STUDIA MATHEMATICA 144 (3) (2001)

Intersection properties for cones of monotone and convex functions with respect to the couple (L_p, BMO)

by

INNA KOZLOV (Haifa)

Abstract. The paper is devoted to some aspects of the real interpolation method in the case of triples (X_0, X_1, Q) where $\overline{X} := (X_0, X_1)$ is a Banach couple and Q is a convex cone. The first fundamental result of the theory, the *interpolation theorem*, holds in this situation (for linear operators preserving the cone structure). The second one, the *reiteration theorem*, holds only under some conditions on the triple. One of these conditions, the so-called *intersection property*, is studied for cones with respect to (L_p, BMO).

1. Introduction. Let Y be a linear space over the field of reals. Suppose that $X \subset Y$ is a linear subspace and that $Q \subset Y$ is a cone. A *norm* on $X \cap Q$ is a map $\|\cdot\|$ of $X \cap Q$ to $[0, \infty)$ having the properties usually required for a norm on a linear space, except that the formula $\|\lambda x\| = |\lambda| \cdot \|x\|$ is only required to hold for $\lambda \geq 0$.

DEFINITION 1.1. A cone Q has the *intersection property* (IP) with respect to the Banach couple $\overline{X} = (X_0, X_1)$ if for all $t > 0$,

$$(X_0 + tX_1) \cap Q = (X_0 \cap Q) + t(X_1 \cap Q) \tag{1.1}$$

where the norms are equivalent up to constants independent of t.

Here the norm of $(X_0 + tX_1) \cap Q$ is taken to be simply the restriction to Q of the natural norm (K-functional) on $X_0 + tX_1$, and the norm on $(X_0 \cap Q) + t(X_1 \cap Q)$ is

$$K(f, t; \overline{X} \cap Q) = \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} \mid f = f_0 + f_1,\ f_i \in X_i \cap Q\},$$

i.e., it is the K-functional of the couple of cones $\overline{X} \cap Q := (X_0 \cap Q, X_1 \cap Q)$.

Hence the intersection property (1.1) is equivalent to the double inequality

$$K(f, t; \overline{X} \cap Q) \approx K(f, t; \overline{X}) \quad (f \in Q,\ t > 0). \tag{1.2}$$

Here $F \approx G$ means that $C_1 F \leq G \leq C_2 F$ for some constants $0 < C_1, C_2 < \infty$ independent of the arguments of F, G. In particular, (1.2) holds uniformly

2000 *Mathematics Subject Classification*: Primary 46B70.

with respect to $t > 0$ and $f \in Q$. We also use the notation $F \prec G$ (or $G \succ F$) if $F \le C_1 G$ for some constant $C_1 > 0$ independent of the arguments of F, G.

If Q has the intersection property then it also satisfies the formula

$$(1.3)\qquad (X_0 \cap Q, X_1 \cap Q)_{\theta,q} = Q \cap (X_0, X_1)_{\theta,q} \quad (0 < \theta < 1,\ 1 \le q \le \infty)$$

with equivalence of norms.

Let us recall that the norm of the cone on the left is

$$\|f\|_{(X_0\cap Q, X_1\cap Q)_{\theta,q}} := \left(\int_0^\infty \left(\frac{K(t, f; \overline{X} \cap Q)}{t^\theta} \right)^q \frac{dt}{t} \right)^{1/q}.$$

Definition 1.2. The cone Q has the *weak intersection property* (WIP) with respect to the Banach couple (or seminormed couple) $\overline{X} = (X_0, X_1)$ if (1.3) holds with equivalence of norms.

Cones satisfying (1.3) were first introduced and studied by Y. Sagher ([12], [13]). He called them "Marcinkiewicz cones". He also gave some interesting applications of his results to harmonic analysis. Other applications of this property, connected with the sharpness of Sobolev embedding theorems and approximation with constraints, are studied in [10].

Let us now define the main cones which are studied in this paper.

Definition 1.3. (a) For $k \in \mathbb{N}$ the cone $M_k[0,1)$ of *k-monotone functions* consists of all $(k-1)$-times differentiable functions $f : [0,1) \to \mathbb{R}$ which satisfy $f^{(i)} \ge 0$ for $i = 0, 1, \dots, k-1$ and for which $f^{(k-1)}$ is non-decreasing.

(b) The cone M_+ consists of all non-negative non-increasing continuous functions on $\mathbb{R}_+ := (0, \infty)$.

In particular, $M_1[0,1)$ is the cone of non-negative non-decreasing functions and $M_2[0,1)$ is the cone of non-negative non-decreasing everywhere differentiable convex functions. We may also consistently define $M_0[0,1)$ to be the cone of non-negative functions.

We recall that the space $\mathrm{BMO}(A)$, $A \subset \mathbb{R}$, of John–Nirenberg consists of all functions $f \in L_1^{\mathrm{loc}}(A)$ for which

$$|f|_{\mathrm{BMO}} := \sup_I \left\{ \frac{1}{|I|} \int_I |f - f_I|\, dx \right\}$$

is finite, where the supremum is taken over all intervals $I \subset A$, and $f_I := |I|^{-1} \int_I f\, dx$.

Let $\mathbf{BMO}(A)$, $A := [0,1)$, denote the *normed* BMO-*space*, defined by the norm

$$\|f\|_{\mathbf{BMO}(A)} := |f|_{\mathrm{BMO}} + \int_0^1 |f|\, dx.$$

In Section 3 we study the IP for the cone of k-monotone functions $M_k,\ k=1,2$, with respect to couples of L_p-spaces and BMO. For instance, we prove the following theorems.

THEOREM. (i) *The cone* $M_1[0,1)$ *has the WIP with respect to* $(L_p[0,1),$ $\mathbf{BMO}[0,1))$, $1\le p<\infty$.

(ii) $M_1[0,1)$ *does not have the IP with respect to* $(L_p[0,1),\mathrm{BMO}[0,1))$ *for* $1\le p<\infty$.

(iii) *The cone* $M_2[0,1)$ *has the IP with respect to* $(L_p[0,1),\mathrm{BMO}[0,1))$, $1\le p<\infty$.

There is a striking difference between the cases of monotone functions on a bounded and unbounded interval:

THEOREM. *The cone* M_+ *of non-negative non-increasing continuous functions on* $\mathbb{R}_+$ *has the IP with respect to* $(L_p(\mathbb{R}_+),\mathrm{BMO}(\mathbb{R}_+))$, $1\le p<\infty$.

We consider cones of functions for which all derivatives up to a fixed order are non-negative. In certain applications it is also important to consider cones of functions which are differentiable up to some order, and for which derivatives of different orders have fixed prescribed signs. One of the simplest examples is the cone C of concave non-decreasing non-negative functions on $\mathbb{R}_+$. The intersection property for this cone was first studied by I. Asekritova [1]. She proved that C has the IP with respect to a couple of weighted L_∞ spaces where the weights are quasi-concave on $\mathbb{R}_+$.

Recently J. Cerdà and J. Martín [6] have obtained a similar result for the cone of non-negative non-increasing functions on $\mathbb{R}_+$ with respect to (L_p,L_q) and also with respect to couples of Lorentz spaces.

2. Reiteration theorems for couples of cones. One of the basic results in the real method of interpolation is the following formula of Holmstedt for couples of Banach spaces $\overline{X}:=(X_0,X_1)$ (see, for instance, [3]):

THEOREM 2.1. *Let* $\overline{Y}:=(\overline{X}_{\theta_0,q_0},\overline{X}_{\theta_1,q_1})$ *and* $K(s):=K(f,s;\overline{X})$ *for* $f\in\sum(\overline{X}):=X_0+X_1$. *Then*

$$K(f,t^{\theta_1-\theta_0};\overline{Y})\approx\Big(\int_0^t[s^{-\theta_j}K(s)]^{q_0}\,\frac{ds}{s}\Big)^{1/q_0}+t^{\theta_1-\theta_0}\Big(\int_t^\infty[s^{-\theta_j}K(s)]^{q_1}\,\frac{ds}{s}\Big)^{1/q_1}. \tag{2.4}$$

In particular, *the reiteration formula*

$$(\overline{X}_{\theta_0,q_0},\overline{X}_{\theta_1,q_1})_{\theta,q}=\overline{X}_{\theta',q} \tag{2.5}$$

holds with equivalent norms. Here $q_0, q_1, q \in [1,\infty]$, $\theta_0, \theta_1, \theta \in (0,1)$, $\theta_0 < \theta_1$ *and* $\theta' = (1-\theta)\theta_0 + \theta\theta_1$.

Unfortunately, we cannot apply Holmstedt's proof [8] directly to the case of a couple of cones $\overline{X}^Q := (X_0 \cap Q, X_1 \cap Q)$ because we have to avoid taking differences of two functions from a cone. Nevertheless, we can use the following version of the reiteration theorem in our setting.

THEOREM 2.2. *Suppose that for fixed* $\theta_j \in (0,1)$ *and* $q_j \in [1,\infty]$ *we have an isomorphism*

$$(X_0 \cap Q, X_1 \cap Q)_{\theta_i,q_i} = (X_0, X_1)_{\theta_i,q_i} \cap Q \tag{2.6}$$

for $i = 0,1$. *For all* $f \in \sum(\overline{X}^Q) := X_0 \cap Q + X_1 \cap Q$ *and* $j = 0,1$ *let*

$$P_j^Q(t) = \left(\int_0^t [s^{-\theta_j} K(f,s;X_0 \cap Q, X_1 \cap Q)]^{q_j} \,\frac{ds}{s}\right)^{1/q_j},$$

$$R_j^Q(t) = \left(\int_t^\infty [s^{-\theta_j} K(f,s;X_0 \cap Q, X_1 \cap Q)]^{q_j} \,\frac{ds}{s}\right)^{1/q_j}.$$

Let, in addition,

$$\overline{K}^Q(f,s) := K(f,s;(\overline{X}^Q)_{\theta_0,q_0},(\overline{X}^Q)_{\theta_1,q_1}).$$

Then

$$\overline{K}^Q(f,t^\lambda) \approx P_0^Q(t) + t^\lambda R_1^Q(t). \tag{2.7}$$

Here, as above, $\theta_0 < \theta_1$ *and* $\lambda := \theta_1 - \theta_0$.

Proof. We adapt Holmstedt's proof to our case of a couple of cones. The inequality

$$P_0^Q(t) + t^\lambda R_1^Q(t) \le C\overline{K}^Q(f,t^\lambda)$$

is proved as in [8], pp. 180–182.

To obtain the converse inequality, for $f \in \sum(\overline{X}^Q)$ and $t > 0$ we choose a decomposition $f = g_t + h_t \in X_0 \cap Q + X_1 \cap Q$, $g_t \in X_0 \cap Q$, $h_t \in X_1 \cap Q$, such that

$$\|g_t\|_{X_0} + t\|h_t\|_{X_1} \le 2K(f,t;X_0 \cap Q, X_1 \cap Q). \tag{2.8}$$

With this choice we have

$$\begin{aligned}\overline{K}^Q(f,t^\lambda) &\le \|g_t\|_{(\overline{X}^Q)_{\theta_0,q_0}} + t^\lambda \|h_t\|_{(\overline{X}^Q)_{\theta_1,q_1}} \\ &= \left(\int_0^\infty [s^{-\theta_0} K(g_t,s;X_0 \cap Q, X_1 \cap Q)]^{q_0} \,\frac{ds}{s}\right)^{1/q_0} \\ &\quad + t^\lambda \left(\int_0^\infty [s^{-\theta_1} K(h_t,s;X_0 \cap Q, X_1 \cap Q)]^{q_1} \,\frac{ds}{s}\right)^{1/q_1}.\end{aligned} \tag{2.9}$$

To estimate the right hand side of (2.9) we have to apply (2.6):

$$\overline{K}^Q(f,t^\lambda) \le C\Big\{\Big(\int_0^\infty [s^{-\theta_0}K(g_t,s;X_0,X_1)]^{q_0}\,\frac{ds}{s}\Big)^{1/q_0} + t^\lambda\Big(\int_0^\infty [s^{-\theta_1}K(h_t,s;X_0,X_1)]^{q_1}\,\frac{ds}{s}\Big)^{1/q_1}\Big\}$$
$$\le C\Big\{\Big(\int_0^t [s^{-\theta_0}K(g_t,s;X_0,X_1)]^{q_0}\,\frac{ds}{s}\Big)^{1/q_0} + \Big(\int_t^\infty [s^{-\theta_0}K(g_t,s;X_0,X_1)]^{q_0}\,\frac{ds}{s}\Big)^{1/q_0}$$
$$+ t^\lambda\Big(\int_0^t [s^{-\theta_1}K(h_t,s;X_0,X_1)]^{q_1}\,\frac{ds}{s}\Big)^{1/q_1} + t^\lambda\Big(\int_t^\infty [s^{-\theta_1}K(h_t,s;X_0,X_1)]^{q_1}\,\frac{ds}{s}\Big)^{1/q_1}\Big\}. \tag{2.10}$$

For the first term we obtain, by the triangle inequality,

$$\Big(\int_0^t [s^{-\theta_0}K(g_t,s;X_0,X_1)]^{q_0}\,\frac{ds}{s}\Big)^{1/q_0} \le \Big(\int_0^t [s^{-\theta_0}K(f,s;X_0,X_1)]^{q_0}\,\frac{ds}{s}\Big)^{1/q_0} + \Big(\int_0^t [s^{-\theta_0}K(h_t,s;X_0,X_1)]^{q_0}\,\frac{ds}{s}\Big)^{1/q_0}. \tag{2.11}$$

Since for every cone Q,

$$K(f,s;X_0,X_1) \le K(f,s;X_0\cap Q, X_1\cap Q),$$

it follows that

$$\Big(\int_0^t [s^{-\theta_0}K(f,s;X_0,X_1)]^{q_0}\,\frac{ds}{s}\Big)^{1/q_0} \le P_0^Q(t).$$

According to the choice of h_t (see 2.8) the last integral in (2.11) is bounded by

$$\Big(\int_0^t [s^{-\theta_0}s\|h_t\|_{X_1}]^{q_0}\,\frac{ds}{s}\Big)^{1/q_0} = C_1(\|h_t\|_{X_1}t)t^{-\theta_0} \tag{2.12}$$

$$\leq 2C_1 K(f,t;X_0\cap Q, X_1\cap Q)t^{-\theta_0}$$
$$\leq C_2 \frac{K(f,t;X_0\cap Q, X_1\cap Q)}{t} \int_{t/2}^{t} s^{-\theta_0}\, ds.$$

Since the function $t\mapsto K(f,t;X_0\cap Q, X_1\cap Q)$ is concave, we have

$$\frac{K(f,t;X_0\cap Q, X_1\cap Q)}{t} \leq \frac{K(f,s;X_0\cap Q, X_1\cap Q)}{s} \quad \text{if } s\leq t.$$

Therefore the right hand side of (2.12) does not exceed

$$(2.13)\quad C_2 \int_{t/2}^{t} [s^{-\theta_0}K(f,s;X_0\cap Q, X_1\cap Q)]^{q_0}\, \frac{ds}{s}$$
$$\leq C_2\Big(\int_{t/2}^{t} [s^{-\theta_0}K(f,s;X_0\cap Q, X_1\cap Q)]^{q_0}\, \frac{ds}{s}\Big)^{1/q_0}$$
$$\leq C_3\Big(\int_{0}^{t} [s^{-\theta_0}K(f,s;X_0\cap Q, X_1\cap Q)]^{q_0}\, \frac{ds}{s}\Big)^{1/q_0} = C_3 P_0^Q(t).$$

(The first inequality in (2.13) follows from Hölder's inequality.)

The second term of (2.10) is estimated by similar arguments:

$$(2.14)\quad \Big(\int_{t}^{\infty} [s^{-\theta_0}K(g_t,s;X_0,X_1)]^{q_0}\, \frac{ds}{s}\Big)^{1/q_0} \leq \Big(\int_{t}^{\infty} [s^{-\theta_0}\|g_t\|_{X_0}]^{q_0}\, \frac{ds}{s}\Big)^{1/q_0}$$
$$\leq 2\Big(\int_{t}^{\infty} [s^{-\theta_0}K(f,t;X_0\cap Q, X_1\cap Q)]^{q_0}\, \frac{ds}{s}\Big)^{1/q_0}$$
$$\leq CK(f,t;X_0\cap Q, X_1\cap Q)t^{-\theta_0}$$
$$\leq C\Big(\int_{0}^{t} [s^{-\theta_0}K(f,s;X_0\cap Q, X_1\cap Q)]^{q_0}\, \frac{ds}{s}\Big)^{1/q_0} = CP_0^Q(t).$$

The remaining two terms of (2.10) are treated analogously. Summing the four estimates we obtain the required inequality

$$\overline{K}^Q(f,t^\lambda) \leq C(P_0^Q(t) + t^\lambda R_1^Q(t)),$$

completing the proof of the theorem.

The following result is easily proved by an adaptation of the previous proof.

Theorem 2.3. *Suppose that for fixed $\theta\in(0,1)$ and $q\in[1,\infty]$ we have an isomorphism*

$$(X_0\cap Q, X_1\cap Q)_{\theta,q} = (X_0,X_1)_{\theta,q}\cap Q.$$

Then

$$K(f,t^\theta;X_0,(\overline{X}^Q)_{\theta,q}) \approx t^\theta\Big(\int_t^\infty [s^{-\theta}K(f,s;X_0\cap Q,X_1\cap Q)]^q\,\frac{ds}{s}\Big)^{1/q}$$

and

$$K(f,t^{1-\theta};(\overline{X}^Q)_{\theta,q},X_1) \approx \Big(\int_0^t [s^{-\theta}K(f,s;X_0\cap Q,X_1\cap Q)]^q\,\frac{ds}{s}\Big)^{1/q}.$$

Corollary 2.4. *If the cone Q has the WIP with respect to $\overline{X} := (X_0,X_1)$, then*

$$((\overline{X}^Q)_{\theta_0,q_0},(\overline{X}^Q)_{\theta_1,q_1})_{\theta,q} = \overline{X}^Q_{\theta',q} \tag{2.15}$$

with equivalent norms. Here $q_0,q_1,q\in[1,\infty]$, $\theta_0,\theta_1,\theta\in(0,1)$, $\theta_0<\theta_1$ and $\theta' := (1-\theta)\theta_0+\theta\theta_1$.

This result is proved in exactly the same way as Theorem 3.1 of [8].

It is worth noting that since the (strong) IP implies the WIP, the same reiteration theorems also hold if the cone Q has the IP with respect to $\overline{X} := (X_0,X_1)$.

Almost identical arguments to those used above lead to the following variants of the preceding reiteration theorems.

Theorem 2.5. *Suppose that for fixed $\theta_j\in(0,1)$ and $q_j\in[1,\infty]$ we have an isomorphism*

$$Q\cap(X_0,X_1\cap Q)_{\theta_i,q_i} = (X_0,X_1)_{\theta_i,q_i}\cap Q$$

for $i=0,1$. Then for every $f\in[X_0+(X_1\cap Q)]\cap Q$,

$$\begin{aligned} &K(f,t^\lambda;(X_0,X_1\cap Q)_{\theta_0,q_0},(X_0,X_1\cap Q)_{\theta_1,q_1}) \\ &\quad\approx \Big(\int_0^t [s^{-\theta_0}K(f,s;X_0,X_1\cap Q)]^{q_0}\,\frac{ds}{s}\Big)^{1/q_0} \\ &\quad\quad + t^\lambda\Big(\int_t^\infty [s^{-\theta_1}K(f,s;X_0,X_1\cap Q)]^{q_1}\,\frac{ds}{s}\Big)^{1/q_1}. \end{aligned} \tag{2.16}$$

Here $\lambda := \theta_1-\theta_0$ and $\theta_0<\theta_1$.

Theorem 2.6. *Suppose that for fixed $\theta\in(0,1)$ and $q\in[1,\infty]$ we have an isomorphism*

$$Q\cap(X_0,X_1\cap Q)_{\theta,q} = (X_0,X_1)_{\theta,q}\cap Q.$$

Then

$$K(f,t^\theta;X_0,(X_0,X_1\cap Q)_{\theta,q}) \approx t^\theta\Big(\int_t^\infty [s^{-\theta}K(f,s;X_0,X_1\cap Q)]^q\,\frac{ds}{s}\Big)^{1/q}$$

and

$$K(f,t^{1-\theta};(X_0,X_1\cap Q)_{\theta,q},X_1)\approx\left(\int_0^t[s^{-\theta}K(f,s;X_0,X_1\cap Q)]^q\,\frac{ds}{s}\right)^{1/q}.$$

COROLLARY 2.7. *Let the cone Q have the WIP with respect to $\overline{X}:=(X_0,X_1)$. Let $\overline{Y}:=(Y_0,Y_1)$, where $Y_i:=\overline{X}_{\theta_i,q_i}$, $i=0,1$, $\theta_0\neq\theta_1$. Then Q has the WIP with respect to $\overline{Y}$. Here $q_i\in[1,\infty]$ and $\theta_i\in(0,1)$, $\theta_0<\theta_1$, $i=0,1$.*

Proof. Applying the WIP for $\overline{X}$ we get

$$(\overline{Y}^Q)_{\theta,q}:=(Y_0\cap Q,Y_1\cap Q)_{\theta,q}=((\overline{X}^Q)_{\theta_0,q_0},(\overline{X}^Q)_{\theta_1,q_1})_{\theta,q}.$$

But for $\overline{X}^Q$ the reiteration theorem holds true. Therefore the right hand side equals $(\overline{X}^Q)_{\eta,q}$ with $\eta:=(1-\theta)\theta_0+\theta\theta_1$. This cone, in turn, equals $(\overline{X})_{\eta,q}\cap Q$ by the WIP of $\overline{X}$. Finally, $(\overline{X})_{\eta,q}=(\overline{Y})_{\theta,q}$, by the classical reiteration theorem. Putting all this together we get

$$(\overline{Y}^Q)_{\theta,q}=(\overline{Y})_{\theta,q}\cap Q.$$

The corollary is proved.

In a similar way using the IP and both the modified (Theorem 2.2 or Theorem 2.3) and usual (Theorem 2.1, and also Corollary 3.6.2(b) of [3], p. 53) Holmstedt formulas we obtain

COROLLARY 2.8. *Let the cone Q have the IP with respect to $\overline{X}$. Then Q has the IP with respect to $(\overline{X}_{\theta_0,q_0},\overline{X}_{\theta_1,q_1})$ and with respect to $(X_0,\overline{X}_{\theta_1,q_1})$. Here $\theta_0\neq\theta_1$, $q_i\in[1,\infty]$ and $\theta_i\in(0,1)$, $\theta_0<\theta_1$, $i=0,1$.*

3. Monotone functions and the couple (L_p,BMO)

THEOREM 3.1. *The cone M_+ of non-negative non-increasing continuous functions on $\mathbb{R}_+$ has the IP with respect to $(L_p(\mathbb{R}_+),\mathrm{BMO}(\mathbb{R}_+))$ for $1\le p<\infty$.*

Proof. We use the following inequality for the K-functional:

$$t(f_p^{\#})^*(t^p)\le CK(f,t;L_p,\mathrm{BMO})\quad(1\le p<\infty),\tag{3.17}$$

obtained by Bennett and Sharpley (see [2], Lemma 4.3, p. 215, and Remark 6.3, p. 228, where the case of the real line was proved). The proof is based on the weak $(1,1)$-boundedness of the Hardy–Littlewood maximal operator, which clearly also holds for the case of an arbitrary interval in $\mathbb{R}$ (bounded or not).

Here $f_p^{\#}$ is the "sharp" maximal function defined by

$$f_p^{\#}(x) := \sup\bigg\{\bigg(\frac{1}{|I|}\int_I |f-f_I|^p\,dx\bigg)^{1/p}\ \bigg|\ \mathbb{R}_+ \supset I = [a,b] \ni x,\ f_I := \frac{1}{|I|}\int_I f\,dx\bigg\}.$$

For our proof we need two auxiliary lemmas.

LEMMA 3.2. *Let $f \in M_+$. Then for every $0 \le t_1 < t_2$ and every closed interval $I \ni t_2$ there is a closed interval $J \ni t_1$ such that*

$$\frac{1}{|I|}\int_I |f-f_I|^p\,dx \le \frac{2^{p+2}}{|J|}\int_J |f-f_J|^p\,dx. \tag{3.18}$$

Proof. Let f be a decreasing continuous function. In the trivial case $t_1 \in I$ we simply set $J := I$. Let now $t_1 \notin I =: [\xi_1, \xi_2]$. Since $t_1 < t_2 \in I$, we have $t_1 < \xi_1$.

Let e_I be the best constant function approximation to f in L_p on the interval I:

$$E_0(f,I)_{L_p} := \inf_c \|f-c\|_{L_p(I)} = \|f-e_I\|_{L_p(I)}. \tag{3.19}$$

In each interval $\Lambda := [a,b]$ define $x_\Lambda := x_\Lambda(f) \in \Lambda$ to be the infimum of $x \in \Lambda$ satisfying

$$\int_a^x (f-f(x))^p\,ds = \int_x^b (f(x)-f)^p\,ds. \tag{3.20}$$

It is easily seen that the function on the left (resp. right) is continuous and non-decreasing (resp. decreasing) and equals 0 at $x=a$ (resp. at $x=b$). Thus, the set of points satisfying (3.20) is not empty and its infimum x_Λ also satisfies (3.20). Then we set

$$c_\Lambda := c_\Lambda(f) := f(x_\Lambda). \tag{3.21}$$

Let us check that

$$(E_0(f,\Lambda)_{L_p} \le)\ \|f-c_\Lambda(f)\|_{L_p(\Lambda)} \le 2^{1/p}E_0(f,\Lambda)_{L_p}. \tag{3.22}$$

In fact, if, for instance $c_\Lambda \ge e_\Lambda$, then

$$\|f-c_\Lambda\|^p_{L_p(\Lambda)} = 2\int_a^{x_\Lambda}(f-c_\Lambda)^p\,dx \le 2\int_a^{x_\Lambda}(f-e_\Lambda)^p\,dx \le 2E_0(f,\Lambda)_{L_p}.$$

The case $c_\Lambda < e_\Lambda$ can be considered similarly.

It is well known and easily checked that

$$(E_0(f,\Lambda)_{L_p} \le)\ \|f-f_\Lambda\|_{L_p(\Lambda)} \le 2E_0(f,\Lambda)_{L_p}. \tag{3.23}$$

Together with the previous equivalence this allows us, roughly speaking, to replace f_I and f_J in (3.18) by c_I and c_J, respectively. Bearing this in mind we continue with the proof of (3.18). So, we have $t_1 < \xi_1$ and $I = [\xi_1, \xi_2]$ and our goal is to determine an interval J satisfying (3.18). We set $J = [t_1, \xi_3]$, where ξ_3 is determined by

$$(\xi_1 - t_1)(f(\xi_1) - c_I)^p = (\xi_3 - \xi_2)(c_I - f(\xi_2))^p, \tag{3.24}$$

and check that in this way we do obtain the required interval.

We first consider the case

$$x_I - t_1 \geq \frac{\xi_3 - t_1}{2} := \frac{|J|}{2}. \tag{3.25}$$

Define

$$k := \int_{\xi_1}^{x_I} (f(x) - c_I)^p \, dx / ((f(\xi_1) - c_I)^p (x_I - \xi_1)) \leq 1 \tag{3.26}$$

by monotonicity of f.

From (3.20), (3.23), (3.26) and the monotonicity of f we now get

$$\begin{aligned} \frac{1}{|I|} \|f - f_I\|_{L_p(I)}^p &\leq \frac{2^{p+1}}{|I|} \int_{\xi_1}^{x_I} (f(x) - c_I)^p \, dx \\ &\leq \frac{2^{p+1} k}{|I|} (f(\xi_1) - c_I)^p (x_I - \xi_1) \leq 2^{p+1} k (f(\xi_1) - c_I)^p \\ &= \frac{2^{p+1} k}{x_I - t_1} (f(\xi_1) - c_I)^p (x_I - t_1) \\ &= \frac{2^{p+1} k}{x_I - t_1} \bigg(\int_{t_1}^{\xi_1} (f(\xi_1) - c_I)^p \, dx + \frac{1}{k} \int_{\xi_1}^{x_I} (f(x) - c_I)^p \, dx \bigg). \end{aligned} \tag{3.27}$$

Applying (3.25) to estimate the right hand side we then have

$$\frac{1}{|I|} \|f - f_I\|_{L_p(I)}^p \leq \frac{2^{p+2}}{\xi_3 - t_1} \bigg(k \int_{t_1}^{\xi_1} (f(\xi_1) - c_I)^p \, dx + \int_{\xi_1}^{x_I} (f(x) - c_I)^p \, dx \bigg).$$

It follows, using (3.20) and (3.24), that

$$\begin{aligned} \frac{1}{|I|} \|f - f_I\|_{L_p(I)}^p &\leq \frac{2^{p+2}}{|J|} \bigg(\int_{t_1}^{\xi_1} (f(\xi_1) - c_I)^p \, dx + \int_{\xi_1}^{x_I} (f(x) - c_I)^p \, dx \bigg) \\ &= \frac{2^{p+2}}{|J|} \bigg(\int_{x_I}^{\xi_2} (c_I - f(x))^p \, dx + \int_{\xi_2}^{\xi_3} (c_I - f(\xi_2))^p \, dx \bigg). \end{aligned} \tag{3.28}$$

Let us check that in the remaining case where

$$x_I - t_1 < \frac{\xi_3 - t_1}{2} = \frac{|J|}{2}$$

(or, equivalently, $\xi_3 - x_I \geq |J|/2$) the same inequality (3.28) holds. To this end it suffices to put

$$k := \int_{x_I}^{\xi_2} (c_I - f(x))^p \, dx / ((c_I - f(\xi_2))^p (\xi_2 - x_I)) \leq 1 \tag{3.29}$$

and use a variant of the previous arguments with this choice of k:

$$\begin{aligned}
\frac{1}{|I|}\|f - f_I\|^p_{L_p(I)} &\leq \frac{2^{p+1}}{|I|} \int_{x_I}^{\xi_2} (c_I - f(x))^p \, dx \\
&\leq \frac{2^{p+1}k}{|I|} (c_I - f(\xi_2))^p (\xi_2 - x_I) \leq 2^{p+1} k (c_I - f(\xi_2))^p \\
&= \frac{2^{p+1}k}{\xi_3 - x_I} (c_I - f(\xi_2))^p (\xi_3 - x_I) \\
&= \frac{2^{p+1}k}{\xi_3 - x_I} \left(\frac{1}{k} \int_{x_I}^{\xi_2} (c_I - f(x))^p \, dx + \int_{\xi_2}^{\xi_3} (c_I - f(\xi_2))^p \, dx \right) \\
&\leq \frac{2^{p+1}}{\xi_3 - t_1} \left(\int_{x_I}^{\xi_2} (c_I - f(x))^p \, dx + k \int_{\xi_2}^{\xi_3} (c_I - f(\xi_2))^p \, dx \right) \\
&\leq \frac{2^{p+2}}{|J|} \left(\int_{x_I}^{\xi_2} (c_I - f(x))^p \, dx + \int_{\xi_2}^{\xi_3} (c_I - f(\xi_2))^p \, dx \right) \\
&= \frac{2^{p+2}}{|J|} \left(\int_{t_1}^{\xi_1} (f(\xi_1) - c_I)^p \, dx + \int_{\xi_1}^{x_I} (f(x) - c_I)^p \, dx \right).
\end{aligned} \tag{3.30}$$

Having established (3.28) we now use it to obtain a different estimate where c_I and x_I are replaced by c_J and x_J.

Suppose first that $c_J \leq c_I$. Then $x_J \geq x_I$ by monotonicity of f. By the same reason and by (3.22) and (3.20) we then have

$$\begin{aligned}
&\frac{2^{p+2}}{|J|} \left(\int_{t_1}^{\xi_1} (f(\xi_1) - c_I)^p \, dx + \int_{\xi_1}^{x_I} (f(x) - c_I)^p \, dx \right) \\
&\qquad \leq \frac{2^{p+2}}{|J|} \int_{t_1}^{x_J} (f(x) - c_J)^p \, dx = \frac{2^{p+1}}{|J|} \int_J |f(x) - c_J|^p \, dx \\
&\qquad \leq \frac{2^{p+2}}{|J|} \int_J |f(x) - e_J|^p \, dx \leq \frac{2^{p+2}}{|J|} \int_J |f(x) - f_J|^p \, dx.
\end{aligned} \tag{3.31}$$

So in this case the proof is complete.

In the remaining case $c_J > c_I$ (and, therefore, $x_J \le x_I$) we get by similar arguments

$$\begin{aligned}(3.32)\qquad &\frac{2^{p+2}}{|J|}\Big(\int_{x_I}^{\xi_2}(c_I - f(x))^p\,dx + \int_{\xi_2}^{\xi_3}(c_I - f(\xi_2))^p\,dx\Big)\\ &\le \frac{2^{p+2}}{|J|}\int_{x_J}^{\xi_3}(c_J - f(x))^p\,dx = \frac{2^{p+1}}{|J|}\int_J |c_J - f(x)|^p\,dx.\end{aligned}$$

Lemma 3.2 is proved.

Lemma 3.3. *For each $f \in M_+$ the "sharp" maximal function $f_p^{\#}$ satisfies*

$$f_p^{\#}(x) \le 2^{1+2/p}(f_p^{\#})^*(x).$$

Proof. We first show that

$$(3.33)\qquad f_p^{\#}(x) \ge \frac{f_p^{\#}(t)}{2^{1+2/p}} \quad \text{for every } x \le t.$$

In fact, Lemma 3.2 implies that for every interval $I \ni t$ there exists an interval $J \ni x$ such that

$$\Big(\frac{1}{|I|}\int_I |f - f_I|^p\,dx\Big)^{1/p} \le \Big(\frac{2^{p+2}}{|J|}\int_J |f - f_J|^p\,dx\Big)^{1/p} \le 2^{1+2/p} f_p^{\#}(x).$$

Taking the supremum over all intervals I with $I \ni x$ gives (3.33).

Now define the non-increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$h(t) := \inf_{x\in[0,t]} f_p^{\#}(x).$$

Since $f_p^{\#}$ is lower semicontinuous, h is continuous from the right. To see this let t_n be a sequence tending to t from the right ($t_n \ge t$). Let ξ_n be a point of minimum, that is,

$$h(t) \ge h(t_n) = \inf_{x\in[0,t_n]} f_p^{\#}(x) = f_p^{\#}(\xi_n).$$

Its existence follows from the lower semicontinuity of $f_p^{\#}$. Without loss of generality we can assume that $\xi_n \in [t, t_n]$. Then

$$\lim_{n\to\infty} h(t_n) = \lim_{n\to\infty} f_p^{\#}(\xi_n) = \lim_{\xi_n\to t} f_p^{\#}(\xi_n) \ge f_p^{\#}(t) \ge h(t).$$

Since h is non-increasing and continuous from the right,

$$(3.34)\qquad h(t) = h^*(t).$$

Let now

$$(3.35)\qquad h(t) = f_p^{\#}(\xi)$$

with $\xi \in [0,t]$. Then from (3.33)–(3.35) it follows that

$$(f_p^{\#})^*(t) \geq h^*(t) = h(t) = f_p^{\#}(\xi) \geq \frac{f_p^{\#}(t)}{2^{1+2/p}}. \tag{3.36}$$

This is precisely the assertion of the lemma.

We now return to the proof of Theorem 3.1. Suppose that $f \in (L_p + \mathrm{BMO}) \cap M_+$. Define the cut-off function g_t by

$$g_t(x) = \begin{cases} c_{[0,t^p]}(f) & \text{if } 0 \leq x \leq x_t, \\ f(x) & \text{if } x_t < x \leq \infty, \end{cases} \tag{3.37}$$

where $x_t := \inf\{x \mid f(x) = c_{[0,t^p]}(f)\}$. It is clear that $0 \leq x_t \leq t^p$. In fact $x_t = x_{[0,t^p]}(f)$.

Since g_t is monotone and bounded, it belongs to $\mathrm{BMO} \cap M_+$. It is also readily seen that $f - g_t \in M_+$. Therefore

$$K(f, t; L_p \cap M_+, \mathrm{BMO} \cap M_+) \leq \|f - g_t\|_{L_p} + t|g_t|_{\mathrm{BMO}}. \tag{3.38}$$

Let us check that both terms on the right are controlled by $t(f_p^{\#})^*(t^p)$. Together with (3.17) this will complete the proof. According to the definition of g_t, (3.22) and Lemma 3.3 we get

$$\begin{aligned} \|f - g_t\|_{L_p} &= \|f - c_{[0,t^p]}(f)\|_{L_p[0,t^p]} \leq 2^{1/p}\Big(\int_0^{t^p} |f - f_{[0,t^p]}|^p\,dx\Big)^{1/p} \\ &\leq 2^{1/p} t \sup_{I \ni t^p} \left(\frac{1}{|I|}\int_I |f - f_I|^p\,dx\right)^{1/p} \\ &= 2^{1/p} t f_p^{\#}(t^p) \leq 2^{1+3/p} t (f_p^{\#})^*(t^p). \end{aligned} \tag{3.39}$$

To estimate $|g_t|_{\mathrm{BMO}}$ we first prove that

$$\Big(\int_I |g_t - (g_t)_I|^p\,dx\Big)^{1/p} \leq 2^{1+1/p}\Big(\int_I |f - f_I|^p\,dx\Big)^{1/p} \tag{3.40}$$

for every interval $I \subset \mathbb{R}_+$.

In fact, it is sufficient to consider the case where x_t is an interior point of $I =: [a,b]$. If $x_I(g_t) < x_t$ then since g_t is constant on $[a, x_t)$, we have $x_I(g_t) = a$ and g_t is constant a.e. on I, and there is nothing to prove. So we assume that $x_I(g_t) \geq x_t$. Since $g_t \leq f$ we will show that $c_I(g_t) \leq c_I(f)$ and $x_I(g_t) \geq x_I(f)$.

In general, the following lemma holds true:

Lemma 3.4. *If $u, v \in M_+$ and $u \leq v$ on I, then $c_I(u) \leq c_I(v)$.*

Proof. For every $\lambda \in \mathbb{R}$ we have

$$u - \lambda \leq v - \lambda, \quad \lambda - v \leq \lambda - u,$$

therefore

$$(u-\lambda)_+ \le (v-\lambda)_+, \quad (\lambda - v)_+ \le (\lambda - u)_+.$$

Thus

$$F(\lambda) := \int_I [(u-\lambda)_+^p - (\lambda-u)_+^p]\,dx \le \int_I [(v-\lambda)_+^p - (\lambda-v)_+^p]\,dx =: G(\lambda).$$

But F and G are both strictly decreasing functions of λ and it is easy to check that

$$F(c_I(u)) = 0, \quad G(c_I(v)) = 0.$$

Since $G(c_I(u)) \ge 0$ it follows that $c_I(v) \ge c_I(u)$ and the proof is complete.

In particular, $c_I(g_t) \le c_I(f)$.

Now let us see that $x_I(g_t) \ge x_I(f)$. It follows from the definitions of $x_I(f)$ and $c_I(f)$ that

$$x_I(f) = \inf\{x \in I \mid f(x) = c_I(f)\}$$

and similarly

$$x_I(g_t) = \inf\{x \in I \mid g_t(x) = c_I(g_t)\}.$$

Now $c_I(g_t) \le c_{[0,t^p]}(f)$. If $c_I(g_t) = c_{[0,t^p]}(f)$ then g_t is constant on I, and therefore there is nothing to prove. So, assume $c_I(g_t) < c_{[0,t^p]}(f)$. Then

$$\{x \in I \mid g_t(x) \le c_I(g_t)\} = \{x \in I \mid f(x) \le c_I(g_t)\} \subset \{x \in I \mid f(x) \le c_I(f)\}$$

and therefore $x_I(g_t) \ge x_I(f)$.

From this together with (3.23) and (3.22) we therefore have

$$\begin{aligned}
(3.41) \qquad \Big(\int_I |g_t - (g_t)_I|^p\,dx\Big)^{1/p} &\le 2\Big(\int_I |g_t - c_I(g_t)|^p\,dx\Big)^{1/p} \\
&= 2\Big(2\int_{x_I(g_t)}^{b} |c_I(g_t) - f(x)|^p\,dx\Big)^{1/p} \\
&\le 2\Big(2\int_{x_I(f)}^{b} |c_I(f) - f(x)|^p\,dx\Big)^{1/p} \\
&= 2\Big(\int_I |c_I(f) - f(x)|^p\,dx\Big)^{1/p} \\
&\le 2^{1+1/p}\Big(\int_I |f - f_I|^p\,dx\Big)^{1/p}.
\end{aligned}$$

So, (3.40) is proved.

To complete the proof of the theorem we need to show that for some constant C depending only on p and for all intervals $I \subset \mathbb{R}_+$ we have

$$\left(\frac{1}{|I|}\int_I |g_t - (g_t)_I|\,dx\right)^{1/p} \le C(f_p^{\#})^*(t^p). \tag{3.42}$$

Let us first observe that in view of Lemma 3.2 it suffices to obtain (3.42) for intervals I of the form $[0,b]$. Now define

$$\Omega := \{x \in \mathbb{R}_+ \mid f_p^{\#}(x) > (f_p^{\#})^*(t^p)\} \quad \text{and} \quad F := \mathbb{R}_+ \setminus \Omega.$$

Let $I = [0,b] \subset \mathbb{R}_+$. Consider the following three cases:

1. $I \cap F \neq \emptyset$, that is, there exists an $x^* \in I \cap F$. By the definition of $f_p^{\#}$ and Ω the right hand side of (3.40) does not exceed

$$2^{1+1/p}|I|^{1/p} f_p^{\#}(x^*) \le 2^{1+1/p}|I|^{1/p}(f_p^{\#})^*(t^p).$$

So by (3.40) we get

$$\left(\frac{1}{|I|}\int_I |g_t - (g_t)_I|^p\,dx\right)^{1/p} \le 2^{1+1/p}(f_p^{\#})^*(t^p)$$

and this is the desired estimate (3.42).

2. $I \subset \Omega$ and $I \subset [0,x_t]$. Then $g_t|_I \equiv \text{const}$ and

$$\frac{1}{|I|}\int_I |g_t - (g_t)_I|^p\,dx \equiv 0,$$

so that (3.42) is trivial.

3. In the remaining case we have $I = [0,b] \subset \Omega$ and $x_t \in (0,b)$. Thus, as in our proof of (3.40) we again have $x_t \le x_I(g_t)$. Furthermore $c_I(g_t) \le \sup_I g_t \le c_{[0,t^p]}(f)$. From the definition of Ω it follows that its measure satisfies $|\Omega| < t^p$. Thus also $x_I(g_t) \le b \le t^p$.

We have

$$\begin{aligned} \int_{x_I(g_t)}^{t^p} (c_I(g_t) - f)^p\,dx &\le \int_{x_t}^{t^p} (c_{[0,t^p]}(f) - f)^p\,dx \\ &= \int_0^{x_t} (f - c_{[0,t^p]}(f))^p\,dx \\ &\le \int_0^{x_I(g_t)} (f - c_I(g_t))^p\,dx. \end{aligned} \tag{3.43}$$

Hence we can choose $\xi \in [0, x_I(g_t)]$ such that

$$\int\limits_{\xi}^{x_I(g_t)} (f - c_I(g_t))^p \, dx = \int\limits_{x_I(g_t)}^{t^p} (c_I(g_t) - f)^p \, dx. \tag{3.44}$$

Now we consider the following two alternatives:

$$x_I(g_t) - \xi \geq \frac{t^p - \xi}{2} \quad \text{or} \quad x_I(g_t) - \xi < \frac{t^p - \xi}{2}.$$

In the first case we set

$$k := \frac{\int_0^{x_I(g_t)} (g_t(x) - c_I(g_t))^p \, dx}{\int_0^{x_I(g_t)} (c_{[0,t^p]}(f) - c_I(g_t))^p \, dx} \quad (\leq 1).$$

Then we have

$$\begin{aligned} \frac{1}{|I|} \|g_t - c_I(g_t)\|^p_{L_p(I)} &= \frac{2}{|I|} \int\limits_0^{x_I(g_t)} (g_t(x) - c_I(g_t))^p \, dx \\ &\leq \frac{2k}{x_I(g_t)} \int\limits_0^{x_I(g_t)} (c_{[0,t^p]}(f) - c_I(g_t))^p \, dx \\ &= 2k(c_{[0,t^p]}(f) - c_I(g_t))^p. \end{aligned} \tag{3.45}$$

The last term of (3.45) does not exceed

$$\begin{aligned} &\frac{2k}{x_I(g_t) - \xi} (c_{[0,t^p]}(f) - c_I(g_t))^p (x_I(g_t) - \xi) \\ &\qquad = \frac{2k}{x_I(g_t) - \xi} \int\limits_{\xi}^{x_I(g_t)} (c_{[0,t^p]}(f) - c_I(g_t))^p \, dx \\ &\qquad \leq \frac{2k}{x_I(g_t) - \xi} \int\limits_0^{x_I(g_t)} (c_{[0,t^p]}(f) - c_I(g_t))^p \, dx \\ &\qquad = \frac{2}{x_I(g_t) - \xi} \int\limits_0^{x_I(g_t)} (g_t(x) - c_I(g_t))^p \, dx \\ &\qquad \leq \frac{4}{t^p - \xi} \int\limits_0^{x_I(g_t)} (g_t(x) - c_I(g_t))^p \, dx \\ &\qquad = \frac{4}{t^p - \xi} \int\limits_{x_I(g_t)}^{b} (c_I(g_t) - g_t(x))^p \, dx \end{aligned} \tag{3.46}$$

$$= \frac{4}{t^p - \xi} \int\limits_{x_I(g_t)}^{b} (c_I(g_t) - f(x))^p \, dx$$

$$\le \frac{4}{t^p - \xi} \int\limits_{x_I(g_t)}^{t^p} (c_I(g_t) - f(x))^p \, dx.$$

Since ξ was chosen to satisfy (3.44) the right hand side equals

$$\frac{4}{t^p - \xi} \int\limits_{\xi}^{x_I(g_t)} (f(x) - c_I(g_t))^p \, dx.$$

Since $x_I(g_t) \ge x_t$ and $f = g$ on $[x_t, \infty)$, condition (3.44) amounts to requiring that $c_{[\xi,t^p]}(f) = c_I(g_t)$ and $x_{[\xi,t^p]}(f) = x_I(g_t)$. Consequently, the preceding series of estimates gives

$$\frac{1}{|I|} \|g_t - c_I(g_t)\|_{L_p(I)}^p \le \frac{2}{t^p - \xi} \int\limits_{\xi}^{t^p} |f - c_{[\xi,t^p]}(f)|^p \, dx. \tag{3.47}$$

The term on the right of (3.47) is dominated by $(Cf_p^{\#}(t^p))^p$. So by Lemma 3.3 and Hölder's inequality we have

$$\frac{1}{|I|} \int\limits_I |g_t - (g_t)_I| \, dx \le C(f_p^{\#})^*(t^p) \tag{3.48}$$

where the constant C depends only on p. We have thus established (3.42) in this case.

Let us check that in the remaining case:

$$x_I(g_t) - \xi < \frac{t^p - \xi}{2} \quad \left(\text{or equivalently } t^p - x_I(g_t) \ge \frac{t^p - \xi}{2}\right)$$

the same estimate (3.42) holds. To this end it suffices to put

$$k := \frac{\int_{x_I(g_t)}^{b} (c_I(g_t) - g_t(x))^p \, dx}{\int_{x_I(g_t)}^{b} (c_I(g_t) - f(b))^p \, dx} \qquad (\le 1)$$

and to repeat the previous arguments with this choice of k:

$$\frac{1}{|I|} \|g_t - c_I(g_t)\|_{L_p(I)}^p = \frac{2}{|I|} \int\limits_{x_I(g_t)}^{b} (c_I(g_t) - g_t(x))^p \, dx \tag{3.49}$$

$$\leq \frac{2k}{b - x_I(g_t)} \int\limits_{x_I(g_t)}^{b} (c_I(g_t) - f(b))^p\,dx = 2k(c_I(g_t) - f(b))^p$$

$$\leq \frac{2}{t^p - x_I(g_t)} \Big(\int\limits_{x_I(g_t)}^{b} (c_I(g_t) - g_t(x))^p\,dx + k \int\limits_{b}^{t^p} (c_I(g_t) - f(b))^p\,dx \Big)$$

$$\leq \frac{2}{t^p - x_I(g_t)} \Big(\int\limits_{x_I(g_t)}^{b} (c_I(g_t) - f(x))^p\,dx + \int\limits_{b}^{t^p} (c_I(g_t) - f(x))^p\,dx \Big)$$

$$\leq \frac{4}{t^p - \xi} \int\limits_{x_I(g_t)}^{t^p} (c_I(g_t) - f(x))^p\,dx.$$

By (3.44) this last expression equals $2(t^p - \xi)^{-1} \int_{\xi}^{t^p} |f(x) - c_I(g_t)|^p\,dx$ and by the same arguments as before we again obtain (3.42).

Putting together cases 1–3 we obtain

$$|g_t|_{\mathrm{BMO}} \leq c(f_p^{\#})^*(t^p),$$

where the constant c depends only on p.

The proof of Theorem 3.1 is complete.

There is a striking difference between the cases of monotone functions on a finite and infinite interval. In fact, the following result holds.

Recall that $\mathbf{BMO}(A)$, $A = [0,1)$, denotes the normed BMO-space, defined by the norm

$$\|f\|_{\mathbf{BMO}(A)} := |f|_{\mathrm{BMO}(A)} + \int\limits_0^1 |f|\,dx.$$

THEOREM 3.5. (i) *The cone $M_1[0,1)$ has the WIP with respect to $(L_p[0,1), \mathbf{BMO}[0,1))$, $1 \leq p < \infty$.*

(ii) *$M_1[0,1)$ does not have the IP with respect to $(L_p[0,1), \mathrm{BMO}[0,1))$ for $1 \leq p < \infty$.*

Proof. (i) The first result we need is

$$(L_1(0,1), \mathbf{BMO}(0,1))_{\theta,q} = (L_1(0,1), L_\infty(0,1))_{\theta,q} \tag{3.50}$$

for all $\theta \in (0,1)$ and $q \in [1,\infty]$. One way of proving this is to deduce it from a similar result of Riviere–Sagher [11], namely $(L_1(\mathbb{R}), \mathrm{BMO}(\mathbb{R}))_{\theta,q} = (L_1(\mathbb{R}), L_\infty(\mathbb{R}))_{\theta,q}$. This can be done with the help of a special linear extension operator which maps $L_1(0,1)$ boundedly into $L_1(\mathbb{R})$ and also maps $\mathbf{BMO}(0,1)$ into $\mathbf{BMO}(\mathbb{R})$. This operator is essentially the same as that constructed by Shvartsman ([14], formula (8), p. 31). Here we use the special case where $k = 1$, $P_0 \equiv 0$ and $F = [0,1]$. The proof that this extension operator has the properties mentioned above is contained in [14] (Proposition 1.2.10 and Corollary 1.2.11, p. 36, and Proposition 1.2.14, p. 38). There are also some similarities between this extension operator and another one constructed by Peter Jones [9]. (Although Jones states that his results apply to subsets of $\mathbb{R}^n$ for $n \geq 2$ his methods in fact can also be applied when $n = 1$.)

Alternatively (3.50) can be obtained by modifying another proof of the result of Riviere–Sagher (and also its generalization to $\mathbb{R}^n$ due to Hanks [7]), given by Bennett and Sharpley [2]. For the reader's convenience we briefly indicate how this can be done. We start by formulating the following slight modification of Lemma 1.1 of [2], p. 202.

Suppose that $n = 1$ *and that the open set* Ω *is contained in* $[0,1]$. *Then we can also assume that all dyadic intervals* (*cubes*) *in the sequence having the properties specified in Lemma* 1.1 *are contained in* $[0,1]$.

We show this simply by discarding from the sequence those intervals Q_j which are not contained in $[0,1]$, and therefore also do not overlap $[0,1]$.

The next step is to prove a modified version of Theorem 4.1 of [2], p. 213. Here we assume that f is supported on $(0,1)$. Recall also that Mf and $f^{\#}$ are both redefined by taking the supremum only over intervals contained in $[0,1]$, and are defined to be zero for $x \notin [0,1]$. Thus their rearrangements also vanish off $[0,1]$.

Instead of the estimate (4.8) of Theorem 4.1 we obtain

$$(Mf)^*(t) \leq \int_t^1 (f^{\#})^*(s)\,\frac{ds}{s} + (Mf)^*(1-0) \tag{3.51}$$

for all $t = 2^{-N}$ where $N \in \mathbb{N}$.

The proof begins exactly as in [2], except that we define the set Ω for some arbitrary t restricted to $(0,1/2)$ and apply our *modified* version of Lemma 1.1. All steps are analogous until (4.14). Then letting t tend to 2^{-N} from below gives (3.51) for $t = 2^{-N}$ with left continuous instead of right continuous rearrangements. But the right continuous rearrangement is smaller than the left continuous one so we obtain (3.51) for $t = 2^{-N}$.

Now we need to show that

$$f^{**}(t) \leq 3(Mf)^*(t) \quad \text{for all } t \in (0,1). \tag{3.52}$$

We obtain this by adapting the proof of the first part of Theorem 1.3 of [2], p. 203. Fix $t \in (0,1)$ and suppose that $(Mf)^*(t) < \infty$. The set $\Omega = \{x \in (0,1) \mid (Mf)(x) > (Mf)^*(t)\}$ is open and has measure not exceeding t. We again apply our modified version of Lemma 1.1 and proceed with the sequence of dyadic subintervals ("cubes") of $[0,1]$ that it provides, exactly analogously to the proof in [2].

Next, as a trivial consequence of the Hardy–Littlewood lemma for the maximal function, we find that $(Mf)^*(1-0) \le c\|f\|_{L_1} = cf^{**}(1)$. Combining this with (3.52) and (3.51) gives

$$f^{**}(2^{-N}) \le c\Big(\int_{2^{-N}}^{1} (f^{\#})^*(s)\,\frac{ds}{s} + \|f\|_{L_1}\Big) \quad \text{for } N = 0,1,2,\ldots$$

Now, for any $t \in (0,1]$ choose an integer N so that $2^{-N-1} < t \le 2^{-N}$. Then

$$\begin{aligned}(3.53)\quad f^{**}(t) \le f^{**}(2^{-N-1}) \le 2f^{**}(2^{-N}) &\le 2c\Big(\int_{2^{-N}}^{1} (f^{\#})^*(s)\,\frac{ds}{s} + \|f\|_{L_1}\Big)\\ &\le 2c\Big(\int_{t}^{1} (f^{\#})^*(s)\,\frac{ds}{s} + \|f\|_{L_1}\Big).\end{aligned}$$

We shall again use the estimate (3.17) in the case $p = 1$. As already observed above, this estimate, i.e.

$$(3.54)\qquad t(f^{\#})^*(t) \le cK(t,f;L_1,\mathrm{BMO}),$$

proved in Lemma 4.3 of [2], p. 215, also holds, with almost identical proof, for $t \in (0,1)$, when $(0,1)$ instead of $\mathbb{R}$ is the underlying measure space, and $f^{\#}$ is defined in the way we are using here, i.e. by taking the supremum only over intervals contained in $[0,1]$.

Suppose that $f \in (L_1(0,1),\mathbf{BMO}(0,1))_{\theta,\infty}$ for some $\theta \in (0,1)$. Then

$$K(t,f;L_1(0,1),\mathrm{BMO}(0,1)) \le K(t,f;L_1(0,1),\mathbf{BMO}(0,1)) \le ct^{\theta}.$$

Consequently, from (3.54) and (3.53) we obtain $(f^{\#})^*(t) \le ct^{\theta-1}$ and

$$f^{**}(t) \le c\Big(\int_t^1 s^{\theta-2}\,ds + \|f\|_{L_1}\Big) \le c(t^{\theta-1} + \|f\|_{L_1}) \quad \text{for all } t \in (0,1).$$

From the standard formula for $K(t,f;L_1,L_\infty)$ this shows that $f \in (L_1[0,1], L_\infty[0,1])_{\theta,\infty}$. Since obviously $L_\infty(0,1) \subset \mathbf{BMO}(0,1)$ we have proved (3.50) for the case $q=\infty$. The same formula for general $q \in (0,\infty]$ follows immediately via the reiteration theorem for the real method. A further application of an "endpoint" version of the reiteration theorem shows that

$$(L_p(0,1),\mathbf{BMO}(0,1))_{\theta,q} = (L_p(0,1),L_\infty(0,1))_{\theta,q}$$

for all $p \in [1,\infty)$.

Hence we obtain

$$\begin{aligned}(3.55)\quad (L_p(0,1),\mathbf{BMO}(0,1))_{\theta,q} \cap M_1 &= (L_p(0,1),L_\infty(0,1))_{\theta,q} \cap M_1\\ &= (L_p(0,1)\cap M_1, L_\infty(0,1)\cap M_1)_{\theta,q},\end{aligned}$$

where the latter isomorphism follows from [6] (see Section 1.2.6). This gives the desired embedding

$$(L_p(0,1), \mathbf{BMO}(0,1))_{\theta,q} \cap M_1 \subset (L_p(0,1) \cap M_1, \mathbf{BMO}(0,1) \cap M_1)_{\theta,q},$$

yielding the WIP of $M_1[0,1)$ with respect to $(L_p(0,1), \mathbf{BMO}(0,1))$, since the reverse embedding is trivial.

(ii) To prove the second statement of Theorem 3.5 we have to present an appropriate counterexample. We consider only the case $p = 1$ to omit cumbersome details. For $p > 1$ we can take the same function f_ε, defined below, as a counterexample.

Suppose, on the contrary, that M_1 has the IP with respect to (L_1, BMO), that is,

$$K(f,t; L_1 \cap M_1, \mathrm{BMO} \cap M_1) \le CK(f,t; L_1, \mathrm{BMO}) \tag{3.56}$$

for some $C > 0$ and every $t > 0$ and $f \in M_1$. Put

$$f_\varepsilon(x) := \begin{cases} 0 & \text{if } 0 \le x \le \varepsilon, \\ 1 & \text{if } \varepsilon < x \le 1, \end{cases} \tag{3.57}$$

where $\varepsilon = \min\{1/8, 1/(8C)\}$. By the definition of K-functional,

$$K(f_\varepsilon, t; L_1, \mathrm{BMO}) \le \|f_\varepsilon - g\|_{L_1} = \varepsilon \le \frac{1}{8C} \tag{3.58}$$

for $g \equiv 1$. On the other hand,

$$K(f_\varepsilon, t; L_1 \cap M_1, \mathrm{BMO} \cap M_1) = \inf_{g \in \mathrm{BMO} \cap M_1,\, f_\varepsilon - g \in L_1 \cap M_1} \{\|f_\varepsilon - g\|_{L_1} + t|g|_{\mathrm{BMO}}\}. \tag{3.59}$$

Since $g \in M_1$ and $f_\varepsilon - g \in M_1$, the function g should be as follows:

$$g(x) := \begin{cases} 0 & \text{if } 0 \le x \le \varepsilon, \\ b & \text{if } \varepsilon < x \le 1, \end{cases} \tag{3.60}$$

where $0 \le b \le 1$. Hence

$$\|f_\varepsilon - g\|_{L_1} + t|g|_{\mathrm{BMO}} = (1-b)(1-\varepsilon) + tb/2$$

and therefore

$$K(f_\varepsilon, t; L_1 \cap M_1, \mathrm{BMO} \cap M_1) = \inf_{0 \le b \le 1} \{(1-b)(1-\varepsilon) + tb/2\}. \tag{3.61}$$

Thus for $t > 1/4$ we get

$$K(f_\varepsilon, t; L_1 \cap M_1, \mathrm{BMO} \cap M_1) > \inf_{0 \le b \le 1} \{b/8 + (1-b)(7/8 - \varepsilon)\} \ge 1/8.$$

Together with (3.58) this gives, for $t > 1/4$,

$$K(f_\varepsilon, t; L_1 \cap M_1, \mathrm{BMO} \cap M_1) > CK(f_\varepsilon, t; L_1, \mathrm{BMO}),$$

contrary to (3.56).

The proof of Theorem 3.5 is complete.

4. Convex functions and the couple (L_p, BMO)

THEOREM 4.1. *The cone $M_2[0,1)$ has the IP with respect to $(L_p[0,1), \mathrm{BMO}[0,1))$ for all $p \in [1,\infty)$.*

Proof. Let $M_2 = M_2[0,1)$ be the cone of differentiable non-decreasing convex functions on $[0,1)$ and $f \in M_2$. Given an interval $I = [a,b] \subset [0,1)$ we set

$$\overline{x} = \overline{x}(f,I) := \sup\{x \mid f(x) = f_I\}.$$

LEMMA 4.2. *We have*

$$b - \overline{x} \le (b-a)/2.$$

Proof. Let l be the tangent line to the graph of f at the point $(a+b)/2$. Since f is convex, $l \le f$. Therefore

$$\begin{aligned}(4.62)\qquad \int_a^{(a+b)/2} \left(f\left(\frac{a+b}{2}\right) - f(x)\right) dx &\le \int_a^{(a+b)/2} \left(f\left(\frac{a+b}{2}\right) - l(x)\right) dx \\ &= \int_{(a+b)/2}^{b} \left(l(x) - f\left(\frac{a+b}{2}\right)\right) dx \\ &\le \int_{(a+b)/2}^{b} \left(f(x) - f\left(\frac{a+b}{2}\right)\right) dx.\end{aligned}$$

Now let x^* be the supremum of $y \in [a,b]$ satisfying

$$(4.63)\qquad \int_a^y (f(y) - f(x))\, dx \le \int_y^b (f(x) - f(y))\, dx.$$

Then $x^* \ge (a+b)/2$ by (4.62). Moreover, the left hand side of (4.63) equals the right hand side if $y = x^*$ (by the continuity of f on $[a,b]$). On the other hand,

$$\int_a^{\overline{x}} (f_I - f(x))\, dx = \int_{\overline{x}}^b (f(x) - f_I)\, dx$$

and $f_I = f(\overline{x})$ by the definition of $\overline{x}$. Hence $\overline{x} \ge x^*$ and so $\overline{x} \ge (a+b)/2$.

DEFINITION 4.3. The space $\mathrm{BLO}(I)$ is defined by finiteness of the quantity

$$(4.64)\qquad |f|_{\mathrm{BLO}} = \sup_I \frac{1}{|I|} \int_I (f - \operatorname*{ess\,inf}_I f)\, dx.$$

We can now state the following results for convex functions.

LEMMA 4.4. *Let $f \in L_p[0,1) \cap M_2$ and $1 \le p < \infty$. Then for every interval $I = [a,b] \subset [0,1)$,*

$$\|f - \inf_I f\|_{L_p(I)} \approx \|f - f_I\|_{L_p(I)}$$

with equivalence constants independent of f and I.

Proof. We begin with the inequality

$$\int_I (f - \inf_I f)^p \, dx \le (2^p + 2^{2p-1}) \int_I |f - f_I|^p \, dx. \tag{4.65}$$

To establish this, note that $\inf_I f = f(a)$ and $f_I = f(\overline{x})$. So the left hand side of (4.65) up to a multiplicative constant of 2^{p-1} does not exceed

$$\int_a^{\overline{x}} (f(x) - f(a))^p \, dx + \int_{\overline{x}}^b (f(x) - f(\overline{x}))^p \, dx + \int_{\overline{x}}^b (f(\overline{x}) - f(a))^p \, dx =: J_1 + J_2 + J_3.$$

Then J_2 is clearly majorized by $\int_I |f - f_I|^p \, dx$. Moreover, by the convexity of f,

$$J_1 \le \int_a^{\overline{x}} (f(\overline{x}) - f(x))^p \, dx. \tag{4.66}$$

Finally, according to Lemma 4.2,

$$\begin{aligned} J_3 &\le \int_a^{\overline{x}} (f(\overline{x}) - f(a))^p \, dx \\ &\le 2^{p-1} \Big(\int_a^{\overline{x}} (f(x) - f(a))^p \, dx + \int_a^{\overline{x}} (f(\overline{x}) - f(x))^p \, dx \Big) \\ &\le 2^{p-1} \Big(J_1 + \int_a^{\overline{x}} (f(\overline{x}) - f(x))^p \, dx \Big). \end{aligned} \tag{4.67}$$

From this and (4.66) we get

$$J_3 \le 2^p \int_a^{\overline{x}} (f(\overline{x}) - f(x))^p \, dx \le 2^p \int_I |f - f_I|^p \, dx.$$

Putting together these estimates gives

$$J_1 + J_2 + J_3 \le (2 + 2^p) \int_I |f - f_I|^p \, dx.$$

Now, for the reverse estimate we apply the inequality (see (3.23))

$$\int_I |f - f_I|^p \, dx \le 2^p E_0(f, I)^p_{L_p}$$

and then the evident inequality

$$E_0(f,I)^p_{L_p} \le \int_I (f - \inf_I f)^p \, dx$$

to complete the proof the lemma.

REMARK 4.5. The proof of the latter inequality is valid for an arbitrary function $f \in L_p(I)$. So we have in this case

$$\int_I |f - f_I|^p \, dx \le 2^p \int_I (f - \inf_I f)^p \, dx.$$

On the other hand, analysing the proofs of Lemmas 4.4 and 4.2 one can conclude that inequality (4.65) also holds for functions f in the cone $\widehat{M_2}$ of non-negative non-decreasing convex functions defined on $[0,1)$.

In particular, if we set $p = 1$, we obtain

COROLLARY 4.6. $\mathrm{BLO} \cap M_2 = \mathrm{BMO} \cap M_2$ *with equivalence of "norms".*

LEMMA 4.7. *Let* $f \in L_p(I) \cap M_2$ *and* $g \in W_1^1(I)$. *Suppose that*

$$f'(x) \ge g'(x) \ge 0 \quad \textit{almost everywhere on } I. \tag{4.68}$$

Then there is a constant $C = C(p)$ *such that*

$$\int_I |g - g_I|^p \le C \int_I |f - f_I|^p.$$

Proof. Without loss of generality we can assume that $f(a) = g(a) = 0$ (here $I = [a,b]$). By (4.68) we have (since $g \in W_1^1(I)$ is absolutely continuous)

$$g(x) = \int_a^x g'(t) \, dt \le \int_a^x f'(t) \, dt = f(x) \quad \text{for } x \in I.$$

This and Remark 4.5 give

$$\int_I |g - g_I|^p \le 2^p \int_I (g - g(a))^p \le 2^p \int_I f^p.$$

Since f is convex, the first inequality (4.65) of Lemma 4.4 implies

$$\int_I f^p = \int_I (f - \min_I f)^p \le (2 + 2^p) \int_I |f - f_I|^p.$$

The last two inequalities prove the lemma.

LEMMA 4.8. *If* $f \in L_p \cap M_2$, *then*

$$f_p^{\#}(x) \approx (f_p^{\#})^*(1-x), \quad 0 < x < 1,$$

with equivalence constants independent of f *and* x.

Proof. Fix $f \in M_2[0,1)$. The first step is to show that whenever $0 < x_1 < x_2 < 1$,

$$f_p^{\#}(x_1) \le C f_p^{\#}(x_2) \tag{4.69}$$

where C depends only on p.

Let $I := [\alpha_1, \beta_1]$ be an arbitrary interval such that $x_1 \in [\alpha_1, \beta_1] \subset [0,1)$. Obviously there exists an interval $J := [\alpha_2, \beta_2] \subset [0,1)$ such that $|J| = |I|$ and $x_2 \in J$ and $\alpha_1 \le \alpha_2$.

Since $f \in M_2[0,1)$ we have $f'(\alpha_1 + x) \le f'(\alpha_2 + x)$ for all $x \in [0, |I|)$. So, by Lemma 4.7, for some C depending only on p, we have

$$\left(\frac{1}{|I|}\int_I |f - f_I|^p\right)^{1/p} \le C\left(\frac{1}{|J|}\int_J |f - f_J|^p\right)^{1/p} \le C(f_p^{\#})^*(x_2).$$

Taking the supremum over all I containing x_1 gives (4.69).

Now define the function $g : (0,1) \to \mathbb{R}$ by $g(x) = f(1-x)$. Clearly $g_p^{\#}(x) = f_p^{\#}(1-x)$. So $(g_p^{\#})^*(x) = (f_p^{\#})^*(x)$ for all $x \in (0,1)$. Define $h(x) = \sup_{1 > t \ge x} g_p^{\#}(t)$. By (4.69) we have

$$g_p^{\#}(x) \le h(x) \le C g_p^{\#}(x).$$

Since h is non-increasing and right continuous we have

$$h^* = h \approx (g_p^{\#})^* = (f_p^{\#})^*$$

and the proof of Lemma 4.8 is complete.

We are now in a position to resume the proof of Theorem 4.1 and show that M_2 has the IP with respect to (L_p, BMO), $1 \le p < \infty$. To this end we will once again use the inequality (3.17) of Bennett–Sharpley. As already explained, it must also hold when the spaces are defined on $[0,1)$ or any other interval $[a,b)$ instead of $\mathbb{R}$, and furthermore the constant C appearing in (3.17) is independent of the choice of $[a,b)$.

Thus the desired result will follow from the inequality

$$K(f, t; L_p \cap M_2, \mathrm{BMO} \cap M_2) \le C(p) t (f_p^{\#})^*(t^p). \tag{4.70}$$

We shall assume that f is convex and *non-increasing* (by the substitution $x \mapsto 1-x$). In other words we take the cone $\widetilde{M}_2$ of non-negative differentiable convex non-increasing functions in place of the cone M_2.

Define the function $g_t : [0,1) \to \mathbb{R}_+$ by

$$g_t(x) = \begin{cases} f(t^p) - f'(t^p)(t^p - x) & \text{if } 0 \le x \le t^p, \\ f(x) & \text{if } t^p < x \le 1. \end{cases} \tag{4.71}$$

It is clear that $g_t \in \widetilde{M}_2$ and $f - g_t \in \widetilde{M}_2$. Thus

$$K(f, t; L_p \cap \widetilde{M}_2, \mathrm{BMO} \cap \widetilde{M}_2) \le \|f - g_t\|_{L_p[0,t^p]} + t|g_t|_{\mathrm{BMO}}. \tag{4.72}$$

Lemma 4.4 gives

$$\|f-g_t\|_{L_p[0,t^p]} \le \|f-f(t^p)\|_{L_p[0,t^p]} = \|f-\inf_{[0,t^p]} f\|_{L_p[0,t^p]} \approx \|f-f_{[0,t^p]}\|_{L_p[0,t^p]}. \tag{4.73}$$

The right hand side is, clearly, less than or equal to $tf_p^{\#}(t^p)$.

Now we estimate the second term in (4.72). Because of the convexity of g_t, for each $I=[a,b]\subset(0,1)$ we have

$$g_t'(s) \le g_t'(a+s) \le 0, \quad 0\le s\le |I|.$$

So, applying Lemma 4.7 to the function $g(1-x)$ gives

$$|g_t|_{\mathrm{BMO}} := \sup_I \frac{1}{|I|}\int_I |g_t-(g_t)_I|\,dx \le C\sup_{1>s>0}\frac{1}{s}\int_0^s |g_t-(g_t)_{[0,s]}|\,dx.$$

To estimate the right hand side rewrite it in the form

$$\max\left\{\sup_{s\ge t^p}\frac{1}{s}\int_0^s |g_t-(g_t)_{[0,s]}|\,dx, \sup_{s<t^p}\frac{1}{s}\int_0^s |g_t-(g_t)_{[0,s]}|\,dx\right\}.$$

Since $f'\le g_t'\le 0$, Lemma 4.7 applied to $f(1-t)$ and $g(1-t)$ gives

$$\sup_{s\ge t^p}\frac{1}{s}\int_0^s |g_t-(g_t)_{[0,s]}|\,dx \le C\sup_{s\ge t^p}\frac{1}{s}\int_0^s |f-f_{[0,s]}|\,dx \le C\sup_{I\ni t^p}\frac{1}{|I|}\int_I |f-f_I|\,dx.$$

By Hölder's inequality we have

$$\frac{1}{|I|}\int_I |f-f_I|\,dx \le \left(\frac{1}{|I|}\int_I |f-f_I|^p\,dx\right)^{1/p},$$

and therefore,

$$\sup_{s\ge t^p}\frac{1}{s}\int_0^s |g_t-(g_t)_{[0,s]}|\,dx \le Cf_p^{\#}(t^p). \tag{4.74}$$

Now let $s<t^p$. Taking into account that g_t is linear on $[0,s]$, we have

$$\frac{1}{s}\int_0^s |g_t-(g_t)_{[0,s]}|\,dx = \frac{1}{s}\cdot\frac{s}{2}f'(t^p)\frac{s}{2} = f'(t^p)\frac{s}{4} \le f'(t^p)\frac{t^p}{4}. \tag{4.75}$$

Using (4.75) twice, we therefore get

$$t\sup_{s<t^p}\frac{1}{s}\int_0^s |g_t-(g_t)_{[0,s]}|\,dx \le tf'(t^p)\frac{t^p}{4} = t^{1-p}\int_0^{t^p} |g_t-(g_t)_{[0,t^p]}|\,dx \tag{4.76}$$

$$\le C(p)t^{1-p}\int_0^{t^p}|f-f_{[0,t^p]}|\,dx$$

$$\le C(p)t^{1-p}\Big(\int_0^{t^p}|f-f_{[0,t^p]}|^p\,dx\Big)^{1/p}(t^p)^{(p-1)/p}$$

$$= C(p)\Big(\int_0^{t^p}|f-f_{[0,t^p]}|^p\,dx\Big)^{1/p}\le C(p)tf_p^{\#}(t^p).$$

Putting together (4.72), (4.73), (4.74) and (4.76) gives

$$K(f,t;L_p\cap\widetilde{M}_2,\mathrm{BMO}\cap\widetilde{M}_2)\le Ctf_p^{\#}(t^p).$$

But $f\in\widetilde{M}_2$ (and $f(1-x)\in M_2$). Therefore, we can apply Lemma 4.8 to obtain $(f_p^{\#})^*(t^p)\approx f_p^{\#}(t^p)$. So we finally get

$$K(f,t;L_p\cap M_2,\mathrm{BMO}\cap M_2)\le C(p)t(f_p^{\#})^*(t^p).$$

The proof of Theorem 4.1 is complete.

Remark. The proof shows that essentially the same result is true for the cone $\widetilde{M}_2$.

Using this remark we prove a similar statement for the cone Conv of differentiable non-negative convex functions, defined on $(0,1)$.

Corollary 4.9. *The cone* Conv *has the IP with respect to* (L_p,BMO).

Proof. Let $f\in\mathrm{Conv}$ and let $c\in(0,1)$ be such that $f'(x)\le 0$ for $x\le c$, while $f'(x)>0$ for $x>c$. Without loss of generality we can assume that $f(c)=0$.

Let us write $f=f_1+f_2$, where

$$f_1(x):=\begin{cases} f(x) & \text{if } 0\le x\le c,\\ 0 & \text{otherwise;}\end{cases}\qquad f_2(x):=f(x)-f_1(x).$$

Clearly, $f_1\in\widetilde{M}_2(0,1]$ and $f_2\in M_2[0,1)$. Since each of these cones has the IP with respect to (L_p,BMO) there exist functions f_j^i, $i,j=1,2$, $f_1^i\in\widetilde{M}_2(0,1]$, $f_2^i\in M_2[0,1)$, $f_j=f_j^1+f_j^2$, $i,j=1,2$, such that

$$\begin{aligned}K(f_j,t;L_p\cap Q_j,\mathrm{BMO}\cap Q_j)&\le\|f_j^1\|_{L_p}+t|f_j^2|_{\mathrm{BMO}}\\&\le CK(f_j,t;L_p,\mathrm{BMO}),\end{aligned}\tag{4.77}$$

where $Q_1=\widetilde{M}_2(0,1]$ and $Q_2=M_2[0,1)$. Furthermore, from the proof of Theorem 4.1 (cf. (4.71)) it is clear that f_j^i can be chosen so that $f_j^i(x)=0$ for every $x\ge c$ (resp. $x\le c$) if $j=1$ (resp. $j=2$).

Then we define

$$g^i:=f_1^i+f_2^i.$$

Clearly, $g^i \in \mathrm{Conv}$ and $g^1 + g^2 = f_1 + f_2 = f$. Thus, by (4.77),

$$K(f,t;L_p \cap \mathrm{Conv}, \mathrm{BMO} \cap \mathrm{Conv}) \le \|g^1\|_{L_p} + t\|g^2\|_{\mathrm{BMO}} \le \|f_1^1\|_{L_p} + \|f_2^1\|_{L_p} + t(|f_1^2|_{\mathrm{BMO}} + |f_2^2|_{\mathrm{BMO}}). \tag{4.78}$$

From the proof of Theorem 4.1 we see that these last four terms are dominated by

$$Ct((f_1)_p^{\#})^*(t^p) + Ct((f_2)_p^{\#})^*(t^p). \tag{4.79}$$

All we need then is to show that

$$(f_j)_p^{\#}(t^p) \le Cf_p^{\#}(t^p) \tag{4.80}$$

for all $x \in [0,1]$ and $j = 1,2$. This implies that each of the terms in (4.79) is dominated by $Ctf_p^{\#}(t^p)$ and so we can apply (3.17) to finish the proof.

We prove (4.80) with the help of Lemma 4.4. Given any $I \subset [0,1]$ we have to show that

$$\int_I |f_j(x) - (f_j)_I|^p\,dx \le C \int_I |f(x) - f_I|^p\,dx. \tag{4.81}$$

Obviously this is true (even with $C = 1$) if $c \notin I$. On the other hand, if $I = [a,b]$ and $a \le c \le b$ we have (cf. Lemma 4.4)

$$\begin{aligned}\int_I |f_j(x) - (f_j)_I|^p\,dx &\le C_1 \int_I |f_j(x) - \inf_I f_j|^p\,dx \\ &\le C_1 \int_{[a,c]} |f_1(x) - f_1(c)|^p\,dx \\ &\quad + C_1 \int_{[c,b]} |f_2(x) - f_2(c)|^p\,dx.\end{aligned} \tag{4.82}$$

Now applying Lemma 4.4 once more to the last two integrals, and also using (3.23) we see that the sum of the last two integrals is dominated by

$$\begin{aligned}&C_2\Big(\inf_{\alpha\in\mathbb{R}} \int_a^c |f_1 - \alpha|^p\,dx + \inf_{\beta\in\mathbb{R}} \int_c^b |f_2 - \beta|^p\,dx\Big) \\ &\le C_2 \inf_{\gamma\in\mathbb{R}} \Big(\int_a^c |f_1 - \gamma|^p\,dx + \int_c^b |f_2 - \gamma|^p\,dx\Big) = C_2 \inf_{\gamma\in\mathbb{R}} \Big(\int_a^b |f - \gamma|^p\,dx\Big) \\ &\le C_2 \int_a^b |f - f_{[a,b]}|^p\,dx.\end{aligned} \tag{4.83}$$

So we have proved (4.81) and completed the proof of the corollary.

Acknowledgments. I am deeply grateful to Professor Brudnyĭ and Professor Cwikel for their inspiring advice concerning this research. I would also like to thank the referee for useful remarks.

References

[1] I. Asekritova, *On the K-functional of the couple* $(K_{\Phi_0}(\overline{X}), K_{\Phi_1}(\overline{X}))$, in: Studies in the Theory of Functions of Several Real Variables, Yaroslavl', 1980, 3–31 (in Russian).

[2] C. Bennett and R. Sharpley, *Weak-type inequalities for H_p and BMO*, in: Proc. Sympos. Pure Math. 35, Part 1, Amer. Math. Soc., 1979, 201–229.

[3] J. Bergh and J. Löfström, *Interpolation Spaces*, Springer, Berlin, 1976.

[4] Yu. Brudnyĭ and N. Krugljak, *Interpolation Functors and Interpolation Spaces*, Vol. 1, North-Holland, 1991.

[5] M. J. Carro, S. Ericsson and L.-E. Persson, *Real interpolation for divisible cones*, Proc. Edinburgh Math. Soc. 42 (1999), 285–304.

[6] J. Cerdà and J. Martín, *Interpolation of operators on decreasing functions*, Math. Scand. 78 (1996), 233–245.

[7] R. Hanks, *Interpolation by the real method between BMO, L^α $(0 < \alpha < \infty)$ and H^α $(0 < \alpha < \infty)$*, Indiana Univ. Math. J. 26 (1977), 679–689.

[8] T. Holmstedt, *Interpolation of quasi-normed spaces*, Math. Scand. 26 (1970), 177–199.

[9] P. Jones, *Extension theorems for BMO*, Indiana Univ. Math. J. 29 (1980), 41–66.

[10] I. Kozlov, *Interpolation and approximation in Banach spaces with a cone*, Ph.D. thesis, 1997.

[11] N. M. Riviere and Y. Sagher, *Interpolation between L^∞ and H^1, the real method*, J. Funct. Anal. 14 (1973), 401–409.

[12] Y. Sagher, *Some remarks on interpolation of operators and Fourier coefficients*, Studia Math. 44 (1972), 239–252.

[13] —, *An application of interpolation theory to Fourier series*, ibid. 41 (1972), 169–181.

[14] P. A. Shvartsman, *Extension theorems with preservation of local polynomial approximation*, Yaroslavl' State University, Yaroslavl', 1986 (in Russian).

Electro-Optics Research and Development Ltd.
Technion
Haifa 32000, Israel
E-mail: eorddik@techunix.technion.ac.il

Received September 11, 2000 (4470)