Shilov boundary for holomorphic functions
on some classical Banach spaces

by

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Abstract. Let $A_{\infty}(B_X)$ be the Banach space of all bounded and continuous functions on the closed unit ball $B_X$ of a complex Banach space $X$ and holomorphic on the open unit ball, with sup norm, and let $A_u(B_X)$ be the subspace of $A_{\infty}(B_X)$ of those functions which are uniformly continuous on $B_X$. A subset $B \subset B_X$ is a boundary for $A_{\infty}(B_X)$ if $\|f\| = \sup_{x \in B} |f(x)|$ for every $f \in A_{\infty}(B_X)$. We prove that for $X = d(w, 1)$ (the Lorentz sequence space) and $X = C_1(H)$, the trace class operators, there is a minimal closed boundary for $A_{\infty}(B_X)$. On the other hand, for $X = S$, the Schreier space, and $X = K(\ell_p, \ell_q)$ ($1 \leq p \leq q < \infty$), there is no minimal closed boundary for the corresponding spaces of holomorphic functions.

1. Introduction. A result of Shilov asserts that if $\mathfrak{A}$ is a unital separating algebra of $C(K)$ ($K$ a compact Hausdorff topological space), then there is a smallest closed subset $S \subset K$ such that every function of $\mathfrak{A}$ attains its norm at some point of $S$ [6, Theorem I.4.2]. Bishop [4] proved that if $K$ is metrizable, then, in fact, there is a minimal subset of $K$ satisfying the above condition for every separating algebra of $C(K)$. That subset is the set of peak points for $\mathfrak{A}$ (see definition below).

Globevnik introduced the corresponding concepts for a subalgebra of $C_b(\Omega)$, the space of bounded continuous functions on a topological space $\Omega$ not necessarily compact [9]. In fact, he considered the case $\Omega = B_X$, where $X$ is a Banach space. If $\mathfrak{A}$ is a subspace of $C_b(\Omega)$, we will say that a subset $B \subset \Omega$ is a boundary for $\mathfrak{A}$ if

$$\|f\| = \sup_{b \in B} |f(b)|, \quad \forall f \in \mathfrak{A}.$$

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If there is a closed boundary \( B \) that is contained in all closed boundaries for \( A \), we will say that \( B \) is the Shilov boundary of \( A \).

If \( X \) is a complex Banach space, we will denote by \( A_u(B_X) \) the space of uniformly continuous functions on the closed unit ball of \( X \) which are holomorphic on the open unit ball. Globevnik [9] described the boundaries of \( A_u(B_{c_0}) \). As a consequence of the description, he showed that this algebra has no Shilov boundary. Aron, Choi, Lourenço and Paques [3] gave examples of boundaries for \( A_u(B_{\ell_\infty}) \) and proved that there is no Shilov boundary for this algebra. They also showed that the unit sphere of \( \ell_1 \) is the Shilov boundary for \( A_u(B_{\ell_1}) \).

Moraes and Romero [14] gave a characterization of the boundaries of \( A_u(B_{d_\ast(w,1)}) \), where \( d_\ast(w,1) \) is the canonical predual of the Lorentz sequence space \( d(w,1) \) when \( w = (1/n) \). Later Acosta, Moraes and Romero [2] generalized that characterization proving it for any space \( d_\ast(w,1) \) and obtained another one in terms of the strong peak sets of the unit ball. In this case, there is no Shilov boundary. Choi, García, Kim and Maestre [5] proved that there is no Shilov boundary for \( A_u(B_{C(K)}), \) when \( K \) is infinite and scattered. Acosta showed the same result for every infinite \( K \) and also proved that for this space the set of extreme points of the unit ball of \( C(K) \) is a boundary for \( A_u(B_{C(K)}) \) (see [1]).

Before going on it is convenient to recall some definitions. Let \( A \) be a function space on a metric space \( \Omega \). An element \( y \in \Omega \) is called a peak point for \( A \) if there is some \( f \in A \) such that \( f(y) = 1 \) and \( |f(x)| < 1 \) for all \( x \in \Omega \setminus \{y\} \). In this case we say that \( f \) peaks at \( y \). An element \( y \in \Omega \) is called a strong peak point for \( A \) if there is some \( f \in A \) satisfying \( f(y) = 1 \) and such that given any \( \varepsilon > 0 \) there is some \( \delta > 0 \) such that \( \text{dist}(x,y) > \varepsilon \) implies that \( |f(x)| < 1 - \delta \). It is clear that every closed boundary for \( A \) contains all the strong peak points.

In this paper we prove that there is no Shilov boundary for \( A_u(B_X) \) when \( X \) is the Schreier space or the space \( K(\ell_p, \ell_q) \) \((1 \leq p \leq q < \infty) \). For the spaces \( X = C_1(H) \), the trace class operators on a complex Hilbert space \( H \), or \( X = d(w,1) \), the Shilov boundary for \( A_u(B_X) \) exists. In fact, all the points in the unit sphere of \( d(w,1) \) are strong peak points for \( A_u(B_{d(w,1)}) \), and so in this case the Shilov boundary is the unit sphere. For \( \ell_1 \) the same result holds. That fact was proved in [3] for the finitely supported sequences in the unit sphere. If \( K \) is infinite, we also prove that there are no strong peak points for \( A_u(B_{C(K)}) \). The set of peak points for \( A_u(B_{C(K)}) \) is the set of extreme points of \( B_{C(K)} \) if \( K \) is separable.

Throughout this paper, all the Banach spaces considered are complex. For a Banach space \( X \), \( B_X \) and \( S_X \) will be the closed unit ball and the unit sphere of \( X \), respectively. We will denote by \( A_{\infty}(B_X) \) the Banach space of all bounded and continuous functions on \( B_X \) which are holomorphic on the
open unit ball, and by $A_u(B_X)$ the space of all functions in $A_\infty(B_X)$ which are uniformly continuous.

2. Existence of the Shilov boundary on the Lorentz sequence space. Given a decreasing sequence $w$ of positive real numbers satisfying $w_1 = 1$ and $w \in c_0 \setminus \ell_1$, the complex Lorentz sequence space $d(w, 1)$ is given by

$$d(w, 1) = \left\{ x : \mathbb{N} \to \mathbb{C} : \sup\left\{ \sum_{n=1}^{\infty} |x(\sigma(n))|w_n : \sigma : \mathbb{N} \to \mathbb{N} \text{ injective} \right\} < \infty \right\}.$$ 

The norm is given by

$$\|x\| = \sup\left\{ \sum_{n=1}^{\infty} w_n|x(\sigma(n))| : \sigma : \mathbb{N} \to \mathbb{N} \text{ injective} \right\} \quad (x \in d(w, 1)).$$

It is well known and easy to verify that the above supremum is attained for the decreasing rearrangement of $x$. The usual vector basis $(e_n)$ is a monotone Schauder basis (see [12]).

A canonical predual $d_\ast(w, 1)$ of $d(w, 1)$ is given by

$$d_\ast(w, 1) = \left\{ x \in c_0 : \lim_n \frac{\sum_{k=1}^{n} x^*(k)}{W_n} = 0 \right\}$$

where $W_n = \sum_{k=1}^{n} w_k$ and $x^*$ is the decreasing rearrangement of $x$. This space is a Banach space endowed with the norm

$$\|x\| = \sup_n\left\{ \frac{\sum_{k=1}^{n} x^*(k)}{W_n} \right\}$$

(see [16] and [7]). $d_\ast(w, 1)$ has a Schauder basis whose sequence of biorthogonal functionals is, in fact, the canonical basis of $d(w, 1)$.

We begin by presenting some useful lemmas.

**Lemma 2.1.** If $(z_n)$ is a bounded sequence of complex numbers such that the sequence $(1 + |z_n| - |1 + z_n|)$ converges to zero, then so does $(|z_n| - z_n)$.

**Proof.** We consider the following identity for a complex number $z$:

$$(1 + |z| - |1 + z|)^2 = 1 + |z|^2 + 2|z| + |1 + z|^2 - 2(1 + |z|) |1 + z|$$

$$= 2(\text{Re } z - |z|) + 2(1 + |z|)(1 + |z| - |1 + z|).$$

If we apply the above identity to the sequence $(z_n)$ and use the assumption, we find that the sequence $(|z_n| - \text{Re } z_n)$ converges to zero.

Now if we consider the expression

$$(|z| - \text{Re } z)^2 = 2(\text{Re } z)^2 + (\text{Im } z)^2 - 2|z| \text{Re } z$$

$$= (\text{Im } z)^2 + 2(\text{Re } z - |z|) \text{Re } z,$$
and we apply the identity to the sequence \((z_n)\), we deduce that \(\text{Im} \, z_n \to 0\). Hence
\[
|z_n - z| = |z_n| - \text{Re} \, z_n - i \text{Im} \, z_n \to 0.
\]

**Lemma 2.2 ([3, Lemma 9]).** Let \(0 < a < 1\). The real-valued function given by
\[
g_a(x) = \left(1 + \frac{x}{1-a}\right) \left(1 + \frac{1-x}{a}\right) \quad (x \in \mathbb{R})
\]
attains its maximum at \(x = a\) and
\[
g_a(x) < g_a(a) = \frac{1}{a(1-a)}, \quad \forall x \in \mathbb{R} \setminus \{a\}.
\]

**Lemma 2.3.** The set of peak points in \(S_X\) for \(A_\infty(B_X)\) is invariant under surjective linear isometries on \(X\). The same holds for the set of strong peak points in \(S_X\).

By the maximum modulus theorem, every peak point for a subspace of \(A_\infty(B_X)\) belongs to \(S_X\). As a consequence, so does every strong peak point. The following result shows the converse for the subspace of all polynomials on \(d(w,1)\).

**Theorem 2.4.** The set of strong peak points for the space of polynomials of degree less than or equal to 2 on \(d(w,1)\) contains the unit sphere of \(d(w,1)\).

**Proof.** Let \(y_0 \in S_{d(w,1)}\). By Lemma 2.3 we can assume that \(\text{supp} \, y_0\) is an interval of positive integers containing \(\{1\}\) and
\[
y_0(j) \in \mathbb{R}^+, \quad \forall j \in \text{supp} \, y_0, \quad y_0(n) \geq y_0(n+1), \quad \forall n \in \mathbb{N}.
\]
We will prove that \(y_0\) is a strong peak point for \(A_\infty(d(w,1))\).

If the support of \(y_0\) contains just one element, then \(y_0 = e_1\) and it is sufficient to consider the first-degree polynomial given by
\[
f(x) = 1 + x(1) \quad (x \in d(w,1)).
\]
Clearly \(\|f\| = 2 = f(y_0)\). By using the fact that in \(S_{d(w,1)}\) the weak and \(\sigma(d(w,1),d_*(w,1))\) convergences coincide ([16, Proposition 2.2] and [10, Corollary III.2.15]) and that every point of the unit sphere is a point of weak-norm continuity of the unit ball [13, Proposition 4], it is easily checked that \(f\) strongly peaks in the unit ball at \(y_0\).

Now assume that \(J := \text{supp} \, y_0\) has at least two elements. Since \(\|y_0\| = 1\), by (1), we know that \(\sum_{i \in J} w_i y_0(i) = 1\) and so \(0 < w_i y_0(i) < 1\) for every \(i \in J\).

For every \(k \in J\) we define
\[
f_k(x) = \frac{1}{M_k} \left(1 + \frac{w_k x(k)}{1 - w_k y_0(k)}\right) \left(1 + \frac{1}{w_k y_0(k)} \sum_{j \in J \setminus \{k\}} w_j x(j)\right) \quad (x \in d(w,1))
\]
where

\[ M_k = \frac{1}{w_k y_0(k)(1 - w_k y_0(k))}. \]

Then \( f_k \) is clearly a non-homogeneous polynomial on \( d(w, 1) \) of degree 2 and \( f_k(y_0) = 1 \). We will check that \( \|f_k\| = 1 \).

If \( x \in B_{d(w, 1)} \), then

\[
|f_k(x)| = \frac{1}{M_k} \left| 1 + \frac{w_k x(k)}{1 - w_k y_0(k)} \right| \left| 1 + \frac{1}{w_k y_0(k)} \sum_{j \in J \setminus \{k\}} w_j x(j) \right|
\]

\[
\leq \frac{1}{M_k} \left( 1 + \frac{w_k |x(k)|}{1 - w_k y_0(k)} \right) \left( 1 + \frac{1}{w_k y_0(k)} \sum_{j \in J \setminus \{k\}} |w_j x(j)| \right)
\]

\[
\leq \frac{1}{M_k} \left( 1 + \frac{w_k y_0(k)}{1 - w_k y_0(k)} \right) \left( 1 + \frac{1 - w_k |x(k)|}{w_k y_0(k)} \right) \quad \text{(since } x \in B_X)\]

\[
= 1.
\]

Hence \( \|f_k\| = 1 \).

Our intention is to show that \( y_0 \) is a strong peak point for the space of second-degree polynomials. To this end, we will prove that

\[
x_n \in B_{d(w, 1)}, \forall n, \quad |f_k(x_n)| \to 1 \Rightarrow x_n(k) \to y_0(k).
\]

For every fixed \( k \), we write

\[
u_n = \sum_{j \in J \setminus \{k\}} w_j x_n(j), \quad \nu_n = \sum_{j \in J \setminus \{k\}} w_j x_n(j).
\]

We rewrite the inequality (2) in terms of the above sequences:

\[
|f_k(x_n)| = \frac{1}{M_k} |1 + u_n| |1 + v_n| \leq \frac{1}{M_k} (1 + |u_n|)(1 + |v_n|) \leq 1.
\]

If we assume that \( |f_k(x_n)| \to 1 \) as \( n \to \infty \), then the sequence \( (1 + v_n) \) has no subsequence converging to zero. From the above inequality we deduce that

\[
|1 + u_n| - 1 - |u_n| \to 0.
\]

Since \( k \) is fixed, Lemma 2.1 implies that \( (|u_n| - u_n) \) converges to zero, that is, \( |x_n(k)| - x_n(k) \to 0 \) as \( n \to \infty \). Also by Lemma 2.2, we know that

\[
w_k |x_n(k)| \to w_k y_0(k) \quad \text{as } n \to \infty.
\]

Hence we deduce that \( x_n(k) \to y_0(k) \) as \( n \to \infty \).
Now we choose a sequence \((\alpha_n)\) in \(\ell_1\) such that \(\text{supp} \alpha = J\), \(\alpha_n > 0\) for all \(n \in J\) and \(\sum_{n \in J} \alpha_n = 1\). Define
\[
f(x) = \sum_{k \in J} \alpha_k f_k(x) \quad (x \in B_{d(w,1)}).
\]
Then \(f\) is a polynomial of degree at most 2 in \(d(w,1)\) and \(\|f\| \leq 1 = f(y_0)\).

We now prove that this function strongly peaks in the unit ball of \(d(w,1)\) at \(y_0\). So assume that \(|f(x_n)| \to 1\) for some sequence \((x_n)\) in the unit ball. Then clearly \(f_k(x_n) \to 1\) as \(n \to \infty\) for every \(k \in J\).

Since \(y_0 \in S_{d(w,1)}\), by condition (3), we know that \((x_n)\) converges pointwise to \(y_0\). All the elements involved in the argument are in the unit ball of \(d(w,1)\) and \((x_n)\) converges to \(y_0\) in the \(\sigma(d(w,1), d_*(w,1))\)-topology. Since \(d_*(w,1)\) is an M-ideal in its dual (see [16, Proposition 2.2] or [10, Examples III.1.4c]), in the unit ball of \(d(w,1)\), the weak and weak* topologies coincide on the unit sphere, in view of [10, Corollary III.2.15]. By applying this to the element \(y_0\), which is the \(w^*\)-limit of \((x_n)\), we see that in fact \((x_n)\) converges weakly to \(y_0\). Since all the points of the unit sphere of \(d(w,1)\) are points of weak-norm continuity [13, Proposition 4], we conclude that \((x_n)\) converges in norm to \(y_0\) and \(y_0\) is a strong peak point, as we wanted to show. ■

**Corollary 2.5.** The Shilov boundary for the space of second-degree polynomials on \(d(w,1)\) is \(S_{d(w,1)}\). Hence \(S_{d(w,1)}\) is also the Shilov boundary for \(A_u(B_{d(w,1)})\) and \(A_\infty(B_{d(w,1)})\).

It is known that all the finitely supported elements in \(S_{\ell_1}\) are strong peak points for the space of second-degree polynomials on \(\ell_1\) [3, Theorem 10]. We now extend that result.

**Theorem 2.6.** \(S_{\ell_1}\) is the set of strong peak points for the space of second-degree polynomials on \(\ell_1\).

**Proof.** If \(y_0 \in S_{\ell_1}\), then, by Lemma 2.3, we can assume that \(y_0(n) \geq 0\) for every \(n\). If \(|\text{supp} y_0| = 1\) and \(\{n\} = \text{supp} y_0\), the function \(x \mapsto 1 + x(n)\) strongly peaks in the unit ball of \(\ell_1\) at \(y_0\). Otherwise, if \(J := \text{supp} y_0\) satisfies \(|J| \geq 2\), then the second-degree polynomial given by
\[
f_k(x) := \frac{1}{y_0(k)(1-y_0(k))} \left(1 + \frac{x(k)}{1-y_0(k)} \right) \left(1 + \sum_{i \neq k} \frac{x(i)}{y_0(k)} \right) \quad (x \in \ell_1)
\]
satisfies \(f_k(y_0) = 1\). In view of Lemma 2.2, also \(\|f_k\| = 1\) and now we can follow the argument in the proof of Theorem 2.4. ■

**3. Boundaries for the Schreier space and \(C(K)\).** A subset \(E = \{n_1 < \cdots < n_k\}\) of the natural numbers \(\mathbb{N}\) is said to be admissible if \(k \leq n_1\). The Schreier space \(S\) is the completion of the space \(c_{00}\) of all scalar sequences
of finite support with respect to the norm $\|x\| = \sup\sum_{j \in E} |x_j|$, where the supremum is taken over all admissible sets $E$ of natural numbers.

The following theorem shows in particular that the intersection of all boundaries for $A_\infty(B_S)$ is empty.

**Theorem 3.1.** Let $S$ be the Schreier space and $B$ be a boundary for $A_\infty(B_S)$. If $x_0 \in B$ and $0 < r < 1$, then $B \setminus (x_0 + rB_S)$ is a boundary for $A_\infty(B_S)$. As a consequence, there is no Shilov boundary for $A_\infty(B_S)$.

**Proof.** Assume that $h \in A_\infty(B_S)$. For every $0 < \varepsilon < (1 - r)/2$, there is $y_0 \in c_{00}$ such that $\|y_0\| < 1$ and

$$|h(y_0)| > \|h\| - \varepsilon.$$ 

We write $k = \max \text{supp} y_0$ and denote by $(P_n)$ the sequence of canonical projections associated to the usual basis of $S$. Choose a positive integer $n$ such that $n > k/(1 - \|y_0\|)$ and $\|(I - P_n)(x_0)\| < \varepsilon$. We will check that $y_0 + \lambda y \in B_S$ for every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and $y = \sum_{j=n+1}^{2n} (1/n) e_j$.

Let $A = E \cup F$ be an admissible set such that $E \subset \{1, \ldots, k\}$ and $\min F > k$. If $E \neq \emptyset$, then $|E| + |F| \leq k$ and

$$\sum_{i \in E \cup F} |y_0 + \lambda y(i)| \leq \sum_{i \in E} |y_0(i)| + \sum_{i \in F} |y(i)| \leq \|y_0\| + \frac{k}{n} \leq 1.$$ 

If $E = \emptyset$, then $\sum_{i \in F} |(y_0 + \lambda y)(i)| = \sum_{i \in F} |y(i)| \leq 1$. So $\|y_0 + \lambda y\| \leq 1$.

By the maximum modulus theorem, there is $\lambda_0 \in \mathbb{C}$ with $|\lambda_0| = 1$ such that

$$|h(y_0 + \lambda_0 y)| \geq |h(y_0)| > \|h\| - \varepsilon.$$ 

Fix $\lambda_1 \in \mathbb{C}$ satisfying $|\lambda_1| = 1$ and

$$|h(y_0 + \lambda_0 y) + \lambda_1| = |h(y_0 + \lambda_0 y)| + 1.$$ 

Since $\|y\| = 1$ and $P_n(y) = 0$, there is $y^* \in S_S$ such that $y^*(\lambda_0 y) = 1$, $y^*(e_j) = 0$ for all $j \leq n$ and so $y^*(y_0) = 0$. Now, we define a holomorphic function $g$ by

$$g(x) := h(x) + \lambda_1 y^*(x) \quad (x \in B_S).$$ 

Clearly $g \in A_\infty(B_S)$ and

$$\|h\| - \varepsilon + 1 < |h(y_0)| + 1 \leq |h(y_0 + \lambda_0 y)| + y^*(\lambda_0 y) = |g(y_0 + \lambda_0 y)| \leq \|g\| \leq \|h\| + 1.$$ 

Since $B$ is a boundary there is $z_0 \in B$ such that

$$|g(z_0)| > \|h\| - \varepsilon + 1.$$ 

On the other hand,

$$|g(z_0)| \leq |h(z_0)| + |y^*(z_0)| \leq \|h\| + |y^*(z_0)| \leq \|h\| + 1.$$
This implies $|y^*(z_0)| > 1 - \varepsilon$. Hence
\[ \| (I - P_n)(z_0) \| \geq |y^*(z_0)| > 1 - \varepsilon. \]
Consequently,
\[ \| z_0 - x_0 \| \geq \| (I - P_n)(z_0 - x_0) \| \]
\[ \geq \| (I - P_n)(z_0) \| - \| (I - P_n)x_0 \| \geq 1 - 2\varepsilon > r. \]
Also $|h(z_0)| + 1 \geq \| h \| + 1 - \varepsilon$ and hence $|h(z_0)| > \| h \| - \varepsilon$. Therefore $z_0 \in B \setminus (x_0 + rB_S)$ and this set is a boundary for $A_\infty(B_S)$. As a consequence, the Shilov boundary of this space does not exist.

We recall that a point $x \in B_X$ is a $C$-extreme point of the unit ball if
\[ (y \in X, \| x + \lambda y \| \leq 1, \forall \lambda \in C, \| \lambda \| = 1) \Rightarrow y = 0. \]

**Theorem 3.2.** If $K$ is any infinite compact Hausdorff topological space, then there are no strong peak points for $A_\infty(B_{C(K)})$. If $K$ is separable, then all the extreme points in $B_{C(K)}$ are peak points for the space of first-degree polynomials on $C(K)$.

**Proof.** It is known that every peak point is a $C$-extreme point [8, Theorem 4]. So we will prove that $C$-extreme points of $B_{C(K)}$ are not strong peak points. Assume that $x_0 \in S_{C(K)}$ is an extreme point of the unit ball. Since $K$ is infinite, there is a sequence $(x_n) \subset C(K)$ satisfying
\[ 0 \leq x_n \leq 1, \| x_n \| = 1, \forall n, \quad \text{supp } x_n \cap \text{supp } x_m = \emptyset, \forall n \neq m. \]
Assume that $h \in B_{A_\infty(B_{C(K)})}$ with $h(x_0) = 1$. Since $(x_n)$ is equivalent to the $c_0$-basis, it converges weakly to zero. Then the sequence $(x_0(1-x_n))$ is in the unit ball of $C(K)$ and converges weakly to $x_0$. Since $C(K)$ has the Dunford–Pettis property, it also has the polynomial Dunford–Pettis property [15], and so the argument in the proof of [1, Proposition 4.1] shows that
\[ h(x_0(1-x_n)) \to 1. \]
Since $x_n$ are non-negative elements in the unit sphere, for every $n$ there is $t_n \in K$ such that $x_n(t_n) = 1$ and so
\[ \| x_0(1-x_n) - x_0 \| \geq \| x_0x_n \| \geq |x_0(t_n)x_n(t_n)| = 1. \]
Hence $x_0$ is not a strong peak point for $A_\infty(B_{C(K)})$.

If $K$ is separable and $\{t_n : n \in \mathbb{N}\}$ is a dense set in $K$, we will prove that the function $u$ such that $u(K) = \{1\}$ is a peak point for the space of first-degree polynomials. In view of Lemma 2.3, this proves the stated assertion.

Define
\[ f(x) := \sum_{n=1}^{\infty} \alpha_n(1 + x(t_n)) \quad (x \in C(K)), \]
where \((\alpha_n) \subset S_{\ell_1}\) with \(\alpha_n > 0\) for every \(n\). Then \(f\) is clearly a first-degree polynomial on \(C(K)\) and \(f(u) = \|f\| = 2\). If \(x \in B_{C(K)}\) and \(|f(x)| = 2\), then \(|1 + x(t_n)| = 2\) for every \(n\) and so \(x(t_n) = 1\) for all \(n\), that is, \(x = u\).

Since \(\ell_\infty\) has a countable subset of functionals that separate points and attain the norm at the same element of the unit ball, we can also obtain:

**Corollary 3.3 ([3]).** All the extreme points in \(B_{\ell_\infty}\) are peak points for the space of first-degree polynomials on \(\ell_\infty\).

### 4. Shilov boundary on the trace class operators.

Let \(H\) be a complex Hilbert space. An operator \(T : H \to H\) is called a *trace class operator* if there are orthonormal sequences \((e_n)\) and \((f_n)\) in \(H\) such that \(T(x) = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle f_n\) for every \(x \in H\) and the sequence \((\lambda_n)\) is in \(\ell_1\). In that case, the norm of \(T\) is given by \(\|T\| = \sum_{n=1}^{\infty} |\lambda_n|\). We denote by \(C_1(H)\) the Banach space of all trace class operators on \(H\).

**Theorem 4.1.** If \(H\) is a complex Hilbert space, then the Shilov boundaries for \(A_u(C_1(H))\) and \(A_\infty(C_1(H))\) both exist and coincide.

**Proof.** Assume that \(\{e_i : i \in I\}\) is an orthonormal basis of \(H\) and \(F \subset I\) is any subset. Then the operator \(\Pi_F\) given by

\[
\Pi_F(T) := P_F TP_F \quad (T \in C_1(H)),
\]

where \(P_F(x) = \sum_{i \in F} x(i) e_i\) \((x \in H)\), is a norm one projection on \(C_1(H)\). Since \(\text{Lin}\{e_i \otimes e_j : i, j \in I\}\) is dense in \(C_1(H)\), for every \(h \in A_\infty(B_{C_1(H)})\) we have

\[
\|h\| = \sup_{F \subset I, F \text{ finite}} \|h \circ \Pi_F\|.
\]

For every complex finite-dimensional space \(Y\), the subset of peak points of \(B_Y\) is a boundary for \(A_u(B_Y)\) [4, Theorem 1]. We will prove that for every finite subset \(F \subset I\), every peak point of the unit ball of \(\Pi_F(C_1(H))\) for the space of bounded and continuous functions on the unit ball of \(\Pi_F(C_1(H))\) which are holomorphic on the open unit ball, is a strong peak point for \(A_u(B_{C_1(H)})\).

Let \(T_0 \in S_{C_1(H)} \cap \Pi_F(C_1(H))\) be a peak point. Then there is a continuous function \(g\) on the unit ball of \(\Pi_F(C_1(H))\), which is holomorphic on the open unit ball and satisfies

\[
g(T_0) = \|g\| = 1 \quad \text{and} \quad |g(T)| < 1, \forall T \in (B_{C_1(H)} \cap \Pi_F(C_1(H))) \setminus \{T_0\}.
\]

Now we extend \(g\) to \(B_{C_1(H)}\) by

\[
\tilde{g}(T) = g(\Pi_F(T)) \quad (T \in B_{C_1(H)}).
\]

Clearly \(\tilde{g} \in A_u(B_{C_1(H)}), \quad \|\tilde{g}\| \leq \|g\| = 1\) and \(\tilde{g}(T_0) = 1\). Assume that \((T_n) \subset B_{C_1(H)}\) with \(\|\tilde{g}(T_n)\| \to 1\), that is, \(\|g(\Pi_F(T_n))\| \to 1\). Since \(\Pi_F(C_1(H))\)
is a finite-dimensional space and $T_0$ is a peak point, we have $\Pi_F(T_n) \to T_0$. Since $\|T_0\| = 1$, it follows that $\|\Pi_F(T_n)\| \to 1$. By using [11, Proposition 2.2], we have
\[
\|P_F T_n P_F\|^2 + \|P_F T_n (I - P_F)\|^2 + \|(I - P_F) T_n P_F\|^2 + \|(I - P_F) T_n (I - P_F)\|^2 \\
\leq \|T_n\|^2 \leq 1,
\]
and so $\|\Pi_F(T_n) - T_n\| = \|P_F T_n P_F - T_n\| \to 0$. Since we know that $(\Pi_F(T_n))$ converges to $T_0$, so does $(T_n)$, and $T_0$ is a strong peak point, as we wanted to show. Since the strong peak points are contained in any closed boundary and in this case the set of strong peak points is a boundary for $A_u(B_{C_1(H)})$, the Shilov boundary for this space is the closure of the set of strong peak points of $A_u(B_{C_1(H)})$. The same argument works for $A_\infty(B_{C_1(H)})$. 

5. Boundaries for $K(\ell_p, \ell_q)$. We now study the properties of the boundaries for $A_\infty(B_X)$, where $X$ is the space of all compact operators on $\ell_p$ for $1 \leq p < \infty$.

**Theorem 5.1.** If $1 \leq p \leq q < \infty$, then there is no Shilov boundary for $A_\infty(B_{K(\ell_p, \ell_q)})$. In fact, if $B$ is a boundary for $A_\infty(B_{K(\ell_p, \ell_q)})$, $0 < r < 1$ and $S_0 \in B$, then $B \setminus (S_0 + r B_{K(\ell_p, \ell_q)})$ is also a boundary for $A_\infty(B_{K(\ell_p, \ell_q)})$. There are closed boundaries $A, B$ for $A_\infty(B_{K(\ell_p, \ell_q)})$ such that $\text{dist}(A, B) \geq 1$. The same assertions hold for $A_u(B_{K(\ell_p, \ell_q)})$.

**Proof.** We denote by $(P_n)$ and $(Q_n)$ the sequences of canonical projections associated to the usual bases of $\ell_p$ and $\ell_q$, respectively.

Assume that $B \subset B_{K(\ell_p, \ell_q)}$ is a boundary for $A_\infty(B_{K(\ell_p, \ell_q)})$, $0 < r < 1$ and $S_0 \in B$. If $h \in A_\infty(B_{K(\ell_p, \ell_q)})$ and $0 < \varepsilon < (1 - r)/3$, then there are $N \in \mathbb{N}$ and $F \in B_{K(\ell_p, \ell_q)}$ which satisfy $Q_N F P_N = F$ and
\[
|h(F)| > \|h\| - \varepsilon.
\]
Since $S_0$ is a compact operator, there exists $n > N$ with
\[
\|(I - Q_n) S_0 (I - P_n)\| < \varepsilon.
\]
Choose $R \in S_{K(\ell_p, \ell_q)}$ such that
\[
(I - Q_n) R (I - P_n) = R,
\]
and $x_0 \in S_{\ell_p}$ satisfying $P_n x_0 = 0$ and $\|R(x_0)\| = 1$. Then there exists $y^* \in S_{\ell_q}$ with $Q_n^*(y^*) = 0$ and $y^*(R(x_0)) = 1$. Notice that $\|F + \lambda R\| \leq 1$ for every complex number $\lambda$ with $|\lambda| = 1$. By the maximum modulus theorem, there is $\lambda_0 \in \mathbb{C}$ such that $|\lambda_0| = 1$ and
\[
|h(F)| \leq |h(F + \lambda_0 R)| \leq \sup_{|\lambda| = 1} |h(F + \lambda R)|.
\]
If $\lambda_1 \in \mathbb{C}$ is a modulus one scalar satisfying
\[
|h(F + \lambda_0 R) + \lambda_1 y^*(\lambda_0 R(x_0))| = |h(F + \lambda_0 R)| + 1,
\]
we define a holomorphic function $g$ by

$$g(T) := h(T + \lambda_1 y^*(Tx_0)) \quad (T \in B_K(\ell_p, \ell_q)).$$

Clearly $g \in \mathcal{A}_\infty(B_K(\ell_p, \ell_q))$ and

$$\|g\| \geq |g(F + \lambda_0 R)| = |h(F + \lambda_0 R) + \lambda_1 y^*(\lambda_0 Rx_0)|$$

$$= |h(F + \lambda_0 R)| + 1 \geq |h(F)| + |y^*(Rx_0)| > \|h\| - \varepsilon + 1.$$

Since $B$ is a boundary for $\mathcal{A}_\infty(B_K(\ell_p, \ell_q))$, there is $S \in B$ such that $|g(S)| > \|g\| - \varepsilon$. Hence

$$|y^*(Sx_0)| \geq 1 - 2\varepsilon.$$

By the choice of $x_0$ and $y^*$,

$$\|(I - Q_n)S(I - P_n)\| \geq |y^*(I - Q_n)S(I - P_n)x_0| = |y^*(Sx_0)| \geq 1 - 2\varepsilon.$$

Finally, we deduce that

$$\|S - S_0\| \geq \|(I - Q_n)(S - S_0)(I - P_n)\|$$

$$\geq \|(I - Q_n)S(I - P_n)\| - \|(I - Q_n)S_0(I - P_n)\| \geq 1 - 3\varepsilon > r.$$

From inequality (4), we also obtain

$$|h(S)| \geq \|h\| - 2\varepsilon.$$

We have just proved that $B \setminus (S_0 + rB_K(\ell_p, \ell_q))$ is a boundary for $\mathcal{A}_\infty(B_K(\ell_p, \ell_q))$. As a consequence, the Shilov boundary of this space does not exist.

Now we give a procedure to construct boundaries for $\mathcal{A}_\infty(B_K(\ell_p, \ell_q))$. Since $\text{Lin}\{x \otimes y : x \in (\ell_p)^*, y \in \ell_q, \text{supp} \ x, \text{supp} \ y \text{ are finite}\}$ is dense in $K(\ell_p, \ell_q)$, for every $h \in \mathcal{A}_\infty(B_K(\ell_p, \ell_q))$ we have

$$\|h\| = \sup\{|\|h_F\| : F \subset \mathbb{N} \text{ finite}\},$$

where $h_F(T) := h(Q_FTP_F)$ for $T \in K(\ell_p, \ell_q)$ and $P_F, Q_F$ are the projections given by

$$P_F(x) = \sum_{n \in F} x(n)e_n \quad (x \in \ell_p), \quad Q_F(x) = \sum_{n \in F} x(n)e_n \quad (x \in \ell_q).$$

Note also that $\|h_F\| \leq \|h_G\|$ for $F \subset G$.

Assume that $(F_n)$ is an increasing sequence of finite subsets of $\mathbb{N}$ such that $G_n := F_{n+1} \setminus F_n$ is non-empty and $\bigcup_n F_n = \mathbb{N}$. We consider the subsets $A_n$ whose elements are those operators $T \in B_K(\ell_p, \ell_q)$ that admit a decomposition $T = R + S$ satisfying

$$\|R\| = \|S\| = 1, \quad R = Q_{F_n}R_P F_n, \quad Q_{F_n}S P_{F_n} = 0, \quad Q_{G_n}S P_{G_n} = S.$$

Note that $A_n$ is closed for every $n$. 

We now check that \( B = \bigcup_n A_n \) is a closed boundary for \( A_\infty(B_{K(\ell_p, \ell_q)}) \). Given \( h \in A_\infty(B_{K(\ell_p, \ell_q)}) \) and \( \varepsilon > 0 \), there is some finite subset \( F \subset \mathbb{N} \) such that \( \|h_F\| > \|h\| - \varepsilon \). If \( F \subset F_m \), then also \( \|h_{F_m}\| \geq \|h\| - \varepsilon \). Hence there is an operator \( R \in S_{K(\ell_p, \ell_q)} \) such that \( Q_{F_m} R P_{F_m} = R \) where \( h_{F_m} \) attains its norm and so
\[
|h(R)| \geq \|h\| - \varepsilon.
\]
If \( S \in S_{K(\ell_p, \ell_q)} \) satisfies \( Q_{F_m} S P_{F_m} = 0 \) and \( Q_{G_m} S P_{G_m} = S \), then the operator \( R + \lambda S \) is in the unit ball of \( K(\ell_p, \ell_q) \), for every complex number \( \lambda \) in the unit disk. The maximum modulus theorem applied to the function \( \lambda \mapsto h(R + \lambda S) \) defined on the complex unit disk shows that there is a complex number \( \lambda_0 \) with \( \|\lambda_0\| = 1 \) and such that
\[
|h(R + \lambda_0 S)| \geq |h(R)| \geq \|h\| - \varepsilon.
\]
Since \( R + \lambda_0 S \in A_m \), \( B \) is a boundary for \( A_\infty(B_{K(\ell_p, \ell_q)}) \).

Note that for two positive integers \( n < m \), if \( T_n \in A_n \) and \( T_m \in A_m \), then
\[
\|T_m - T_n\| \geq \|Q_{G_m} (T_m - T_n) P_{G_m}\| = \|Q_{G_m} T_m P_{G_m}\| = 1.
\]
Since every \( A_n \) is closed, the above inequality shows that \( B \) is also closed.

By the same argument, \( \bigcup_n A_{2n} \) and \( \bigcup_n A_{2n-1} \) are also closed boundaries for \( A_\infty(B_{K(\ell_p, \ell_q)}) \). In view of (5), the distance between them is at least 1. \( \blacksquare \)

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