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Ascent, descent and roots of Fredholm operators

by

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Abstract. Let T be a Fredholm operator on a Banach space. Say T is rootless if there is no bounded linear operator S and no positive integer $m \geq 2$ such that $S^m = T$. Criteria and examples of rootlessness are given. This leads to a study of ascent and descent whether finite or infinite for T with examples having infinite ascent and descent.

1. Introduction. Let X be a linear space of sequences, say $X = \ell_2$. This paper originated with the realization that the familiar shift operators $T_1(x) = (0, x, x_2, \ldots)$ and $T_2(x) = (x_2, x_3, \ldots)$ are *rootless* in the sense that there is no bounded linear operator S on X and no positive integer $m \geq 2$ where $S^m = T_k$. We first sought general criteria for rootlessness and other examples. One such result is that a Fredholm linear operator T on a Banach space is rootless if T has infinite ascent and one-dimensional null space. This study is intimately concerned with the notions of index, ascent and descent for a Fredholm linear operator T. The question of when the ascent and descent of T are finite or infinite is of interest. We seek criteria which do not involve the powers T^m. For example every T with negative index has infinite descent. Every T with finite ascent and descent has index zero. The set of such T is characterized algebraically in Section 3 with no mention of the powers T^m.

Next let T be any bounded linear operator. In Section 4 we examine the essential resolvent set for T, the set of complex λ for which $\lambda I - T$ is a Fredholm operator. The subset of such λ where $\lambda I - T$ has finite ascent and descent is open. Examples with all possible indices show that the subset where $\lambda I - T$ has infinite ascent and descent can be open and not void.

For early work on ascent and descent see the article of A. E. Taylor [6]. For later results we cite [3], [4] and [5].

2. Rootless operators. Henceforth X will denote an infinite-dimensional Banach space. Let $\mathfrak{B}(X)$ (resp. $\mathfrak{K}(X)$) denote the algebra of all bounded linear (resp. compact) operators on X and $\Phi(X)$ be the set of all Fred-

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holm operators in $\mathfrak{B}(X)$. For $T \in \mathfrak{B}(X)$ let $N(T)$ and $R(T)$ denote, respectively, the null space and range space of T. Let $\alpha(T)$ be the dimension of $N(T)$ and $\beta(T)$ that of $X/R(T)$. For $T \in \Phi(X)$ the index of T, $\operatorname{ind}(T)$, is given by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$. Let $S, T \in \Phi(X)$. Then $ST \in \Phi(X)$ and we have the index formula

$$\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T). \tag{2.1}$$

See for example [1, Th. 3.2.7]. Also $T^* \in \Phi(X^*)$ and $\alpha(T^*) = \beta(T)$, $\alpha(T) = \beta(T^*)$ by [1, Prop. 1.2.7].

We say that $T \in \Phi(X)$ is *rootless* if there is no $S \in B(X)$ and no integer $m \geq 2$ such that $S^m = T$. If any such S exists then $S \in \Phi(X)$ by [1, p. 9].

Lemma 2.1. *Let $T \in \Phi(X)$. There is a positive integer N and non-negative integers $C_{\mathrm{a}}(T)$ and $C_{\mathrm{d}}(T)$ where $\alpha(T^{n+1}) - \alpha(T^n) = C_{\mathrm{a}}(T)$ and $\beta(T^{n+1}) - \beta(T^n) = C_{\mathrm{d}}(T)$ for all $n \geq N$. Also* $\operatorname{ind}(T) \leq C_{\mathrm{a}}(T) \leq \alpha(T)$ *and* $-\operatorname{ind}(T) \leq C_{\mathrm{d}}(T) \leq \beta(T)$ *and $\alpha(T^n) \leq n\alpha(T)$ for all n.*

Proof. For each positive integer j we see, by (2.1), that $\operatorname{ind}(T^j) = j\operatorname{ind}(T)$. Hence

$$\alpha(T^{n+1}) - \alpha(T^n) - [\beta(T^{n+1}) - \beta(T^n)] = \operatorname{ind}(T). \tag{2.2}$$

For $T_1, T_2 \in \Phi(X)$ we have the relation

$$\alpha(T_1T_2) = \alpha(T_2) + \dim[T_2(X) \cap T_1^{-1}(0)]. \tag{2.3}$$

We use this with $T_1 = T$ and $T_2 = T^n$ to obtain

$$\alpha(T^{n+1}) - \alpha(T^n) = \dim[T^n(X) \cap T_1^{-1}(0)]. \tag{2.4}$$

Therefore, setting $\Delta_n = \alpha(T^{n+1}) - \alpha(T^n)$, we see by (2.4) that $\Delta_{n+1} \leq \Delta_n$. Each Δ_n is a non-negative integer. Hence there is a non-negative integer $C_{\mathrm{a}}(T)$ and a positive integer N such that $\Delta_n = C_{\mathrm{a}}(T)$ for all $n \geq N$. Then $\alpha(T^{N+r}) = \alpha(T^N) + rC_{\mathrm{a}}(T)$ for all positive integers r so that also

$$\alpha(T^{N+r})/(N+r) = \alpha(T^N)/(N+r) + rC_{\mathrm{a}}(T)/(N+r).$$

Since $\lim_r 1/(N+r) = 0$ and $\lim_r r/(N+r) = 1$ we see that $\lim \alpha(T^j)/j = C_{\mathrm{a}}(T)$. Next we use (2.3) with $T_1 = T^{n-1}$ and $T_2 = T$ to see that $\alpha(T^n) \leq \alpha(T) + \alpha(T^{n-1})$ for each positive integer n. Thus $\alpha(T^2) \leq 2\alpha(T)$. We proceed by mathematical induction. Suppose that $\alpha(T^j) \leq j\alpha(T)$ for a positive integer j. Then

$$\alpha(T^{j+1}) \leq \alpha(T) + \alpha(T^j) \leq (j+1)\alpha(T).$$

Therefore $\alpha(T^n)/n \leq \alpha(T)$ for all positive integers n. But as $\alpha(T^n)/n \to C_{\mathrm{a}}(T)$ we have $C_{\mathrm{a}}(T) \leq \alpha(T)$. Also, by (2.2), we see that $\operatorname{ind}(T) \leq C_{\mathrm{a}}(T)$.

Likewise there is a positive integer m such that, for $n \geq m$, $\operatorname{ind}(T^*) \leq \alpha(T^{*(n+1)}) - \alpha(T^{*n}) \leq \alpha(T^*)$. Therefore $-\operatorname{ind}(T) \leq C_{\mathrm{d}}(T) \leq \beta(T)$. ∎

PROPOSITION 2.2. *Let $T \in \Phi(X)$. If T has finite ascent (resp. descent) then* $\operatorname{ind}(T) \le 0$ *(resp. ≥ 0). If T has both finite ascent and descent then* $\operatorname{ind}(T) = 0$.

Proof. Finite ascent (resp. descent) for T is equivalent to $C_{\mathrm{a}}(T) = 0$ (resp. $C_{\mathrm{d}}(T) = 0$). We apply Lemma 2.1. ∎

THEOREM 2.3. *Let $T \in \Phi(X)$. If* $\operatorname{ind}(T) = 0$ *then T has finite ascent if and only if it has finite descent. If* $\operatorname{ind}(T) > 0$ *(resp. < 0) then T has infinite ascent (resp. descent). If T has finite ascent (resp. descent) and infinite descent (resp. ascent) then* $\operatorname{ind}(T) < 0$ *(resp. > 0).*

Proof. Suppose $\operatorname{ind}(T) = 0$. By (2.1) we have $\operatorname{ind}(T^n) = n \operatorname{ind}(T) = 0$ for all positive integers n.

If $\operatorname{ind}(T) > 0$ we must, by Lemma 2.1, have $C_{\mathrm{a}}(T) > 0$, and if $\operatorname{ind}(T) < 0$ we have $C_{\mathrm{d}}(T) > 0$. Suppose that T has finite ascent and infinite descent. Then $\operatorname{ind}(T) \neq 0$. But we cannot have $\operatorname{ind}(T) > 0$ for otherwise T would have infinite ascent. ∎

THEOREM 2.4. *Let $T \in \Phi(X)$. Then $\alpha(T^n) = n\alpha(T)$ for all positive integers n if and only if $\alpha(T) = C_{\mathrm{a}}(T)$. If $\alpha(T) = 1$ and T has infinite ascent then $\alpha(T^n) = n$ for all n.*

Proof. Suppose that $\alpha(T^n) = n\alpha(T)$ for all positive integers n. As seen in the proof of Lemma 2.1, $\alpha(T^n)/n \to C_{\mathrm{a}}(T)$. Therefore $C_{\mathrm{a}}(T) = \alpha(T)$.

Conversely, suppose that $C_{\mathrm{a}}(T) = \alpha(T)$. As seen in Lemma 2.1, $\alpha(T^n) \le n\alpha(T)$ for all positive integers n. But $\alpha(T^{n+1}) - \alpha(T^n) \ge C_{\mathrm{a}}(T)$ for all n. From this we see that $\alpha(T^n) \ge n\alpha(T)$ for all n.

If $\alpha(T) = 1$ then, by Lemma 2.1, either $C_{\mathrm{a}}(T) = 1$ or $C_{\mathrm{a}}(T) = 0$. If also T has infinite ascent then $C_{\mathrm{a}}(T) = 1$. ∎

LEMMA 2.5. *Let $T \in \Phi(X)$. For a positive integer p, $C_{\mathrm{a}}(T^p) = pC_{\mathrm{a}}(T)$. Also $C_{\mathrm{a}}(T) - C_{\mathrm{d}}(T) = \operatorname{ind}(T)$.*

Proof. We have

$$\alpha(T^{p(n+1)}) - \alpha(T^{pn}) = \sum_{j=1}^{p} [\alpha(T^{pn+j}) - \alpha(T^{pn+j-1})].$$

By Lemma 2.1 we see that $C_{\mathrm{a}}(T^p) = pC_{\mathrm{a}}(T)$.

By the index formula $\operatorname{ind}(T^n) = n \operatorname{ind}(T)$ for all positive integers n. For n sufficiently large $C_{\mathrm{a}}(T) = \alpha(T^n) - \alpha(T^{n-1})$ and also $C_{\mathrm{d}}(T) = \beta(T^n) - \beta(T^{n-1})$. Then $C_{\mathrm{a}}(T) - C_{\mathrm{d}}(T) = \operatorname{ind}(T^n) - \operatorname{ind}(T^{n-1}) = \operatorname{ind}(T)$. ∎

THEOREM 2.6. *Let $T \in \Phi(X)$. Then T is rootless if* $\operatorname{ind}(T) = \pm 1$ *or if* $C_{\mathrm{a}}(T) = 1$.

Proof. Suppose $T = S^n$ for $S \in \mathfrak{B}(X)$. Then $\operatorname{ind}(T) = n \operatorname{ind}(S)$ by (2.1) and $C_{\mathrm{a}}(T) = nC_{\mathrm{a}}(S)$ by Lemma 2.5. Therefore $T = S^n$ is impossible for $n \geq 2$. ∎

COROLLARY 2.7. *$T \in \Phi(X)$ is rootless if $\alpha(T) = 1$ and T has infinite ascent.*

Proof. As noted above, we have $C_{\mathrm{a}}(T) = 1$. ∎

THEOREM 2.8. *Let $T \in \Phi(X)$. Suppose $\operatorname{ind}(T) \neq 0$ and $C_{\mathrm{a}}(T) > 0$. Then T is rootless if $\operatorname{ind}(T)$ and $C_{\mathrm{a}}(T)$ are relatively prime.*

Proof. Suppose $T = S^n$ for $S \in \mathfrak{B}(X)$. As above $n \operatorname{ind}(S) = \operatorname{ind}(T)$ and $nC_{\mathrm{a}}(S) = C_{\mathrm{a}}(T)$. Therefore we cannot have $n \geq 2$. ∎

We show that there exist $T \in \Phi(X)$ of all possible indices which are rootless. These examples will have infinite ascent and descent. For index zero we examine an operator V_1 already considered in [7, p. 599] and [1, pp. 13, 14]. Here and below X may be taken as a linear space of sequences $x = (x_1, x_2, \ldots)$, say $X = \ell_2$. We set

$$V_1(x) = (0, x_4, x_1, x_6, x_3, \ldots)$$

where, after the x_j on the right side, the next entries are those with successive odd and even indices interlacing.

For any linear operator T of this sort where the x_j on the right have no repetitions and only a finite number of x_j fail to appear on the right, $\alpha(T)$ is the number of missing x_j's. There are only a finite number of zeros on the right and $\beta(T)$ is that number of zeros. Here $\alpha(V_1) = \beta(V_1)$ and $\operatorname{ind}(V_1) = 0$. It was shown ([7, p. 599] or [1, p. 14]) that V_1 has infinite ascent. By Theorem 2.3 we see that V_1 has infinite ascent. By the same theorem, V_1 has infinite descent. Also V_1 is rootless by Corollary 2.7.

Let n be a positive integer. Our definition of $V_n(x)$ starts off on the right with successive pairs $0, x_{2j}$ beginning with $0, x_4$ and ending with $0, x_{2(n+1)}$. This is then followed by x_1, and thereafter the next available x_j occur, odd and even indices interlacing. In our formal definition of $V_n(x)$, x_j with j negative is to be read as zero. We set $V_n(x) = \{y_j\}$ where $y_j = x_{j+2}$ if $j \geq 2$ is even, and $y_j = x_{j-2n}$ if $j \geq 1$ is odd. Specifically,

$$V_n(x) = (0, x_4, 0, \ldots, 0, x_{2(n+1)}, x_1, \ldots).$$

There are n zeros on the right and every x_j appears except x_2. Thus $\alpha(V_n) = 1$ and $\beta(V_n) = n$ so that $\operatorname{ind}(V_n) = 1 - n$. Then V_n has infinite descent by Theorem 2.3.

Next we consider the powers V_n^k of V_n. Each application of V_n moves the entries with even index two spaces to the left, those with odd index $2n$ spaces to the right. For example

$$V_n^2(x) = (0, x_6, 0, \ldots, 0, x_{4n}, x_1, \ldots).$$

Let $\delta_j = \{x_k\}$ where $x_j = 1$ and $x_k = 0$ otherwise. Then $V_n^k(\delta_{2r}) = \delta_{2r-2k}$. This shows that $V_n^{n-1}(\delta_{2n}) = \delta_2 \neq 0$ and $V_n^n(\delta_{2n}) = 0$. Therefore V_n has infinite ascent and, by Corollary 2.7, it is rootless.

The operators V_n^* are rootless and have infinite ascent and descent and have positive indices of index $n-1$.

All the examples of rootless operators presented above have infinite ascent and/or descent. There are examples with finite ascent and descent. The simplest example is the matrix

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

which is rootless. To see this suppose that $T = S^n$ where

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The null space of T as a linear operator on the vectors $\binom{x}{y}$ is the set of all $\binom{x}{0}$. The null space of S is contained in that of T so is either zero or one-dimensional. It cannot be zero for otherwise S and so T would be invertible. Then $S\binom{1}{0} = \binom{0}{0}$ so that $a = c = 0$. Likewise $(0,1)S = (0,0)$ so that $c = d = 0$. Then

$$S = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$$

so that $S^2 = 0$. Hence T is rootless.

A study of rootless matrices is under preparation.

3. On finite ascent and descent. Here we examine $\Gamma(X)$, the set of all $T \in \Phi(X)$ which have finite ascent and descent. Given $T \in \Phi(X)$ there exist infinitely many $U \in \Phi(X)$ so that $TU = I + W_1$ and $UT = I + W_2$ where each $W_k \in \mathfrak{K}(X)$. We show that $T \in \Gamma(X)$ if and only if there exists $U \in \Phi(X)$ such that $TU = UT$.

LEMMA 3.1. *Let $T, U \in \Phi(X)$ be such that $TU = UT$. Then TU has finite ascent (descent) if and only if T and U have finite ascent (descent).*

Proof. As $TU = UT$, we have $\alpha((TU)^n) = \alpha(T^nU^n)$. But $\alpha(T^nU^n) = \alpha(U^n) + \dim[R(U^n) \cap N(T^n)]$. Then $\alpha(U^n) \le \alpha((TU)^n) \le \alpha(T^n) + \alpha(U^n)$. Thus TU has finite ascent if and only if both T and U have finite ascent. As $T^*U^* = U^*T^*$ we may apply this result to T^* and U^* and use [1, Prop. 1.2.7] to have the conclusion on descents. ■

PROPOSITION 3.2. *Let $T \in \Phi(X)$, $V \in \mathfrak{B}(X)$ where $T = V^n$. Then T has finite ascent (descent) if and only if V does.*

Proof. First of all $V \in \Phi(X)$ by [1, Cor. 1.3.6]. Clearly V has finite ascent (descent) if T does. The converse follows from Lemma 3.1. ■

Theorem 3.3. *Let $T \in \Phi(X)$. The following are equivalent*:

(a) $T \in \Gamma(X)$.

(b) *There exist $U \in \mathfrak{B}(X)$, $W \in \mathfrak{K}(X)$ such that $UT = TU = I + W$.*

(c) *There exists $U \in \Phi(X)$ such that $UT = TU$ and $UT \in \Gamma(X)$.*

Proof. Assume (a). By [1, Th. 1.4.5] we can write $T = V_1 + V_2$ where $V_1 \in \Phi(X)$ is an isomorphism of X onto X, $V_2 \in \mathfrak{K}(X)$ and $V_1V_2 = V_2V_1$. Clearly V_1 and hence V_1^{-1} permutes with T. We have $V_1^{-1}T = I + V_1^{-1}V_2$ and $TV_1^{-1} = I + V_2V_1^{-1}$ with $V_1^{-1}V_2 = V_2V_1^{-1} \in \mathfrak{K}(X)$.

Assume (b). We have $U \in \Phi(X)$ by [1, Cor. 1.3.6] so that (b) implies (c). That (c) implies (a) follows from Lemma 3.1. ∎

Corollary 3.4. *Let $T \in \Phi(X)$. Suppose that $UT = TU = I + W$ as in* (b) *of Theorem* 3.3. *Let $S \in \mathfrak{B}(X)$ permute with T and U. Then there is $\varepsilon > 0$ so that $T + \lambda S \in \Gamma(X)$ for all complex λ, $|\lambda| < \varepsilon$.*

Proof. As $W = UT - I$ and $UT = TU$ we see that $UW = WU$. Also $U(T+\lambda S) = (T+\lambda S)U$ for all complex λ. For some $\varepsilon_1 > 0$, $T+\lambda S \in \Phi(X)$ for all $|\lambda| < \varepsilon_1$, as $\Phi(X)$ is an open set [1, Th. 4.4.1]. We have $U(T+\lambda S) = I + \lambda US + W$. As S permutes with U and T it follows that $SW = WS$. Thus $(I + \lambda US)W = W(I + \lambda US)$. There exists $\varepsilon_2 > 0$ so that $I + \lambda US$ is an isomorphism of X onto X for $|\lambda| < \varepsilon_2$. Let $|\lambda| < \min(\varepsilon_1, \varepsilon_2)$. By [1, Th. 1.4.5], we have $I + \lambda US + W \in \Gamma(X)$. Then $T + \lambda S \in \Gamma(X)$ by Theorem 3.3. ∎

4. On the essential resolvent set. As in the standard text [2, p. 358], by the *essential spectrum* of $T \in \mathfrak{B}(X)$ we mean the set of complex λ for which $\lambda I - T \notin \Phi(X)$. By the *essential resolvent set* $\mathfrak{E}(T)$ of T we mean its complement, the set of λ for which $\lambda I - T \in \Phi(X)$. The set $\mathfrak{E}(T)$ is open in $\mathfrak{B}(X)$ by [1, Th. 4.4.1].

Theorem 4.1. *The set of complex $\lambda \in \mathfrak{E}(T)$ for which $\lambda I - T$ has finite ascent and descent is open.*

Proof. The choice $S = I$ in Corollary 3.4 shows that if $\lambda_0 I - T \in \Gamma(X)$ then, for some $\varepsilon > 0$, so does $\lambda I - T$ if $|\lambda - \lambda_0| < \varepsilon$. ∎

Let T be a fixed element of $\mathfrak{B}(X)$. We adopt the following notation. Let $\mathfrak{F}_{\mathrm{a}}$ (resp. $\mathfrak{F}_{\mathrm{d}}$) be the set of $\lambda \in \mathfrak{E}(T)$ for which $\lambda I - T$ has finite ascent (resp. descent). Let $\mathfrak{I}_{\mathrm{a}}$ (resp. $\mathfrak{I}_{\mathrm{d}}$) be the set of $\lambda \in \mathfrak{E}(T)$ for which $\lambda I - T$ has infinite ascent (resp. descent).

Theorem 4.2. *Suppose that $\mathfrak{I}_{\mathrm{a}} \cap \mathfrak{I}_{\mathrm{d}}$ is relatively closed as a subset of $\mathfrak{E}(T)$. Then $\mathfrak{F}_{\mathrm{a}}$, $\mathfrak{F}_{\mathrm{d}}$, $\mathfrak{F}_{\mathrm{a}} \cap \mathfrak{I}_{\mathrm{d}}$ and $\mathfrak{F}_{\mathrm{d}} \cap \mathfrak{I}_{\mathrm{a}}$ are open subsets of $\mathfrak{B}(X)$.*

Proof. Let $\lambda_0 \in \mathfrak{F}_{\mathrm{a}} \cap \mathfrak{I}_{\mathrm{d}}$. By hypothesis there exists $\varepsilon_1 > 0$ so that $\lambda \in \mathfrak{F}_{\mathrm{a}} \cup \mathfrak{F}_{\mathrm{d}}$ for $|\lambda - \lambda_0| < \varepsilon_1$. By Theorem 2.3, $\operatorname{ind}(\lambda_0 I - T) < 0$. Then, by [1, Th. 4.4.1], there is $\varepsilon_2 > 0$ so that $\operatorname{ind}(\lambda I - T) < 0$ if $|\lambda - \lambda_0| < \varepsilon_2$. By Theorem 2.3, $\lambda \in \mathfrak{I}_{\mathrm{d}}$ for these λ. Therefore $\lambda \in \mathfrak{F}_{\mathrm{a}} \cap \mathfrak{I}_{\mathrm{d}}$ if $|\lambda - \lambda_0| < \min(\varepsilon_1, \varepsilon_2)$. Now $\mathfrak{F}_{\mathrm{a}} \cap \mathfrak{F}_{\mathrm{d}}$ is open by Theorem 4.4.1 of [1]. This also follows by Corollary 3.4. Therefore $\mathfrak{F}_{\mathrm{a}}$ is open. Likewise $\mathfrak{F}_{\mathrm{d}}$ and $\mathfrak{F}_{\mathrm{d}} \cap \mathfrak{I}_{\mathrm{a}}$ are open. ■

We show, by example, that we can have $\mathfrak{I}_{\mathrm{a}} \cap \mathfrak{I}_{\mathrm{d}}$ non-void as well as being open and closed in $\mathfrak{E}(T)$.

Let $X = \ell_2$. For $x = (\xi_1, \xi_2, \ldots)$ in X we treat the operator $V_1(x) = (0, \xi_4, \xi_1, \xi_6, \xi_3, \ldots)$ of index zero discussed above. As $\|V_1^n\| = 1$ for all positive integers n, its spectrum is contained in the unit disc of the complex plane. For each λ with $|\lambda| < 1$, $N(\lambda I - V_1)$ is the set of scalar multiples of

$$z = (0, 1, 0, \lambda, 0, \lambda^2, \ldots, 0, \lambda^n, \ldots).$$

Hence $\operatorname{sp}(V_1)$ is the entire unit disc. For $|\lambda| = 1$ we show that $R(\lambda I - V_1)$ is not closed. First note that $\lambda I - V_1$ is one-to-one. Then $R(\lambda I - V_1) \neq X$ as $\lambda \notin \operatorname{sp}(V_1)$. Suppose $R(\lambda I - V_1) \neq X$ is closed. Then, by [1, Th. 2.5.6], there is a neighborhood of λ in which $\mu I - V_1$ has closed range $\neq X$. This is impossible as the neighborhood contains μ, $|\mu| > 1$. In particular $\lambda I - V_1 \notin \Phi(X)$ if $|\lambda| = 1$.

We consider the adjoint V_1^* of V_1,

$$V_1^*(x) = (\xi_3, 0, \xi_5, \xi_2, \xi_7, \xi_4, \ldots),$$

where ξ_j with odd and even j interlace. We find

$$V_1^* V_1(x) = (\xi_1, 0, \xi_3, \xi_4, \xi_5, \ldots), \qquad V_1 V_1^*(x) = (0, \xi_2, \xi_3, \xi_4, \xi_5, \ldots).$$

Thus each of $V_1^* V_1$ and $V_1 V_1^*$ is of the form $I + W$ where $W \in \mathfrak{K}(X)$. Let π be the canonical homomorphism of $\mathfrak{B}(X)$ onto the C^*-algebra $\mathfrak{B}(X)/\mathfrak{K}(X)$. Then $\pi(V_1)$ is a unitary element in that C^*-algebra. Therefore the essential spectrum of V_1 is contained in $\{\lambda : |\lambda| = 1\}$. As $R(\lambda I - V_1)$ is not closed for these values of λ we see that the unit circle is the essential spectrum. Therefore $\lambda I - V_1 \in \Phi(X)$ if $|\lambda| < 1$, and by [1, Th. 4.4.1], $\operatorname{ind}(\lambda I - V_1) = 0$.

In view of Theorem 2.3, to show that $\lambda I - V_1$ has infinite ascent and descent, it is enough to show that z is in the range of every $(\lambda I - V_1)^n$. Let Q denote the set of all $x = (\xi_1, \xi_2, \ldots)$ where $\xi_j = 0$ for all j odd. Note that $z \in Q$. Let $x = (0, a_2, 0, a_4, \ldots)$ be a general element of Q. Then

$$(\lambda I - V_1)(x) = (0, \lambda a_2 - a_4, 0, \lambda a_4 - a_6, \ldots).$$

Now $z \in Q$ has $a_{2n} = \lambda^{n-1}$. Then

$$(\lambda I - V_1)(w_1) = z \quad \text{where} \quad w_1 \in Q \text{ with } a_{2n} = -(n-1)\lambda^{n-2}.$$

Likewise $(\lambda I - V_1)(w_2) = w_1$ where $w_2 \in Q$ with $a_{2n} = (n-1)(n-2)\lambda^{n-3}/2!$.

More generally we define $w_k \in Q$ by

$$a_{2n} = (-1)^k (n-1)(n-2)\dots(n-k)\lambda^{n-k-1}/k!.$$

Then $(\lambda I - V_1)(w_k) = w_{k-1}$.

Hence $\lambda I - V_1$ has infinite ascent and infinite descent by Theorem 2.3. Thus the set $\mathfrak{I}_{\mathrm{a}} \cap \mathfrak{I}_{\mathrm{d}}$ for V_1 is the open unit disc. Now $\mathfrak{E}(V_1)$ is the complex plane with the unit circle deleted. Thus $\mathfrak{I}_{\mathrm{a}} \cap \mathfrak{I}_{\mathrm{d}}$ is open and closed as a set in $\mathfrak{E}(V_1)$ as well as being non-void.

Examples with the same conclusion can be found with any preassigned index. Consider the operators V_n and V_n^* introduced in Section 2. For these, $\mathfrak{I}_{\mathrm{a}} \cap \mathfrak{I}_{\mathrm{d}}$ have the properties shown for V_1 while $\lambda I - V_n$ has index $1-n$, and $\lambda I - V_n^*$ has index $n-1$ for all λ, $|\lambda| < 1$. We omit the details which parallel those for V_1.

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