (i) $X_0 \subset X_1$; (ii) $X_1 \subset X_0$; (iii) $\Delta(\vec{X})$ is closed neither in X_0 nor in X_1. For each of these categories the K-envelopes of X_0°, X_1°, $\Sigma^\circ(\vec{X})$ are equivalent to the K-envelopes of X_0, X_1, $\Sigma(\vec{X})$ respectively.

In order to state the equivalence of $\mu(t, X, \vec{X})$ and $\mu(t, X^\circ, \vec{X})$ for every intermediate space X we need the following lemma (which is also important for many other results of this paper). For simplicity, we do not mention henceforth the couple $\vec{X}$ in the notation of K-functionals, when it does not lead to ambiguity.

Lemma 1. *Let $f \in \Sigma(\vec{X})$, $t_0 > 0$ and suppose that for some $\varepsilon \in (0,1)$ there exist $a \in (0, t_0)$ and $b \in (t_0, \infty)$ such that*

$$K(a,f) = \frac{1-\varepsilon}{2\gamma} K(t_0, f), \qquad K(b,f) = \frac{1-\varepsilon}{2\gamma} K(t_0,f)\frac{b}{t_0}, \tag{2}$$

where γ is the constant of K-divisibility for the Banach couple $\vec{X}$ (see [BK], p. 325). *Then there exists $g \in \Delta^c(\vec{X})$ such that*

$$K(t,g) \le K(t,f) \quad \textit{for all } t > 0, \qquad K(t_0, g) \ge \frac{\varepsilon}{\gamma} K(t_0, f).$$

Proof. Consider three concave functions:

$$\varphi_1(t) = \min(K(a,f), K(t,f)), \qquad \varphi_2(t) = \min(tK(b,f)/b, K(t,f)),$$
$$\varphi_3(t) = \min(tK(a,f)/a, K(t,f), K(b,f)).$$

We have $K(t,f) \le \varphi_1(t) + \varphi_2(t) + \varphi_3(t)$ for all $t > 0$, thus by K-divisibility, there exist $f_1, f_2, f_3 \in \Sigma(\vec{X})$ such that $f = f_1 + f_2 + f_3$ and $K(t, f_i) \le$

$\gamma\varphi_i(t)$, $i = 1,2,3$, for all $t > 0$. Moreover, $K(t, f_3) \sim \min(1,t)$, and thus $f_3 \in \Delta^c(\vec{X})$. On the other hand, $K(t_0, f)$ is a norm on $\Sigma(\vec{X})$, hence

$$\begin{aligned} K(t_0, f_3) &\geq K(t_0, f) - K(t_0, f_1) - K(t_0, f_2) \\ &\geq K(t_0, f) - \gamma\varphi_1(t_0) - \gamma\varphi_2(t_0) \\ &= K(t_0, f) - \gamma K(a, f) - \gamma t_0 \frac{K(b, f)}{b}. \end{aligned}$$

Owing to conditions (2) we obtain $K(t_0, f_3) \geq \varepsilon K(t_0, f)$ and the proof is completed by setting $g = (1/\gamma) f_3$. ∎

Remark. In what follows we also use some other properties of the element g constructed in this lemma:

(3) $$K(t, g) \leq K(b, f) \quad \text{for } t \geq b, \quad K(t, g) \leq \frac{tK(a, f)}{a} \quad \text{for } t \leq a.$$

An element g having all the above mentioned properties will be called an *ε-contraction* of f to the point t_0.

Now we are able to compare the K-envelopes $\mu(t, X, \vec{X})$ and $\mu(t, X^\circ, \vec{X})$.

Theorem 1. *Suppose an intermediate space X has K-monotone norm. Then $\mu(t, X, \vec{X}) \sim \mu(t, X^\circ, \vec{X})$ for any Banach couple $\vec{X}$ which does not have improper components. The same is true if X_0 is an improper component of $\vec{X}$ but $\lim_{t\to 0} \mu(t, X, \vec{X}) = 0$ and/or if X_1 is an improper component but $\lim_{t\to\infty} \mu(t, X, \vec{X})/t = 0$.*

Proof. The definition of K-envelopes implies that for any $t_0 > 0$, there exists an element f from the unit ball of X such that $K(t, f) \leq \mu(t, X, \vec{X})$ and $K(t_0, f) \geq \frac{1}{2}\mu(t_0, X, \vec{X})$. If at the same time

(4) $$\min(1, 1/t) K(t, f) \to 0 \quad \text{as } t \to 0 \text{ and as } t \to \infty,$$

then for any $\varepsilon \in (0,1)$ we can find a, b satisfying (2). Therefore there exists a $\frac{1}{2}$-contraction of f to the point t_0, which will be denoted by g. Recall that $g \in \Delta^c(\vec{X}) \subset (X^\circ)^c$. On the other hand $K(t, g) \leq K(t, f)$ and thus $\|g\|_X \leq \|f\|_X \leq 1$, and we see that g belongs to the unit ball of $(X^\circ)^c$. As a result,

$$\mu(t_0, X^\circ, \vec{X}) = \mu(t_0, (X^\circ)^c, \vec{X}) \geq K(t_0, g) \geq \frac{1}{2\gamma} K(t_0, f) \geq \frac{1}{4\gamma} \mu(t_0, X, \vec{X}).$$

Since the reverse inequality $\mu(t_0, X^\circ, \vec{X}) \leq \mu(t_0, X, \vec{X})$ is obvious, this proves the equivalence of the K-envelopes at all points $t_0 > 0$ where the corresponding function f satisfies the conditions (4).

Suppose now that for some $f \in X$ one of the conditions (4) is not fulfilled, e.g. $\lim_{t\to 0} K(t, f) > 0$ (recall that this is possible only when X_0 is not an improper component of $\vec{X}$). From the norm monotonicity in X, it follows that X contains all $g \in \Sigma(\vec{X})$ for which $K(t,g) \sim$ const, i.e. $X \supset X_0$. In this case $\mu(t,X,\vec{X}) \sim 1$ on $(0,1)$. At the same time $X^\circ \supset X_0^\circ$ and X_0 is not improper, thus also $\mu(t,X^\circ,\vec{X}) \sim 1$ on $(0,1)$ and both K-envelopes are equivalent on this interval. Analogously the inequality $\lim_{t\to\infty} K(t,f)/t > 0$ implies that $X \supset X_1$ and $\mu(t,X^\circ,\vec{X}) \sim \mu(t,X,\vec{X}) \sim t$ on $(1,\infty)$. In particular, $\mu(t,\Sigma^\circ(\vec{X}),\vec{X}) \sim \mu(t,\Sigma(\vec{X}),\vec{X})$ on the whole $(0,\infty)$.

Assume now that $X \supset X_0$ but $X \not\supset X_1$. It remains to compare our K-envelopes on $(1,\infty)$, where $\mu(t,X,\vec{X})$ is not equivalent to t. In the case $\lim_{t\to\infty} \mu(t,X,\vec{X}) < \infty$ we have $X = X_0$ and thus $\mu(t,X,\vec{X}) \sim \mu(t,X^\circ,\vec{X})$, since X_0 is not an improper component of $\vec{X}$. If $\lim_{t\to\infty} \mu(t,X,\vec{X}) = \infty$, then we can find $a,\ b$ satisfying (2) for any sufficiently large t_0 and a suitable f. Repeating the first part of the proof, we obtain $\mu(t,X,\vec{X}) \sim \mu(t,X^\circ,\vec{X})$ just on the required interval $(1,\infty)$. The case $X \supset X_1,\ X \not\supset X_0$ can be investigated analogously. ■

The following example shows that the condition on the norm in X to be K-monotone is essential not only for the proof. Take $f \in \Sigma(\vec{X})$ such that $K(t,f)$ is not equivalent to $\min(1,t)$, i.e. $f \notin \Delta^c(\vec{X})$. Set $X = \Delta(\vec{X}) \oplus f$. Then $\mu(t,X^\circ,\vec{X}) = \mu(t,\Delta(\vec{X}),\vec{X}) \sim \min(1,t)$. At the same time $\mu(t,X,\vec{X}) \sim K(t,f)$.

2. Comparison of generalized Marcinkiewicz spaces. The generalized Marcinkiewicz space $M_\varphi = M_\varphi(\vec{X})$ can be defined for any function $\varphi : (0,\infty) \to (0,\infty)$ as the collection of all $f \in \Sigma(\vec{X})$ such that

$$\|f\|_{M_\varphi} = \sup_{0<t<\infty} K(t,f)/\varphi(t) < \infty.$$

It turns out that all such spaces can be obtained if we take only quasiconcave functions φ. It is easy to check that always

$$\|f\|_{X_0^c} = \sup_{t>0} K(t,f), \qquad \|f\|_{X_1^c} = \sup_{t>0} K(t,f)/t,$$

and thus $M_{\varphi_0} = X_0^c$ and $M_{\varphi_1} = X_1^c$ for $\varphi_0(t) = 1$, $\varphi_1(t) = t$. Just as easily it can be shown that $M_{\min(1,t)} = \Delta^c(\vec{X})$ and $M_{\max(1,t)} = \Sigma(\vec{X})$. The generalized Marcinkiewicz spaces have an important *reiteration* property:

$$\varphi = \varphi_0\theta(\varphi_1/\varphi_0) \ \Rightarrow\ M_\varphi = M_\theta(M_{\varphi_0}, M_{\varphi_1})$$

(see e.g. [O], p. 428), which gives, in particular,

$$M_{\varphi_0} \cap M_{\varphi_1} = M_{\min(\varphi_0,\varphi_1)}, \qquad M_{\varphi_0} + M_{\varphi_1} = M_{\max(\varphi_0,\varphi_1)}.$$

The spaces M_φ have K-monotone norms, hence they are interpolation in their Banach couples. If the couple is trivial (i.e. $\Delta(\vec{X})$ is closed in $\Sigma(\vec{X})$), it has only 4 interpolation spaces: $X_0, X_1, \Delta(\vec{X}), \Sigma(\vec{X})$, thus for any parameter function φ the corresponding space M_φ should coincide with one of these spaces. And in fact, for any trivial couple $\vec{X}$, we have the relation

$$M_\varphi(\vec{X}) = \Delta(\vec{X}) + \lim_{t\to 0} \varphi(t)\cdot X_0 + \lim_{t\to\infty} \frac{\varphi(t)}{t}\cdot X_1,$$

which easily solves all problems of comparison of the spaces M_φ for trivial Banach couples. This enables us to consider only nontrivial couples in what follows.

As an important example, we find the form of M_φ-spaces for the Banach couple of sequence spaces $X_0 = l_\infty^a$, $X_1 = l_\infty^b$ (as usual, the norm of a sequence $f = (f(1), f(2), \ldots)$ in the weight space l_∞^a is defined as $\|fa\|_{l_\infty} = \sup_n |f(n)|a(n)$). It can be easily shown that $K(t, f, l_\infty^a, l_\infty^b) \sim \|f\min(a, tb)\|_{l_\infty}$, hence

$$\begin{aligned}\|f\|_{M_\varphi} &\sim \sup_t \left\| f\frac{\min(a,tb)}{\varphi(t)}\right\|_{l_\infty} = \sup_n |f(n)| \sup_t \min\left(\frac{a(n)}{\varphi(t)},\ \frac{tb(n)}{\varphi(t)}\right)\\ &= \sup_n |f(n)|\ \Phi(a(n), b(n)), \quad \text{where} \quad \Phi(a,b) = \frac{a}{\varphi(a/b)}.\end{aligned}$$

Thus we obtain

$$M_\varphi(l_\infty^a,\ l_\infty^b) = l_\infty^{\Phi(a,b)}. \tag{5}$$

The correspondence between M_φ-spaces and their parameters φ is, in general, not one-to-one; the same space can be generated by different and even inequivalent functions φ. As shown in [P4], for couples of weight sequence spaces, two spaces M_φ and M_ψ are different for any pair of inequivalent functions φ, ψ if and only if the couple is K-abundant. Ibidem there are also some criteria for such couples to be non-K-abundant; now we shall use them in order to illustrate the possibility of an equality $M_\varphi(\vec{X}) = M_\psi(\vec{X})$ for inequivalent parameter functions $\varphi(t)$ and $\psi(t)$.

Consider again the couple $\vec{X} = (l_\infty^a, l_\infty^b)$ with all $b(n) = 1$ and $a(n) = a_n$ such that $a_n\uparrow$ and $\lim_{n\to\infty} a_{n+1}/a_n = \infty$. Let $\varphi(t) = \sqrt{t}$ and $\psi(t) = \sup_n \sqrt{a_n}\min(1, t/a_n)$. From (5) we obtain

$$M_\varphi = l_\infty^{\Phi(a,1)} = l_\infty^{\sqrt{a}}, \qquad M_\psi = l_\infty^{\Psi(a,1)},$$

where

$$\Psi(t,1) = \frac{t}{\psi(t)} = \left(\sup_n \sqrt{a_n}\min\left(\frac{1}{t},\ \frac{1}{a_n}\right)\right)^{-1}.$$

Taking $t = a_k$, we obtain

$$\Psi(a_k, 1) = \inf_n \frac{1}{\sqrt{a_n}}\max(a_k, a_n) = \sqrt{a_k},$$

i.e. $M_\psi = l_\infty^{\sqrt{a}}$ just like M_φ.

It remains to show that φ and ψ are not equivalent. Take $t_k = \sqrt{a_k a_{k+1}}$; then $t_k \to \infty$ as $k \to \infty$. Furthermore

$$\frac{\psi(t_k)}{\varphi(t_k)} = \frac{1}{\sqrt[4]{a_k a_{k+1}}} \sup_n \sqrt{a_n} \min\left(1, \frac{\sqrt{a_k a_{k+1}}}{a_n}\right)$$
$$= \max\left(\sup_{n\le k} \frac{\sqrt{a_n}}{\sqrt[4]{a_k a_{k+1}}}, \sup_{n\ge k+1} \frac{\sqrt[4]{a_k a_{k+1}}}{\sqrt{a_n}}\right) = \sqrt[4]{\frac{a_k}{a_{k+1}}} \to 0$$

as $k \to \infty$, which proves inequivalence of φ and ψ.

If $\|f\|_{M_\varphi} \le 1$ then $K(t,f) \le \varphi(t)$ for all $t > 0$, hence the corresponding K-envelope $\omega(t) = \mu(t, M_\varphi, \vec{X}) \le \varphi(t)$. Moreover, $M_\varphi = M_\omega$ and ω is the minimal possible parameter function defining the same generalized Marcinkiewicz space (up to equivalence of norms). For this reason the K-envelope will be called the *optimal parameter* of this space. Using optimal parameters, we immediately obtain a one-to-one correspondence between parameters and spaces, namely, $M_{\omega_1} = M_{\omega_2}$ if and only if $\omega_1 \sim \omega_2$. Therefore in order to find to what extent two parameter functions of the same generalized Marcinkewicz space could be different, it is sufficient to study relations between the usual and optimal parameters.

LEMMA 2. *Let ω be the K-envelope of a space $M_\varphi(\vec{X})$ with parameter function φ not equivalent to* $\min(1,t)$ *on either of the intervals* $(0,1), (1,\infty)$. *If $X_1^\circ \not\subset X_0^\circ$ and* $\lim_{t\to\infty} \varphi(t)/t = 0$ *then there exists a sequence $t_n \to \infty$ such that*

$$\omega(t_n) = \varphi(t_n) \quad \textit{for all } n = 1, 2, \ldots$$

If $X_0^\circ \not\subset X_1^\circ$ and $\lim_{t\to 0} \varphi(t) = 0$ *then there exists a sequence $t_n \to 0$ with the same property.*

Proof. Notice, first of all, that $M_\varphi(\vec{X}) = M_{\varphi^*}(\vec{X}^T)$ for $\varphi^*(t) = t\varphi(1/t)$, hence it suffices to prove the first assertion. Without loss of generality we may assume that $\varphi(1) = 1$ and consequently $\varphi(t) \ge t$ for $t \le 1$. Defining $\psi(t) = \varphi(t)$ for $t \ge 1$, $\psi(t) = t$ for $t \le 1$, we obtain $M_\psi(\vec{X}) \subset M_\varphi(\vec{X})$ and $\mu(t, M_\psi, \vec{X}) \le \mu(t, M_\varphi, \vec{X})$. If we could find a sequence $t_n \ge 1$ with $t_n \to \infty$ such that $\mu(t_n, M_\psi, \vec{X}) = \psi(t_n)$, we automatically obtain $\mu(t_n, M_\varphi, \vec{X}) = \varphi(t_n)$ because the inequality $\mu(t_n, M_\varphi, \vec{X}) > \varphi(t_n)$ is impossible. So, we may consider from the beginning only functions $\varphi(t)$ which are equal to t for $t \le 1$ and thus $\varphi(t) \le t$ for all $t > 0$. We also have $\lim_{t\to\infty} \varphi(t) = \infty$, since otherwise $\varphi(t) \sim \min(1,t)$.

We now compare the spaces M_φ and X_1. The condition $\varphi(t) \le t$ implies that $M_\varphi \subset X_1$, and $X_1^\circ \not\subset X_0^\circ$ implies that $\mu(t, X_1, \vec{X}) = t$. At the same time the functions $\varphi(t)$ and t are inequivalent and $\omega(t) \le \varphi(t)$, hence the functions $\omega(t)$ and t are also inequivalent. But the last two functions are the optimal parameters of their spaces, thus these spaces cannot be equal.

All these arguments can be applied to the couple (X_0°, X_1°), which implies that also $M_\varphi(\vec{X}^\circ) \neq X_1^\circ$. In consequence, the norms of M_φ and X_1 are not equivalent even on elements from $\Delta(\vec{X})$.

For every $a \geq 1$, define now the space $Y_a \supset M_\varphi$ with the norm

$$\|f\|_{Y_a} = \sup_{t\leq a} K(t,f)/\varphi(t).$$

The inequality $\varphi(t) \leq t$ implies that

$$\|f\|_{Y_a} \geq \sup_{t\leq a} K(t,f)/t = \lim_{t\to 0} K(t,f)/t = \|f\|_{X_1}.$$

On the other hand, $t/\varphi(t) \leq a/\varphi(a)$ for $t \leq a$, thus

$$\|f\|_{Y_a} \leq \frac{a}{\varphi(a)} \|f\|_{X_1},$$

which shows that $Y_a = X_1$. Therefore the norms $\|f\|_{Y_a}$ and $\|f\|_{M_\varphi}$ are not equivalent even on $\Delta(\vec{X})$, i.e. there exists $f_a \in \Delta(\vec{X})$ having norm 1 in M_φ and an arbitrarily small norm in Y_a. This means that

$$1 = \|f_a\|_{M_\varphi} = \sup_t K(t,f_a)/\varphi(t) = \sup_{t\geq a} K(t,f_a)/\varphi(t).$$

At the same time $f_a \in \Delta(\vec{X})$ implies that $K(t,f_a) \sim \min(1,t)$, therefore

$$\lim_{t\to\infty} K(t,f_a)/\varphi(t) = 0 \ \Rightarrow\ \sup_{t\geq a} K(t,f_a)/\varphi(t) = \sup_{a\leq t\leq b} K(t,f_a)/\varphi(t)$$

for some $b > a$. From the continuity of all functions considered, we immediately see that the supremum is attained at some point $t_a \in [a,b]$.

Now we start with $a = 1$ and denote the corresponding t_a by t_1. The next value of a should be taken greater than the b obtained for the previous a. We denote the new value of t_a by t_2. Continuing, we obtain a sequence $f_n \in \Delta(\vec{X})$ with $\|f_n\|_{M_\varphi} = 1$ and a sequence $t_n \to \infty$ for which $K(t_n, f_n) = \varphi(t_n)$. Thus $\mu(t_n, M_\varphi, \vec{X}) \geq \varphi(t_n)$, i.e. $\omega(t_n) = \varphi(t_n)$ for all $n = 1, 2, \ldots$ ∎

COROLLARY. *Let φ, ψ be arbitrary quasiconcave functions. If $X_1^\circ \not\subset X_0^\circ$ and $\lim_{t\to\infty} \varphi(t)/\psi(t) = 0$, then $M_\varphi(\vec{X}) \neq M_\psi(\vec{X})$. The same is true if $X_0^\circ \not\subset X_1^\circ$ and $\lim_{t\to 0} \varphi(t)/\psi(t) = 0$.*

Proof. Once again we consider only the first assertion and assume, on the contrary, that $M_\varphi = M_\psi$. For their K-envelopes, we now obtain $\mu(t, M_\varphi, \vec{X}) \sim \mu(t, M_\psi, \vec{X})$. Obviously $\psi(t)$ is not equivalent to $\min(1,t)$. Therefore if $\lim_{t\to\infty} \psi(t)/t = 0$ then, as shown in Lemma 2, there exists a sequence $t_n \to \infty$ such that $\mu(t_n, M_\psi, \vec{X}) = \psi(t_n)$ and thus $\mu(t_n, M_\varphi, \vec{X})/\psi(t_n) \not\to 0$. But $\varphi(t_n) \geq \mu(t_n, M_\varphi, \vec{X})$, hence also $\varphi(t_n)/\psi(t_n) \not\to 0$, and we obtain a contradiction. If $\lim_{t\to\infty} \psi(t)/t > 0$ then $\psi(t) \sim t$ on $(1,\infty)$, i.e. $M_\psi \supset X_1$. Being equal to M_ψ, the space M_φ should also have this property, so

$\mu(t, M_\varphi, \vec{X}) \sim t$ on the same interval $(1,\infty)$. Once again $\mu(t, M_\varphi, \vec{X})/\psi(t) \not\to 0$ as $t \to \infty$, yielding the same contradiction. ∎

Now we are able to give a full answer to the problem of comparison of generalized Marcinkiewicz spaces with inequivalent parameters.

THEOREM 2. *Let φ, ψ be quasiconcave functions. If there exists a sequence $t_n \to \infty$ (or $t_n \to 0$) as $n \to \infty$ such that $\varphi(t_n) \sim \psi(t_n)$, then there exists a Banach couple $\vec{X}$ for which $M_\varphi(\vec{X}) = M_\psi(\vec{X})$. If such a sequence exists neither on the interval $(0,1)$ for the case $X_0^\circ \not\subset X_1^\circ$ nor on the interval $(1,\infty)$ for the case $X_1^\circ \not\subset X_0^\circ$, then $M_\varphi(\vec{X}) \neq M_\psi(\vec{X})$ for any nontrivial Banach couple.*

Proof. As before, we consider only the case $t_n \to \infty$. If $\varphi(t_n) \sim \psi(t_n)$, then we set $a = (t_1, t_2, \ldots)$ and consider the couple (l_∞^a, l_∞). By (5), $M_\varphi(l_\infty^a, l_\infty) = l_\infty^c$, where $c = (c_1, c_2, \ldots)$ with $c_n = t_n/\varphi(t_n)$. Similarly $M_\psi(l_\infty^a, l_\infty) = l_\infty^d$, where $d = (d_1, d_2, \ldots)$ with $d_n = t_n/\psi(t_n)$. By assumption, $c \sim d$, thus $l_\infty^c = l_\infty^d$, which we had to show.

If $X_1^\circ \not\subset X_0^\circ$ and the required sequence $t_n \to \infty$ does not exist, then either $\lim_{t\to\infty} \varphi(t)/\psi(t)$ or $\lim_{t\to\infty} \psi(t)/\varphi(t)$ exists and equals 0. Then the inequality $M_\varphi \neq M_\psi$ immediately follows from the last Corollary. The case $X_0^\circ \not\subset X_1^\circ$ can be considered analogously. ∎

3. Characterization of generalized Marcinkiewicz spaces. Every Banach space X can be taken as a component of some Banach couple, either as X_0 or as X_1. So it can be regarded as a generalized Marcinkiewicz space M_φ with the parameter $\varphi(t) = 1$ or $\varphi(t) = t$. Moreover, taking the second space of this couple larger or smaller than X, we may attribute to X either of the functions $\min(1,t)$, $\max(1,t)$. A much more difficult problem is to find a Banach couple $\vec{X}$ for which $X = M_\varphi$ for some parameter φ different from the four extreme functions indicated above. When does such a couple exist, and which functions φ can appear? All these questions will be answered in this section.

THEOREM 3. *Let φ be a quasiconcave function which is not equivalent to any of the functions $1, t, \min(1,t), \max(1,t)$. A Banach space X is equal to the space $M_\varphi(\vec{X})$ for some nontrivial Banach couple $\vec{X}$ if and only if it contains a subspace isomorphic to l_∞.*

Proof. Sufficiency. As is known (see e.g. [LT], p. 105), any subspace isomorphic to l_∞ is complemented, thus we have a representation $X = Y \oplus Z$, where Y is isomorphic to l_∞. This means that there exists a bounded invertible linear operator T such that $Tl_\infty = Y$.

Since any equality of Banach spaces is meant as equivalence of their norms, we may assume the function φ to be concave and strictly increasing on $(0,\infty)$, i.e. having a single-valued inverse function $\varphi^{-1}(t)$ not equivalent to t, at least on one of the intervals $(0,1),\ (1,\infty)$. As usual, let it be $(1,\infty)$. In this case we can find two infinite sequences $a=\{a_n\}$ and $b=\{b_n\}$, connected by the relation $b_n=a_n/\varphi^{-1}(a_n),\ n=1,2,\ldots$, and such that $a_n\to\infty,\ b_n\to 0$ as $n\to\infty$. As a result, the Banach couple (l^a_∞, l^b_∞) is nontrivial and ordered. The formula (5) gives $M_\varphi(l^a_\infty, l^b_\infty)=l_\infty$.

Define now $Y_0=Tl^a_\infty$; this is possible, since the operator T can be applied to $l^a_\infty\subset l_\infty$. As shown in [S], p. 166, any isomorphism can be extended to larger spaces, so we may also define $Y_1=Tl^b_\infty$. Now we set $X_0=Y_0\oplus Z$, $X_1=Y_1\oplus Z$. Recall that constructing the space $M_\varphi(\vec{X})$ with a given parameter φ may be considered as an interpolation functor (see e.g. [DKO], p. 48). For any interpolation functor $\mathcal{F}$, we have $\mathcal{F}(TX_0,TX_1)=T\mathcal{F}(X_0,X_1)$ and $\mathcal{F}(Y_0\oplus Z, Y_1\oplus Z)=\mathcal{F}(Y_0,Y_1)\oplus Z$. For our functor $M_\varphi(\vec{X})$, this implies that

$$M_\varphi(X_0,X_1)=M_\varphi(Y_0,Y_1)\oplus Z=TM_\varphi(l^a_\infty,l^b_\infty)\oplus Z=Tl_\infty\oplus Z=X,$$

which proves the sufficiency part of our theorem.

Necessity. We take an arbitrary generalized Marcinkiewicz space $M_\varphi(\vec{X})$ in a nontrivial Banach couple $\vec{X}$ with a parameter φ not equivalent to the functions $1, t, \min(1,t), \max(1,t)$ and construct a subspace $Y\subset M_\varphi$ isomorphic to l_∞. Nontriviality of $\vec{X}$ implies that the K-envelope $\mu(t, M_\varphi, \vec{X})$ is also not equivalent to the indicated functions at least on one of the intervals $(0,1),\ (1,\infty)$. For definiteness, let it be $(1,\infty)$ (the second case can be derived from this by passing to the transposed couple $\vec{X}^T$). Without loss of generality, we may assume that φ is optimal, i.e. $\varphi=\mu(t,M_\varphi,\vec{X})$. Then $\varphi(t)\to\infty$ and $\varphi(t)/t\to 0$ as $t\to\infty$, and for any $t_n\geq 1$ there exists $f_n\in M_\varphi$ such that $K(t,f_n)\leq\varphi(t)$ for all $t>0$ but $K(t_n,f_n)\geq\frac{1}{2}\varphi(t_n)$.

If t_n is sufficiently large then, by our agreement about the properties of φ, there exists a $\frac{1}{2}$-contraction of f_n to the point t_n; denote it by g_n. By definition, the g_n is associated with some interval $[a_n,b_n]$, defined by the relations (2), which in our case take the form

$$K(a_n,g_n)=\frac{1}{4\gamma}K(t_n,f_n),\qquad K(b_n,g_n)=\frac{b_n}{4\gamma t_n}K(t_n,f_n).$$

Recall that also $K(t,g_n)\leq K(t,f_n)$ for all $t>0,\ K(t_n,g_n)\geq\frac{1}{2\gamma}K(t_n,f_n)$, and

$$\begin{aligned} K(t,g_n)&\leq K(b_n,f_n) &&\text{for } t\geq b_n,\\ K(t,g_n)&\leq tK(a_n,f_n)/a_n &&\text{for } t\leq a_n.\end{aligned}$$

Now choose $t_n,\ n=1,2,\ldots,$ so far apart that for all n the following inequalities are satisfied:

$$\varphi(a_{n+1}) \geq C\varphi(b_n), \qquad \frac{\varphi(a_{n+1})}{a_{n+1}} \leq \frac{\varphi(b_n)}{Cb_n}, \tag{6}$$

where one may take an arbitrary constant $C > 8\gamma+1$. The possibility of such a choice is once again ensured by the properties of $\varphi(t)$ for $t \to \infty$. Taking some fixed n, define $h_n = \sum_{k\neq n} g_k$. We show that this series converges in $\Sigma(\vec{X})$, by estimating $K(t, h_n)$ for $t \in [a_n, a_{n+1}]$. Considering separately the terms with $k < n$ and $k > n$, we obtain

$$\begin{aligned}\sum_{k<n} K(t, g_k) &\leq \sum_{k<n} K(b_k, f_k) \leq \sum_{k<n} \varphi(b_k) \\ &\leq \varphi(t) \sum_{k=1}^{\infty} \frac{1}{C^k} = \frac{1}{C-1}\varphi(t), \\ \sum_{k>n} K(t, g_k) &\leq t \sum_{k>n} \frac{K(a_k, f_k)}{a_k} \leq t \sum_{k>n} \frac{\varphi(a_k)}{a_k} \\ &\leq \varphi(t) \sum_{k=1}^{\infty} \frac{1}{C^k} = \frac{1}{C-1}\varphi(t),\end{aligned}$$

hence $h_n \in \Sigma(\vec{X})$ and

$$K(t, h_n) \leq \frac{2}{C-1}\varphi(t) \qquad \text{for all } t \in [a_n, a_{n+1}].$$

Define now the space Y as the set of all $f \in \Sigma(\vec{X})$ which can be represented in the form

$$f = \sum_{n=1}^{\infty} c_n g_n, \qquad c = \{c_n\} \in l_\infty.$$

It is easy to see that such a series converges for any bounded sequence of coefficients, considering, for example,

$$\begin{aligned}K(t_1, f) &\leq |c_1| K(t_1, g_1) + \sum_{k\neq 1} |c_k| K(t_1, g_k) \\ &\leq \|c\|_{l_\infty} \left(\varphi(t_1) + \frac{1}{C-1}\varphi(t_1) \right) < \infty.\end{aligned}$$

The formula $Tc = f$ defines a mapping of l_∞ onto Y; we show that this mapping is one-to-one, i.e. $Tc = 0$ if and only if $c = 0$.

Indeed, let $c \neq 0$ and let n be such that

$$|c_n| \geq (1-\varepsilon)\|c\|_{l_\infty} \qquad \text{for some } \varepsilon < \frac{C - 8\gamma - 1}{C-1}.$$

Then

$$K(t_n, f) \geq |c_n| K(t_n, g_n) - \sum_{k \neq n} |c_k| K(t_n, g_k)$$
$$\geq \|c\|_{l_\infty} \varphi(t_n) \left(\frac{1-\varepsilon}{4\gamma} - \frac{2}{C-1} \right) > 0,$$

hence $f \neq 0$.

It remains to show that the space Y, endowed with the norm $\|f\|_Y = \|c\|_{l_\infty}$, is a subspace of M_φ. Actually, the comparison of norms in one direction has already been made because

$$\|f\|_{M_\varphi} \geq \frac{K(t_n, f)}{\varphi(t_n)} \geq \|c\|_{l_\infty} \left(\frac{1-\varepsilon}{4\gamma} - \frac{2}{C-1} \right).$$

For an estimate in the other direction, consider an arbitrary $t > 0$. If $t \in (0, a_2)$ then the calculations are similar to the estimation of $K(t_1, f)$:

$$K(t, f) \leq |c_1| K(t, g_1) + \|c\|_{l_\infty} K(t, h_1) \leq \frac{C}{C-1} \varphi(t) \|c\|_{l_\infty}.$$

If otherwise $t \in [a_n, a_{n+1}]$ for some $n \geq 2$, then

$$K(t, f) \leq \|c\|_{l_\infty} (K(t, g_n) + K(t, h_n)) \leq \frac{C+1}{C-1} \varphi(t) \|c\|_{l_\infty}.$$

Thus we have proved that the norms of Y and of M_φ are equivalent on all functions from Y. ∎

The analogy between generalized Marcinkiewicz spaces and weight spaces l_∞^a can also be seen if one considers other properties of these spaces, for example, nondensity of mutual embeddings.

THEOREM 4. *Let $X_i \not\subset M_\varphi(\vec{X})$, $i = 0, 1$. Then no embedding $M_\psi(\vec{X}) \subsetneq M_\varphi(\vec{X})$ is dense.*

Proof. Assume, for simplicity, that both parameters φ, ψ are optimal for their spaces. Then $M_\varphi \neq M_\psi$ implies that the functions φ and ψ are not equivalent at least on one of the intervals $(0, 1)$, $(1, \infty)$. Both cases can be studied analogously and, moreover, can be derived from each other. So, we assume that $\varphi(t) \not\sim \psi(t)$ on $(1, \infty)$. This means that $\lim \psi(t_n)/\varphi(t_n) = 0$ for some sequence $t_n \to \infty$. The assumptions of the theorem ensure that $\varphi(t)$ is not equivalent to 1 and to t, thus omitting some t_n (if necessary), we can always get a sequence $\{t_n\}$ satisfying inequalities (6). Then we take the same elements g_n as before and define $f = \sum_{n=1}^{\infty} g_n$ which corresponds to the coefficients $c_n = 1$ and thus belongs to M_φ. Moreover, the previous estimates show that

$$K(t_n, f) \geq \lambda \varphi(t_n), \quad \text{where} \quad \lambda = \frac{1}{4\gamma} - \frac{2}{C-1} > 0,$$

and thus for any $g \in M_\psi$,

$$\|f-g\|_{M_\varphi} = \sup_t \frac{K(t,f-g)}{\varphi(t)} \ge \sup_t \left(\frac{K(t,f)}{\varphi(t)} - \frac{K(t,g)}{\varphi(t)}\right) \ge \limsup_{n\to\infty} \left(\frac{K(t_n,f)}{\varphi(t_n)} - \frac{K(t_n,g)}{\varphi(t_n)}\right).$$

But $K(t_n,g) \le \|g\|_{M_\psi}\psi(t_n)$, hence $\lim_{n\to\infty} K(t_n,g)/\varphi(t_n) = 0$ for any $g \in M_\psi$. At the same time

$$\limsup_{n\to\infty} \frac{K(t_n,f)}{\varphi(t_n)} \ge \lambda \ \Rightarrow\ \|f-g\|_{M_\varphi} \ge \lambda$$

for all possible g, which proves nondensity of M_ψ in M_φ. ∎

The requirements on the space M_φ in Theorem 4 are essential, since without them an embedding $M_\psi \subset M_\varphi$ may happen to be dense. For example, taking $\varphi(t) = \max(\psi(t), t)$ for some ψ with $\psi(t)/t \to 0$ as $t \to \infty$, we obtain an embedding $M_\psi \subsetneq M_\varphi = M_\psi + X_1$, which is dense for every Banach couple $\vec{X}$ where $\Delta(\vec{X})$ is dense in X_1. Some partial extensions of Theorem 4 can be obtained under additional conditions, for example, when $M_\varphi \supset X_1$ but $\varphi(t) \to 0$ as $t \to 0$ and $\psi(t) \not\sim \varphi(t)$ on $(0,1)$, or when $M_\varphi \supset X_0$ but $\varphi(t)/t \to 0$ as $t \to \infty$ and $\psi(t) \not\sim \varphi(t)$ on $(1,\infty)$.

If $\Delta(\vec{X})$ is dense neither in X_0 nor in X_1, the assertion of Theorem 4 is true without any restrictions. Indeed, M_ψ and M_φ are interpolation spaces for the couple $\vec{X}$, and we may use the general theorem of Aronszajn and Gagliardo ([AG], see also [BK], p. 131) which says that any interpolation space X satisfies one of the following conditions: (i) $X \subset X_0^\circ + X_1^\circ$; (ii) $X_0 \subset X \subset X_0 + X_1^\circ$; (iii) $X_1 \subset X \subset X_0^\circ + X_1$; (iv) $X = \Sigma(\vec{X})$. No space from one group can be dense in a space from another group, so we should only consider those spaces M_ψ and M_φ which both belong to one of the groups. In case (i) we may apply Theorem 4, case (iv) is impossible, and cases (ii), (iii) are just those which were mentioned in the preceding paragraph as extensions of Theorem 4.

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