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On L_1-subspaces of holomorphic functions

by

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Abstract. We study the spaces

$$H_\mu(\Omega) = \Big\{ f : \Omega \to \mathbb{C} \text{ holomorphic} : \int_0^R \int_0^{2\pi} |f(re^{i\varphi})| \, d\varphi \, d\mu(r) < \infty \Big\}$$

where Ω is a disc with radius R and μ is a given probability measure on $[0, R[$. We show that, depending on μ, $H_\mu(\Omega)$ is either isomorphic to l_1 or to $(\sum \oplus A_n)_{(1)}$. Here A_n is the space of all polynomials of degree $\le n$ endowed with the L_1-norm on the unit sphere.

1. Introduction. Let $R > 0$ and $\Omega = R\cdot\mathbb{D} = \{z \in \mathbb{C} : |z| < R\}$, or $\Omega = \mathbb{C}$ and $R = \infty$. We want to study Banach spaces of holomorphic functions endowed with a norm $\int_\Omega |f(z)| \, d\nu(z)$ where ν is a given bounded positive measure on Ω. In the present note we consider the case $d\nu(re^{i\varphi}) = d\varphi d\mu(r)$ where μ is a given bounded positive measure on $[0, R[$. We put

$$M_1(f, r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\varphi})| \, d\varphi, \qquad \|f\|_\mu = \int_0^R M_1(f, r) \, d\mu(r)$$

and

$$H_\mu = H_\mu(\Omega) = \{f : \Omega \to \mathbb{C} \text{ holomorphic} : \|f\|_\mu < \infty\}.$$

Recall that $M_1(f, r)$ is increasing in r if f is holomorphic. It is easily seen that H_μ is a Banach space. We can assume without loss of generality that

$$\mu([r, R[) > 0 \quad \text{for any } r < R. \tag{1.1}$$

(Otherwise we restrict our functions to $\rho\mathbb{D}$ where $\rho = \sup\{\tau : \mu([r, \tau[) > 0$ for all $r < \tau\}$ and put $R = \rho$.) Moreover we assume

$$\int_0^R r^n \, d\mu(r) < \infty \quad \text{for any } n \ge 0. \tag{1.2}$$

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Certainly (1.2) is automatically satisfied if $R < \infty$. But in the case of entire functions, without (1.2), H_μ might be finite-dimensional.

We easily see that the polynomials are dense in H_μ. Indeed, fix $f \in H_\mu$. Let $\sigma_n f$ be the nth Cesàro mean of f (see Section 4 below for definition). Then $\sigma_n f \to f$ pointwise as $n \to \infty$, and $M_1(\sigma_n f, r) \le M_1(f, r)$ for all n. Moreover $\sigma_n f$ is a polynomial. The dominated convergence theorem implies $\lim_{n\to\infty} \|\sigma_n f - f\|_\mu = 0$.

Examples. (i) $\Omega = \mathbb{D}$: $d\mu_1 = r\,dr$. H_{μ_1} is the classical Djrbashian or Bergman space ([2, 3]). It is known to be isomorphic to l_1 ([10, 5]). Even if we consider $d\mu_2 = (1-r)^\alpha r^\beta dr$ for some $\alpha \ge 0$ and $\beta \ge 0$ we have $H_{\mu_2} \sim l_1$ (see e.g. [11]; "$\sim$" means "is isomorphic to"). Furthermore consider

$$d\mu_3 = \frac{dr}{(1-r)\log^\gamma(e/(1-r))} \quad \text{for some } \gamma > 1, \quad d\mu_4 = \sum_{k=1}^{\infty} \frac{1}{k(k+1)}\, \delta_{1-2^{-k}}$$

(δ_a is the Dirac measure at a). It was shown in [7] that in both cases H_μ is not isomorphic to l_1.

(ii) $\Omega = \mathbb{C}$: Consider $d\mu_5 = e^{-r}dr$ and $d\mu_6 = e^{-\log^2 r}dr$. μ_5 was investigated e.g. in [4]. There it was shown that $H_{\mu_5} \sim l_1$ (see Section 2).

We want to give a complete isomorphic classification of the Banach spaces $H_\mu(\Omega)$. To this end let A_n be the space of all polynomials of degree $\le n$ endowed with the norm $M_1(\cdot, 1)$.

1.1. Theorem. *Each H_μ is isomorphic to either l_1 or $(\sum_{n=1}^\infty \oplus A_n)_{(1)}$.*

Theorem 1.1 is an extension of [8] where a similar result was shown only for measures on $[0,1[$ under additional rather restrictive assumptions on μ excluding many examples. To decide to which isomorphism class a given space H_μ belongs we focus on purely non-atomic measures μ. This is no restriction since we have

1.2. Proposition. *Let μ be any probability measure on $[0,R[$ and $\epsilon > 0$. Then there is a purely non-atomic bounded measure μ_0 on $[0,R[$ such that $H_\mu = H_{\mu_0}$ and*

$$(1-\epsilon)\|f\|_\mu \le \|f\|_{\mu_0} \le \|f\|_\mu, \qquad f \in H_\mu.$$

Let us assume now that μ is purely non-atomic. Fix $b \ge 5$. Then we use induction to define $0 \le m_1 < m_2 < \cdots$ and $0 \le s_1 < s_1 < \cdots < R$ as follows. Put $m_1 = 0$. If we already have m_n, consider s_n with

$$\int_0^{s_n} r^{m_n}\, d\mu = b \int_{s_n}^{R} r^{m_n}\, d\mu. \tag{1.3}$$

Then find $m_{n+1} > m_n$ with

$$\int_0^{s_n} r^{m_{n+1}}\,d\mu = \int_{s_n}^{R} r^{m_{n+1}}\,d\mu. \tag{1.4}$$

It is easily seen that $\lim_{n\to\infty} s_n = R$ and $\lim_{n\to\infty} m_n = \infty$. We have

1.3. THEOREM. *There are $c_1 > 0$, $c_2 > 0$ and $t_{n,k} \ge 0$ such that*

$$c_1\|f\|_\mu \le \sum_{n=1}^{\infty} M_1(T_n f, s_n)\biggl(\int_0^{s_n}\Bigl(\frac{r}{s_n}\Bigr)^{m_{n-1}} d\mu + \int_{s_n}^{R}\Bigl(\frac{r}{s_n}\Bigr)^{m_{n+1}} d\mu\biggr) \le c_2\|f\|_\mu$$

for all $f \in H_\mu$ where $T_n(\sum_{k=0}^{\infty}\alpha_k z^k) = \sum_{m_{n-1}\le k<m_{n+1}} \alpha_k t_{n,k} z^k$.

Moreover:

1.4. THEOREM. *$H_\mu \sim l_1$ if and only if there are $\alpha,\beta,\gamma > 0$ such that, for each n,*

$$\alpha \le \frac{m_{n+1}-m_n}{m_n-m_{n-1}} \le \beta \quad \text{or} \quad m_{n+1}-m_{n-1} \le \gamma. \tag{1.5}$$

The paper is organized as follows. In Section 2 we discuss the two examples on $\mathbb{C}$ that we already mentioned, and compute explicitly the indices m_n. In Section 3 we prove Proposition 1.2 while in Section 4 we collect a few technical lemmas. Then we prove Theorem 1.3 in Section 5. Section 6 is dedicated to the proofs of Theorems 1.1, 1.4 and 1.5 (below).

Our results have many similarities with the isomorphic classification of weighted sup-norm spaces of holomorphic functions ([9]). However, they cannot be inferred directly from those results via duality. This follows e.g. from [11, Theorem 2] which states that, if H_μ with a "weighting" measure μ is the dual of a weighted sup-norm space, then H_μ is complemented in an L_1-space.

Finally, we note that the isomorphic classification for the spaces

$$H_{p,\mu} = \Bigl\{f : \Omega \to \mathbb{C} \text{ holomorphic} : \int_0^R\int_0^{2\pi} |f(re^{i\varphi})|^p\,d\varphi\,d\mu(r) < \infty\Bigr\}$$

is much easier if $1 < p < \infty$.

1.5. THEOREM. *If $1 < p < \infty$ then $H_{p,\mu}$ is always isomorphic to l_p.*

For the proof see end of Section 6.

2. Two examples. Here we construct explicitly the indices m_n mentioned in Theorem 1.3 and 1.4 for two examples.

(a) Put $d\mu(r) = \exp(-\log^2 r)dr$. Then, using the substitution

$$r = \exp(s/\sqrt{2} + (m+1)/2),$$

for any $x \ge 0$ and $m \ge 0$ we obtain

$$\int_0^x r^m e^{-\log^2 r} dr = \frac{e^{(m+1)^2/4}}{\sqrt{2}} \int_{-\infty}^{(\log x-(m+1)/2)\sqrt{2}} e^{-s^2/2}\, ds.$$

In particular, $\int_0^\infty r^m \exp(-\log^2 r)\, dr = \sqrt{\pi}\exp((m+1)^2/4)$. Using the tables of the normal distribution ([1]) we get, for fixed m_n and $s_n = \exp(1.3/\sqrt{2} + (m_n+1)/2)$,

$$\int_0^{s_n} r^{m_n} e^{-\log^2 r}\, dr = c\sqrt{\pi}\, e^{(m_n+1)^2/4} \quad \text{where} \quad c \ge 0.9.$$

Hence

$$\int_0^{s_n} r^{m_n} e^{-\log^2 r} dr = b \int_{s_n}^\infty r^{m_n} e^{-\log^2 r}\, dr \quad \text{where} \quad b = \frac{c}{1-c}, \ \text{i.e. } b \ge 9.$$

Now if

$$m_{n+1} = m_n + \sqrt{2}\cdot 1.3 \tag{2.1}$$

we have $\exp(1.3/\sqrt{2} + (m_n+1)/2) = \exp((m_{n+1}+1)/2)$. Hence

$$\int_0^{s_n} r^{m_{n+1}} e^{-\log^2 r}\, dr = \frac{e^{(m_{n+1}+1)^2/4}}{\sqrt{2}} \int_{-\infty}^0 e^{-s^2/2}\, ds$$
$$= \frac{\sqrt{\pi}}{2}\, e^{(m_{n+1}+1)^2/4} = \int_{s_n}^\infty r^{m_{n+1}} e^{-\log^2 r}\, dr.$$

Now (2.1) tells us that the assumptions of Theorem 1.4 are satisfied. Hence $H_\mu \sim l_1$. Moreover, since $\sup_n (m_{n+1} - m_n) < \infty$ the "summands" in the equivalent norm in Theorem 1.3 have uniformly bounded length. This cannot happen for any measure on $[0, R[$ if $R < \infty$ (see Proposition 2.1).

(b) We next consider the measure $d\mu(r) = \exp(-r)dr$ on $[0,\infty[$. Here

$$\int_0^\infty r^m \exp(-r)\, dr = \Gamma(m+1)$$

is the gamma function. Using the substitution $t = 2r$ we obtain, for any $x > 0$, $\int_0^x r^m \exp(-r)\, dr = 2^{-m-1}\int_0^{2x} t^m \exp(-t/2)\, dt$, which is the distribution function (up to the factor $\Gamma(m+1)^{-1}$) of a χ^2-distribution. A well-known limit theorem ([1, 26.4.11]) tells us that

$$\lim_{m\to\infty} \left(\frac{1}{2^{m+1}\Gamma(m+1)} \int_0^{2x} t^m e^{-t/2}\, dt\right)\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x-m-1)/\sqrt{m+1}} e^{-t^2/2}\, dt\right)^{-1} = 1.$$

So, if $s_n = 1.3\sqrt{m_n+1} + m_n + 1$ we have $(s_n - m_n - 1)/\sqrt{m_n+1} = 1.3$ and $\int_0^{s_n} r^{m_n} \exp(-r)\, dr \sim c\Gamma(m_n+1)$ where $c \ge 0.9$. Hence $\int_0^{s_n} r^{m_n} \exp(-r)\, dr \sim b\int_{s_n}^\infty r^{m_n} \exp(-r)\, dr$ where $b \ge 9$ if n is large enough. If we put

(2.2) $$m_{n+1} = m_n + 1.3\sqrt{m_n + 1}$$

then

$$\int_0^{s_n} r^{m_{n+1}} e^{-r}\, dr \sim \frac{\Gamma(m_{n+1}+1)}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-t^2/2}\, dt = \frac{\Gamma(m_{n+1}+1)}{2}.$$

Thus $\int_0^{s_n} r^{m_{n+1}} \exp(-r)\, dr \sim \int_{s_n}^{\infty} r^{m_{n+1}} \exp(-r)\, dr$. Using this and (2.2), Theorem 1.4 again shows that $H_\mu \sim l_1$.

Next we prove that for $R < \infty$ the length of the summands in Theorem 1.3 necessarily tends to ∞.

2.1. PROPOSITION. *Let μ be a purely non-atomic probability measure on $[0, R[$ where $R < \infty$. Fix $b > 1$ and, for any $m > 0$, pick t_m with $\int_0^{t_m} r^m\, d\mu = b \int_{t_m}^{R} r^m\, d\mu$.*

(a) *For any a with $0 < a < R$ we have*

$$\lim_{m\to\infty} \frac{\int_0^{t_m} r^m\, d\mu}{\int_a^{t_m} r^m\, d\mu} = 1.$$

(b) *If $n = n(m)$ is such that $\int_0^{t_m} r^n\, d\mu = \int_{t_m}^{R} r^n\, d\mu$ then $\lim_{m\to\infty}(n(m) - m) = \infty$.*

Proof. (a) First we observe

$$\frac{\int_0^{t_m} r^m\, d\mu}{\int_a^{t_m} r^m\, d\mu} = \frac{\int_0^{a} (r/a)^m\, d\mu + \int_a^{t_m} (r/a)^m\, d\mu}{\int_a^{t_m} (r/a)^m\, d\mu}.$$

Clearly, $\lim_{m\to\infty} \int_0^a (r/a)^m\, d\mu = 0$ and $\lim_{m\to\infty} \int_a^{t_m} (r/a)^m\, d\mu = \infty$ since $\lim_{m\to\infty} t_m = R$. This proves (a).

(b) Assume that there are a constant $c > 0$ and, for all k, numbers $m_k > k$ with $n(m_k) - m_k \le c$. Fix $a < R$ and $\epsilon > 0$ such that $(b-\epsilon)(a/R)^c > 1$. Let k be large enough and pick $t = t_{m_k}$, $n = n(m_k)$ such that $\int_a^t r^{m_k}\, d\mu \ge (b-\epsilon) \int_t^R r^{m_k}\, d\mu$ and $\int_a^t r^n\, d\mu \ne 0$. This is possible in view of (a). Since $n > m_k$ we obtain

$$\begin{aligned}
\int_a^t \left(\frac{r}{R}\right)^n d\mu &\ge \left(\frac{a}{R}\right)^c \int_a^t \left(\frac{r}{R}\right)^{m_k} d\mu \ge \left(\frac{a}{R}\right)^c (b-\epsilon) \int_t^R \left(\frac{r}{R}\right)^{m_k} d\mu \\
&\ge \left(\frac{a}{R}\right)^c (b-\epsilon) \int_t^R \left(\frac{r}{R}\right)^{n} d\mu \\
&\ge \left(\frac{a}{R}\right)^c (b-\epsilon) \int_0^t \left(\frac{r}{R}\right)^{n} d\mu \ge \left(\frac{a}{R}\right)^c (b-\epsilon) \int_a^t \left(\frac{r}{R}\right)^{n} d\mu.
\end{aligned}$$

This is a contradiction since $(a/R)^c(b-\epsilon) > 1$. ■

3. Approximation by purely non-atomic measures. First we show

3.1. LEMMA. *Let* $0 < r < s$ *and* $0 < m < n$.

(a) *If* $f(z) = \sum_{m<k\le n, k\in\mathbb{Z}} \alpha_k z^k$ *then* $M_1(f,r) \le (r/s)^m M_1(f,s)$.
(b) *If* $g(z) = \sum_{0\le k\le n, k\in\mathbb{Z}} \alpha_k z^k$ *then* $M_1(g,s) \le (s/r)^n M_1(g,r)$.

Proof. (a) Put $h(z) = \sum_{0\le k\le n-[m]-1,\, k\in\mathbb{Z}} \alpha_{k+[m]+1} z^k$ where $[m]$ is the largest integer $\le m$. Then $f(z) = z^{[m]+1}h(z)$ and

$$\begin{aligned} M_1(f,r) = r^{[m]+1}M_1(h,r) &\le r^{[m]+1}M_1(h,s) = (r/s)^{[m]+1}M_1(f,s) \\ &\le (r/s)^m M_1(f,s). \end{aligned}$$

(b) Put $h_1(z) = g(1/z)$ and $h_2(z) = z^{[n]}g(1/z)$. Then

$$\begin{aligned} M_1(g,s) &= M_1(h_1,1/s) = s^{[n]}M_1(h_2,1/s) \\ &\le s^{[n]}M_1(h_2,1/r) = (s/r)^{[n]}M_1(h_1,1/r) \\ &= (s/r)^{[n]}M_1(g,r) \le (s/r)^n M_1(g,r). \ \blacksquare \end{aligned}$$

3.2. LEMMA. *Let* $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k \in H_\mu$.

(a) $|\alpha_k|s^k\mu([s,R[) \le \|f\|_\mu$ *for any* k *and* $s \in [0,R[$.
(b) *For any* $r_0 \in [0,R[$, $n_0 > 0$ *and* $\epsilon > 0$ *there is* $n \ge n_0$ (*independent of* f) *such that* $M_1(f-f_n,r) \le \epsilon\|f\|_\mu$ *if* $r \le r_0$ *where* $f_n(z) = \sum_{k=0}^{n} \alpha_k z^k$.
(c) *For any* $r_0 \in]0,R[$ *and any* $\epsilon > 0$ *there is* $r_1 < r_0$, *independent of* f, *such that* $r_0 - r_1 < \epsilon$ *and*

$$(1-\epsilon)M_1(f,r_0) - \epsilon\|f\|_\mu \le \frac{1}{r_0-r_1}\int_{r_1}^{r_0} M_1(f,r)\,dr \le M_1(f,r_0).$$

Proof. (a) Clearly we have $|\alpha_k|s^k \le M_1(f,s)$. Hence

$$|\alpha_k|s^k\mu([s,R[) \le \int_s^R M_1(f,r)\,d\mu \le \|f\|_\mu.$$

(b) Fix s with $r_0 < s < R$. Let $n > n_0$ be such that

$$\left(\frac{r_0}{s}\right)^{n+1}\frac{s}{s-r_0} \le \epsilon\mu([s,R[).$$

Then, for any $r \le r_0$,

$$\begin{aligned} M_1(f-f_n,r) \le M_1(f-f_n,r_0) &\le \sum_{k=n+1}^{\infty} |\alpha_k|s^k\left(\frac{r_0}{s}\right)^k \\ &\le \left(\frac{r_0}{s}\right)^{n+1}\left(\frac{s}{s-r_0}\right)\frac{\|f\|_\mu}{\mu([s,R[)} \le \epsilon\|f\|_\mu. \end{aligned}$$

(c) The second inequality is trivial. To prove the first inequality we use (b) to obtain, for any $\delta > 0$, some n with

$$M_1(f_n,r) - \delta\|f\|_\mu \le M_1(f,r) \le M_1(f_n,r) + \delta\|f\|_\mu \quad \text{if } r \le r_0.$$

Hence, if $r_1 < r_0$ then, by Lemma 3.1(b),

$$\begin{aligned}\frac{1}{r_0 - r_1}\int_{r_1}^{r_0} M_1(f,r)\,dr &\ge \frac{1}{r_0 - r_1}\int_{r_1}^{r_0} M_1(f_n,r)\,dr - \delta\|f\|_\mu \\ &\ge \frac{M_1(f_n,r_0)}{r_0 - r_1}\int_{r_1}^{r_0}\left(\frac{r}{r_0}\right)^n dr - \delta\|f\|_\mu \\ &\ge \left(\frac{r_1}{r_0}\right)^n M_1(f_n,r_0) - \delta\|f\|_\mu \\ &\ge \left(\frac{r_1}{r_0}\right)^n M_1(f,r_0) - \left(1 + \left(\frac{r_1}{r_0}\right)^n\right)\delta\|f\|_\mu.\end{aligned}$$

Now put $\delta = \epsilon/2$ and take r_1 so close to r_0 that $(r_1/r_0)^n \ge 1 - \epsilon$. ■

3.3. *Proof of Proposition 1.2.* Split μ into $\mu = \nu + \mu_1$ where ν is purely non-atomic and $\mu_1 = \sum_k \alpha_k \delta_{s_k}$ for some positive α_k with $\sum_k \alpha_k \le 1$ and some s_k with $0 \le s_k < R$. Fix $\epsilon > 0$ and let $0 < \epsilon' < \epsilon$ be such that $1 - 2\epsilon' \ge 1 - \epsilon$. Find $r_k < s_k$ with

$$(1 - \epsilon')M_1(f,s_k) - \epsilon'\|f\|_\mu \le \frac{1}{s_k - r_k}\int_{r_k}^{s_k} M_1(f,r)\,dr \le M_1(f,s_k),$$

which is possible according to Lemma 3.2. Put

$$d\mu_0 = d\nu + \sum_k \frac{\alpha_k}{s_k - r_k}\,1_{[r_k,s_k]}dr.$$

Then we obtain $(1 - 2\epsilon')\|f\|_\mu \le \|f\|_{\mu_0} \le \|f\|_\mu$ for all $f \in H_\mu$. This implies Proposition 1.2. ■

4. Classical convolution operators. For a harmonic function $f : \Omega \to \mathbb{C}$ with $f(re^{i\varphi}) = \sum_{k\in\mathbb{Z}} \alpha_k r^{|k|} e^{ik\varphi}$ and $n > m > 0$ let

$$(\sigma_n f)(re^{i\varphi}) = \sum_{|k|<n,k\in\mathbb{Z}} \frac{[n] - |k|}{[n]}\alpha_k r^{|k|} e^{ik\varphi} \tag{4.1}$$

and

$$V_{n,m}f = \frac{[n]\sigma_n f - [m]\sigma_m f}{[n] - [m]} \quad \text{if } [m] < [n].$$

Hence

$$(4.2)\qquad (V_{n,m}f)(re^{i\varphi}) = \sum_{|k|\le m,\, k\in\mathbb{Z}} \alpha_k r^{|k|} e^{ik\varphi} + \sum_{m<|k|<n,\, k\in\mathbb{Z}} \frac{[n]-|k|}{[n]-[m]}\, \alpha_k r^{|k|} e^{ik\varphi}.$$

(4.2) also makes sense if $[m] = [n]$. Then $V_{n,m}$ is a Dirichlet projection. Finally put $(Rf)(z) = \sum_{0\le k} \alpha_k z^k$.

In the following lemma fix $r > 0$ and let $\|T\|$ be the norm of a bounded operator on the space of all harmonic functions f with $M_1(f,r) < \infty$ (endowed with the norm $M_1(\cdot, r)$).

4.1. Lemma. *We have*

(a) $\|V_{n,m}\| \le \dfrac{[n]+[m]}{[n]-[m]}$.

(b) $M_1(Rh,r) \le \left(1 + \dfrac{[n]-[m]}{[m]}\right) M_1(h,r)$ *for any* $r > 0$ *and* $h \in \operatorname{span}\{r^{|k|}\exp(ik\varphi) : k \in \mathbb{Z},\, m < |k| \le n\}$.

(c) $\|V_{n_4,n_3} - V_{n_2,n_1}\| \le 4\left(\dfrac{[n_4]-[n_1]}{[n_2]-[n_1]}\right)\left(3 + 4\dfrac{[n_4]-[n_1]}{[n_4]-[n_3]}\right)$ *if* $0 < n_1 < n_2 < n_3 < n_4$.

(d) $\|V_{n_4,n_3} - V_{n_2,n_1}\| \le 2([n_4]-[n_1])$ *and* $\|R(V_{n_4,n_3} - V_{n_2,n_1})\| \le [n_4]-[n_1]$. *for any* n_k, $k = 1,\dots,4$, *with* $0 < n_1 < n_2 < n_3 < n_4$.

The proof is literally the same as the proof of [9, 3.3. Lemma].

In the following lemma we restrict the preceding operators to holomorphic functions.

4.2. Lemma. *Fix* $b > 0$, $c > 1/b$ *and* $0 < m < n$, $0 < s < R$ *such that*

$$(4.3)\qquad \int_0^s r^m\, d\mu \ge b \int_s^R r^m\, d\mu \quad \text{and} \quad \int_s^R r^n\, d\mu \ge c \int_0^s r^n\, d\mu$$

(a) *Consider* $f(z) = \sum_{0\le k\le m,\, k\in\mathbb{Z}} \alpha_k z^k$ *and* $g(z) = \sum_{k\ge n,\, k\in\mathbb{Z}} \alpha_k z^k$. *Then*

$$\|f\|_\mu \le \frac{b+1}{bc_1 - c_2}\, \|f+g\|_\mu \quad \textit{with} \quad c_1 = \min(c,1),\ c_2 = \min(1/c,1).$$

(b) *We have*

$$\|V_{n,m}h\|_\mu \le \left(181\, \frac{b+1}{bc_1 - c_2} + 88\right)\|h\|_\mu \quad \textit{for all } h \in H_\mu.$$

Proof. (a) For $s \le r$ we have $M_1(f,r) \le (r/s)^m M_1(f,s)$ according to Lemma 3.1. Then (4.3) implies

$$\int_s^R M_1(f,r)\,d\mu \le M_1(f,s)\int_s^R \left(\frac{r}{s}\right)^m d\mu \le \frac{1}{b}\,M_1(f,s)\int_0^s \left(\frac{r}{s}\right)^m d\mu$$
$$\le \frac{1}{b}\int_0^s M_1(f,r)\left(\frac{s}{r}\right)^m \left(\frac{r}{s}\right)^m d\mu = \frac{1}{b}\int_0^s M_1(f,r)\,d\mu.$$

Hence $\int_0^R M_1(f,r)\,d\mu \le (1+1/b)\int_0^s M_1(f,r)\,d\mu$. Similarly we obtain

$$c\int_0^s M_1(g,r)\,d\mu \le cM_1(g,s)\int_0^s \left(\frac{r}{s}\right)^n d\mu \le M_1(g,s)\int_s^R \left(\frac{r}{s}\right)^n d\mu$$
$$\le \int_s^R M_1(g,r)\left(\frac{s}{r}\right)^n \left(\frac{r}{s}\right)^n d\mu = \int_s^R M_1(g,r)\,d\mu.$$

This implies

$$\int_0^R M_1(f+g,r)\,d\mu \ge c_1\int_0^s M_1(f+g,r)\,d\mu + c_2\int_s^R M_1(f+g,r)\,d\mu$$
$$\ge c_1\int_0^s M_1(f,r)\,d\mu - c_1\int_0^s M_1(g,r)\,d\mu + c_2\int_s^R M_1(g,r)\,d\mu - c_2\int_s^R M_1(f,r)\,d\mu$$
$$\ge \left(c_1 - \frac{c_2}{b}\right)\int_0^s M_1(f,r)\,d\mu \ge \frac{bc_1-c_2}{b+1}\int_0^R M_1(f,r)\,d\mu.$$

This proves (a).

(b) If $[n]\ge 2[m]$ then $\|V_{n,m}\| \le ([n]+[m])([n]-[m])^{-1} \le 3$ in view of Lemma 4.1.

Now let $[n] < 2[m]$, i.e. $2[m]-[n] > 0$. Put

$$h(z) = \sum_{k=0}^{\infty} \alpha_k z^k, \qquad \tilde f(z) = \sum_{k\le m} \alpha_k z^k, \qquad \tilde g(z) = \sum_{k\ge n} \alpha_k z^k.$$

Moreover, let $T = V_{2n-m,n} - V_{m,2m-n}$ and $S = V_{n,m}T$. In view of (4.2) this means $S = V_{n,m} - V_{m,2m-n}$. Lemma 4.1 implies $\|T\| \le 180$ and $\|S\| \le 88$. Finally, put $f = (\mathrm{id}-T)\tilde f$ and $g = (\mathrm{id}-T)\tilde g$. Then we obtain $h = f+g+Th$, $V_{n,m}f = f$ and $V_{n,m}g = 0$. Now (a) yields

$$\|V_{n,m}h\|_\mu = \|f+Sh\|_\mu \le \|f\|_\mu + \|Sh\|_\mu$$
$$\le \frac{b+1}{bc_1-c_2}\,\|f+g\|_\mu + \|Sh\|_\mu$$
$$\le \frac{b+1}{bc_1-c_2}\,\|f+g+Th\|_\mu + \|Sh\|_\mu + \frac{b+1}{bc_1-c_2}\,\|Th\|_\mu$$
$$\le \left(181\,\frac{b+1}{bc_1-c_2} + 88\right)\|h\|_\mu. \ \blacksquare$$

4.3. Lemma. *Let* $0 \le m < n < p$ *and* $f(z) = \sum_{m \le k \le p,\, k \in \mathbb{Z}} \alpha_k z^k$. *Then*

$$M_1(V_{p,n} f, r) \le 2M_1(f, r) \quad \textit{and} \quad M_1(V_{n,m} f, r) \le M_1(f, r)$$

for any $r > 0$.

Proof. Let $(U_j f)(re^{i\varphi}) = e^{ij\varphi} f(re^{i\varphi})$. Then we have

$$V_{n,m} f = U_{[m]} \sigma_{[n]-[m]} U_{-[m]} f \quad \text{and} \quad V_{p,n} f = U_{[p]}(id - \sigma_{[p]-[n]}) U_{-[p]} f.$$

This implies Lemma 4.3 since the Cesàro means as well as the operators U_j are all contractive. ∎

5. Proof of Theorem 1.3. We need a few lemmas.

5.1. Lemma. *Let* $0 \le m \le n$ *and* $s \in [0, R[$. *Assume there is* $c > 0$ *with*

$$\int_0^s r^m \, d\mu \le c \int_s^R r^m \, d\mu \quad \textit{and} \quad \int_s^R r^n \, d\mu \le c \int_0^s r^n \, d\mu.$$

Then, for any $f(z) = \sum_{m \le k \le n, k \in \mathbb{Z}} \alpha_k z^k$, *we have*

$$\|f\|_\mu \le \left(\int_0^s \left(\frac{r}{s}\right)^m d\mu + \int_s^R \left(\frac{r}{s}\right)^n d\mu \right) M_1(f, s) \le c\|f\|_\mu.$$

Proof. Using Lemma 3.1 we get

$$\begin{aligned}
\int_0^R M_1(f, r) \, d\mu &\le M_1(f, s) \left(\int_0^s \left(\frac{r}{s}\right)^m d\mu + \int_s^R \left(\frac{r}{s}\right)^n d\mu \right) \\
&\le c M_1(f, s) \left(\int_s^R \left(\frac{r}{s}\right)^m d\mu + \int_0^s \left(\frac{r}{s}\right)^n d\mu \right) \\
&\le c \int_s^R M_1(f, r) \left(\frac{s}{r}\right)^m \left(\frac{r}{s}\right)^m d\mu + c \int_0^s M_1(f, r) \left(\frac{s}{r}\right)^n \left(\frac{r}{s}\right)^n d\mu \\
&= c \int_0^R M_1(f, r) \, d\mu. \ \blacksquare
\end{aligned}$$

5.2. Lemma. *Fix* $b > 1$ *and* $0 < c < b$. *Let* $0 \le m < n$ *and* $0 \le s < t < R$ *be such that*

$$\int_0^s r^m \, d\mu \le c \int_s^R r^m \, d\mu \quad \textit{and} \quad \int_0^t r^n \, d\mu \ge b \int_t^R r^n \, d\mu.$$

Then, for any $f(z) = \sum_{m \le k \le n,\, k \in \mathbb{Z}} \alpha_k z^k$, *we have*

$$\|f\|_\mu \le (1 + c) \left(\frac{b + 1 + c}{b - c} \right) \int_s^t M_1(f, r) \, d\mu.$$

Proof. First we obtain

$$\begin{aligned}
\|f\|_\mu &\le M_1(f,s)\int_0^s \left(\frac{r}{s}\right)^m d\mu + \int_s^R M_1(f,r)\,d\mu \\
&\le cM_1(f,s)\int_s^R \left(\frac{r}{s}\right)^m d\mu + \int_s^R M_1(f,r)\,d\mu \\
&\le c\int_s^R M_1(f,r)\left(\frac{s}{r}\right)^m \left(\frac{r}{s}\right)^m d\mu + \int_s^R M_1(f,r)\,d\mu \\
&= (1+c)\int_s^R M_1(f,r)\,d\mu.
\end{aligned}$$

Moreover

$$\begin{aligned}
\int_s^R M_1(f,r)\,d\mu &\le \int_s^t M_1(f,r)\,d\mu + M_1(f,t)\int_t^R \left(\frac{r}{t}\right)^n d\mu \\
&\le \int_s^t M_1(f,r)\,d\mu + \frac{M_1(f,t)}{b}\int_0^t \left(\frac{r}{t}\right)^n d\mu \\
&\le \left(1+\frac{1}{b}\right)\int_s^t M_1(f,r)\,d\mu + \frac{M_1(f,t)}{b}\int_0^s \left(\frac{r}{t}\right)^n d\mu \\
&\le \left(1+\frac{1}{b}\right)\int_s^t M_1(f,r)\,d\mu + \frac{M_1(f,s)}{b}\left(\frac{t}{s}\right)^n\int_0^s \left(\frac{r}{t}\right)^n d\mu \\
&\le \left(1+\frac{1}{b}\right)\int_s^t M_1(f,r)\,d\mu + \frac{M_1(f,s)}{b}\int_0^s \left(\frac{r}{s}\right)^m d\mu \\
&\le \left(1+\frac{1}{b}\right)\int_s^t M_1(f,r)\,d\mu + c\frac{M_1(f,s)}{b}\int_s^t \left(\frac{r}{s}\right)^m d\mu \\
&\quad + c\,\frac{M_1(f,s)}{b}\int_t^R \left(\frac{r}{s}\right)^m d\mu \\
&\le \left(1+\frac{1+c}{b}\right)\int_s^t M_1(f,r)\,d\mu + \frac{c}{b}\int_t^R M_1(f,r)\,d\mu \\
&\le \frac{b+c+1}{b}\int_s^t M_1(f,r)\,d\mu + \frac{c}{b}\int_s^R M_1(f,r)\,d\mu.
\end{aligned}$$

This implies

$$\int_s^R M_1(f,r)\,d\mu \le \frac{b+c+1}{b-c}\int_s^t M_1(f,r)\,d\mu$$

and hence

$$\|f\|_\mu \le (1+c)\left(\frac{b+c+1}{b-c}\right)\int_s^t M_1(f,r)\,d\mu. \ \blacksquare$$

5.3. Lemma. *Let $b>1$, $0<m<n$, $0<s<t<R$ and assume that*

$$\int_0^s r^m\,d\mu \le b\int_s^R r^m\,d\mu,\quad \int_0^s r^n\,d\mu = \int_s^R r^n\,d\mu,\quad \int_0^t r^n\,d\mu = b\int_t^R r^n\,d\mu.$$

Then there is $N=N(b)$ with $\int_0^t r^m\,d\mu \le 3^N b\int_t^R r^m\,d\mu$; N does not depend on m, n, s, t.

Proof. For $j=0,1,\ldots,$ put $b_j=3^j b$, $c_j=(2b_j)^{-1}$. Moreover put $t_0=s$. Find $t_0<t_1<t_2<\cdots$ with

$$\int_{t_{j-1}}^{t_j} r^n\,d\mu = c_{j-1}\int_0^{t_{j-1}} r^n\,d\mu. \tag{5.1}$$

We actually take

$$t_j=\sup\Big\{u>t_{j-1}: \int_{t_{j-1}}^{u} r^n\,d\mu = c_{j-1}\int_0^{t_{j-1}} r^n\,d\mu\Big\}.$$

Then we claim

$$\int_0^{t_j} r^m\,d\mu \le 3^j b\int_{t_j}^R r^m\,d\mu. \tag{5.2}$$

We prove (5.2) by induction. (5.2) is clear if $j=0$. Assume it holds for some j. Then we obtain

$$\int_{t_j}^{t_{j+1}}\left(\frac{r}{t_j}\right)^m d\mu \le \int_{t_j}^{t_{j+1}}\left(\frac{r}{t_j}\right)^n d\mu = c_j\int_0^{t_j}\left(\frac{r}{t_j}\right)^n d\mu$$
$$\le c_j\int_0^{t_j}\left(\frac{r}{t_j}\right)^m d\mu.$$

Hence

$$\int_{t_j}^{t_{j+1}} r^m\,d\mu \le c_j\int_0^{t_j} r^m\,d\mu \le b_j c_j\int_{t_j}^R r^m\,d\mu$$
$$=\frac{1}{2}\int_{t_{j+1}}^R r^m\,d\mu + \frac{1}{2}\int_{t_j}^{t_{j+1}} r^m\,d\mu.$$

This implies $\int_{t_j}^{t_{j+1}} r^m\,d\mu \le \int_{t_{j+1}}^{R} r^m\,d\mu$ and

$$\begin{aligned}\int_0^{t_{j+1}} r^m\,d\mu &\le \int_0^{t_j} r^m\,d\mu + \int_{t_j}^{t_{j+1}} r^m\,d\mu \\ &\le 3^j b\int_{t_j}^{R} r^m\,d\mu + \int_{t_j}^{t_{j+1}} r^m\,d\mu \\ &= 3^j b \int_{t_{j+1}}^{R} r^m\,d\mu + (3^j b+1)\int_{t_j}^{t_{j+1}} r^m\,d\mu \\ &\le (2\cdot 3^j b+1)\int_{t_{j+1}}^{R} r^m\,d\mu \le 3^{j+1} b\int_{t_{j+1}}^{R} r^m\,d\mu.\end{aligned}$$

We claim that there is N, depending only on b, such that $t_N \ge t$, which proves the lemma in view of (5.2). Indeed, (5.1) implies

$$\begin{aligned}\int_0^{t_{j+1}} r^n\,d\mu &= \int_0^{t_j} r^n\,d\mu + \int_{t_j}^{t_{j+1}} r^n\,d\mu = (c_j+1)\int_0^{t_j} r^n\,d\mu \\ &= (c_j+1)(c_{j-1}+1)\int_0^{t_{j-1}} r^n\,d\mu = \dots = \prod_{j=0}^{j}(c_j+1)\int_0^{s} r^n\,d\mu.\end{aligned}$$

On the other hand we have

$$\int_0^t r^n\,d\mu = \frac{b}{b+1}\int_0^R r^n\,d\mu = \frac{2b}{b+1}\int_0^s r^n\,d\mu.$$

To prove the claim we need to show $\prod_{j=0}^{\infty}(c_j+1) > 2b(b+1)^{-1}$ since $f(u) = (\int_0^u r^n\,d\mu)(\int_0^s r^n\,d\mu)^{-1}$ is increasing. Indeed,

$$\prod_{j=0}^{\infty}(c_j+1) = \left(\frac{2b+1}{2b}\right)\left(\frac{2\cdot 3b+1}{2\cdot 3b}\right)\left(\frac{2\cdot 3^2 b+1}{2\cdot 3^2 b}\right)\dots \ge 2b+1 > \frac{2b}{b+1}. \ \blacksquare$$

Conclusion of the proof of Theorem 1.3. Consider m, n, s_n with (1.3) and (1.4) for $b \ge 5$. Take a polynomial $f \in H_\mu$ and put

$$T_n f = (V_{m_{n+1},m_n} - V_{m_n,m_{n-1}})f. \tag{5.3}$$

Here take $V_{m_1,m_{-1}} = 0$, i.e. $T_1 = V_{m_2,m_1} f$. (Recall that only finitely many summands are different from zero since f is a polynomial.)

Then we have $f = \sum_n T_n f$. An application of Lemma 5.2 with $s = s_{n-2}$ and $t = s_{n+1}$ yields $\|T_n f\|_\mu \le d_1 \int_{s_{n-2}}^{s_{n+1}} M_1(T_n f, r)\,d\mu$ for a universal constant d_1 (independent of f and n). We claim that there is another universal constant d_2 with

$$\int_{s_{n-2}}^{s_{n+1}} M_1(T_n f, r)\,d\mu \le d_2 \int_{s_{n-2}}^{s_{n+1}} M_1(f, r)\,d\mu. \tag{5.4}$$

Then we conclude

$$\begin{aligned}\|f\|_\mu &\le \sum_n \|T_n f\|_\mu \le d_1 \sum_n \int_{s_{n-2}}^{s_{n+1}} M_1(T_n f, r)\,d\mu \\ &\le d_1 d_2 \sum_n \int_{s_{n-2}}^{s_{n+1}} M_1(f, r)\,d\mu \le 3 d_1 d_2 \int_0^R M_1(f, r)\,d\mu.\end{aligned} \tag{5.5}$$

Now we apply Lemma 5.3 with $s = s_{n-1}$ and $t = s_n$ to obtain $\int_0^{s_n} r^{m_{n-1}}\,d\mu \le 3^N b \int_{s_n}^R r^{m_{n-1}}\,d\mu$. Since we also have $\int_{s_n}^R r^{m_{n+1}}\,d\mu = \int_0^{s_n} r^{m_{n+1}}\,d\mu$ Lemma 5.1 implies

$$\|T_n f\|_\mu \le \left(\int_0^{s_n} \left(\frac{r}{s_n}\right)^{m_{n-1}} d\mu + \int_{s_n}^R \left(\frac{r}{s_n}\right)^{m_{n+1}} d\mu\right) M_1(T_n f, s_n) \le d_3 \|T_n f\|_\mu$$

for some universal constant d_3. Since the polynomials are dense in H_μ this together with (5.5) proves Theorem 1.3.

It remains to show (5.4). To this end we apply Lemma 4.2 for the measure $d\nu = 1_{[s_{n-2}, s_{n+1}]}\,d\mu$. We prove

$$\begin{aligned}\int_{s_n}^{s_{n+1}} r^{m_{n+1}}\,d\mu &\ge \frac{b-1}{b+1} \int_{s_{n-2}}^{s_n} r^{m_{n+1}}\,d\mu, \\ \frac{b-1}{2} \int_{s_n}^{s_{n+1}} r^{m_n}\,d\mu &\le \int_{s_{n-2}}^{s_n} r^{m_n}\,d\mu.\end{aligned} \tag{5.6}$$

Then V_{m_{n+1}, m_n} is uniformly bounded on H_ν since $(b-1)^2(2b+2)^{-1} > 1$ if $b \ge 5$.

Moreover we show

$$\begin{aligned}\frac{b-1}{b+1} \int_{s_{n-2}}^{s_{n-1}} r^{m_n}\,d\mu &\le \int_{s_{n-1}}^{s_{n+1}} r^{m_n}\,d\mu, \\ \frac{b-1}{2} \int_{s_{n-1}}^{s_{n+1}} r^{m_{n-1}}\,d\mu &\le \int_{s_{n-2}}^{s_{n-1}} r^{m_{n-1}}\,d\mu.\end{aligned} \tag{5.7}$$

By Lemma 4.2, $V_{m_n, m_{n-1}}$ is uniformly bounded on H_ν since we have $(b-1)^2(2b+2)^{-1} > 1$ if $b \ge 5$. Hence T_n is uniformly bounded on H_ν, which proves (5.4).

To show (5.6) we note that, by (1.4), $\int_0^{s_n} r^{m_{n+1}}\,d\mu = 2^{-1} \int_0^R r^{m_{n+1}}\,d\mu$, and by (1.3), $\int_0^{s_{n+1}} r^{m_{n+1}}\,d\mu = b(b+1)^{-1} \int_0^R r^{m_{n+1}}\,d\mu$. Hence

$$\int_{s_n}^{s_{n+1}} r^{m_{n+1}}\,d\mu = \frac{b-1}{2b+2}\int_0^R r^{m_{n+1}}\,d\mu = \frac{b-1}{b+1}\int_0^{s_n} r^{m_{n+1}}\,d\mu \ge \frac{b-1}{b+1}\int_{s_{n-2}}^{s_n} r^{m_{n+1}}\,d\mu.$$

Similarly we have $\int_{s_{n-1}}^{s_n} r^{m_n}\,d\mu = (b-1)(2b+2)^{-1}\int_0^R r^{m_n}\,d\mu$ and therefore

$$\begin{aligned}\int_{s_n}^{s_{n+1}} r^{m_n}\,d\mu &\le \int_{s_n}^{R} r^{m_n}\,d\mu = \frac{1}{b}\int_0^{s_n} r^{m_n}\,d\mu \\ &= \frac{1}{b}\int_0^{s_{n-1}} r^{m_n}\,d\mu + \frac{1}{b}\int_{s_{n-1}}^{s_n} r^{m_n}\,d\mu \\ &= \frac{1}{2b}\int_0^R r^{m_n}\,d\mu + \frac{1}{b}\int_{s_{n-1}}^{s_n} r^{m_n}\,d\mu \\ &= \left(\frac{2(b+1)}{2b(b-1)} + \frac{1}{b}\right)\int_{s_{n-1}}^{s_n} r^{m_n}\,d\mu \le \frac{2}{b-1}\int_{s_{n-2}}^{s_n} r^{m_n}\,d\mu,\end{aligned}$$

which shows (5.6).

To prove (5.7) we start with

$$\begin{aligned}\int_{s_{n-2}}^{s_{n-1}} r^{m_n}\,d\mu &\le \int_0^{s_{n-1}} r^{m_n}\,d\mu = \frac{1}{2}\int_0^R r^{m_n}\,d\mu \\ &= \frac{1}{2}\cdot\frac{2b+2}{b-1}\int_{s_{n-1}}^{s_n} r^{m_n}\,d\mu \le \frac{b+1}{b-1}\int_{s_{n-1}}^{s_{n+1}} r^{m_n}\,d\mu.\end{aligned}$$

Furthermore

$$\begin{aligned}\int_{s_{n-2}}^{s_{n-1}} r^{m_{n-1}}\,d\mu &= \frac{b-1}{2b+2}\int_0^R r^{m_{n-1}}\,d\mu \\ &= \frac{b-1}{2}\int_{s_{n-1}}^{R} r^{m_{n-1}}\,d\mu \ge \frac{b-1}{2}\int_{s_{n-1}}^{s_{n+1}} r^{m_{n-1}}\,d\mu,\end{aligned}$$

which completes the proof of (5.7). ∎

6. Final proofs. Now we consider sequences (m_n) and (s_n) satisfying (1.3) and (1.4) for some $b \ge 5$. Let T_n be as in Theorem 1.3 (see (5.3)). Using (4.2) we see that

$$T_n f = 0 \quad \text{if } f \in \operatorname{span}\{z^k : |k| \le m_{n-1} \text{ or } |k| \ge m_{n+1}\}.$$

In particular

$$(6.1)\quad T_nT_{n'} = 0 \text{ if } |n-n'| \ge 2, \qquad T_n(T_{n-1}+T_n+T_{n+1}) = T_n \text{ for all } n.$$

Put $X = (\sum_n \oplus A_n)_{(1)}$. In [8] it was shown that

$$(6.2)\qquad X \sim (X \oplus X \oplus \ldots)_{(1)} \sim \Big(\sum_k \oplus A_{n_k}\Big)_{(1)} \quad \text{if } \sup_k n_k = \infty.$$

Moreover put

$$(6.3)\qquad a_n = \int_0^{s_n} \left(\frac{r}{s_n}\right)^{m_{n-1}} d\mu + \int_{s_n}^{R} \left(\frac{r}{s_n}\right)^{m_{n+1}} d\mu.$$

Let $B_n = \operatorname{span}\{z^k : m_{n-1} \le k \le m_{n+1}\}$ be endowed with the norm $M_1(\cdot, s_n)a_n$.

For any function h and $s > 0$ put $h_s(z) = h(sz)$. If $0 \le m < n$, $f(z) = \sum_{0 \le k \le n-[m]} \alpha_k z^k$ and $g(z) = z^{[m]} f(z)$ then we have $s^{[m]} M_1(f_s, 1) = M_1(g, s)$. This implies that $A_{[m_{n+1}]-[m_{n-1}]}$ and B_n are isometrically isomorphic.

6.1. PROPOSITION.

(a) *H_μ is isomorphic to a complemented subspace of $(\sum_n \oplus A_n)_{(1)}$.*
(b) *If (m_n) satisfies* (1.5) *then H_μ is isomorphic to a complemented subspace of l_1.*

Proof. (a) It suffices to show that H_μ is isomorphic to a complemented subspace of $(\sum_n \oplus B_n)_{(1)}$.

Define $Sf = (T_n f)$, $f \in H_\mu$. Then S is an isomorphism into $(\sum_n \oplus B_n)_{(1)}$ according to Theorem 1.3. For $(g_n) \in (\sum_n \oplus B_n)_{(1)}$ put

$$P(g_n) = \left(T_n\left(\frac{a_{n-1}}{a_n} g_{n-1} + g_n + \frac{a_{n+1}}{a_n} g_{n+1}\right)\right).$$

Then Lemma 4.3 tells us that P is well-defined and bounded. (Recall that $V_{m_{n+1},m_n}|_{B_{n-1}} = \mathrm{id}_{B_{n-1}}$ and $V_{m_n,m_{n-1}}|_{B_{n+1}} = 0$.) Using (6.1) we see that P is a projection onto SH_μ.

(b) Let L_n be the completion of $\{f : f \text{ a trigonometric polynomial}\}$ with respect to $M_1(\cdot, s_n)a_n$. Then L_n is an L_1-space which contains B_n. All B_n are finite-dimensional. Hence we find finite-dimensional spaces $C_n \supset B_n$ consisting of trigonometric polynomials, where $C_n \subset L_n$, such that $\sup_n d(C_n, l_1^{\dim C_n}) < \infty$. Here $d(X_1, X_2) = \inf\{\|T\| \cdot \|T^{-1}\| : T : X_1 \to X_2 \text{ an onto-isomorphism}\}$ is the Banach–Mazur distance. Then clearly $(\sum_n \oplus C_n)_{(1)} \sim l_1$. By definition all $V_{n,m}$ are well-defined on the C_k (see (4.2)) and the norm estimates of Lemma 4.1 hold for $M_1(\cdot, s_n)a_n$ instead of $M_1(\cdot, r)$. So the operators T_n are well-defined on all C_k. Again (6.1) holds.

For $(h_n) \in (\sum_n \oplus C_n)_{(1)}$ put

$$Q(h_n) = \left(RT_n\left(\frac{a_{n-1}}{a_n}\, h_{n-1} + h_n + \frac{a_{n+1}}{a_n}\, h_{n+1}\right)\right).$$

(Recall R is the Riesz projection.) By (1.5), according to Lemma 4.1, the operators RT_n are uniformly bounded. Indeed, for any $r > 0$ we have

$$\begin{aligned} M_1(RT_nh, r) &\le \left(1 + \frac{m_{n+1} - m_n + m_n - m_{n-1}}{m_{n-1}}\right) M_1(T_nh, r) \\ &\le \left(1 + (\beta^2 + \beta)\, \frac{m_{n-1} - m_{n-2}}{m_{n-1}}\right) M_1(T_nh, r) \\ &\le (1 + \beta + \beta^2) M_1(T_nh, r) \end{aligned}$$

unless $m_{n+1} - m_{n-1} \le \gamma$ or $m_n - m_{n-2} \le \gamma$. In the latter cases we get similar estimates. In any case we obtain

$$M_1(RT_nh, r) \le \max((1 + \beta + \beta^2), (1 + (\beta + 1)\gamma)) M_1(T_nh, r).$$

Hence in view of (6.1), Q is a well-defined bounded projection onto SH_μ. ■

We need another lemma.

6.2. Lemma.

(a) *Fix $p, q \in \mathbb{Z}_+$ and let $N = \{p + jq : j \in \mathbb{Z}\} \cap \mathbb{Z}_+$. For $f(z) = \sum_{k=0}^\infty \alpha_k z^k$ put $(P_N f)(z) = \sum_{k\in N} \alpha_k z^k$. Then $M_1(P_N f, r) \le M_1(f, r)$ for any $r > 0$.*

(b) *Let $n_1, n_2 \in \mathbb{Z}_+$ and $m \le \min(n_1, n_2)$. Then there is an isometry $i : A_m \to (A_{n_1} \oplus A_{n_2})_{(1)}$ and a projection $Q : (A_{n_1} \oplus A_{n_2})_{(1)} \to i(A_m)$ with $\|Q\| \le 2$ and $Q(z^j, 0) = 0 = Q(0, z^j)$ for all $j \ge m$.*

(c) *Let $n_1, n_2 \in \mathbb{Z}_+$ and $m \le \min(n_1, n_2)$. Then there is an isometry $j : A_m \to (A_{n_1} \oplus A_{n_2})_{(1)}$ and a projection $P : (A_{n_1} \oplus A_{n_2})_{(1)} \to j(A_m)$ with $\|P\| \le 2$ and $P(z^l, 0) = 0 = P(0, z^l)$ for all $l \le \min(n_1, n_2) - m$.*

Proof. (a) The proof is the same as the proof of [9, Lemma 4.3].

(b) This is [8, Lemma 2.1].

(c) Let i and Q be as in (b). Let $S : (A_{n_1} \oplus A_{n_2})_{(1)} \to (A_{n_1} \oplus A_{n_2})_{(1)}$ be the isometry with $S(z^l, z^k) = (z^{n_1-l}, z^{n_2-k})$. Then put $j = S \circ i$ and $P = SQS^{-1}$. ■

Recall that if $|n_1 - n_2| \ge 2$ then $(B_{n_1} \oplus B_{n_2})_{(1)}$ is isomorphic to a subspace of H_μ. Indeed, take $f_k \in B_{n_k}$, $k = 1, 2$. Then, by Lemmas 5.1 and 5.3,

$$c_1\|f_k\|_\mu \le M_1(f_k, s_{n_k}) a_{n_k} \le c_2\|f_k\|_\mu, \quad k = 1, 2, \tag{6.4}$$

for universal constants c_1, c_2. We have

$$\sum_n T_n(f_1 + f_2) = T_{n_1-1}f_1 + T_{n_1}f_1 + T_{n_1+1}f_1 + T_{n_2-1}f_2 + T_{n_2}f_2 + T_{n_2+1}f_2$$

in view of (6.1). Hence Theorem 1.3 implies

$$d_1\|f_1+f_2\|_\mu \le \|f_1\|_\mu + \|f_2\|_\mu \le d_2\|f_1+f_2\|_\mu \tag{6.5}$$

for universal constants d_1, d_2.

6.3. PROPOSITION. *Assume that* (m_n) *does not satisfy* (1.5). *Then* H_μ *contains a complemented subspace isomorphic to* $(\sum_n \oplus A_n)_{(1)}$.

Proof. Case 1: There are $0 < n_1 < n_2 < \cdots$ with

$$(m_{n_k} - m_{n_k-1})k \le m_{n_k+1} - m_{n_k} \quad \text{and} \quad k \le m_{n_k+1} - m_{n_k-1}$$

for all k. Put $q_k = [m_{n_k}] - [m_{n_k-1}]$ and $N_k = \{[m_{n_k}] + jq_k : j \in \mathbb{Z}\} \cap \mathbb{Z}_+$. Recall that in view of (4.2) we have

$$\begin{aligned} P_{N_k}(T_{n_k} + T_{n_k+1})\Big(\sum_{k=0}^{\infty}\alpha_k z^k\Big) \\ = \sum_{\substack{[m_{n_k}]\le j\le[m_{n_k+1}] \\ j\in N_k}} \alpha_j z^j + \sum_{\substack{m_{n_k+1}<j<m_{n_k+2} \\ j\in N_k}} \alpha_j\gamma_j z^j \end{aligned} \tag{6.6}$$

for some γ_j. Let $p_k = \max\{j \in N_k : [m_{n_k}] + jq_k < m_{n_k+2}\}$ and $p'_k = \max\{j \in N_k : [m_{n_k}] + jq_k \le [m_{n_k+1}]\}$. Moreover let $S_k : (A_{p_k} \oplus A_{p_{k+1}})_{(1)} \to (B_{n_k} \oplus B_{n_{k+1}})_{(1)}$ be defined by

$$S_k(f,g) = \bigg(\Big(\frac{z}{s_{n_k}}\Big)^{[m_{n_k}]} f\Big(\Big(\frac{z}{s_{n_k}}\Big)^{q_k}\Big), \Big(\frac{z}{s_{n_{k+1}}}\Big)^{[m_{n_{k+1}}]} f\Big(\Big(\frac{z}{s_{n_{k+1}}}\Big)^{q_{k+1}}\Big)\bigg),$$

which is an isometry. Put $l_k = \min(p'_k, p'_{k+1})$. Let $i : A_{l_k} \to (A_{p_k} \oplus A_{p_{k+1}})_{(1)}$ be an isometry and $\tilde{Q}_k : (A_{p_k} \oplus A_{p_{k+1}})_{(1)} \to i(A_{l_k})$ a projection with $\|\tilde{Q}_k\| \le 2$ and $\tilde{Q}_k(z^j, 0) = 0 = \tilde{Q}_k(0, z^j)$ if $j \le l_k$ (Lemma 6.2(b)). Then put, for $f \in H_\mu$,

$$Q_k f = S_k\tilde{Q}_k S_k^{-1}(T_{n_k}f, T_{n_{k+1}}f) \in (B_{n_k} \oplus B_{n_{k+1}})_{(1)}.$$

The latter space can be identified with a subspace of H_μ (by (6.4) and (6.5)). Taking (6.6) into account we see that Q_k is a projection onto a space which is isomorphic to A_{l_k}. Then $Qf = \sum_k Q_{2k}f$ is a bounded projection onto a subspace of H_μ which is isomorphic to $(\sum_k \oplus A_{l_{2k}})_{(1)}$. This proves the proposition in Case 1 (in view of (6.2)).

Case 2: There are $0 < n_1 < n_2 < \cdots$ with

$$(m_{n_k+1} - m_{n_k})k \le m_{n_k} - m_{n_k-1} \quad \text{and} \quad k \le m_{n_k+1} - m_{n_k-1}.$$

Then proceed exactly as in Case 1 and use Lemma 6.2(c) instead of (b). ■

Concluding remarks. If (1.5) is satisfied then H_μ is complemented in l_1, hence isomorphic to l_1 ([6]). If (1.5) is not satisfied then, using Pełczyński's decomposition method, (6.2), Proposition 6.1(b) and Proposition 6.3

we see that $H_\mu \sim (\sum_n \oplus A_n)_{(1)}$. The spaces A_n are never uniformly complemented in l_1. Therefore $(\sum_n \oplus A_n)_{(1)}$ cannot be isomorphic to l_1. This finishes the proofs of Theorems 1.1 and 1.4.

For $H_{p,\mu}$, $1 < p < \infty$, we proceed exactly as before. Here we can replace $V_{n,m}$ by the Dirichlet projections $V_{m,m}$ and use that $\int_0^{2\pi} |(V_{m,m}f)(re^{i\varphi})|^p\, d\varphi \le c\int_0^{2\pi} |f(re^{i\varphi})|^p\, d\varphi$ where c is independent of r, m and f. Then we conclude that $H_{p,\mu}$ is always complemented in l_p and hence isomorphic to l_p.

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