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Uniform convergence of the greedy algorithm with respect to the Walsh system

by

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Abstract. For any $0 < \epsilon < 1$, $p \geq 1$ and each function $f \in L^p[0,1]$ one can find a function $g \in L^\infty[0,1)$ with $\operatorname{mes}\{x \in [0,1) : g \neq f\} < \epsilon$ such that its greedy algorithm with respect to the Walsh system converges uniformly on $[0,1)$ and the sequence $\{|c_k(g)| : k \in \operatorname{spec}(g)\}$ is decreasing, where $\{c_k(g)\}$ is the sequence of Fourier coefficients of g with respect to the Walsh system.

1. Introduction. In this paper we will consider the uniform convergence of the greedy algorithm with respect to the Walsh system after modification of functions.

The Walsh system, an extension of the Rademacher system, may be obtained in the following manner. Let r be the periodic function, of least period 1, defined on $[0,1)$ by

$$r = \chi_{[0,1/2)} - \chi_{[1/2,1)}.$$

The Rademacher system, $\{r_n : n = 0,1,\ldots\}$, is defined by the conditions

$$r_n(x) = r(2^n x), \qquad \forall x \in \mathbb{R},\ n = 0,1,\ldots,$$

and, in the ordering employed by Payley (see [3], [18] and [21]), the nth element of the Walsh system is given by

$$\varphi_n = \prod_{k=0}^{\infty} r_k^{n_k}, \tag{1}$$

where $\sum_{k=0}^{\infty} n_k 2^k$ is the unique binary expansion of n, with each n_k either 0 or 1.

Let $\Phi = \{\varphi_k(x)\}$ be the Walsh system and let $f \in L^p[0,1]$, $p \geq 1$. We denote by $c_k(f)$ the Fourier–Walsh coefficients of f, i.e.

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$$c_k(f) = \int_0^1 f(x)\varphi_k(x)\,dx.$$

We call a permutation $\sigma = \{\sigma(n)\}_{n=1}^{\infty}$ of nonnegative integers *decreasing*, and write $\sigma \in D(f, \Phi)$, if

$$|c_{\sigma(n)}(f)| \geq |c_{\sigma(n+1)}(f)|, \qquad n = 1, 2, \ldots.$$

If the inequalities are strict, $D(f, \Phi)$ consists of only one permutation. We define the mth *greedy approximant* of f with regard to the Walsh system Φ by

$$G_m(f) = G_m(f, \Phi, \sigma) = \sum_{n=1}^{m} c_{\sigma(n)}(f)\varphi_{\sigma(n)}.$$

This nonlinear method of approximation is known as greedy algorithm (see [20]). We say that the greedy algorithm of f with respect to Φ *converges* if for some $\sigma \in D(f, \Phi)$ we have

$$\lim_{m\to\infty} \|G_m(f, \Phi, \sigma) - f\|_{L^\infty} = 0.$$

The above-mentioned definitions are not given in the most general form, but only in the generality in which they will be applied in this paper.

Greedy algorithms in Banach spaces with respect to normalized bases have been considered in [1], [2], [4], [8], [9–12], [20], [22].

The *spectrum* of f (denoted by $\operatorname{spec}(f)$) is the support of $\{c_k(f)\}$, i.e. the set of integers k for which $c_k(f)$ is nonzero.

We now present some results having a direct bearing on the present work.

T. W. Körner answering a question raised by Carleson and Coifman constructed in [11] an L^2 function and then in [12] a continuous function whose greedy algorithm with respect to the trigonometric systems diverges almost everywhere.

In [20], V. N. Temlyakov constructed a function f that belongs to all L^p, $1 \leq p < 2$ (respectively $p > 2$), whose greedy algorithm with respect to the trigonometric system diverges in measure (respectively in L^p, $p > 2$).

In [4] R. Gribonval and M. Nielsen proved that for any $p \neq 2$ there exists a function from $L^p[0,1]$ whose greedy algorithm with respect to the Walsh system diverges in $L^p[0,1]$.

The following question arises naturally: Is there a measurable set e of arbitrarily small measure such that a suitable change of the values of any function f of class $L^p[0,1]$ on e leads to a new modified function $g \in L^p$ whose greedy algorithm with respect to the Walsh system (or the trigonometric system) converges to g almost everywhere or in the $L^p[0,1]$ norm?

We denote by $L^\infty[0,1)$ the space of all bounded measurable functions on $[0,1)$ with the norm $\|\cdot\|_\infty = \sup_{x\in[0,1)} |\cdot|$.

In the present work we prove the following theorems.

Theorem 1. *For any* $0 < \epsilon < 1$, $p \geq 1$ *and each function* $f \in L^p[0,1]$ *one can find a function* $g \in L^\infty[0,1)$ *with* $\mathrm{mes}\{x \in [0,1) : g \neq f\} < \epsilon$ *whose greedy algorithm* $\{G_m(g)\}$ *with respect to the Walsh system converges to* g *uniformly on* $[0,1)$.

Theorem 2. *For any* $0 < \epsilon < 1$, $p \geq 1$ *and each function* $f \in L^p[0,1]$ *one can find a function* $g \in L^\infty[0,1)$ *with* $\mathrm{mes}\{x \in [0,1) : g \neq f\} < \epsilon$ *such that the sequence* $\{|c_k(g)| : k \in \mathrm{spec}(g)\}$ *is decreasing.*

Theorems 1 and 2 follow from the more general Theorem 3.

Theorem 3. *For any* $0 < \epsilon < 1$, $p \geq 1$ *and each* $f \in L^p[0,1]$ *one can find* $g \in L^\infty[0,1)$ *with* $\mathrm{mes}\{x \in [0,1) : g \neq f\} < \epsilon$ *such that the sequence* $\{|c_k(g)| : k \in \mathrm{spec}(g)\}$ *is decreasing and the Fourier–Walsh series of* g *converges uniformly on* $[0,1)$.

Remark. It must be pointed out that in this theorem the "exceptional" set on which the function f is modified depends on f.

The following problem remains open:

Question 1. *Is it possible to construct in Theorems* 1–3 *the "exceptional" set independent of* f*?*

Question 2. *Is Theorem* 3 *or Theorems* 1 *and* 2 *true for the trigonometric system?*

Note that in [7] it is proved that there exist a complete orthonormal system $\{\varphi_k\}$ and a function $f \in L^p$, $p > 2$, such that if g is any function in $L^p[0,1]$ with

$$\mathrm{mes}\{x \in [0,1] : f(x) = g(x)\} > 0,$$

then the greedy algorithm of g with respect to the system $\{\varphi_k\}$ diverges in $L^p[0,1]$.

Note also that in 1939, Men'shov [14] proved the following fundamental theorem.

Theorem (Men'shov's C-strong property). *Let* f *be an a.e. finite measurable function on* $[0,2\pi]$. *Then for each* $\epsilon > 0$ *one can define a continuous function* g *coinciding with* f *on a subset* E *of measure* $|E| > 2\pi - \epsilon$ *such that the Fourier series of* g *with respect to the trigonometric system converges uniformly on* $[0,2\pi]$.

Further interesting results in this direction were obtained by many famous mathematicians (see for example [13–17], [19]). We also mention our papers [5]–[8].

2. Proofs of main lemmas. For $m = 1, 2, \dots$ and $1 \le j \le 2^m$ we put

$$I_m^{(j)}(x) = \begin{cases} 1 & \text{if } x \in [0,1] \setminus \Delta_m^{(j)}, \\ 1 - 2^m & \text{if } x \in \Delta_m^{(j)} = ((j-1)/2^m, j/2^m), \end{cases} \tag{2}$$

and periodically extend these functions on $\mathbb{R}$ with period 1.

We denote by $\chi_E(x)$ the characteristic function of the set E, i.e.

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases} \tag{3}$$

Then clearly

$$I_m^{(j)}(x) = \varphi_0(x) - 2^m \cdot \chi_{\Delta_m^{(j)}}(x), \tag{4}$$

and for natural numbers $m \ge 1$, and $j \in [1, 2^m]$ let

$$b_i(\chi_{\Delta_m^{(j)}}) = \int_0^1 \chi_{\Delta_m^{(j)}}(x)\varphi_i(x)\,dx = \pm\frac{1}{2^m}, \qquad 0 \le i < 2^m, \tag{5}$$

$$a_i(I_m^{(j)}) = \int_0^1 I_m^{(j)}(x)\varphi_i(x)\,dx = \begin{cases} 0 & \text{if } i = 0 \text{ or } i \ge 2^m, \\ \pm 1 & \text{if } 1 \le i < 2^m. \end{cases} \tag{6}$$

Hence

$$\chi_{\Delta_m^{(j)}}(x) = \sum_{i=0}^{2^m-1} b_i(\chi_{\Delta_m^{(j)}})\varphi_i(x), \tag{7}$$

$$I_m^{(j)}(x) = \sum_{i=1}^{2^m-1} a_i(I_m^{(j)})\varphi_i(x). \tag{8}$$

LEMMA 1. *Let $\Delta = \Delta_m^{(k)} = ((k-1)/2^m, k/2^m)$, $k \in [1, 2^m]$ and let $N_0 \in \mathbb{N}$, $\gamma \ne 0$, $\epsilon \in (0,1)$ be given. Then there exists a measurable set $E \subset \Delta$ and a polynomial Q in the Walsh system $\{\varphi_k\}$ of the form*

$$Q = \sum_{k=N_0}^{N} a_k \varphi_k,$$

which satisfy the following conditions:

(i) *the coefficients $\{a_k\}_{k=N_0}^N$ are 0 or $\pm\gamma|\Delta|$,*

(ii) $|E| > (1-\varepsilon)|\Delta|$,

(iii) $Q(x) = \begin{cases} \gamma & \textit{if } x \in E, \\ 0 & \textit{if } x \notin \Delta, \end{cases}$

(iv) $\max_{N_0 \le M \le N} \|\sum_{k=N_0}^M a_k\varphi_k\|_\infty \le 3|\gamma|\epsilon^{-1}$.

Proof. Let

$$\nu_0 = [\log_2(1/\epsilon)] + 1, \qquad s = [\log_2 N_0] + m. \tag{9}$$

We define the polynomial $Q(x)$ and the numbers c_n, a_i and b_j by

$$Q(x) = \gamma \cdot \chi_{\Delta_m^{(k)}}(x) \cdot I_{\nu_0}^{(1)}(2^s x), \qquad x \in [0,1], \tag{10}$$

$$c_n = c_n(Q) = \int_0^1 Q(x)\varphi_n(x)\,dx, \qquad \forall n \ge 0, \tag{11}$$

$$b_i = b_i(\chi_{\Delta_m^{(k)}}), \quad 0 \le i < 2^m, \qquad a_j = a_j(I_{\nu_0}^{(1)}), \quad 0 < j < 2^{\nu_0}. \tag{12}$$

Taking into account the equation

$$\varphi_i(x) \cdot \varphi_j(2^s x) = \varphi_{j\cdot 2^s + i}(x) \quad \text{ if } 0 \le i, j < 2^s \text{ (see (1))},$$

and relations (5)–(8) and (10)–(12), we find that

$$\begin{aligned} Q(x) &= \gamma \cdot \sum_{i=0}^{2^m-1} b_i\varphi_i(x) \cdot \sum_{j=1}^{2^{\nu_0}-1} a_j\varphi_j(2^s x) \\ &= \gamma \cdot \sum_{j=1}^{2^{\nu_0}-1} a_j \cdot \sum_{i=0}^{2^m-1} b_i \varphi_{j\cdot 2^s+i}(x) = \sum_{k=N_0}^{\bar N} c_k\varphi_k(x), \end{aligned} \tag{13}$$

where

(14)
$$c_k = c_k(Q) = \begin{cases} \pm\gamma/2^m \text{ or } 0 & \text{if } k \in [N_0, \bar N], \\ 0 & \text{if } k \notin [N_0, \bar N], \end{cases} \qquad \bar N = 2^{s+\nu_0} + 2^m - 2^s - 1.$$

Then let

$$E = \{x : Q(x) = \gamma\}.$$

Clearly (see (2) and (10)),

$$|E| = 2^{-m}(1 - 2^{-\nu_0}) > (1-\epsilon)|\Delta|, \tag{15}$$

$$Q(x) = \begin{cases} \gamma & \text{if } x \in E, \\ \gamma(1 - 2^{\nu_0}) & \text{if } x \in \Delta \setminus E, \\ 0 & \text{if } x \notin \Delta. \end{cases} \tag{16}$$

Let $M \in [N_0, N]$. Then for some $j_0 \in [1, 2^{\nu_0}]$, $i_0 \in [0, 2^m]$ (see (12), (13)) we have

$$\sum_{k=N_0}^{M} c_k\varphi_k(x) = \gamma \sum_{j=0}^{j_0-1} a_j \Big(\sum_{i=0}^{2^m-1} b_i\varphi_i(x)\Big)\varphi_j(2^s x) + \gamma a_{j_0}\varphi_{j_0}(2^s x) \sum_{i=0}^{i_0} b_i\varphi_i(x).$$

From this and (3)–(8), for all $x \in [0,1)$ we obtain

$$\Big|\sum_{k=N_0}^{M} c_k\varphi_k(x)\Big| = |\gamma| \Big[(j_0 - 1)\cdot \chi_\Delta(x) + \frac{i_0}{2^m}\Big] \le \begin{cases} 2^{\nu_0+1}\cdot|\gamma|, & x \in \Delta, \\ |\gamma|, & x \in [0,1) \setminus \Delta. \end{cases}$$

Thus

$$\max_{N_0\le M\le N}\Big\|\sum_{k=N_0}^{M} a_k\varphi_k\Big\|_\infty \le 3|\gamma|\epsilon^{-1}.$$

Lemma 1 is proved.

LEMMA 2. *Let $m_0 > 1$, $\epsilon, \delta > 0$ and a Walsh polynomial $f(x)$ be given. Then one can find a set $E \subset [0,1)$ with $|E| > 1-\epsilon$ and a polynomial in the Walsh system*

$$Q(x) = \sum_{k=m_0}^{N} a_k\varphi_k(x)$$

satisfying the following conditions:

(i) $0 \le |a_k| < \delta$ *and the nonzero coefficients in* $\{|a_k|\}_{k=m_0}^N$ *are in decreasing order,*
(ii) $Q(x) = f(x)$ *for all* $x \in E$,
(iii) $\max_{m_0\le n\le N} \|\sum_{k=m_0}^{n} a_k\varphi_k\|_\infty < (3/\epsilon)\|f\|_\infty$.

Proof. Let

$$f(x) = \sum_{k=0}^{K} b_k\varphi_k(x) = \sum_{\nu=1}^{\nu_0} \gamma_\nu\cdot\chi_{\Delta_\nu}(x), \qquad \sum_{\nu=1}^{\nu_0}|\Delta_\nu| = 1, \tag{17}$$

where the Δ_ν are dyadic intervals of the form $\Delta_m^{(k)} = ((k-1)/2^m, k/2^m)$, $k \in [1, 2^m]$. Without loss of generality, one may assume that

$$0 < |\gamma_1|\,|\Delta_1| < \cdots < |\gamma_{\nu_0}|\,|\Delta_{\nu_0}| < \delta. \tag{18}$$

Successively applying Lemma 1, we determine sets $E_\nu \subset \Delta_\nu$ and polynomials

$$Q_\nu = \sum_{j=m_{\nu-1}}^{m_\nu-1} a_j\varphi_j,\ a_j = 0 \text{ or } \pm\gamma_j|\Delta_j| \text{ if } j\in[m_{\nu-1}, m_\nu),\ \nu=1,\dots,\nu_0, \tag{19}$$

which satisfy the following conditions:

$$|E_\nu| > (1-\epsilon)|\Delta_\nu|, \tag{20}$$

$$Q_\nu = \begin{cases}\gamma_\nu & \text{if } x\in E_\nu,\\ 0 & \text{if } x\notin\Delta_\nu,\end{cases} \tag{21}$$

$$\max_{m_{\nu-1}\le m< m_\nu}\Big\|\sum_{j=m_{\nu-1}}^{m} a_j\varphi_{j\nu}\Big\|_\infty < \frac{3|\gamma_\nu|}{\epsilon}. \tag{22}$$

We define

$$Q = \sum_{\nu=1}^{\nu_0} Q_\nu = \sum_{k=m_0}^{N} a_k \varphi_k, \qquad N = m_{\nu_0} - 1, \tag{23}$$

$$E = \bigcup_{\nu=1}^{\nu_0} E_\nu. \tag{24}$$

By (19)–(24) we obtain

$$Q(x) = f(x) \quad \text{for } x \in E, \qquad |E| > 1 - \epsilon,$$

$0 \le |a_k| < \delta$ and the nonzero coefficients in $\{|a_k|\}_{k=m_0}^{N}$ are in decreasing order. Taking into account (17), (21)–(23) we obtain

$$\max_{m_0 \le n \le N} \Big\| \sum_{k=m_0}^{n} a_k \varphi_k \Big\|_\infty < \frac{3}{\epsilon} \, \|f\|_\infty.$$

Lemma 2 is proved.

3. Proof of Theorem 3. Let $f \in L^\infty[0,1)$ and $\varepsilon \in (0,1)$. It is easy to see that one can choose a sequence $\{f_n\}_{n=1}^\infty$ of polynomials in the Walsh system such that

$$\lim_{N\to\infty} \Big\| \sum_{n=1}^{N} f_n(x) - f(x) \Big\|_\infty = 0, \tag{25}$$
$$\|f_n(x)\|_\infty \le \varepsilon \cdot 2^{-2(n+1)}, \;\; n \ge 2.$$

Applying repeatedly Lemma 2, we obtain sequences of sets $\{E_n\}_{n=1}^\infty$ and polynomials in the Walsh system

$$Q_n(x) = \sum_{k=m_{n-1}}^{m_n - 1} a_k \varphi_{s_k}(x), \qquad n \ge 1, \, m_n \nearrow, \tag{26}$$

which for all $n \ge 1$ satisfy the following conditions:

$$Q_n(x) = f_n(x) \quad \text{for } x \in E_n, \tag{27}$$

$$|E_n| > 1 - \varepsilon 2^{-n}, \tag{28}$$

$$\max_{m \in [m_{n-1}, m_n)} \Big\| \sum_{k=m_{n-1}}^{m} a_k \varphi_{s_k}(x) \Big\|_\infty \le 3\varepsilon^{-1} 2^n \cdot \|f_n\|_\infty, \tag{29}$$

$$|a_k| < |a_{k-1}| < \min\{|a_{m_{n-1}-1}|, 2^{-n}\} \quad \text{for all } k \in [m_{n-1}, m_n). \tag{30}$$

We put

$$\sum_{k=1}^{\infty} a_k \varphi_{s_k}(x) = \sum_{n=1}^{\infty} \sum_{k=m_{n-1}}^{m_n - 1} a_k \varphi_{s_k}(x), \tag{31}$$

$$g(x) = \sum_{n=1}^{\infty} Q_n(x). \tag{32}$$

From (27)–(29), (32) we get $g \in L^\infty[0,1)$ and

$$g(x) = f(x) \quad \text{for } x \in \bigcap_{n=1}^{\infty} E_n, \quad \Big| \bigcap_{n=1}^{\infty} E_n \Big| > 1 - \varepsilon.$$

It is easy to see that the series (31) converges to g uniformly on $[0,1)$ (see (26), (27), (29)) and therefore

$$a_k = \int_0^1 g(x) \varphi_{s_k}(x)\, dx = c_{s_k}(g), \quad k = 1, 2, \ldots.$$

Obviously, $\{|c_k(g)| : k \in \operatorname{spec}(g)\}$ is decreasing (see (30)).

Theorem 3 is proved.

Remark. Note that a more general result is true: let $\{\beta_k\}_{k=1}^{\infty}$ be a sequence of positive numbers with $\beta_k \to 0$. There exists a sequence $\{A_k\}_{k=1}^{\infty}$ of real numbers with $|A_k| \searrow 0$, $\sum_{n=1}^{\infty} |A_k| \beta_k < \infty$, with the following property: for any $0 < \epsilon < 1$, $p \geq 1$ and each $f \in L^p[0,1]$ one can find $g \in L^\infty[0,1)$ with $\operatorname{mes}\{x \in [0,1) : g \neq f\} < \epsilon$ such that $\{|c_k(g)| : k \in \operatorname{spec}(g)\} \subset \{A_k\}_{k=1}^{\infty}$, the Fourier–Walsh series of g converges uniformly on $[0,1)$, and for all $1 \leq p \leq \infty$,

$$\Big\| \sum_{k=0}^{n} c_k(g) \varphi_k(x) \Big\|_p \leq 2\|g\|_p \leq \frac{6}{\epsilon^{1-1/p}} \|f\|_p, \quad n = 1, 2, \ldots, \tag{33}$$

where $\{c_k(g)\}$ is the sequence of Fourier–Walsh coefficients of g.

From this we have

Corollary. *For any $0 < \epsilon < 1$, $p > 2$ and each $f \in L^p[0,1]$ one can find $g \in L^\infty[0,1)$ with $\operatorname{mes}\{x \in [0,1) : g \neq f\} < \epsilon$ such that the greedy algorithm $\{G_m(g)\}$ of g with respect to the Walsh system converges uniformly on $[0,1)$, and for all $1 \leq p \leq \infty$,*

$$\|G_m(g)\|_p \leq 2\|g\|_p \leq \frac{6}{\epsilon^{1-1/p}} \|f\|_p, \quad m = 1, 2, \ldots. \tag{34}$$

Note that in [22] P. Wojtaszczyk proved that for any basis of a Banach space X the uniform boundedness of $\{G_m(f)\}$ in X is equivalent to the convergence of $\{G_m(f)\}$ to f in the norm of X, but (34) does not imply the convergence of the greedy algorithm $\{G_m(g)\}$ to the new modified function g.

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