Bounded evaluation operators from $H^p$ into $\ell^q$

by

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Abstract. Given $0 < p, q < \infty$ and any sequence $z = \{z_n\}$ in the unit disc $\mathbb{D}$, we define an operator from functions on $\mathbb{D}$ to sequences by $T_{z,p}(f) = \{(1 - |z_n|^2)^{1/p} f(z_n)\}$. Necessary and sufficient conditions on $\{z_n\}$ are given such that $T_{z,p}$ maps the Hardy space $H^p$ boundedly into the sequence space $\ell^q$. A corresponding result for Bergman spaces is also stated.

1. Introduction. For $0 < p < \infty$ let $\ell^p$ denote the classical sequence space and $H^p$ denote the classical Hardy space of the unit disc, $\mathbb{D}$. It is well known that, for all $f \in H^p$ and $z \in \mathbb{D}$,

\begin{equation}
|f(z)| \leq \|f\|_{H^p} (1 - |z|^2)^{-1/p}
\end{equation}

(see e.g. [4, p. 36]), and that this gives a sharp rate of growth for $H^p$ functions. Given any sequence $z = \{z_n\}$ in $\mathbb{D}$ we define the operator $T_{z,p}$ by

\begin{equation}
T_{z,p}(f) = \{(1 - |z_n|^2)^{1/p} f(z_n)\} \quad \text{for } f \text{ a function on } \mathbb{D}.
\end{equation}

The operator plays a key role in interpolation theory, indeed, $z$ is said to be an interpolating sequence for $H^p$ if $T_{z,p}$ maps $H^p$ onto $\ell^p$. Note that (1) trivially implies that $\|T_{z,p}(f)\|_{\ell^\infty} \leq \|f\|_{H^p}$ for all $f \in H^p$. It is also straightforward to show that for an infinite sequence $z$, $T_{z,p}$ maps $H^p$ into $c_0$, the space of sequences which tend to zero, if and only if $|z_n| \to 1$ as $n \to \infty$.

The aim of this paper is as follows: given $0 < p, q < \infty$, describe all sequences $z$ such that there exists a constant $C$ such that

\begin{equation}
\|T_{z,p}(f)\|_{\ell^q} \leq C \|f\|_{H^p} \quad \text{for all } f \in H^p.
\end{equation}

Given $z, w \in \mathbb{D}$, let $\phi_w$ denote the corresponding Möbius transform and $d(z,w)$ the pseudohyperbolic distance, i.e.

\[ \phi_w(z) = \frac{z - w}{1 - \overline{w}z} \quad \text{and} \quad d(z,w) = |\phi_w(z)|. \]

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A sequence of points \( \{z_n\} \) in \( \mathbb{D} \) is said to be \textit{uniformly discrete} if
\[
\inf_{n \neq m} d(z_n, z_m) > \delta > 0 \quad \text{for some} \ \delta,
\]
and \textit{uniformly separated} if
\[
\inf_n \prod_{m \neq n} d(z_n, z_m) > \delta > 0 \quad \text{for some} \ \delta.
\]

Perhaps surprisingly, the characterisation of sequences \( z \) such that (3) holds forms a trichotomy depending only upon whether \( p \) is less than, equal to or greater than \( q \):

\textbf{Theorem 1.} Given \( 0 < p, q < \infty \) and a sequence \( \{z_n\} \) in \( \mathbb{D} \), the following are equivalent:

1. There exists a constant \( C \) such that
\[
\sum_n (1 - |z_n|^2)^{q/p} |f(z_n)|^q \leq C \|f\|_{H^p}^q \quad \text{for all} \ f \in H^p.
\]

2. (a) \( p < q \) and \( \{z_n\} \) is a finite union of uniformly discrete sequences;
   (b) \( p = q \) and \( \{z_n\} \) is a finite union of uniformly separated sequences;
   (c) \( p > q \) and \( \{z_n\} \) is a finite sequence.

The conclusion of Theorem 1 when \( p = q \) is well known; see [5], [8] and [9]. It is closely related to the fact that \( T_{z,p} \) maps \( H^p \) onto \( \ell^p \) if and only if the infinite sequence \( \{z_n\} \) is uniformly separated; this was proved by Carleson [2] when \( p = \infty \), Shapiro and Shields [12] when \( 1 \leq p < \infty \) and Kabaľa [7] when \( 0 < p < 1 \); see e.g. [4, Chapter 9].

We shall therefore concentrate on the cases when \( p \neq q \), where the characterisations given do not appear to be stated in the literature.

\textbf{2. The case} \( p < q \). Our main tool is the following generalisation of Carleson’s measure theorem due to Duren [3]. Given \( \theta_0 \in [0, 2\pi) \) and \( 0 \leq \theta \leq \theta_0 + h \), let
\[
S(\theta_0, h) = \{re^{i\theta} : 1 - h \leq r < 1, \theta_0 \leq \theta \leq \theta_0 + h\}
\]
be the corresponding Carleson square.

\textbf{Theorem 2.} Given a finite positive Borel measure \( \mu \) on \( \mathbb{D} \) and \( 0 < p \leq q < \infty \), there exists a constant \( C \) such that
\[
\int_{\mathbb{D}} |f(z)|^q d\mu(z) \leq C \|f\|_{H^p}^q \quad \text{for all} \ f \in H^p
\]
if and only if there exists a constant \( \tilde{C} \) such that \( \mu(S(\theta_0, h)) \leq \tilde{C}h^{q/p} \) for all Carleson squares \( S(\theta_0, h) \).

We can now prove Theorem 1 in the case that \( p < q \).
**Theorem 3.** Given a sequence \( \{z_n\} \) in \( \mathbb{D} \), the following are equivalent:

1. For all \( 0 < p < q < \infty \), there exists a constant \( C \) such that
   \[
   \sum_n (1 - |z_n|^2)^{q/p} |f(z_n)|^q \leq C \|f\|_{H^p}^q
   \]
   for all \( f \in H^p \).

2. \( \{z_n\} \) is a finite union of uniformly discrete sequences.

3. For some \( r > 1 \),
   \[
   \sup_{z \in \mathbb{D}} \sum_n (1 - |\phi_{z_n}(z)|^2)^r < \infty.
   \]

4. For all \( r > 1 \),
   \[
   \sup_{z \in \mathbb{D}} \sum_n (1 - |\phi_{z_n}(z)|^2)^r < \infty.
   \]

**Proof.** (1) \( \Rightarrow \) (4). Given any \( 0 < p < \infty \), let \( q = rp \). For all \( z \in \mathbb{D} \), let
   \[
   f_z(w) = \frac{(1 - |z|^2)^{1/p}}{(1 - \overline{z}w)^{2/p}},
   \]
so \( \|f_z\|_{H^p} \equiv 1 \). Consequently,
   \[
   \sup_{z \in \mathbb{D}} \sum_n \left(1 - |\phi_{z_n}(z)|^2\right)^r \leq C.
   \]
   The result now follows from the identity
   \[
   1 - |\phi_{z_n}(z)|^2 = \frac{(1 - |z|^2)(1 - |z_n|^2)}{|1 - \overline{z}z_n|^2}.
   \]

(4) \( \Rightarrow \) (3) is trivial so we show that (3) \( \Rightarrow \) (2), following a method from [5] and [9]. For any point \( z \in \mathbb{D} \), let \( N(z) \) denote the number of points of \( \{z_n\} \) contained in \( \Delta(z, 1/2) := \{w \in \mathbb{D} : d(w, z) < 1/2\} \). Then there exist \( K > 0 \) such that, for all \( z \in \mathbb{D} \),
   \[
   K \geq \sum_n (1 - |\phi_{z_n}(z)|^2)^r \geq \sum_{z_n \in \Delta(z, 1/2)} (1 - |\phi_{z_n}(z)|^2)^r \geq (3/4)^r N(z),
   \]
so \( N(z) \leq K(4/3)^r \). Since there exists an integer \( N \) such that \( N(z) \leq N \) for all \( z \in \mathbb{D} \), it follows that \( \{z_n\} \) can be split into the union of at most \( N \) uniformly discrete sequences (see e.g. [6, p. 69]).

(2) \( \Rightarrow \) (1). We may as well suppose that \( \{z_n\} \) is uniformly discrete. Then, letting
   \[
   Q(\theta_0, h) = \{re^{i\theta} : 1 - h \leq r < 1 - h/2, \theta_0 \leq \theta \leq \theta_0 + h\}
   \]
be the top half of the Carleson square \( S(\theta_0, h) \), it is easily shown that there exists an integer \( M \) such that every set \( Q(\theta_0, h) \) contains at most \( M \) points of the sequence \( \{z_n\} \). So, letting \( \mu \) be the discrete measure
   \[
   \mu = \sum_n (1 - |z_n|^2)^{q/p} \delta_{z_n},
   \]
we have for any $S(\theta_0, h)$,
\[
\mu(S(\theta_0, h)) = \sum_{k=0}^{\infty} \sum_{j=0}^{2^{k-1}} \mu(Q(\theta_0 + 2^{-k}j, 2^{-k}h))
\]
\[
\leq \sum_{k=0}^{\infty} \sum_{j=0}^{2^{k-1}} M(1 - (1 - 2^{-k-1}h)^2)^{q/p}
\]
\[
\leq M \sum_{k=0}^{\infty} 2^k(2^{-k}h)^{q/p} = M h^{q/p} \sum_{k=0}^{\infty} 2^{k(q-1)/p} = C h^{q/p},
\]
for some $C$ as $q > p$. Now (1) follows from Theorem 2. 

The surprising arithmetic fact that (3) implies (4) in Theorem 3 generalises [9, Theorem 4].

3. The case $p > q$. Using (1), it is easily shown that (3) holds when \{\{z_n\}\} is a finite sequence.

**Proposition 4.** Let $0 < q < p < \infty$ and \{\{z_n\}\} be a sequence in $\mathbb{D}$. Suppose that there exists a constant $C$ such that
\[
\sum_n (1 - |z_n|^2)^{q/p}|f(z_n)|^q \leq C \|f\|_{H^p}^q \quad \text{for all } f \in H^p.
\]
Then \{\{z_n\}\} is a finite sequence.

**Proof.** Suppose that (4) holds for an infinite sequence \{\{z_n\}\}. Then, for all $f \in H^p$,
\[
\sum_n (1 - |z_n|^2)^{q/p}|f(z_n)|^q \leq \left( \sum_n (1 - |z_n|^2)^{q/p}|f(z_n)|^q \right)^{p/q} \leq C^{p/q} \|f\|_{H^p}^p.
\]
So, by Theorem 1, \{\{z_n\}\} is a finite union of uniformly separated sequences. By removing superfluous terms if necessary, we may suppose that \{\{z_n\}\} is an infinite uniformly separated sequence. Then the map $T_{z,p} : H^p \to \ell^p$ as defined in (2) is onto (see the comments after Theorem 1). By Banach's open mapping theorem there exists a constant $N$ such that for all $\{\alpha_n\} \in \ell^p$, there exists $f \in H^p$ with $T_{z,p} f = \{\alpha_n\}$ and $\|f\|_{H^p} \leq N \|\{\alpha_n\}\|_{\ell^p}$ (see e.g. [4, p. 149]). So, in view of (4), $\|\{\alpha_n\}\|_{\ell^q} \leq C^{1/q} \|f\|_{H^p} \leq C^{1/q} N \|\{\alpha_n\}\|_{\ell^p}$ for all $\{\alpha_n\} \in \ell^p$, which gives a contradiction.

4. Remarks and acknowledgements. The inequality (3) has a dual formulation. For $1 < p, q < \infty$, let $p' = p/(p-1)$ and $q' = q/(q-1)$. Then we may identify the dual space of $\ell^q$ with $\ell^{q'}$ and the dual space of $H^p$ with $H^{p'}$ (under the pairing induced by the inner product in $H^2$; see e.g. [4, p. 113]). Given $z \in \mathbb{D}$, let $k_z$ denote the corresponding Cauchy kernel, so
\[ k_z(w) = 1/(1 - z w). \] The following reproducing property holds: for \( f \in H^p \), \( f(z) = \langle f, k_z \rangle \). By considering the adjoint of \( T_{z,p} \), it is easily shown that for \( p, q \) as above, (3) holds if and only if there exists a constant \( \tilde{C} \) such that

\[
\left\| \sum_n \alpha_n (1 - |z_n|^2)^{1/p} k_{z_n} \right\|_{H^{p'}} \leq \tilde{C} \| \{ \alpha_n \} \|_{\ell^q'} \quad \text{for all } \{ \alpha_n \} \in \ell^q'.
\]

Using this equivalent formulation, an application for Theorem 1 in the classification of Schatten class Hankel operators has been found in [10].

We can also consider an analogous problem for Bergman spaces. For \( 0 < p < \infty \) let \( A^p \) denote the classical Bergman space of the unit disc. It is well known that \( |f(z)| \leq \|f\|_{A^p} (1 - |z|^2)^{-2/p} \) for all \( f \in A^p \) and \( z \in \mathbb{D} \) (see e.g. [6, p. 36]. Given any sequence \( z = \{ z_n \} \) in \( \mathbb{D} \) we define the operator \( R_{z,p} \) by \( R_{z,p}(f) = \{(1 - |z_n|^2)^{2/p} f(z_n)\} \).

**Theorem 5.** Given \( 0 < p, q < \infty \) and a sequence \( \{ z_n \} \) in \( \mathbb{D} \), the following are equivalent:

1. There exists a constant \( C \) such that
   \[
   \sum_n (1 - |z_n|^2)^{2n/p} |f(z_n)|^q \leq C \|f\|_{A^p}^q \quad \text{for all } f \in A^p.
   \]

2. (a) \( p \leq q \) and \( \{ z_n \} \) is a finite union of uniformly discrete sequences;  
   (b) \( p > q \) and \( \{ z_n \} \) is a finite sequence.

The conclusion when \( p = q \) may be found in [13]; see also [6, p. 70]. It is closely related to Amar’s result that, if \( \{ z_n \} \) is uniformly discrete, then \( \{ z_n \} \) is the finite union of sequences \( \{ z_n^{(i)} \} \) such that each \( R_{z^{(i)},p} \) maps \( A^p \) onto \( \ell^p \) (see [1, Theorem 2.1.1], also [11]). The proofs when \( p \neq q \) are similar to the Hardy space cases but simpler, and so are omitted.

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