STUDIA MATHEMATICA 158 (3) (2003)

The density of states of a local almost periodic operator in $\mathbb{R}^\nu$

by

Andrzej Krupa (Warszawa)

Abstract. We prove the existence of the density of states of a local, self-adjoint operator determined by a coercive, almost periodic quadratic form on $H^m(\mathbb{R}^\nu)$. The support of the density coincides with the spectrum of the operator in $L^2(\mathbb{R}^\nu)$.

Differential operators with almost periodic (briefly a.p.) coefficients appear in some fields of mathematical physics, such as quantum theories of crystals or disordered systems. For example, the physical states of an electron moving in an array of ions can be described as the eigenfunctions of a Schrödinger operator with an a.p. potential. The integrated density of states for such a system is understood, roughly speaking, as the amount of states, corresponding to energy levels not exceeding a given value λ, per volume unit. To give a rigorous mathematical definition of this notion we cannot refer to the sole properties of a given a.p. operator in $L^2(\mathbb{R}^\nu)$, since its spectrum usually has no discrete component, and because the euclidean space $\mathbb{R}^\nu$ has infinite volume. The most common measure used to overcome these difficulties is to approximate our operator by its restrictions to compact subsets of $\mathbb{R}^\nu$.

Let $\mathcal{L} = \sum_\alpha c_\alpha D^\alpha$ be a uniformly elliptic, formally self-adjoint differential operator in $\mathbb{R}^\nu$ with c_α almost periodic, and denote by L the corresponding self-adjoint operator in $L^2(\mathbb{R}^\nu)$. Assuming that the coefficients c_α are C^∞, M. A. Shubin [6] proved the existence of the integrated density of states $\varrho(\lambda, L)$, defined as the weak limit of the distribution functions $\varrho(\lambda, L_k)$ of the eigenfunctions of $\mathcal{L}$ in domains $G_k \subset \mathbb{R}^\nu$ (subject to some boundary conditions), divided by the volume of G_k, as G_k "converge" to $\mathbb{R}^\nu$ in some sense. Moreover, Shubin proved that the support of the measure $d\varrho/d\lambda$ coincides with the spectrum $\sigma(L)$. In the present paper we show that the regularity conditions on c_α can be essentially weakened, at least for the Dirichlet boundary conditions. We prove this result for $\mathcal{L}$ determined by a

2000 *Mathematics Subject Classification*: 47F05, 35P20, 47B25.

local, almost periodic sesquilinear form l, coercive on $H^m(\mathbb{R}^\nu)$ (which allows $\mathcal{L} = \sum_{\alpha,\beta} D^\beta c_{\alpha,\beta} D^\alpha$ with $c_{\alpha,\beta}$ non-smooth for all α, β and discontinuous for lower order multiindices).

1. Preliminaries. Almost periodic functions, operators and forms. We shall use the following notation. The symbol D denotes the ν-tuple of operators $-i(\partial/\partial x_1, \ldots, \partial/\partial x_n)$ in $\mathbb{R}^\nu$. If $\alpha \in \mathbb{Z}_+^\nu$, $\alpha = (\alpha_1, \ldots, \alpha_\nu)$, then $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $D^\alpha = (-i)^{|\alpha|}\partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_\nu^{\alpha_\nu}$. If $G \subset \mathbb{R}^\nu$ is a domain, then $\|f\|_0$ denotes the L^2-norm of a function f on G. For $m \in \mathbb{N}$ we write

$$\|f\|_m^2 = \|f\|_0^2 + \sum_{|\alpha|=m} \|D^\alpha f\|_0^2$$

and $H^m(G)$ is the corresponding Sobolev space. Let $C_{\mathrm{comp}}^\infty(G)$ stand for the space of all compactly supported, infinitely differentiable complex functions on G. We denote by $H_0^m(G)$ the closure of $C_{\mathrm{comp}}^\infty(G)$ in $H^m(G)$ (for the case $G = \mathbb{R}^\nu$ we have $H_0^m(\mathbb{R}^\nu) = H^m(\mathbb{R}^\nu)$), and by H^{-m} its antidual. The symbol $(\cdot,\cdot)$ stands for the antiduality bracket between H^{-m} and H_0^m; in particular, if $m = 0$, it coincides with the scalar product in L^2.

$|G|$ denotes the Lebesgue volume of a domain G.

T_ξ will denote translation by ξ: if Γ is a commutative group and f is a function on Γ then $(T_\xi f)(\gamma) = f(\gamma - \xi)$.

Let us now recall the notion of almost periodicity. Let X be a Banach space and Γ a group of isometries of X onto itself. A vector $x \in X$ is called *almost periodic* (*a.p.*) with respect to Γ if the set $\{Ux : U \in \Gamma\}$ has compact closure in X. The set of all almost periodic vectors is a closed linear subspace of X. In this paper the notion of almost periodicity will be applied to functions on $\mathbb{R}^\nu$, operators and sesquilinear forms.

Denote by $C_{\mathrm{b}}(\mathbb{R}^\nu)$ the space of bounded continuous complex functions on $\mathbb{R}^\nu$ with sup-norm. Elements of C_{b} which are almost periodic with respect to $\{T_\xi\}_{\xi\in\mathbb{R}^\nu}$ are called *uniformly a.p. functions* and the space of all such functions is denoted by $\mathrm{CAP}(\mathbb{R}^\nu)$. The space of trigonometric polynomials is dense in CAP (Bochner's theorem). The intersection of $\mathrm{CAP}(\mathbb{R}^\nu)$ with $C_{\mathrm{b}}^\infty(\mathbb{R}^\nu)$ (the space of functions which are infinitely differentiable and bounded together with all derivatives) is denoted by $\mathrm{CAP}^\infty(\mathbb{R}^\nu)$.

Given $p \geq 1$, put

$$|||f|||_p = \sup_{\xi\in\mathbb{R}^\nu} \Big(\int_{\|x\|\leq 1} |f(x-\xi)|^p \, dx \Big)^{1/p}.$$

If $X = \{f \in L_{\mathrm{loc}}^p(\mathbb{R}^\nu) : |||f|||_p < \infty\}$ then vectors of X which are a.p. with respect to $\{T_\xi\}_{\xi\in\mathbb{R}^\nu}$ are called *Stepanov almost periodic functions.* The space of all such functions, denoted by $\mathrm{S}^p\mathrm{AP}(\mathbb{R}^\nu)$, contains $\mathrm{CAP}(\mathbb{R}^\nu)$ as a dense subspace. There also exist unbounded and discontinuous Stepanov a.p. functions, e.g. every periodic, locally L^p function belongs to $\mathrm{S}^p\mathrm{AP}(\mathbb{R}^\nu)$.

For $m, k \in \mathbb{Z}$ denote by $B(H^m, H^k)$ the space of bounded linear mappings from $H^m(\mathbb{R}^\nu)$ into $H^k(\mathbb{R}^\nu)$. For $A \in \mathrm{BAP}(H^m, H^k)$ and $\xi \in \mathbb{R}^\nu$ put $\mathcal{T}_\xi A = T_\xi A T_{-\xi}$. Elements of $B(H^m, H^k)$ that are a.p. with respect to $\{\mathcal{T}_\xi\}$ will be called *almost periodic operators*; the space of all such operators will be denoted by $\mathrm{BAP}(H^m, H^k)$ (or $\mathrm{BAP}(L^2)$ for $m = k = 0$). Below we list some simple properties of a.p. operators.

PROPOSITION 1.1. (a) *If* $A \in \mathrm{BAP}(H^m, H^k)$ *and* $B \in \mathrm{BAP}(H^k, H^l)$ *then* $BA \in \mathrm{BAP}(H^m, H^l)$.

(b) *If* $A \in \mathrm{BAP}(H^m, H^k)$ *is bijective then* $A^{-1} \in \mathrm{BAP}(H^k, H^m)$.

(c) $\mathrm{BAP}(L^2)$ *is a* C^**-algebra.*

Evidently, all operators commuting with translation are a.p., in particular $D^\alpha \in \mathrm{BAP}(H^m, H^k)$ if $|\alpha| \le m - k$. Here is another example.

EXAMPLE 1.1. (a) Let $u \in \mathrm{CAP}(\mathbb{R}^\nu)$. Then the multiplication operator $f \mapsto uf$ belongs to $\mathrm{BAP}(L^2)$.

(b) Let k, l be nonnegative integers, $k + l \ge 1$. Put $d = \min\{k, \nu/2\} + \min\{l, \nu/2\}$. Let

$$(1.1)\qquad \begin{aligned} &p = \frac{\nu}{d} \quad \text{if} \quad k, l > \frac{\nu}{2} \quad \text{or} \quad k = 0,\ l > \frac{\nu}{2} \quad \text{or} \quad l = 0,\ k > \frac{\nu}{2},\\ &p > \frac{\nu}{d} \quad \text{in other cases,} \end{aligned}$$

and suppose $u \in \mathrm{S}^p\mathrm{AP}(\mathbb{R}^\nu)$. Then, by the inequality $\|u\|_{-l} \le \mathrm{const} \|\!|u|\!\|_p \|f\|_k$ (cf. [7] or [4, Theorem XIII.96]), we have $u \in \mathrm{BAP}(H^k, H^{-l})$.

Denote by $\mathrm{QF}(H^m)$ the space of bounded sesquilinear forms on $H^m(\mathbb{R}^\nu)$ with norm $\|s\| = \sup\{|s(f,h)| : \|f\|_m = \|h\|_m = 1\}$. For $s \in \mathrm{QF}(H^m)$ and $\xi \in \mathbb{R}^\nu$ put $(\mathcal{T}_\xi s)(f, h) = s(T_{-\xi} f, T_{-\xi} h)$. The space of all forms that are a.p. with respect to $\{\mathcal{T}_\xi\}$ will be denoted by $\mathrm{QAP}(\mathbb{R}^\nu)$. Consider the canonical one-to-one correspondence κ between $\mathrm{QF}(H^m)$ and $B(H^m, H^{-m})$, given by the formula $s(f, h) = (\kappa(s) f, h)$. It is an isometry and satisfies $\mathcal{T}_\xi \circ \kappa = \kappa \circ \mathcal{T}_\xi$ for $\xi \in \mathbb{R}^\nu$, so we have

$$(1.2)\qquad \kappa(\mathrm{QAP}(H^m)) = \mathrm{BAP}(H^m, H^{-m}).$$

This equality together with Proposition 1.1(a) and Example 1.1 enables us to construct an example of an a.p. form:

EXAMPLE 1.2. Let $m \in \mathbb{Z}_+$ and let $\alpha, \beta \in \mathbb{Z}_+^\nu$ be such that $|\alpha|, |\beta| \le m$. Let $u \in \mathrm{CAP}(\mathbb{R}^\nu)$ if $|\alpha| = |\beta| = m$ and $u \in \mathrm{S}^p\mathrm{AP}(\mathbb{R}^\nu)$ otherwise, where p satisfies (1.1) with $k = m - |\alpha|$ and $l = m - |\beta|$. Then the formula $s(f, h) = (uD^\alpha f, D^\beta h)$ defines an a.p. sesquilinear form on $H^m(\mathbb{R}^\nu)$.

2. The result. Before the formulation of our theorem we recall some facts and notions concerning sesquilinear forms.

Let $\Omega \subset \mathbb{R}^\nu$ be a domain and m a positive integer. A sesquilinear form s on $H_0^m(\Omega)$ is called *symmetric* if $s(f,h) = \overline{s(h,f)}$ for all f, h, and *coercive* if $\operatorname{Re} s(f,f) \geq \gamma \|f\|_m^2 - M\|f\|_0^2$ for some positive constants γ, M independent of f. An integral form

$$s(f,g) = \sum_{|\alpha|,|\beta| \leq m} (c_{\alpha,\beta} D^\alpha f, D^\beta g)$$

with coefficients $c_{\alpha,\beta}$ bounded and uniformly continuous on Ω is coercive iff it is *uniformly elliptic*, i.e. iff

$$\operatorname{Re} \sum_{|\alpha|,|\beta| = m} c_{\alpha,\beta}(x) \xi^\alpha \xi^\beta \geq \varepsilon \|\xi\|^{2m} \tag{2.1}$$

for fixed $\varepsilon > 0$ and all $x \in \Omega$, $\xi \in \mathbb{R}^\nu$. (Sufficiency is guaranteed by Gårding's inequality; for necessity see [8].) Any form s on $H_0^m(\Omega)$ can be treated as a densely defined form on $L^2(\Omega)$. If s is symmetric and coercive on H_0^m then it is semi-bounded and closed as a form on L^2. Thus s determines a unique self-adjoint operator S in $L^2(\Omega)$ such that $(Sf, g) = s(f,g)$ for $f \in D(S)$, $g \in H_0^m(\Omega)$.

Now fix a symmetric, coercive form s on $H^m(\mathbb{R}^\nu)$ and denote by S the corresponding self-adjoint operator in $L^2(\mathbb{R}^\nu)$. Suppose that Ω is a bounded domain in $\mathbb{R}^\nu$. Since $H_0^m(\Omega)$ is a subspace of $H^m(\mathbb{R}^\nu)$, we can define the restriction s_Ω of s to $H_0^m(\Omega)$; the corresponding operator in $L^2(\Omega)$ will be denoted by S_Ω. Let E_Ω be the spectral resolution of S_Ω. We can define a measure on $\mathbb{R}$ by

$$\mu(\Delta; S_\Omega) = \operatorname{tr} E_\Omega(\Delta), \tag{2.2}$$

where $\Delta \subset \mathbb{R}$ is a Borel set. By the Rellich–Kondrashov theorem ([1]) and the inclusion $D(S_\Omega) \subset H_0^m(\Omega)$, the spectrum of S_Ω is purely discrete. Thus $\mu(\Delta; S_\Omega)$ counts eigenvalues of S_Ω in Δ (respecting multiplicities). Moreover S_Ω is bounded from below, so $\mu(\cdot\,; S_\Omega)$ has a finite distribution function

$$\varrho(\lambda; S_\Omega) = \operatorname{tr} E_\Omega((-\infty, \lambda]). \tag{2.3}$$

The quantity $\mu(\Delta; S_\Omega)/|\Omega|$ measures the density of eigenstates of S_Ω, i.e. the number of eigenvalues in Δ per volume unit. If $\mu(\cdot\,; S_\Omega)/|\Omega|$ converges weakly as Ω tends to $\mathbb{R}^\nu$ (in some sense), then the limit measure $\mu(\cdot\,; S)$ is called the *density of states* of S, and the corresponding distribution function $\varrho(\cdot\,; S)$ the *integrated density of states*. Our aim is to establish the existence of $\mu(\cdot\,; S)$ and to determine its support for some almost periodic forms s.

THEOREM 2.1. *Let $m > 0$ be an integer and l a symmetric, coercive, almost periodic sesquilinear form on $H^m(\mathbb{R}^\nu)$. Denote by L the self-adjoint operator in $L^2(\mathbb{R}^\nu)$ determined by l and assume that L is* local *in the sense* *that $l(f,g)=0$ for $f,g\in C^\infty_{\mathrm{comp}}$ with disjoint supports. Let G be a bounded domain in $\mathbb{R}^\nu$ such that $|\partial G|=0$ and denote by rG the image of G under the mapping $x\mapsto rx$, where r is a positive number. Define operators L_{rG} in $L^2(rG)$ and measures $\mu(\cdot;L_{rG})$ as above. Then*

$$\frac{\mu(\cdot;L_{rG})}{|rG|}\to\mu(\cdot;L)\quad \text{as } r\to\infty \tag{2.4}$$

weakly, where $\mu(\cdot;L)$ is a Borel measure on $\mathbb{R}$ with support $\sigma(L)$.

The locality of L is an important assumption in our theorem. It is known that local operators are "close" to differential ones. For example, a continuous local operator from C^∞_{comp} into C^∞ coincides with a differential operator on every fixed compact set (Peetre's theorem). In our proof the locality of L will allow us to approximate it by differential operators with smooth coefficients, for which the result is known. Namely, in [6] Shubin proved the above theorem assuming additionally that

(2.5)
- L is a differential operator with CAP^∞ coefficients,
- G has C^∞-smooth boundary.

Under these assumptions, the domain of L_{rG} coincides with $H^m_0(rG)\cap H^{2m}(rG)$ (Dirichlet boundary conditions ([1])).

To complete this section we present an example of a form satisfying the assumptions of our theorem.

Example 2.1. Put

$$l(f,g)=\sum_{|\alpha|,|\beta|\le m}(c_{\alpha,\beta}D^\alpha f,D^\beta g)$$

where the $c_{\alpha,\beta}$ are complex functions on $\mathbb{R}^\nu$ such that $c_{\alpha,\beta}=\overline{c}_{\beta,\alpha}$. Suppose that if $|\alpha|=|\beta|=m$ then $c_{\alpha,\beta}$ belongs to $\mathrm{CAP}(\mathbb{R}^\nu)$ and satisfies the uniform ellipticity condition (2.1). Suppose further that the lower order coefficients (i.e. $c_{\alpha,\beta}$ with $|\alpha|+|\beta|<2m$) belong to $\mathrm{S}^p\mathrm{AP}(\mathbb{R}^\nu)$, where p is related to α,β as in Example 1.2. Then l is a local, symmetric, coercive and almost periodic form on $H^m(\mathbb{R}^\nu)$.

Indeed, locality and symmetry are obvious, and almost periodicity was proved in Example 1.2. We now prove coercivity. Since the principal part $\sum_{|\alpha|,|\beta|=m}(c_{\alpha,\beta}D^\alpha\cdot,D^\beta\cdot)$ is coercive by Gårding's inequality, it is enough to show that for every $\varepsilon>0$ there exists $M>0$ such that

$$\Big|\sum_{|\alpha|+|\beta|<2m}(c_{\alpha,\beta}D^\alpha f,D^\beta f)\Big|\le\varepsilon\|f\|^2_m+M\|f\|^2_0$$

([1]) Shubin's result covers also other types of boundary conditions; moreover, the sense of convergence $\Omega\to\mathbb{R}^\nu$ is wider there. We shall not consider those cases.

for all $f \in H^m(\mathbb{R}^\nu)$. This can be done by approximating $c_{\alpha,\beta}$ by trigonometric polynomials in $\mathrm{S}^p\mathrm{AP}(\mathbb{R}^\nu)$ (hence in $B(H^{m-|\alpha|}, H^{|\beta|-m})$, see Example 1.2) and employing Ehrling's inequality

$$\|f\|_{m-1} \le \delta\|f\|_m + C(\delta)\|f\|_0,$$

which holds for arbitrarily small $\delta > 0$.

3. Some auxiliary constructions. Denote by $b\mathbb{R}^\nu$ the Bohr compactification of $\mathbb{R}^\nu$, i.e. the dual group of $\mathbb{R}^\nu_{\mathrm{disc}}$ ($\mathbb{R}^\nu$ endowed with discrete topology). Identifying $\xi \in \mathbb{R}^\nu$ with the function e_ξ, $e_\xi(x) = e^{ix\xi}$, we see that $\mathbb{R}^\nu$ is densely embedded in $b\mathbb{R}^\nu$. Next, $b\mathbb{R}^\nu$ is a maximal compactification of $\mathbb{R}^\nu$ in the sense that, given a compact group Γ, any continuous group homomorphism from $\mathbb{R}^\nu$ into Γ can be extended by continuity to a homomorphism from $b\mathbb{R}^\nu$ into Γ. In particular, if $\{U_\xi\}_{\xi\in\mathbb{R}^\nu}$ is a ν-parameter group of isometries in a Banach space X, such that $\xi \mapsto U_\xi$ is strongly continuous, and if every $x \in X$ is almost periodic with respect to $\{U_\xi\}$, then $\xi \mapsto U_\xi$ can be extended to a strongly continuous representation U_ω of $b\mathbb{R}^\nu$ in X. This property implies, in particular, that every $f \in \mathrm{CAP}(\mathbb{R}^\nu)$ admits a continuous extension on $b\mathbb{R}^\nu$, taking the form $f(\omega) = (T_{-\omega}f)(0)$, $\omega \in b\mathbb{R}^\nu$. In what follows we shall also use the extensions $\mathcal{T}_\omega$ of representations $\mathcal{T}_\xi$ in $\mathrm{QAP}(H^m)$ and $\mathrm{BAP}(L^2)$, defined in Section 1. (Notice that if $s \in \mathrm{QAP}(H^m)$ then $\xi \mapsto \mathcal{T}_\xi s$ is norm-continuous, though this is not the case of an arbitrary $s \in \mathrm{QF}(H^m)$; the same holds true for the operators.)

Put $\mathcal{H}^m = L^2(b\mathbb{R}^\nu) \otimes H^m$, where the Lebesgue space $L^2(b\mathbb{R}^\nu)$ is based on the normalized Haar measure $d\omega$ on $b\mathbb{R}^\nu$. We can also represent $\mathcal{H}^m$ as $L^2(b\mathbb{R}^\nu; H^m)$ or a direct integral of a constant field of Hilbert spaces $H_\omega \equiv H^m$ over $b\mathbb{R}^\nu$. For an a.p. sesquilinear form s on H^m we define a form on $\mathcal{H}^m$ by

$$s^\#(\varphi,\psi) = \int\limits_{b\mathbb{R}^\nu} (\mathcal{T}_\omega s)(\varphi(\omega), \psi(\omega))\, d\omega$$

for $\varphi, \psi \in \mathcal{H}^m$. If $A \in \mathrm{BAP}(L^2(\mathbb{R}^\nu))$, we put analogously

$$A^\# = \int\limits_{b\mathbb{R}^\nu}^{\oplus} \mathcal{T}_\omega A\, d\omega.$$

Suppose now that s is symmetric and coercive on $H^m(\mathbb{R}^\nu)$ and let S be the corresponding self-adjoint operator in $L^2(\mathbb{R}^\nu)$. Then $s^\#$, considered as a form in $\mathcal{H}^0$, is densely defined, closed and bounded below, so it determines a self-adjoint operator in $\mathcal{H}^0$. We shall denote this operator by $S^\#$.

PROPOSITION 3.1. *Let S be as above. Then $\sigma(S^\#) = \sigma(S)$.*

Proof. First assume that $m = 0$, i.e. $S \in B(L^2)$. Then the conclusion is obvious, since $^{\#}$ is an isomorphism of the C^*-algebra $\mathrm{BAP}(L^2)$ onto its image in $B(\mathcal{H}^0)$. Now let S be arbitrary. With no loss of generality we can assume that $s(f,f) \geq \|f\|_m^2$ for $f \in H^m$. Then $\kappa(s)$ (as defined in Section 1) is an isomorphism of H^m onto H^{-m}. Thus, by (1.2) and Proposition 1.1(b), $\kappa(s)^{-1} \in \mathrm{BAP}(H^{-m}, H^m)$. Denote by I^m, I_m the natural embeddings of H^m into L^2 and of L^2 into H^{-m}, respectively. Then $S^{-1} = I^m \kappa(s)^{-1} I_m$, so by Proposition 1.1(a), $S^{-1} \in \mathrm{BAP}(L^2)$ and $(S^{-1})^{\#}$ makes sense. We have proven that $\sigma((S^{-1})^{\#}) = \sigma(S^{-1})$, so noting that $(S^{-1})^{\#} = (S^{\#})^{-1}$, we have $\sigma(S^{\#}) = \sigma(S)$. ∎

The transformation $^{\#}$ was first used in [2] to study a.p. pseudodifferential operators in $\mathbb{R}^\nu$. Its usefulness stems from the fact that it maps isometrically $\mathrm{BAP}(L^2)$ into a von Neumann algebra $\mathfrak{A}$ in $B(\mathcal{H}^0)$ with a normal, faithful trace defined on the cone of positive operators of $\mathfrak{A}$, which takes finite values at sufficiently many arguments (for the theory of von Neumann algebras and traces see e.g. [3]). This algebra is defined as the commutant of the family $\{T_\xi \otimes T_{-\xi},\, e_\lambda \otimes I : \xi, \lambda \in \mathbb{R}^\nu\}$ of operators, where e_λ is the multiplication operator determined by the function $x \mapsto e^{ix\lambda}$, extended to $b\mathbb{R}^\nu$. If $\mathcal{A} \in \mathfrak{A}$ then $\mathcal{A}$ takes the form $\int_{b\mathbb{R}^\nu}^{\oplus} A_\omega \, d\omega$ for some weakly measurable, bounded function $b\mathbb{R}^\nu \ni \omega \mapsto A_\omega \in B(L^2(\mathbb{R}^\nu))$. The integral $\int_{b\mathbb{R}^\nu} A_\omega \, d\omega$ is then a Fourier multiplier in $L^2(\mathbb{R}^\nu)$, i.e. an operator of the form $a(D)$, where $a \in L^\infty(\mathbb{R}^\nu)$. If $\mathcal{A} \in \mathfrak{A}_+$ (the cone of positive operators in $\mathfrak{A}$), then $a \geq 0$ a.e. in $\mathbb{R}^\nu$ and the formula

$$\mathrm{Tr}(\mathcal{A}) = \int_{\mathbb{R}^\nu} a \tag{3.1}$$

defines a normal, faithful trace Tr on $\mathfrak{A}_+$ (see [2]).

Observe that the operators $S^{\#}$ defined above are affiliated to $\mathfrak{A}$, which means that they commute with the $T_\xi \otimes T_{-\xi}$ and $e_\lambda \otimes I$. Thus their spectral projections $\mathcal{E}(\varDelta)$ belong to $\mathfrak{A}$ and $\mathrm{Tr}(\mathcal{E}(\varDelta))$ is well defined. We shall use this property in the final section.

4. Proof of the theorem. Let $l, L, r, G, \ldots$ be as in the statement of the theorem. The limit measure μ will be defined by the formula

$$\mu(\varDelta; L) = \mathrm{Tr}(\mathcal{E}(\varDelta)),$$

where Tr is the trace defined in the previous section and $\mathcal{E}$ is the spectral resolution of $L^{\#}$. Observe that by the fidelity of the trace we have $\operatorname{supp} \mu(\cdot; L) = \sigma(L^{\#})$, which, in view of Proposition 3.1, implies that

$$\operatorname{supp} \mu(\cdot; L) = \sigma(L).$$

Thus it remains to show (2.4). First we prove that the distribution function $\varrho(\lambda; L) := \mu((-\infty, \lambda]; L)$ is finite. We shall exploit the fact that if $\mathfrak{I} \subset \mathfrak{A}$ is a two-sided ideal, then so is its "square root", the set of all $\mathcal{A} \in \mathfrak{A}$ such that $\mathcal{A}^*\mathcal{A} \in \mathfrak{I}$ (denoted by $\mathfrak{I}^{1/2}$, see [3]). Put $\mathcal{E}_\lambda = \mathcal{E}((-\infty, \lambda])$ and let $\mathfrak{A}_n$ be the 2^nth "root" of the ideal $\{\mathcal{A} \in \mathfrak{A} : \operatorname{Tr} |\mathcal{A}| < \infty\}$. It suffices to prove that $\mathcal{E}_\lambda \in \mathfrak{A}_n$ for some $n \in \mathbb{N}$. Put $a(\xi) = (|\xi|+1)^{-1}$, $\xi \in \mathbb{R}^\nu$, and let $a(D)$ be the corresponding Fourier multiplier in $L^2(\mathbb{R}^\nu)$. By (3.1), $a(D)^\# \in \mathfrak{A}_n$ for $n > \log_2 \nu$. The range of $a(D)^\#$ coincides with $\mathcal{H}^1$, while $\operatorname{Ran} \mathcal{E}_\lambda \subset D(L^\#) \subset \mathcal{H}^m$. Therefore, by the closed graph theorem, the operator $\mathcal{B} := (a(D)^\#)^{-1}\mathcal{E}_\lambda$ is bounded; it is obvious that $\mathcal{B} \in \mathfrak{A}$. Hence $\mathcal{E}_\lambda = a(D)^\#\mathcal{B}$ belongs to $\mathfrak{A}_n$ for $n > \log_2 \nu$.

Now it suffices to prove that $\varrho(\lambda; L_{rG})/|rG|$ tends to $\varrho(\lambda; L)$ for every λ which is a point of continuity of $\varrho(\cdot\,; L)$. (Under the additional assumptions (2.5) this was proved by Shubin in [6].) To make use of Shubin's result we need the following lemma:

Lemma 4.1. *There exists a sequence of differential operators L^n of order $\leq 2m$, with coefficients in* CAP^∞, *such that the corresponding sesquilinear forms l^n on $H^m(\mathbb{R}^\nu)$ (defined by $l^n(f,g) = (L^n f, g)$) are symmetric and converge to l in the norm of* $\mathrm{QF}(H^m)$.

Proof. Fix $n \in \mathbb{N}$. The function $b\mathbb{R}^\nu \ni \omega \mapsto \mathcal{T}_\omega l \in \mathrm{QAP}(H^m)$ is norm-continuous. Let $\mathcal{U}$ be a neighbourhood of 0 in $b\mathbb{R}^\nu$ such that $\|\mathcal{T}_\omega l - l\| < 1/(2n)$ for $\omega \in \mathcal{U}$. Choose $\varphi \in C(b\mathbb{R}^\nu)$ such that $\varphi \geq 0$, $\operatorname{supp} \varphi \subset \mathcal{U}$ and $\int \varphi = 1$. Find a trigonometric polynomial ψ on $b\mathbb{R}^\nu$, $\psi = \sum_j a_j e_{\lambda_j}$, such that $\|\varphi - \psi\|_\infty < 1/(2n\|l\|)$. We may assume that ψ is real. Define

$$l^n = \int\limits_{b\mathbb{R}^\nu} \psi(\omega)\mathcal{T}_\omega l \, d\omega.$$

Then l^n is symmetric and $\|l - l^n\| < 1/n$. It remains to prove that the corresponding operator $L^n := \kappa(l^n)$ is differential with CAP^∞ coefficients. Put

$$l_j^n(f,h) = a_j \int\limits_{b\mathbb{R}^\nu} e_{\lambda_j}(\omega)(\mathcal{T}_\omega l)(f, e_{\lambda_j} h) \, d\omega$$

for $f, h \in H^m$. Then $\mathcal{T}_\xi l_j^n = l_j^n$, so $\kappa(l_j^n)$ commutes with the T_ξ ($\xi \in \mathbb{R}^\nu$). Thus there exists a distribution u on $\mathbb{R}^\nu$ such that $\kappa(l_j^n) f = u * f$ for all $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^\nu)$ (see [5]). The locality of L implies that $\kappa(l_j^n)$ preserves supports, whence $\operatorname{supp} u \subset \{0\}$. Thus $\kappa(l_j^n)$ is a differential operator with constant coefficients (of order $\leq 2m$, as $\kappa(l_j^n) \in B(H^m, H^{-m})$). But $L^n = \sum_j e_{\lambda_j} \kappa(l_j^n)$, which completes the proof. ■

Observe that with no loss of generality we may assume that

$$l(f,f) \geq \gamma \|f\|_m^2 \tag{4.1}$$

for some $\gamma > 0$ and all f. Then for large n we have

$$l^n(f,f) \ge \frac{\gamma}{2}\,\|f\|_m^2, \tag{4.2}$$

whence L^n are precisely of order $2m$ and are uniformly elliptic for large n.

To prove our theorem we will also need the following version of the min-max principle:

LEMMA 4.2. *Let $\mathfrak{B}$ be a von Neumann algebra in a Hilbert space H with a normal, faithful trace* Sp *on $\mathfrak{B}_+$. Let s be a positive-definite, closed sesquilinear form with domain $Q(s)$ dense in H, such that the corresponding self-adjoint operator S is affiliated to $\mathfrak{B}$. Denote by E the spectral resolution of S and put $E_\lambda := E((-\infty,\lambda])$. Then, given any $x \ge 0$, we have*

$$\begin{aligned}(4.3)\qquad & \mathrm{Sp}(E_\lambda) \ge x \quad \textit{iff} \\ & \qquad \forall P : \mathrm{Sp}(P) < x\ \exists h \in Q(s)\cap \ker P : \quad \|h\| = 1,\ s(h,h) \le \lambda\end{aligned}$$

where P runs over all orthogonal projections in $\mathfrak{B}$.

Proof. The "if" part is rather obvious: if the right-hand side of (4.3) holds true then $\mathrm{Sp}(E_\lambda)$ cannot be less than x, since the quadratic form $s(\cdot,\cdot) - \lambda\|\cdot\|^2$ is strictly positive on $\ker E_\lambda = \operatorname{Ran} E((\lambda,\infty])$.

Let us prove the "only if" part. Suppose that $\mathrm{Sp}(E_\lambda) \ge x$ and take an orthogonal projection $P \in \mathfrak{B}$ such that $\mathrm{Sp}(P) < x$. It is sufficient to show that $\ker P \cap \operatorname{Ran} E_\lambda$ is non-null (observe that $\operatorname{Ran} E_\lambda \subset Q(s)$, because s is bounded from below). Suppose the opposite and consider the operator $R := PE_\lambda$. We then have $\ker R = \ker E_\lambda$; moreover $\operatorname{Ran} R \subset \operatorname{Ran} P$. Let $R = U|R|$ be the polar decomposition of R. Then U is a partial isometry with $\operatorname{Ran} E_\lambda$ as initial space and with range contained in $\operatorname{Ran} P$. Therefore $U^*U = E_\lambda$ and $UU^* \le P$. Taking into account that $\mathrm{Sp}(U^*U) = \mathrm{Sp}(UU^*)$ (see [3, Chapter 1]) we obtain

$$x \le \mathrm{Sp}(E_\lambda) = \mathrm{Sp}(UU^*) \le \mathrm{Sp}(P) < x,$$

which is a contradiction. ∎

Now we can prove our main result. Assume first that G has C^∞ boundary. Let $\lambda > 0$ be a point of continuity of $\varrho(\cdot;L)$. Fix $x \ge 0$ and suppose that $\varrho(\lambda;L) > x$. Then there exist $\varepsilon \in (0,\lambda/2)$ and $\delta > 0$ such that

$$\varrho(\lambda - 2\varepsilon; L) \ge x + 2\delta. \tag{4.4}$$

Let L^n and l^n be as in Lemma 4.1. Taking estimate (4.1) into account we find that

$$(l^n)^{\#}(\varphi,\varphi) \le (\lambda - 2\varepsilon)(1 + \gamma^{-1}\|l^n - l\|)$$

for all $\varphi \in \mathcal{H}^m$ such that $l^{\#}(\varphi,\varphi) \le \lambda - 2\varepsilon$. Thus, by (4.3),

$$\varrho(\lambda - \varepsilon; L^n) \ge x + 2\delta \tag{4.5}$$

for n sufficiently large. By letting ε decrease if necessary, we may assume that $\lambda-\varepsilon$ is a point of continuity of $\varrho(\cdot\,;L^n)$ for all n. Hence, by the Shubin's result quoted above,

$$\frac{\varrho(\lambda-\varepsilon;L^n_{rG})}{|rG|}\ge x+\delta$$

for $r>r(n)$. Now fix n large and repeat the proof of the implication (4.4)⇒(4.5), replacing L with L^n_{rG} and L^n with L_{rG} (use estimate (4.2) instead of (4.1)). We obtain

$$\frac{\varrho(\lambda;L_{rG})}{|rG|}\ge x+\delta>x$$

for $r>r(n)$. Analogously $\varrho(\lambda;L)<y$ implies $\varrho(\lambda;L_{rG})/|rG|<y$ for large r, which completes the proof in the case of regular G.

Now let G be an arbitrary (nonempty) bounded domain in $\mathbb{R}^\nu$ such that $|\partial G|=0$. For fixed $q>1$ choose domains G_+ and G_- with C^∞ boundaries such that $G_+\supset G\supset G_-$ and $|G_+|/|G_-|\le q$. By the mini-max principle (4.3) we have $\varrho(\cdot\,;L_{rG_+})\ge\varrho(\cdot\,;L_{rG})\ge\varrho(\cdot\,;L_{rG_-})$. Thus, if λ is a point of continuity of $\varrho(\cdot\,;L)$, we have

$$\limsup_{r\to\infty}\frac{\varrho(\lambda;L_{rG})}{|rG|}\le q\varrho(\lambda;L)\quad\text{and}\quad\liminf_{r\to\infty}\frac{\varrho(\lambda;L_{rG})}{|rG|}\ge q^{-1}\varrho(\lambda;L).$$

Letting $q\downarrow 1$ we obtain the conclusion. ■

References

[1] R. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] L. Coburn, R. Moyer and I. Singer, *C^*-algebras of almost periodic pseudo-differential operators*, Acta Math. 130 (1973), 279–307.

[3] J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien* (*algèbres de von Neumann*), Gauthier-Villars, Paris, 1969.

[4] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Vol. 4, Academic Press, New York, 1978.

[5] L. Schwartz, *Théorie des distributions*, Hermann, Paris, 1966.

[6] M. A. Shubin, *The spectral theory and the index of elliptic operators with almost periodic coefficients*, Uspekhi Mat. Nauk 34 (1979), no. 2 (206), 95–135 (in Russian); English transl.: Russian Math. Surveys 34 (1979), no. 2, 109–158.

[7] R. Strichartz, *Multipliers on fractional Sobolev spaces*, J. Math. Mech. 16 (1967), 1031–1060.

[8] F. Treves, *Basic Linear Partial Differential Equations*, Academic Press, New York, 1975.

Institute of Applied Mathematics and Mechanics
Warsaw University
Banacha 2
02-097 Warszawa, Poland
E-mail: krupa@mimuw.edu.pl

Received May 10, 2002
Revised version January 29, 2003 (4947)