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On the Hermite expansions of functions from the Hardy class

by

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Abstract. Considering functions f on $\mathbb{R}^n$ for which both f and $\hat{f}$ are bounded by the Gaussian $e^{-\frac{1}{2}a|x|^2}, 0 < a < 1$, we show that their Fourier–Hermite coefficients have exponential decay. Optimal decay is obtained for $O(n)$-finite functions, thus extending a one-dimensional result of Vemuri.

1. Introduction. Consider the normalised Hermite functions Φ_α, $\alpha \in \mathbb{N}^n$, on $\mathbb{R}^n$ which are eigenfunctions of the Hermite operator $H = -\Delta + |x|^2$ with eigenvalues $2|\alpha| + n$. They form an orthonormal basis for $L^2(\mathbb{R}^n)$ so that every $f \in L^2(\mathbb{R}^n)$ has the expansion

$$f = \sum_{\alpha \in \mathbb{N}^n} (f, \Phi_\alpha)\Phi_\alpha.$$

When the Hermite coefficients of f have exponential decay, say $|(f, \Phi_\alpha)| \le Ce^{-(2|\alpha|+n)t}$ for some $t > 0$, then by Mehler's formula (see [Th1]) it can be easily shown that f satisfies the estimate

$$|f(x)| \le Ce^{-\frac{1}{2}\tanh(t)|x|^2}.$$

As Φ_α are also eigenfunctions of the Fourier transform with eigenvalues $(-i)^{|\alpha|}$, it follows that $|(\hat{f}, \Phi_\alpha)| \le Ce^{-(2|\alpha|+n)t}$ and hence $\hat{f}$ also satisfies the same estimate as f.

However, it is possible to prove better estimates for f and $\hat{f}$. The assumption on (f, Φ_α) together with the asymptotic properties of holomorphically extended Hermite functions lead us to the fact that f extends to $\mathbb{C}^n$ as an entire function and satisfies

$$|f(x + iy)| \le C_m(1 + |x|^2 + |y|^2)^{-m} e^{-\frac{1}{2}\tanh(2s)|x|^2 + \frac{1}{2}\coth(2s)|y|^2}$$

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for every $m \in \mathbb{N}$ and $0 < s < t$, and a similar estimate holds for $\hat{f}$ as well. Indeed, under the assumption on (f, Φ_α) the entire function $f(z)$ belongs to the Hermite Bergman space $\mathcal{H}_s(\mathbb{C}^n)$ consisting of entire functions which are square integrable with respect to the weight function

$$U_s(x, y) = e^{\tanh(2s)|x|^2 - \coth(2s)|y|^2}$$

for every $s < t$ and hence as shown in [RT] the functions $f(z)$ and $\hat{f}(z)$ both satisfy the above estimate.

Suppose we only know that f and $\hat{f}$ are bounded on $\mathbb{R}^n$ by the Gaussian $e^{-\frac{1}{2}\tanh(2t)|x|^2}$. We would like to know if these conditions in turn imply some exponential decay of the Hermite coefficients of f. It will be so if we can prove that $f(z)$ satisfies

$$|f(x + iy)|^2 \le Ce^{-\tanh(2s)|x|^2 + \coth(2s)|y|^2}$$

for some $s > 0$. Under the assumption on f and $\hat{f}$ it is clear, from the Fourier inversion formula, that f extends to $\mathbb{C}^n$ as an entire function which satisfies

$$|f(x + iy)| \le Ce^{\frac{1}{2}\coth(2t)|y|^2}.$$

But a priori it is not at all clear if $f(x + iy)$ has any decay in x. In this article we address the problem of estimating f on $\mathbb{C}^n$.

This problem has connections with a classical theorem of Hardy [H] proved in 1933 which says that a function f and its Fourier transform $\hat{f}$ cannot both have arbitrary Gaussian decay. The precise statement is as follows. For a function $f \in L^1(\mathbb{R}^n)$, let

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\, dx$$

be its Fourier transform. Suppose

$$|f(x)| \le Ce^{-a|x|^2}, \quad |\hat{f}(\xi)| \le Ce^{-b|\xi|^2}$$

for some positive constants a and b. Then $f = 0$ when $ab > 1/4$ and $f(x) = Ce^{-a|x|^2}$ when $ab = 1/4$. Moreover, there are infinitely many linearly independent functions satisfying both conditions when $ab < 1/4$. Examples of such functions are provided by the Hermite functions Φ_α.

Hardy's theorem has received considerable attention over the last fifteen years or so as can be seen from the large number of papers pertaining to the theorem (see e.g. the monograph [Th3] and the references therein). However, all the works so far have treated only the case $ab \ge 1/4$ in various set-ups. The case $ab < 1/4$ did not receive any closer study until recently when Vemuri [V] has looked at functions satisfying Hardy's conditions with $a = b < 1/2$. By a very clever use of the Bargmann transform he has proved the following characterisation of such functions.

THEOREM 1.1. *Suppose $f \in L^1(\mathbb{R})$ satisfies the conditions*

$$|f(x)| \le Ce^{-\frac{1}{2}ax^2}, \qquad |\hat{f}(\xi)| \le Ce^{-\frac{1}{2}a\xi^2}$$

for some $0 < a < 1$. Then the Fourier–Hermite coefficients of f satisfy $|(f, \Phi_k)| \le C(2k+1)^{-1/4} e^{-(2k+1)t/2}$, where t is determined by the condition $a = \tanh(2t)$.

Vemuri has considered functions of one variable only. A natural question is whether a similar result is true for functions on $\mathbb{R}^n$. The proof in [V], like many other proofs of Hardy-type theorems, depends on the Phragmén–Lindelöf maximum principle which is essentially a theorem in one complex variable. If we consider functions f which are tensor products of one-dimensional functions, then an analogue of Theorem 1.1 follows easily. More generally, the arguments in [V] can be used to prove the following result. We state the result in terms of the Hermite projection operators P_k which are defined by

$$P_k f = \sum_{|\alpha|=k} (f, \Phi_\alpha)\Phi_\alpha$$

for any $f \in L^2(\mathbb{R}^n)$. We refer to [Th1] for more about Hermite expansions.

THEOREM 1.2. *Suppose $|f(x)| \le Ce^{-\frac{1}{2}a|x|^2}$ and for any $j = 1, \ldots, n$, $|\mathcal{F}_j f(\xi)| \le Ce^{-\frac{1}{2}a|\xi|^2}$ where $\mathcal{F}_j f$ is the partial Fourier transform of f in any set of j variables. Then $\|P_k f\|_2 \le C(2k+n)^{(n-2)/4} e^{-(2k+n)t/2}$ where* $a = \tanh(2t)$.

There are strong reasons to believe that the result is true for all functions satisfying the Hardy conditions. However, at present we do not know how to prove this. Nevertheless, we have the following slightly weaker result.

THEOREM 1.3. *Suppose $f \in L^1(\mathbb{R}^n)$ satisfies the estimates*

$$|f(x)| \le Ce^{-\frac{1}{2}a|x|^2}, \qquad |\hat{f}(x)| \le Ce^{-\frac{1}{2}a|x|^2}$$

for some $0 < a < 1$. Then $\|P_k f\|_2 \le C(2k+n)^{(n-1)/2} e^{-(2k+n)s/2}$ where s is determined by the condition $\tanh(2s) = a/2$.

We prove this theorem in Section 4 by relating the Hermite projections $P_k f$ with the Fourier–Wigner transform $V(f, f)$ and appealing to a version of Hardy's theorem for the Hankel transform. Since the Fourier transform of a radial function reduces to a Hankel transform, the multi-dimensional analogue of Theorem 1.1 can be shown to be true for all radial functions. More generally, we can prove the same for all $O(n)$-finite functions in $L^2(\mathbb{R}^n)$. In other words, the multi-dimensional analogue of Theorem 1.1 remains true for all functions whose restrictions to the unit sphere S^{n-1} have only finitely many terms in their spherical harmonic expansions.

THEOREM 1.4. *Suppose $f \in L^1(\mathbb{R}^n)$ satisfies the same conditions as in the previous theorem. If we further assume that f is $O(n)$-finite, then $\|P_k f\|_2 \le C(2k+n)^{(n-2)/4} e^{-(2k+n)t/2}$ where $a = \tanh(2t)$.*

We prove this theorem in Section 5 by studying a vector valued Bargmann transform. Let us define the *Hardy class* $H(a)$, $0 < a < 1$, as the set of all functions f satisfying the Hardy conditions in Theorem 1.3. We are interested in estimating the Hermite coefficients of f from $H(a)$. This problem has been completely solved in the one-dimensional case by Vemuri [V]. In a work closely related to this article, Janssen and Eijndhoven [JE] have studied the growth of Hermite coefficients in one dimension. Here we treat the higher dimensional case. It would also be interesting to find a precise relation between the Hardy conditions and the membership in Hermite–Bergman spaces $\mathcal{H}_t(\mathbb{C}^n)$.

2. Preliminaries. In this section we set up the notations and collect relevant results about Hermite functions, Fourier–Wigner and Hankel transforms. We closely follow the notations used in [Th1] and [F] and we refer to the same for the proofs and any unexplained terminology. If we write the Hermite expansion of $f \in L^2(\mathbb{R}^n)$ as $f = \sum_{k=0}^{\infty} P_k f$, then the Plancherel theorem reads $\|f\|_2^2 = \sum_{k=0}^{\infty} \|P_k f\|_2^2$.

The *Fourier–Wigner transform* of two functions $f, g \in L^2(\mathbb{R}^n)$ is a function on $\mathbb{C}^n$ defined by

$$V(f,g)(x+iy) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(x\cdot\xi + \frac{1}{2}x\cdot y)} f(\xi + y)\, \overline{g(\xi)}\, d\xi.$$

We make use of the identity (see [Th1])

$$\int_{\mathbb{C}^n} V(f_1, g_1)(z)\, \overline{V(f_2, g_2)(z)}\, dz = (f_1, f_2)(g_2, g_1)$$

for any $f_i, g_i \in L^2(\mathbb{R}^n)$.

The special Hermite functions $\Phi_{\alpha\beta} = V(\Phi_\alpha, \Phi_\beta)$ form an orthonormal basis for $L^2(\mathbb{C}^n)$. We observe that

$$\sum_{|\alpha|=k} |(f, \Phi_\alpha)|^2 = \int_{\mathbb{C}^n} V(f,f)(z) \sum_{|\alpha|=k} \overline{\Phi_{\alpha\alpha}(z)}\, dz.$$

If we let $\varphi_k^{n-1}(z)$ stand for the Laguerre function $L_k^{(n-1)}\big(\frac{1}{2}|z|^2\big) e^{-\frac{1}{4}|z|^2}$ then we know that

$$\sum_{|\alpha|=k} \Phi_{\alpha,\alpha}(z) = (2\pi)^{-n/2} \varphi_k^{n-1}(z)$$

and therefore we get the useful relation

$$\|P_k f\|_2^2 = (2\pi)^{-n/2} \int_{\mathbb{C}^n} V(f,f)(z) \varphi_k^{n-1}(z)\, dz,$$

which will be used in Section 4. The same idea has been used in [JE] in the study of the growth of Hermite coefficients.

In Section 5 we will make use of a Hecke–Bochner type formula for the Hermite projection operators. Recall first the Hecke–Bochner identity for the Fourier transform on $\mathbb{R}^n$. Let P be a harmonic polynomial which is homogeneous of degree m, called a *solid harmonic*. If f is radial on $\mathbb{R}^n$ such that $fP \in L^2(\mathbb{R}^n)$, then the Fourier transform of fP is again of the same form, viz. $\widehat{fP} = FP$ where F is given by a Hankel transform of f. A similar result is true for the Hermite projections. Let L_k^δ stand for Laguerre polynomials of type δ which are defined by the generating function identity

$$\sum_{k=0}^{\infty} L_k^\delta(x) e^{-\frac{1}{2}x} r^k = (1-r)^{-\delta-1} e^{-\frac{1}{2}\frac{1+r}{1-r}x}$$

for $|r| < 1$, $x > 0$. Define

$$R_k^\delta(f) = 2\,\frac{\Gamma(k+1)}{\Gamma(k+\delta+1)} \int_0^\infty f(s)\psi_k^\delta(s) s^{2\delta+1}\, ds$$

where the Laguerre functions ψ_k^δ are defined by

$$\psi_k^\delta(s) = L_k^\delta(s^2) e^{-\frac{1}{2}s^2}.$$

With these notations we have (see [Th1, Theorem 3.4.1, p. 82])

PROPOSITION 2.1. *Let $f \in L^2(\mathbb{R}^n)$ be such that $f = gP$ where g is radial and P is a solid harmonic of degree m. Then $P_j f = 0$ unless $j = 2k + m$ in which case*

$$P_{2k+m} f(x) = R_k^{n/2+m-1}(g) P(x) \psi_k^{n/2+m-1}(|x|).$$

The restrictions of solid harmonics to S^{n-1} are called *spherical harmonics*. Let $\{Y_{mj} : 1 \le j \le d_m,\ m \in \mathbb{N}\}$ be an orthonormal basis for $L^2(S^{n-1})$ consisting of spherical harmonics. Given $f \in L^2(\mathbb{R}^n)$ we have the expansion

$$f(r\omega) = \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} f_{mj}(r) Y_{mj}(\omega)$$

where f_{mj} are the spherical harmonic coefficients of f defined by

$$f_{mj}(r) = \int_{S^{n-1}} f(r\eta) Y_{mj}(\eta)\, d\eta.$$

The above proposition leads to the formula

$$P_{2k} f(x) = \sum_{m=0}^{k} \sum_{j=1}^{d_{2m}} R_{k-m}^{n/2+2m-1}(\tilde{f}_{2m,j}) \psi_{k-m}^{n/2+2m-1}(r) r^{2m} Y_{2m,j}(\omega)$$

where $\tilde{f}_{m,j}(r) = r^{-m} f_{mj}(r)$. A similar formula can be written for $P_{2k+1}f$ as well. The functions ψ_k^δ form an orthogonal system in $L^2(\mathbb{R}^+, r^{2\delta+1}dr)$ and suitably normalised they form an orthonormal basis.

3. Bargmann transform and Hardy's theorem. For the convenience of the readers we briefly recall the argument used by Vemuri [V] in proving Theorem 1.1. As we have already mentioned, we will be using variants of the same arguments, so it will help fixing the ideas. Recall that the *Bargmann transform* B defined by

$$Bf(z) = \pi^{-n/2} e^{-\frac{1}{4}z^2} \int_{\mathbb{R}^n} f(x) e^{-\frac{1}{2}|x|^2} e^{z\cdot x}\, dx$$

for $z \in \mathbb{C}^n$ is an isometric isomorphism from $L^2(\mathbb{R}^n)$ onto the Fock space consisting of the entire functions on $\mathbb{C}^n$ that are square integrable with respect to the Gaussian measure $(4\pi)^{-n/2} e^{-\frac{1}{2}|z|^2}$ (see [B] and [Th4]). It takes the Hermite functions Φ_α to the monomials $\zeta_\alpha(z) = (2^\alpha \alpha! \pi^{n/2})^{-1/2} z^\alpha$. Moreover, it has the interesting property that $B\widehat{f}(z) = Bf(-iz)$.

If f satisfies the Gaussian estimate $f(x) = O(e^{-\frac{1}{2}a|x|^2})$ then from the definition of B it follows that

$$|Bf(w)| \le C(1+a)^{-n/2} \exp\left(\frac{v^2 + \mu u^2}{4}\right)$$

where $\mu = \frac{1-a}{1+a}$, $w = u + iv$ and $u^2 = \sum_{j=1}^n u_j^2$ etc. The relation $B\widehat{f}(z) = Bf(-iz)$ then leads to

$$|Bf(w)| \le C(1+a)^{-n/2} \exp\left(\frac{u^2 + \mu v^2}{4}\right).$$

When $n = 1$, taking $w = re^{i\theta}$ we get

$$|Bf(w)| \le C(1+a)^{-1/2} \exp\left(\frac{(\mu + (1-\mu)\sin^2\theta) r^2}{4}\right)$$

and

$$|Bf(w)| \le C(1+a)^{-1/2} \exp\left(\frac{(\mu + (1-\mu)\cos^2\theta) r^2}{4}\right).$$

A Phragmén–Lindelöf argument then leads to the estimate

$$|Bf(w)| \le C(1+a)^{-1/2} \exp\left(\frac{\sqrt{\mu}}{4}\, r^2\right).$$

If c_k are the Taylor coefficients of Bf then Cauchy's estimates lead to

$$|c_k| \le C(1+a)^{-1/2} \exp\left(\frac{\sqrt{\mu}}{4}\, r^2\right) r^{-k}$$

and optimizing with respect to r we can get

$$|c_k| \le C(1+a)^{-1/2}\left(\frac{e\sqrt{\mu}}{2k}\right)^{k/2}.$$

Since c_k are related to the Hermite coefficients of f, we get a slightly weaker form of Theorem 1.1. For the argument leading to Theorem 1.1 we refer to [V]. In the n-dimensional case it is possible to use the same arguments to prove Theorem 1.2 under the extra assumptions made in the hypothesis. We leave the details to the reader.

4. Hardy's theorem for the Hankel transform and a proof of Theorem 1.3. For any $\delta > -1/2$ we define the *Hankel transform* H_δ on $L^1(\mathbb{R}^+, r^{2\delta+1}dr)$ by

$$H_\delta f(r) = \int_0^\infty f(s)\,\frac{J_\delta(rs)}{(rs)^\delta}\,s^{2\delta+1}\,ds$$

where J_δ stands for the usual Bessel function. It is well known that J_δ has the following power series expansion:

$$J_\delta(z) = \left(\frac{z}{2}\right)^\delta \sum_{k=0}^\infty \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\delta+1)}\left(\frac{z}{2}\right)^{2k}.$$

From this it is clear that $J_\delta(z)/z^\delta$ extends to $\mathbb{C}$ as an even entire function. It is also well known that H_δ extends to $L^2(\mathbb{R}^+, r^{2\delta+1}dr)$ as a unitary operator and the inversion formula is given by

$$f(r) = \int_0^\infty H_\delta f(s)\,\frac{J_\delta(rs)}{(rs)^\delta}\,s^{2\delta+1}\,ds$$

for all f for which $H_\delta f$ is integrable with respect to $s^{2\delta+1}ds$. Moreover, it is known that $H_\delta\psi_k^\delta = (-1)^k\psi_k^\delta$. We will make use of this fact in what follows.

An analogue of Hardy's theorem (i.e. the case $ab \ge 1/4$) is known for the Hankel transform as well (see [Tu]). We now prove an analogue of Theorem 1.1 for the Hankel transform.

THEOREM 4.1. *Let $f \in L^1(\mathbb{R}^+, r^{2\delta+1}dr)$ be such that both f and $H_\delta f$ satisfy the Hardy condition with $a = \tanh(2t)$. Then the Laguerre coefficients of f satisfy the following estimate:*

$$|(f,\psi_k^\delta)| \le C(1+a)^{-\delta}(4k+2\delta+1)^\delta e^{-2tk}.$$

The proof of this theorem is similar to that of Theorem 1.1 given in [V]. We just need to replace the Bargmann transform by another transform adapted to the Hankel transform. We now proceed to define this transform which we denote by U_δ. For $f \in L^2(\mathbb{R}^+, r^{2\delta+1}\,dr)$ we let

$$U_\delta f(w) = e^{w^2/4} \int_0^\infty f(s)\, \frac{J_\delta(iws)}{(iws)^\delta}\, e^{-\frac{1}{2}(w^2+s^2)} s^{2\delta+1}\, ds$$

for any $w \in \mathbb{C}$. We also have the Poisson integral representation of Bessel functions J_δ, for every $\delta > -1/2$ (see [SW, Lemma 3.1, p. 153]) as

$$J_\delta(w) = \frac{(w/2)^\delta}{\Gamma(\delta+1/2)\Gamma(1/2)} \int_{-1}^{1} e^{iws}(1-s^2)^{\delta-1/2}\, ds.$$

With this representation of J_δ, an easy application of Morera's theorem shows that $U_\delta f$ extends to $\mathbb{C}$ as an even entire function of w. Moreover, the generating function identity

$$\sum_{k=0}^\infty \frac{L_k^\delta(x)}{\Gamma(k+\delta+1)}\, w^k = e^w (xw)^{-\delta/2} J_\delta(2(xw)^{1/2})$$

satisfied by the Laguerre polynomials can be rewritten as

$$\sum_{k=0}^\infty \frac{(-1)^k 2^{-2k}}{\Gamma(k+\delta+1)}\, \psi_k^\delta(r) w^{2k} = 2^\delta e^{w^2/4}\, \frac{J_\delta(irw)}{(irw)^\delta}\, e^{-\frac{1}{2}(r^2+w^2)}.$$

In view of this we have

$$U_\delta f(w) = 2^{-\delta} \sum_{k=0}^\infty \frac{(-1)^k 2^{-2k}}{\Gamma(k+\delta+1)}\, (f, \psi_k^\delta) w^{2k}.$$

This shows that the transformation U_δ takes the Laguerre functions ψ_k^δ to constant multiples of the monomials w^{2k}. We also have the relation $U_\delta H_\delta f(w) = U_\delta f(-iw)$, which follows from the fact that $H_\delta \psi_k^\delta = (-1)^k \psi_k^\delta$.

The image of $L^2(\mathbb{R}^+, r^{2\delta+1}dr)$ under the transform U_δ is known to be a weighted Bargmann space (see [C]). Indeed, if we let

$$h(w) = \frac{2^\delta}{\pi} \left(\frac{|w|^2}{2}\right)^{2\delta+1} K_{\delta+1/2}\left(\frac{|w|^2}{2}\right)$$

where

$$K_\delta(z) = \left(\frac{\pi}{2z}\right)^{1/2} \frac{e^{-z}}{\Gamma(\delta+1/2)} \int_0^\infty e^{-t} t^{\delta-1/2} \left(1+\frac{t}{2z}\right)^{\delta-1/2} dt$$

then the image is precisely the Hilbert space of even entire functions that are square integrable with respect to $h(w)dw$ (see Cholewinski [C]). As $h(w)$ is radial it is clear that the w^{2k} form an orthogonal system with respect to $h(w)dw$. Moreover, it can be shown that (see [C])

$$\int_{\mathbb{C}} |w|^{4k} h(w)\, dw = 2^{1+2\delta} 2^{4k} \Gamma(k+1)\Gamma(k+\delta+1).$$

Thus, if we let

$$\zeta_k(w) = (-1)^k (2^{1+2\delta+4k}\Gamma(k+1)\Gamma(k+\delta+1))^{-1/2} w^{2k}$$

then the ζ_k form an orthonormal basis for the image of $L^2(\mathbb{R}^+, r^{2\delta+1}dr)$ under U_δ, and

$$U_\delta\bigg(\bigg(2\,\frac{\Gamma(k+1)}{\Gamma(k+\delta+1)}\bigg)^{1/2}\psi_k^\delta\bigg)(w) = \zeta_k(w).$$

We can now proceed as in Vemuri [V] with U_δ playing the role of the Bargmann transform to prove Theorem 4.1.

We now use Theorem 4.1 to prove Theorem 1.3. Let $\mathfrak{F}_{\mathrm{s}}$ stand for the symplectic Fourier transform defined for suitable functions on $\mathbb{C}^n$ by $\mathfrak{F}_{\mathrm{s}}F(z) = \int_{\mathbb{C}^n} F(w)e^{-\frac{i}{2}\Im(z.\overline{w})}\,dw$. We need the following estimate on the Fourier–Wigner transform $V(f,f)$ when f and $\hat{f}$ satisfy the Hardy conditions.

Proposition 4.2. *Let $f \in L^2(\mathbb{R}^n)$ satisfy $|f(x)| \le Ce^{-a|x|^2}$ and $|\widehat{f}(\xi)| \le Ce^{-a|\xi|^2}$ for some $C > 0$ and $a > 0$. Then $|V(f,f)(z)| \le C_n a^{-n/2} e^{-\frac{1}{4}a|z|^2}$ and $|\mathfrak{F}_{\mathrm{s}}V(f,f)(z)| \le C_n a^{-n/2} e^{-\frac{1}{4}a|z|^2}$, where $C_n > 0$ depends only on C and n.*

Proof. By definition, for $z = x+iy \in \mathbb{C}^n$,

$$V(f,f)(z) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\frac{1}{2}x\cdot y)} f(\xi+y)\,\overline{f(\xi)}\,d\xi.$$

An easy calculation using Fourier inversion shows that

$$V(\widehat{f},\widehat{f})(z) = V(f,f)(iz).$$

From the definition it follows that $|V(f,f)(z)|$ is bounded by

$$(2\pi)^{-n/2}\int_{\mathbb{R}^n} |f(\xi+y)|\,|\overline{f(\xi)}|\,d\xi \le C\int_{\mathbb{R}^n} e^{-a|\xi+y|^2}e^{-a|\xi|^2}\,d\xi.$$

The last integral is equal to

$$Ce^{-\frac{1}{2}a|y|^2}\int_{\mathbb{R}^n} e^{-2a|\xi+y/2|^2}\,d\xi = C_n a^{-n/2}e^{-\frac{1}{2}a|y|^2}.$$

Replacing f by $\widehat{f}$, we also get

$$|V(\widehat{f},\widehat{f})(z)| \le C_n a^{-n/2}e^{-\frac{1}{2}a|y|^2}.$$

And thus the relation $V(f,f)(z) = V(\widehat{f},\widehat{f})(-iz)$ gives

$$|V(f,f)(z)| \le C_n a^{-n/2}e^{-\frac{1}{2}a|x|^2}.$$

Combining these two, we get

$$|V(f,f)(z)| \le C_n a^{-n/2}e^{-\frac{1}{4}a|z|^2}.$$

With the notation $\widetilde{f}(x) = f(-x)$, the above calculation together with the relation $\mathfrak{F}_{\mathrm{s}} V(f,f)(z) = (4\pi)^n V(f,\widetilde{f})(z)$ implies that

$$|\mathfrak{F}_{\mathrm{s}} V(f,f)(z)| \le C_n a^{-n/2} e^{-\frac{1}{4}a|z|^2}.$$

This completes the proof of the proposition. ∎

In view of the expression for the norm of P_k in terms of the Laguerre coefficients of $V(f,f)$, in order to prove Theorem 1.3 we only need to prove the following result.

THEOREM 4.3. *Suppose $f \in L^2(\mathbb{R}^n)$ is such that $|V(f,f)(z)| \le Ce^{-\frac{1}{8}a|z|^2}$ and $|\mathfrak{F}_{\mathrm{s}} V(f,f)(z)| \le Ce^{-\frac{1}{8}a|z|^2}$. Then*

$$\Big| \int_{\mathbb{C}^n} V(f,f)(z)\varphi_k^{n-1}(z)\,dz \Big| \le C(2k+n)^{n-1} e^{-(2k+n)s}$$

where s is determined by $a/2 = \tanh(2s)$.

Proof. As $\varphi_k^{n-1}(z)$ is radial, recalling the definition of ψ_k^{n-1}, the integral we want to estimate reduces to $2^n \int_0^\infty F(\sqrt{2}\,r)\psi_k^{n-1}(r) r^{2n-1}\,dr$ where

$$F(r) = \int_{S^{2n-1}} V(f,f)(r\omega)\,d\omega,$$

which clearly satisfies the estimate $|F(r)| \le Ce^{-\frac{1}{8}ar^2}$. If we can show that the function $G(r) = F(\sqrt{2}\,r)$ satisfies the estimate $|H_{n-1}G(r)| \le Ce^{-\frac{1}{4}ar^2}$, then we can appeal to Theorem 4.1 to get the required estimate.

We now perform the following calculations:

$$\begin{aligned}
\int_{S^{2n-1}} \mathfrak{F}_{\mathrm{s}} V(f,f)(\sqrt{2}\,r\omega)\,d\omega &= C_n \int_{S^{2n-1}} \widehat{V(f,f)}\Big(-\frac{i}{2}\sqrt{2}\,r\omega\Big)\,d\omega \\
&= C_n \int_{S^{2n-1}} \widehat{V(f,f)}\Big(\frac{r}{\sqrt{2}}\,\omega\Big)\,d\omega = C_n \int_0^\infty F(s)\,\frac{J_{n-1}\big(\frac{r}{\sqrt{2}}s\big)}{\big(\frac{r}{\sqrt{2}}s\big)^{n-1}}\, s^{2n-1}\,ds \\
&= 2^n C_n \int_0^\infty F(\sqrt{2}\,s)\,\frac{J_{n-1}(rs)}{(rs)^{n-1}}\, s^{2n-1}\,ds \\
&= 2^n C_n \int_0^\infty G(s)\,\frac{J_{n-1}(rs)}{(rs)^{n-1}}\, s^{2n-1}\,ds = 2^n C_n (H_{n-1}G)(r),
\end{aligned}$$

which proves our claim on $H_{n-1}G(r)$. ∎

If we can improve the estimates in Proposition 4.2 to $|V(f,f)(z)| \le Ce^{-\frac{1}{2}a|z|^2}$ and $|\mathfrak{F}_{\mathrm{s}} V(f,f)(z)| \le Ce^{-\frac{1}{2}a|z|^2}$ then we could prove Theorem 4.3

with $\tanh(2s) = a$. In fact, one needs only

$$\Big| \int\limits_{S^{2n-1}} V(f,f)(r\omega)\,d\omega \Big| \le Ce^{-\frac{1}{2}ar^2}$$

and a similar estimate for $\mathfrak{F}_{\mathrm{s}} V(f,f)$ which is good enough to improve Theorem 1.3. But there are some limitations on the decay of the Fourier–Wigner transform due to the uncertainty principle proved in [G]. The following example shows that improving the estimates in Proposition 4.2 is not always possible, which means that the proof via the Fourier–Wigner transform is not robust enough to lead to Theorem 1.3.

Example 4.4. Let $a = 1/\sqrt{2}$ and consider the function $f \in L^2(\mathbb{R}^2)$ defined by

$$f(x_1,x_2) = e^{-\frac{a}{2}(x_1^2+x_2^2+2ix_1x_2)}.$$

An easy calculation (using $a = \frac{1}{2a}$) shows that

$$\widehat{f}(\xi,\eta) = \frac{\pi\sqrt{2}}{a}\, e^{-\frac{1}{4a}(\xi^2+\eta^2-2i\xi\eta)} = 2\pi\, e^{-\frac{a}{2}(\xi^2+\eta^2-2i\xi\eta)}$$

and with $z = (x_1+iy_1, x_2+iy_2)$

$$V(f,f)(z) = \frac{1}{2a}\, e^{-\frac{a}{2}|z|^2} e^{\frac{1}{2}(x_1y_2+x_2y_1)}.$$

From the above expression for $V(f,f)$, it is clear that the estimate

$$|V(f,f)(z)| \le Ce^{-\frac{a}{4}|z|^2}$$

is not valid.

For any $0 < b < 1/2$, set

$$R_b = \{\omega = (\omega_1,\omega_2,\omega_3,\omega_4) \in S^3 \mid \omega_1\omega_4 > b,\ \omega_2\omega_3 > 0\}.$$

Clearly, R_b is a subset of S^3 of positive measure. Let us denote its measure by $|R_b|$. Then

$$\begin{aligned}\int\limits_{S^3} V(f,f)(r\omega)\,d\omega &= \frac{1}{2a}\, e^{-\frac{a}{2}r^2} \int\limits_{S^3} e^{\frac{1}{2}r^2(\omega_1\omega_4+\omega_2\omega_3)}\,d\omega \\ &\ge \frac{1}{2a}\, e^{-\frac{a}{2}r^2} \int\limits_{R_b} e^{\frac{1}{2}r^2(\omega_1\omega_4+\omega_2\omega_3)}\,d\omega \\ &\ge \frac{1}{2a}\,|R_b| e^{-\frac{a}{2}r^2} e^{\frac{b}{2}r^2} = \frac{1}{2a}\,|R_b| e^{-\frac{a}{4}r^2} e^{\frac{1}{2}(b-a/2)r^2}.\end{aligned}$$

Since $a = 1/\sqrt{2}$, one can choose $0 < b < 1/2$ such that $b - a/2 > 0$. Therefore, $\int_{S^3} V(f,f)(r\omega)\,d\omega$ cannot be bounded by $e^{-\frac{a}{4}r^2}$.

However, we can show that $\|P_k f\|_2$ has the required decay. Indeed, for any $0 < r < 1$, we have

$$\sum_{k=0}^{\infty} r^k \|P_k f\|_2^2 = (2\pi)^{-1} \int_{\mathbb{C}^2} V(f,f)(z)\Big(\sum_{k=0}^{\infty} r^k \varphi_k^1(z)\Big)\,dz$$

and hence using the generating function identity

$$\sum_{k=0}^{\infty} r^k \varphi_k^1(z) = (1-r)^{-2} e^{-\frac{1}{4}\frac{1+r}{1-r}|z|^2}$$

and the explicit expression for $V(f,f)$ we can calculate that

$$\sum_{k=0}^{\infty} r^k \|P_k f\|_2^2 = \frac{8\pi a(1-r)^{-2}}{\big(2a+\frac{1+r}{1-r}\big)\big(\big(2a+\frac{1+r}{1-r}\big)-\frac{1}{2a+\frac{1+r}{1-r}}\big)}.$$

Writing $\mu = \frac{1-a}{1+a}$ and simplifying, we obtain

$$\sum_{k=0}^{\infty} r^k \|P_k f\|_2^2 = \frac{2\pi}{1+a}\sum_{k=0}^{\infty} \mu^k r^{2k}.$$

Comparing the coefficients of r^k we see that $P_{2k+1}f = 0$ and $\|P_{2k}f\|_2^2 = \frac{2\pi}{1+a}\mu^k$, which is the expected decay.

5. A vector-valued Bargmann transform and a proof of Theorem 1.4. In order to prove Theorem 1.4 we need to study a vector-valued Bargmann transform. For functions f from $L^2(\mathbb{R}^n)$ consider

$$Bf(z,\omega) = \pi^{-n/2} e^{-\frac{1}{4}z^2} \int_{\mathbb{R}^n} f(x)\, e^{-\frac{1}{2}|x|^2} e^{zx\cdot\omega}\,dx$$

where $z \in \mathbb{C}$ and $\omega \in S^{n-1}$. We think of Bf as an entire function of one complex variable taking values in the vector space $L^2(S^{n-1})$. As before, one can easily verify that $B\widehat{f}(z,\omega) = Bf(-iz,\omega)$. We consider functions satisfying the conditions

$$\Big(\int_{S^{n-1}} |f(s\eta)|^2\,d\eta\Big)^{1/2} \le Ce^{-\frac{a}{2}s^2}, \tag{5.1}$$

$$\Big(\int_{S^{n-1}} |\widehat{f}(s\eta)|^2\,d\eta\Big)^{1/2} \le Ce^{-\frac{a}{2}s^2}, \tag{5.2}$$

for some $a > 0$. The basic estimates on the Bargmann transforms of such functions are given below.

PROPOSITION 5.1. *Let $f \in L^2(\mathbb{R}^n)$ be such that* (5.1) *and* (5.2) *are valid for some $a > 0$. Then for every $z = u + iv \in \mathbb{C}$,*

$$\int_{S^{n-1}} |Bf(z,\omega)|^2\, d\omega \le C(1+a)^{-n} \ \exp\biggl(\frac{v^2 + \mu u^2}{2}\biggr),$$

$$\int_{S^{n-1}} |Bf(z,\omega)|^2\, d\omega \le C(1+a)^{-n} \exp\biggl(\frac{u^2 + \mu v^2}{2}\biggr),$$

where $\mu = \frac{1-a}{1+a}$ as before.

Proof. For $z \in \mathbb{C}$ and $\omega \in S^{n-1}$,

$$\begin{aligned} Bf(z,\omega) &= \pi^{-n/2} e^{-\frac{1}{4}z^2} \int_{\mathbb{R}^n} f(x) e^{-\frac{1}{2}|x|^2} e^{zx\cdot\omega}\, dx \\ &= \pi^{-n/2} e^{-\frac{1}{4}z^2} \int_0^\infty \Bigl(\int_{S^{n-1}} f(s\eta) e^{sz\eta\cdot\omega}\, d\eta \Bigr) e^{-\frac{1}{2}s^2} s^{n-1}\, ds. \end{aligned}$$

Thus we get the estimate

$$|Bf(z,\omega)| \le \pi^{-n/2} |e^{-\frac{1}{4}z^2}| \int_0^\infty |T_{sz} f(\omega)| e^{-\frac{1}{2}s^2} s^{n-1}\, ds$$

where

$$T_{sz} f(\omega) = \int_{S^{n-1}} f(s\eta) e^{sz\eta\cdot\omega}\, d\eta.$$

If we write $z = u + iv$, then

$$|T_{sz} f(\omega)| \le \int_{S^{n-1}} |f(s\eta)| e^{su\eta\cdot\omega}\, d\eta$$

and consequently,

$$\begin{aligned} \Bigl(\int_{S^{n-1}} |T_{sz} f(\omega)|^2\, d\omega \Bigr)^{1/2} &\le \Bigl(\int_{S^{n-1}} |f(s\eta)|^2\, d\eta \Bigr)^{1/2} \int_{S^{n-1}} e^{su\eta\cdot\omega}\, d\eta \\ &\le C\ e^{-\frac{a}{2}s^2} \frac{J_{n/2-1}(isu)}{(isu)^{n/2-1}}. \end{aligned}$$

Notice that $(\int_{S^{n-1}} |Bf(z,\omega)|^2\, d\omega)^{1/2}$ is bounded by

$$\begin{aligned} &e^{-\frac{1}{4}(u^2-v^2)} \Bigl(\int_{S^{n-1}} \Bigl(\int_0^\infty |T_{sz} f(\omega)|\ e^{-\frac{1}{2}s^2} s^{n-1}\, ds \Bigr)^2 d\omega \Bigr)^{1/2} \\ &\qquad \le e^{-\frac{1}{4}(u^2-v^2)} \int_0^\infty \Bigl(\int_{S^{n-1}} |T_{sz} f(\omega)|^2\, d\omega \Bigr)^{1/2} e^{-\frac{1}{2}s^2} s^{n-1}\, ds \end{aligned}$$

where the last inequality is achieved using Minkowski's integral inequality.

Now, using the above estimates, we get

$$\Big(\int_{S^{n-1}} |Bf(z,\omega)|^2\,d\omega\Big)^{1/2} \le Ce^{-\frac{1}{4}(u^2-v^2)}\int_0^\infty e^{-\frac{1}{2}(1+a)s^2}\frac{J_{n/2-1}(isu)}{(isu)^{n/2-1}}\,s^{n-1}\,ds$$
$$= \frac{C}{(1+a)^{n/2}}\exp\Big(-\Big(\frac{u^2-v^2}{4}\Big)\Big)\exp\Big(\frac{u^2}{2(1+a)}\Big)$$
$$= \frac{C}{(1+a)^{n/2}}\exp\Big(\frac{v^2+\mu u^2}{4}\Big)$$

where the second last equality is obtained using the fact that the Hankel transform of the Gaussian $e^{-\frac{1}{2}r^2}$ is the Gaussian $e^{-\frac{1}{2}r^2}$ itself (see the proof of Theorem 1.1.2 in [Th3]). Replacing f by $\widehat{f}$ and using the fact that $B\widehat{f}(z,\omega)=(2\pi)^{n/2}Bf(-iz,\omega)$, we also get

$$\Big(\int_{S^{n-1}} |Bf(z,\omega)|^2\,d\omega\Big)^{1/2} \le \frac{C}{(1+a)^{n/2}}\exp\Big(\frac{u^2+\mu v^2}{4}\Big).$$

Hence the proposition is proved. ∎

THEOREM 5.2. *For a function f on $\mathbb{R}^n$ satisfying the conditions* (5.1) *and* (5.2) *let $Bf(z,\omega)=\sum_{k=0}^{\infty} d_k(\omega)z^k$ be the Taylor series expansion of the Bargmann transform. Then*

$$\int_{S^{n-1}} |d_k(\omega)|^2\,d\omega \le C2^{-k}\,\frac{k^{-1/2}}{\Gamma(k+1)}\,\mu^{k/2}.$$

Proof. The proof of this theorem can be reduced to the scalar-valued case treated in [V]. Indeed, for any normalised $g\in L^2(S^{n-1})$ the scalar-valued function

$$F_g(z)=\int_{S^{n-1}} Bf(z,\omega)g(\omega)\,d\omega$$

is an entire function satisfying the estimates stated in Proposition 5.1. The arguments in [V] lead to estimates for the integral

$$\int_{S^{n-1}} d_k(\omega)g(\omega)\,d\omega.$$

Taking the supremum over all such g we get the required estimates. ∎

In order to apply the above estimates to prove Theorem 1.4 we need the following result which shows that the $L^2(S^{n-1})$ norms of $d_k(\omega)$ can be expressed in terms of Laguerre coefficients of the spherical harmonic components of f restricted to the unit sphere.

THEOREM 5.3. *For $f \in L^2(\mathbb{R}^n)$, if $Bf(z,\omega) = \sum_{k=0}^{\infty} d_k(\omega) z^k$ is the Taylor series expansion, then for all $k \geq 1$,*

$$\int\limits_{S^{n-1}} |d_{2k}(\omega)|^2 \, d\omega = 2^{-4k+2} \sum_{m=0}^{k} \sum_{j=1}^{d_{2m}} \frac{|(\widetilde{f}_{2m,j}, \psi_{k-m}^{n/2+2m-1})|^2}{(\Gamma(n/2+k+m))^2},$$

$$\int\limits_{S^{n-1}} |d_{2k+1}(\omega)|^2 \, d\omega = 2^{-4k} \sum_{m=0}^{k} \sum_{j=1}^{d_{2m+1}} \frac{|(\widetilde{f}_{2m+1,j}, \psi_{k-m}^{n/2+2m})|^2}{\big(\Gamma(n/2+k+m+1)\big)^2}.$$

Proof. It is known ([He, Lemma 3.6, p. 25]) that for every $z_1, z_2 \in \mathbb{C}$,

$$e^{iz_1 z_2 \eta\cdot\omega} = (2\pi)^{n/2} \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} i^m \frac{J_{n/2+m-1}(z_1 z_2)}{(z_1 z_2)^{n/2-1}} Y_{mj}(\eta) Y_{mj}(\omega),$$

which in particular implies that

$$e^{sz\eta\cdot\omega} = (2\pi)^{n/2} \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} i^m \frac{J_{n/2+m-1}(-isz)}{(-isz)^{n/2-1}} Y_{mj}(\eta) Y_{mj}(\omega)$$

and thus

$$\int\limits_{S^{n-1}} f(s\eta) e^{sz\eta\cdot\omega} \, d\eta = (2\pi)^{n/2} \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} i^m f_{mj}(s) \frac{J_{n/2+m-1}(-isz)}{(-isz)^{n/2-1}} Y_{mj}(\omega).$$

Now,

$$\begin{aligned}
Bf(z,\omega) &= \pi^{-n/2} e^{-\frac{1}{4}z^2} \int\limits_{\mathbb{R}^n} f(x) e^{-\frac{1}{2}|x|^2} e^{zx\cdot\omega} \, dx \\
&= \pi^{-n/2} e^{-\frac{1}{4}z^2} \int\limits_0^{\infty} \int\limits_{S^{n-1}} f(s\eta) e^{-\frac{1}{2}s^2} e^{sz\eta\cdot\omega} s^{n-1} \, d\eta \, ds \\
&= 2^{n/2} e^{-\frac{1}{4}z^2} \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} i^m \bigg(\int\limits_0^{\infty} f_{mj}(s) \frac{J_{n/2+m-1}(-isz)}{(-isz)^{n/2-1}} e^{-\frac{1}{2}s^2} s^{n-1} \, ds \bigg) Y_{mj}(\omega) \\
&= 2^{n/2} \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} z^m (U_m \widetilde{f}_{mj})(-z) Y_{mj}(\omega) \\
&= 2^{n/2} \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} z^m (U_m \widetilde{f}_{mj})(z) Y_{mj}(\omega)
\end{aligned}$$

where for simplicity we have written U_m in place of $U_{n/2+m-1}$.

If we write the power series expansion of $U_m \widetilde{f}_{mj}$ as

$$(U_m \widetilde{f}_{mj})(z) = \sum_{k=0}^{\infty} b_{k;m,j} z^{2k}$$

then, as we saw in Section 4,

$$b_{k;m,j} = 2^{-(n/2+m-1)} \frac{(-1)^k 2^{-2k}}{\Gamma(n/2+k+m)} (\widetilde{f}_{mj}, \psi_k^{n/2+m-1}).$$

Now, $d_k(\omega) = \frac{1}{2\pi i}\int_\gamma \frac{Bf(z,\omega)}{z^{k+1}}\,dz$ implies that

$$d_{2k}(\omega) = 2^{n/2}(-1)^{n/2-1} \sum_{m=0}^{k}\sum_{j=1}^{d_{2m}} (b_{k-m;2m,j}) Y_{2m,j}(\omega),$$

which implies

$$\int\limits_{S^{n-1}} |d_{2k}(\omega)|^2\,d\omega = 2^n \sum_{m=0}^{k}\sum_{j=1}^{d_{2m}} |b_{k-m;2m,j}|^2$$

and therefore

$$\int\limits_{S^{n-1}} |d_{2k}(\omega)|^2\,d\omega = 2^{-4k+2} \sum_{m=0}^{k}\sum_{j=1}^{d_{2m}} \frac{|(\widetilde{f}_{2m,j}, \psi_{k-m}^{n/2+2m-1})|^2}{(\Gamma(n/2+k+m))^2}.$$

A similar calculation shows that

$$\int\limits_{S^{n-1}} |d_{2k+1}(\omega)|^2\,d\omega = 2^{-4k} \sum_{m=0}^{k}\sum_{j=1}^{d_{2m+1}} \frac{|(\widetilde{f}_{2m+1,j}, \psi_{k-m}^{n/2+2m})|^2}{(\Gamma(n/2+k+m+1))^2},$$

which completes the proof of the theorem. ∎

Combining Theorems 5.2 and 5.3 we can prove Theorem 1.4. To see this, we first observe that the Hecke–Bochner formula for the Hermite projections leads to

Proposition 5.4.

$$\|P_{2k}f\|_2^2 = 2\sum_{m=0}^{k}\sum_{j=1}^{d_{2m}} (\Gamma(k-m+1)\Gamma(n/2+k+m)) \frac{|(\widetilde{f}_{2m,j}, \psi_{k-m}^{n/2+2m-1})|^2}{(\Gamma(n/2+k+m))^2}.$$

A similar expression holds for $P_{2k+1}f$.

We note the similarity between the expression for $\|P_{2k}f\|_2^2$ and the $L^2(S^{n-1})$ norms of $d_{2k}(\omega)$. We therefore rewrite the expression for $\|P_{2k}f\|_2^2$ as

$$2\sum_{m=0}^{k} 2^{2m} \sum_{j=1}^{d_{2m}} c(k,m)\left(2^{-2k}\Gamma(2k+1)\frac{|(\widetilde{f}_{2m,j}, \psi_{k-m}^{n/2+2m-1})|^2}{(\Gamma(n/2+k+m))^2}\right)$$

where

$$c(k,m) = 2^{2(k-m)} \frac{\Gamma(k-m+1)\Gamma(n/2+k+m)}{\Gamma(2k+1)}.$$

Stirling's formula for the gamma function shows that $c(k,k) = O(k^{n/2-1})$. In general, we have

Lemma 5.5. *For any* $0 \le m \le k$ *we have* $c(k,m) = O(k^{(n-1)/2})$.

Proof. We show that for all $k \ge 1$ and $0 \le m \le k$,

$$2^{2(k-m)} \frac{\Gamma(k-m+1)\Gamma(k+m+n/2)}{\Gamma(2k+1)} = O(k^{n/2-1/2}).$$

For this, fix $k \in \mathbb{N}$ and consider $c(k,m)$. When $m = k$, $c(k,k) = \frac{\Gamma(2k+n/2)}{\Gamma(2k+1)}$ and by Stirling's formula, for large k it behaves like

$$\frac{(2k+n/2-1)^{2k+n/2-1/2}\ e^{-(2k+n/2-1)}}{(2k)^{2k+1/2}e^{-2k}} = \frac{(1+\frac{n/2-1}{2k})^{2k}(2k+n/2-1)^{n/2-1/2}\ e^{-(n/2-1)}}{(2k)^{1/2}}.$$

As $\left(1+\frac{n/2-1}{2k}\right)^{2k} \le e^{n/2-1}$, we obtain $c(k,k) = O(k^{n/2-1})$.

Now for $0 \le m < k$, consider

$$\begin{aligned}
c(k,m) &= 2^{2(k-m)} \frac{\Gamma(k-m+1)\Gamma(k+m+n/2)}{\Gamma(2k+1)} \\
&\sim 2^{2(k-m)} \frac{(k-m)^{k-m+1/2}e^{-(k-m)}(k+m+n/2-1)^{k+m+n/2-1/2}e^{-(k+m+n/2-1)}}{(2k)^{2k+1/2}e^{-2k}} \\
&= \frac{e^{-(n/2-1)}}{\sqrt{2}}\left(\frac{2^{-2m}(k-m)^{k-m+1/2}(k+n/2+m-1)^{k+n/2+m-1/2}}{k^{2k+1/2}}\right) \\
&\le C_n e^{-(n/2-1)}2^{-2m}\left(1-\frac{m}{k}\right)^k\left(1+\frac{m+n/2-1}{k}\right)^k\left\{\frac{1+\frac{m+n/2-1}{k}}{1-\frac{m}{k}}\right\}^m k^{n/2-1/2} \\
&\le C_n e^{-(n/2-1)}2^{-2m}e^{-m}e^{m+n/2-1}\left\{\frac{1+\frac{m+n/2-1}{k}}{1-\frac{m}{k}}\right\}^m k^{n/2-1/2} \\
&= C_n 2^{-2m}\left\{\frac{1+\frac{m+n/2-1}{k}}{1-\frac{m}{k}}\right\}^m k^{n/2-1/2}.
\end{aligned}$$

But $2^{-2m}\left\{\frac{1+\frac{m+n/2-1}{k}}{1-\frac{m}{k}}\right\}^m \le 1$ if and only if $1+\frac{m+n/2-1}{k} \le 4(1-m/k)$, which happens precisely when $m \le \frac{1}{5}(3k-n/2+1)$. Since for sufficiently large k, $\frac{1}{5}(3k-n/2+1) \ge [(k+1)/2]$, it follows that for $0 \le m \le [(k+1)/2]$, $c(k,m) \le C_n k^{n/2-1/2}$. Now consider $[(k+1)/2] < m < k$. In this case

$$\begin{aligned} c(k,m) &\le C_n e^{-(n/2-1)} 2^{-2m} \left(1-\frac{m}{k}\right)^{k-m} \left(\frac{k+m+n/2-1}{k}\right)^{k+m} k^{n/2-1/2} \\ &= C_n e^{-(n/2-1)} 2^{-2m} \left(1-\frac{m}{k}\right)^{k-m} \left(1+\frac{m}{k}\right)^{k+m} \left(1+\frac{n/2-1}{k+m}\right)^{k+m} k^{n/2-1/2} \\ &\le C_n e^{-(n/2-1)} 2^{-2m} \left(\frac{1}{2}\right)^{k-m} 2^{k+m} e^{n/2-1}\, k^{n/2-1/2} \\ &= C_n k^{n/2-1/2}. \end{aligned}$$

In the second last step, we use the fact that $m > [(k+1)/2]$ implies that $1 - m/k < 1/2$. ∎

The extra factor of 2^{2m} in the expression for $\|P_{2k}f\|_2^2$ suggests that we consider the operator T defined by

$$Tf(r\omega) = \sum_{m=0}^{\infty} 2^{-m/2} \Big(\sum_{j=1}^{d_m} f_{mj}(r) Y_{mj}(\omega) \Big).$$

It is then clear that when f satisfies the Hardy conditions, Tf satisfies

$$\Big(\int_{S^{n-1}} |Tf(r\omega)|^2 \, d\omega \Big)^{1/2} \le C e^{-\frac{1}{2}ar^2}.$$

A similar estimate is true for $\widehat{Tf}$ as well.

THEOREM 5.6. *Suppose f satisfies the Hardy conditions with $a = \tanh(2t)$. Then for any $k = 0, 1, 2, \ldots$ we have*

$$\|P_k(Tf)\|_2 \le C(2k+n)^{(n-2)/4} e^{-(2k+n)t/2}.$$

The theorem follows by using the estimates obtained in Theorem 5.2 along with the above lemma. Theorem 1.4 follows as a corollary to Theorem 5.6 since for such functions $\|P_k f\|_2 \le C \|P_k(Tf)\|_2$, where C depends on the number of spherical harmonic coefficients present in the expansion of f.

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