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Algebraic isomorphisms and Jordan derivations of $\mathcal{J}$-subspace lattice algebras

by

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Abstract. It is shown that every algebraic isomorphism between standard subalgebras of $\mathcal{J}$-subspace lattice algebras is quasi-spatial and every Jordan derivation of standard subalgebras of $\mathcal{J}$-subspace lattice algebras is an additive derivation. Also, it is proved that every finite rank operator in a $\mathcal{J}$-subspace lattice algebra can be written as a finite sum of rank one operators each belonging to that algebra. As an additional result, a multiplicative bijection of a $\mathcal{J}$-subspace lattice algebra onto an arbitrary ring is proved to be automatically additive. Those results can be applied to atomic Boolean subspace lattice algebras and pentagon subspace lattice algebras.

1. Introduction and preliminaries. Let X be a real or complex Banach space. A family $\mathcal{L}$ of subspaces of X is a *subspace lattice* on X if it contains (0) and X, and is closed under the operations $\vee$ and $\cap$ in the sense that $\bigvee_{\gamma\in\Gamma} L_\gamma \in \mathcal{L}$ and $\bigcap_{\gamma\in\Gamma} L_\gamma \in \mathcal{L}$ for every family $\{L_\gamma\}_{\gamma\in\Gamma}$ of elements of $\mathcal{L}$. For a subspace lattice $\mathcal{L}$ on X, the associated *subspace lattice algebra* $\operatorname{Alg}\mathcal{L}$ is the set of operators on X leaving every subspace in $\mathcal{L}$ invariant. Obviously, $\operatorname{Alg}\mathcal{L}$ is a unital weakly closed operator algebra.

The class of $\mathcal{J}$-subspace lattices was defined in [17] and subsequently discussed in [12, 13]. Given a subspace lattice $\mathcal{L}$ on X, put

$$\mathcal{J}(\mathcal{L}) = \{K \in \mathcal{L} : K \neq (0) \text{ and } K_- \neq X\},$$

where $K_- = \bigvee\{L \in \mathcal{L} : K \nsubseteq L\}$. Call $\mathcal{L}$ a *$\mathcal{J}$-subspace lattice* on X if

(1) $\bigvee\{K : K \in \mathcal{J}(\mathcal{L})\} = X$,
(2) $\bigcap\{K_- : K \in \mathcal{J}(\mathcal{L})\} = (0)$,
(3) $K \vee K_- = X$ for every $K \in \mathcal{J}(\mathcal{L})$,
(4) $K \cap K_- = (0)$ for every $K \in \mathcal{J}(\mathcal{L})$.

The relevance of $\mathcal{J}(\mathcal{L})$ is due to the following lemma which is crucial to what follows.

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Lemma 1.1 (Longstaff [11]; see also [9]). *If $\mathcal{L}$ is a subspace lattice on X, then the rank one operator $x \otimes f$ is in* $\operatorname{Alg}\mathcal{L}$ *if and only if there exists some $K \in \mathcal{J}(\mathcal{L})$ such that $x \in K$ and $f \in K_-^\perp$, where $L_-^\perp$ means $(L_-)^\perp$.*

From Lemma 1.1 we can see that if $\mathcal{L}$ is a $\mathcal{J}$-subspace lattice then $\operatorname{Alg}\mathcal{L}$ is rich in rank one operators, and hence deserves some attention. It should be mentioned that both atomic Boolean subspace lattices and pentagon subspace lattices are members of the class of $\mathcal{J}$-subspace lattices [13].

There are two important transformations on operator algebras: "isomorphism" and "derivation". Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be algebras of operators on Banach spaces X_1 and X_2, respectively. An *algebraic isomorphism* $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ from $\mathcal{A}_1$ onto $\mathcal{A}_2$ is a multiplicative linear bijection. Call ϕ *spatial* if there exists a bounded linear bijective operator $T : X_1 \to X_2$ such that $\phi(A) = TAT^{-1}$ for all $A \in \mathcal{A}_1$. Algebraic isomorphisms need not be spatial and need not even preserve rank (see [3, Example 5.1]). It is well known that every algebraic isomorphism from $B(X_1)$ onto $B(X_2)$ is spatial. A partial generalization of this result was obtained by Ringrose [19], namely, if $\mathcal{N}_1$ and $\mathcal{N}_2$ are nests of subspaces of Hilbert spaces H_1 and H_2, respectively, then every algebraic isomorphism from $\operatorname{Alg}\mathcal{N}_1$ onto $\operatorname{Alg}\mathcal{N}_2$ is spatial.

Compared to spatiality, quasi-spatiality is a strictly weaker notion. With $\mathcal{A}_1$, $\mathcal{A}_2$, X_1 and X_2 as in the preceding paragraph, an algebraic isomorphism ϕ from $\mathcal{A}_1$ onto $\mathcal{A}_2$ is said to be *quasi-spatial* if there exists a densely defined, closed, injective linear transformation $T : \mathcal{D}(T) \subseteq X_1 \to X_2$ with dense range, and with domain $\mathcal{D}(T)$ invariant under every element of $\mathcal{A}_1$, such that $\phi(A)Tx = TAx$ for every $A \in \mathcal{A}_1$ and every $x \in \mathcal{D}(T)$. The notion of quasi-spatiality was introduced by Lambrou in [10], where it is shown that every algebraic isomorphism between atomic Boolean subspace lattice algebras is quasi-spatial. This result also holds for pentagon subspace lattice algebras [8] and for $\mathcal{J}$-subspace lattice algebras on reflexive Banach spaces [17]. In addition, every rank preserving algebraic isomorphism from $\operatorname{Alg}\mathcal{L}_1$ onto $\operatorname{Alg}\mathcal{L}_2$ is quasi-spatial, where $\mathcal{L}_i$ is either a finite distributive subspace lattice on X_i (see [18]) or a completely distributive commutative subspace lattice on a Hilbert space H_i (see [3]), for $i = 1, 2$.

Let $\mathcal{A}$ be an algebra of operators on a Banach space X and let $\delta : \mathcal{A} \to B(X)$ be a map. Then δ is called an *additive* (respectively *linear*) *derivation* if δ is additive (respectively linear) and $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathcal{A}$. We say that δ is a *Jordan derivation* if δ is additive and for every $A \in \mathcal{A}$ we have $\delta(A^2) = A\delta(A) + \delta(A)A$. Clearly, an additive derivation is a Jordan derivation. The converse problem of whether a Jordan derivation is an additive derivation has received many mathematicians' attention for many years [1, 4, 20, 21]. In [4], I. N. Herstein proved that every Jordan derivation of a 2-torsion free prime ring is an additive derivation. In [1],

M. Brešar generalized this result to 2-torsion free semisimple rings. It follows immediately that every Jordan derivation of a semisimple Banach algebra is an additive derivation. Recently, J. H. Zhang [21] proved that every linear Jordan derivation of a nest algebra on a Hilbert space is an inner derivation.

In this paper, we show that every algebraic isomorphism between standard subalgebras of $\mathcal{J}$-subspace lattice algebras is quasi-spatial and every Jordan derivation of a standard subalgebra of a $\mathcal{J}$-subspace lattice algebra to $B(X)$ is an additive derivation. Also, it is proved that every finite rank operator in a $\mathcal{J}$-subspace lattice algebra can be written as a finite sum of rank one operators each belonging to that algebra. This result was first proved in [17]. As an additional result, a multiplicative bijection of a standard subalgebra of a $\mathcal{J}$-subspace lattice algebra onto an arbitrary ring is proved to be automatically additive.

Let us introduce the notation and concepts that we will use throughout. All algebras and vector spaces will be over $\mathbb{F}$, where $\mathbb{F}$ is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Given a Banach space X with topological dual X^*, by $B(X)$ we denote the algebra of all bounded linear operators on X. The terms *operator* on X and *subspace* of X will mean "bounded linear map of X into itself" and "norm closed linear manifold in X", respectively. For $A \in B(X)$, denote by A^* the adjoint of A, and by I the identity operator on X. For $x \in X$ and $f \in X^*$, the operator $x \otimes f$ is defined by $y \mapsto f(y)x$ for $y \in X$. For any nonempty subset $L \subseteq X$, $L^\perp$ denotes its annihilator, that is, $L^\perp = \{f \in X^* : f(x) = 0 \text{ for all } x \in L\}$. For every family $\{L_\gamma\}_{\gamma \in \Gamma}$ of subspaces of X, we have $(\bigvee_{\gamma \in \Gamma} L_\gamma)^\perp = \bigcap_{\gamma \in \Gamma} L_\gamma^\perp$ and $\bigvee_{\gamma \in \Gamma} L_\gamma^\perp \subseteq (\bigcap_{\gamma \in \Gamma} L_\gamma)^\perp$; indeed, it is easy to verify that $(\bigcap_{\gamma \in \Gamma} L_\gamma)^\perp$ is the weak* closure of $\bigvee_{\gamma \in \Gamma} L_\gamma^\perp$. Here "$\vee$" and "$\cap$" denote "norm closed linear span" and "set-theoretic intersection". Given a subspace lattice $\mathcal{L}$, we say that a subalgebra of $\operatorname{Alg} \mathcal{L}$ is *standard* if it contains all the finite rank operators in $\operatorname{Alg} \mathcal{L}$. It will be convenient to write $\langle x \rangle = \{\lambda x : \lambda \in \mathbb{F}\}$ for a vector x and denote by $\langle \mathcal{J}(\mathcal{L}) \rangle$ the (not necessarily closed) linear span of $\bigcup \{K : K \in \mathcal{J}(\mathcal{L})\}$ for a subspace lattice $\mathcal{L}$. In general, for a set S of vectors, $\langle S \rangle$ denotes the linear manifold spanned by S.

We close this section by summarizing some basic properties of a $\mathcal{J}$-subspace lattice (algebra), which can be found in [13].

LEMMA 1.2. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space. Then*

(i) $K \subseteq L_-$ *for any* $K, L \in \mathcal{J}(\mathcal{L})$ *with* $K \neq L$;

(ii) $K \cap L = (0)$ *for any* $K, L \in \mathcal{J}(\mathcal{L})$ *with* $K \neq L$;

(iii) $x \otimes f \in \operatorname{Alg} \mathcal{L}$ *if and only if there exists a unique* $K \in \mathcal{J}(\mathcal{L})$ *such that* $x \in K$ *and* $f \in K_-^\perp$;

(iv) *for every nonzero vector* $x \in K$ *with* $K \in \mathcal{J}(\mathcal{L})$, *there exists an* $f \in K_-^\perp$ *such that* $f(x) = 1$; *dually*, *for every nonzero functional* $f \in K_-^\perp$ *with* $K \in \mathcal{J}(\mathcal{L})$, *there exists an* $x \in K$ *such that* $f(x) = 1$.

2. Algebraic isomorphisms. In this section, we study algebraic isomorphisms between standard subalgebras of $\mathcal{J}$-subspace lattice algebras. It will be shown that such maps are quasi-spatial.

We will make a crucial use of the following lemma which assures that every algebraic isomorphism of $\mathcal{J}$-subspace lattice algebras preserves rank one operators, that is, it carries rank one operators to rank one operators. The next result follows from Lemma 3.3.2 and Theorem 3.3.2 of [17]. We include a proof here for the reader's convenience.

LEMMA 2.1. *Let* $\mathcal{L}$ *be a* $\mathcal{J}$*-subspace lattice on a Banach space* X *and suppose that* $T \in \operatorname{Alg}\mathcal{L}$ *is nonzero. Then* T *is of rank one if and only if whenever* $ATB = 0$ *with* $A, B \in \operatorname{Alg}\mathcal{L}$ *being of rank one*, *then either* $AT = 0$ *or* $TB = 0$.

Proof. Necessity is obvious.

Sufficiency. Since $T \neq 0$, there are $K \in \mathcal{J}(\mathcal{L})$ and $x_0 \in K$ such that $Tx_0 \neq 0$. First we show that $Tx \in \langle Tx_0\rangle$ for every $x \in K$. Otherwise, there is $x_1 \in K$ such that Tx_1 is linearly independent of Tx_0. Since both Tx_0 and Tx_1 are in K and $K \cap K_- = (0)$, there is $f_i \in K_-^\perp$ such that $f_i(Tx_j) = \delta_{ij}$ $(i, j = 0, 1)$. Let $A = x_0 \otimes f_0$ and $B = x_1 \otimes f_1$. Then $A, B \in \operatorname{Alg}\mathcal{L}$. It is easy to verfy that $ATB = 0$. But $ATx_0 \neq 0$ and $TBTx_1 \neq 0$.

Next we show that if $E \neq K$ with $E \in \mathcal{J}(\mathcal{L})$, then $Tx = 0$ for every $x \in E$. Otherwise, suppose there is $x \in E$ such that $Tx \neq 0$. Take nonzero functionals $f_0 \in K_-^\perp$ and $f \in E_-^\perp$ such that $f_0(Tx_0) \neq 0$. Since $Tx \in E$ $\subseteq K_-$, it follows that $(x_0 \otimes f_0)T(x \otimes f) = 0$. But $(x_0 \otimes f_0)Tx_0 \neq 0$ and $Tx \otimes f \neq 0$.

Consequently, the range of T restricted to $\langle\mathcal{J}(\mathcal{L})\rangle$ is $\langle Tx_0\rangle$. Since $\langle\mathcal{J}(\mathcal{L})\rangle$ is dense in X, T is of rank one.

For reflexive Banach spaces, the following result can be proved using [17, Theorem 3.3.5] and [10, Theorem 4.2].

THEOREM 2.2. *Let* $\mathcal{L}_i$ *be a* $\mathcal{J}$*-subspace lattice on a Banach space* X_i, $i = 1, 2$. *Let* ϕ *be an algebraic isomorphism from* $\operatorname{Alg}\mathcal{L}_1$ *onto* $\operatorname{Alg}\mathcal{L}_2$. *Then* ϕ *is automatically* (*norm*) *continuous.*

Proof. By the closed graph theorem, it suffices to prove that ϕ is a closed operator from $\operatorname{Alg}\mathcal{L}_1$ into $\operatorname{Alg}\mathcal{L}_2$. Let T_n, T be in $\operatorname{Alg}\mathcal{L}_1$ and S in $\operatorname{Alg}\mathcal{L}_2$ such that $T_n \to T$ and $\phi(T_n) \to S$.

Let F be in $\mathcal{J}(\mathcal{L}_2)$ and x in F. We want to prove that $\phi(T)x = Sx$. Otherwise, since $(\phi(T) - S)x \in F$ and $F \cap F_- = (0)$, there is $g \in F_-^\perp$ such that $g((\phi(T)-S)x)\neq 0$. Take nonzero vectors $y\in F$ and $f\in F_-^{\perp}$. Then $x\otimes f$ and $y\otimes g$ are both in $\operatorname{Alg}\mathcal{L}_2$. Since ϕ preserves rank one operators by Lemma 2.1, there are $u\otimes h$ and $v\otimes w$ in $\operatorname{Alg}\mathcal{L}_1$ such that

$$\phi(u\otimes h)=x\otimes f \quad\text{and}\quad \phi(v\otimes w)=y\otimes g.$$

Thus we have

$$\begin{aligned}g(Sx)(y\otimes f)&=(y\otimes g)S(x\otimes f)=\lim_{n\to\infty}(y\otimes g)\phi(T_n)(x\otimes f)\\&=\lim_{n\to\infty}\phi(v\otimes w)\phi(T_n)\phi(u\otimes h)=\lim_{n\to\infty}w(T_nu)\phi(v\otimes h)\\&=w(Tu)\phi(v\otimes h)=\phi((v\otimes w)T(u\otimes h))\\&=(y\otimes g)\phi(T)(x\otimes f)=g(\phi(T)x)(y\otimes f).\end{aligned}$$

It follows that $g(Sx)=g(\phi(T)x)$. This is a contradiction.

Now since S and $\phi(T)$ are linear and $\langle\mathcal{J}(\mathcal{L}_2)\rangle$ is dense in X_2 we conclude that $\phi(T)=S$.

Our main result in this section is the following.

THEOREM 2.3. *Let $\mathcal{L}_i$ be a $\mathcal{J}$-subspace lattice on a Banach space X_i and $\mathcal{A}_i$ be a standard subalgebra of $\operatorname{Alg}\mathcal{L}_i$, $i=1,2$. Let ϕ be an algebraic isomorphism from $\mathcal{A}_1$ onto $\mathcal{A}_2$. Then ϕ is quasi-spatial.*

For clarity of exposition, we organize the proof in a series of lemmas.

LEMMA 2.4. *For every $K\in\mathcal{J}(\mathcal{L}_1)$, there exists an injective linear map $T_K:K\to X_2$ such that $\phi(A)T_Kx=T_KAx$ for $A\in\mathcal{A}_1$ and $x\in K$.*

Proof. Since $K\cap K_-=(0)$, we can choose $x_K\in K$ and $f_K\in K_-^{\perp}$ such that $f_K(x_K)=1$. Then $x_K\otimes f_K\in\mathcal{A}_1$. Since, by Lemma 2.1, ϕ preserves rank one operators, we can suppose $\phi(x_K\otimes f_K)=y_K\otimes g_K$, where $y_K\in X_2$ and $g_K\in X_2^*$. It follows from $f_K(x_K)=1$ that $g_K(y_K)=1$. Noting that $x\otimes f_K\in\operatorname{Alg}\mathcal{L}_1$ for every $x\in K$, define a map $T_K:K\to X_2$ by

$$T_Kx=\phi(x\otimes f_K)y_K,\qquad x\in K.$$

Then T_K is linear. For $A\in\mathcal{A}_1$, $Ax\in K$ for every $x\in K$. Thus

$$\phi(A)T_Kx=\phi(A)\phi(x\otimes f_K)y_K=\phi(Ax\otimes f_K)y_K=T_KAx,\qquad x\in K.$$

Finally we show that T_K is injective. Suppose that $T_Kx=0$ with $x\in K$. Then $\phi(x\otimes f_K)=\phi((x\otimes f_K)(x_K\otimes f_K))=\phi(x\otimes f_K)y_K\otimes g_K=0$. By the injectivity of ϕ, we conclude that $x=0$.

In what follows, for every $K\in\mathcal{J}(\mathcal{L}_1)$, T_K will denote the map as constructed in Lemma 2.4. Obviously, it depends on the choices of x_K, f_K, y_K and g_K. So it will be assumed that those choices have been made for every $K\in\mathcal{J}(\mathcal{L}_1)$.

LEMMA 2.5. *There exists a surjective map $\widehat{\phi}$ from $\mathcal{J}(\mathcal{L}_1)$ onto $\mathcal{J}(\mathcal{L}_2)$ such that for every $K\in\mathcal{J}(\mathcal{L}_1)$, $T_Kx\in\widehat{\phi}(K)$ for every $x\in K$.*

Proof. Let K be in $\mathcal{J}(\mathcal{L}_1)$. Since $y_K \otimes g_K = \phi(x_K \otimes f_K)$ is in $\operatorname{Alg}\mathcal{L}_2$, there is $\widehat{K} \in \mathcal{J}(\mathcal{L}_2)$ such that $y_K \in \widehat{K}$. Since the intersection of any two distinct elements in $\mathcal{J}(\mathcal{L}_2)$ is (0), such a $\widehat{K}$ is unique. Thus the map $\widehat{\phi}(K) = \widehat{K}$ is well defined. For $x \in K$, since $\phi(x \otimes f_K) \in \mathcal{A}_2$, it follows that $T_K x = \phi(x \otimes f_K) y_K \in \widehat{\phi}(K)$.

It remains to prove that $\widehat{\phi}$ is surjective. Let L be an arbitrary element in $\mathcal{J}(\mathcal{L}_2)$. Take nonzero vectors $y \in L$ and $g \in L_-^\perp$ such that $g(y) = 1$. Then there is a rank one operator $x \otimes f \in \mathcal{A}_1$ with $f(x) = 1$ such that $\phi(x \otimes f) = y \otimes g$. Suppose that $x \in K$ where $K \in \mathcal{J}(\mathcal{L}_1)$. Then

$$T_K x = \phi(x \otimes f_K) y_K = \phi(x \otimes f)\phi(x \otimes f_K) y_K = g(T_K x) y.$$

It follows that $y \in \widehat{\phi}(K)$. Consequently, $L = \widehat{\phi}(K)$.

Lemma 2.6. *For every $K \in \mathcal{J}(\mathcal{L}_1)$, T_K is a surjective linear map from K onto $\widehat{\phi}(K)$.*

Proof. Let K be in $\mathcal{J}(\mathcal{L}_1)$. Let $y \in \widehat{\phi}(K)$ with $y \neq 0$. It is easy to see that $g_K \in \widehat{\phi}(K)_-^\perp$. So there is a rank one operator $x \otimes f \in \operatorname{Alg}\mathcal{L}_1$ with $x \in M$ and $f \in M_-^\perp$ for some $M \in \mathcal{J}(\mathcal{L}_1)$, such that $\phi(x \otimes f) = y \otimes g_K$. Thus

$$\phi(f(x_K)x \otimes f_K) = \phi((x \otimes f)(x_K \otimes f_K)) = (y \otimes g_K)(y_K \otimes g_K) = y \otimes g_K.$$

This implies that $f(x_K) \neq 0$, and hence $M = K$. Moreover,

$$T_K(f(x_K)x) = \phi(f(x_K)x \otimes f_K) y_K = y.$$

Lemma 2.7. *Suppose that $K_1, \ldots, K_n$ are distinct elements in $\mathcal{J}(\mathcal{L}_1)$, and let $x_i \in K_i$. If $x_1 + \ldots + x_n = 0$, then $x_1 = \ldots = x_n = 0$.*

Proof. Indeed, for each i, we have $x_i \in K_i \cap (\bigvee_{j \neq i} K_j) \subseteq K_i \cap (K_i)_- = (0)$. Hence $x_i = 0$.

We can now prove our main result.

Proof of Theorem 2.3. Every nonzero $x \in \langle \mathcal{J}(\mathcal{L}_1) \rangle$ has a representation $x = x_1 + \ldots + x_n$ with $x_i \in K_i$, $1 \leq i \leq n$, where $K_1, \ldots, K_n$ are distinct elements in $\mathcal{J}(\mathcal{L}_1)$. If each x_i is required to be nonzero, by Lemma 2.7, this representation is unique up to permutations of the K_i's. Thus we can define a linear map T_0 from $\langle \mathcal{J}(\mathcal{L}_1) \rangle$ to X_2 by

$$T_0 x = T_{K_1} x_1 + \ldots + T_{K_n} x_n,$$

where $0 \neq x = x_1 + \ldots + x_n$ with $0 \neq x_i \in K_i$, $1 \leq i \leq n$, and where $K_1, \ldots, K_n$ are distinct elements in $\mathcal{J}(\mathcal{L}_1)$. By a routine computation, we get

$$\phi(A) T_0 x = T_0 A x \tag{2.1}$$

for $A \in \mathcal{A}_1$ and $x \in \langle \mathcal{J}(\mathcal{L}_1) \rangle$.

Let $G(T_0)$ be the graph of T_0, that is, $G(T_0) = \{(x, T_0x) : x \in \langle \mathcal{J}(\mathcal{L}_1)\rangle\}$, and let $\overline{G(T_0)}$ be the norm closure of $G(T_0)$. Let $\mathcal{D} = \{x \in X_1 : (x, y) \in \overline{G(T_0)}$ for some $y \in X_2\}$. Then $\mathcal{D}$ is obviously a linear manifold and $\langle \mathcal{J}(\mathcal{L}_1)\rangle \subseteq \mathcal{D}$. Since $\langle \mathcal{J}(\mathcal{L}_1)\rangle$ is dense in X_1, so is $\mathcal{D}$.

For every $x \in \mathcal{D}$, we will show that there exists a unique $y \in X_2$ such that $(x, y) \in \overline{G(T_0)}$. Suppose that $(0, y) \in \overline{G(T_0)}$. Then we have a sequence $\{x_m\}_{m=1}^{\infty}$ of vectors in $\langle \mathcal{J}(\mathcal{L}_1)\rangle$ such that $x_m \to 0$ and $T_0x_m \to y$. If $y \neq 0$, there is $g \in L_-^{\perp}$ with $L \in \mathcal{J}(\mathcal{L}_2)$ such that $g(y) \neq 0$ since $\bigcup\{L_-^{\perp} : L \in \mathcal{J}(\mathcal{L}_2)\}$ is weak* dense in X_2^*. Let z be a nonzero vector in L, and suppose that $\phi(A) = z \otimes g$, where A in $\mathcal{A}_1$ is of rank one. Then T_0A is a bounded operator from X_1 to X_2. Thus, from (2.1), we have $(z \otimes g)y = 0$. This is a contradiction.

Therefore, we can define a map $T : \mathcal{D} \subseteq X_1 \to X_2$ in an obvious way, such that $G(T) = \overline{G(T_0)}$. Clearly, T is linear and injective. Moreover, the range of T contains that of T_0. By Lemmas 2.5 and 2.6, the range of T_0 is $\langle \mathcal{J}(\mathcal{L}_2)\rangle$, so the range of T is dense in X_2.

It remains to prove that $\mathcal{D}$ is invariant under every element in $\mathcal{A}_1$ and $\phi(A)Tx = TAx$ for every $A \in \mathcal{A}_1$ and every $x \in \mathcal{D}$.

Let $A \in \mathcal{A}_1$ and $x \in \mathcal{D}$. Then $(x, Tx) \in \overline{G(T_0)}$. Thus there exists a sequence $\{x_m\}_1^{\infty}$ of elements in $\langle \mathcal{J}(\mathcal{L}_1)\rangle$ such that $x_m \to x$ and $T_0x_m \to Tx$. It follows from (2.1) that $T_0Ax_m \to \phi(A)Tx$. Therefore $(Ax_m, T_0Ax_m) \to (Ax, \phi(A)Tx)$. Since $(Ax_m, T_0Ax_m) \in G(T_0)$, we have $(Ax, \phi(A)Tx) \in \overline{G(T_0)}$. Consequently, $Ax \in \mathcal{D}$ and $\phi(A)Tx = TAx$.

REMARK 2.8. Though Theorem 2.2 ensures that an algebraic isomorphism between $\mathcal{J}$-subspace lattice algebras is bounded, we do not know whether it is necessarily spatial.

REMARK 2.9. If ϕ is only a ring isomorphism in Theorem 2.3, the above proof gives the same result except that T is just additive.

As we know, a ring isomorphism preserves the additive and the multiplicative structures. It is an interesting problem to study when a multiplicative map is additive. The first quite surprising result is due to Martindale [16], who proved the following.

THEOREM M. *Let $\mathcal{R}$ be a ring containing a family $\{e_\alpha : \alpha \in \Lambda\}$ of idempotents which satisfies*:

(1) *$x\mathcal{R} = 0$ implies $x = 0$.*

(2) *If $e_\alpha \mathcal{R} x = 0$ for each $\alpha \in \Lambda$, then $x = 0$.*

(3) *For each $\alpha \in \Lambda$, $e_\alpha x e_\alpha \mathcal{R}(1 - e_\alpha) = 0$ implies $e_\alpha x e_\alpha = 0$.*

Then any multiplicative isomorphism of $\mathcal{R}$ onto an arbitrary ring is additive.

For more information on multiplicative maps, we refer to [14] and its references. The following result can be reformulated by saying that the additivity assumption in the definition of ring isomorphisms of certain standard subalgebras of $\mathcal{J}$-subspace lattice algebras is superfluous.

THEOREM 2.10. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X. Let $\mathcal{A}$ be a standard subalgebra of* $\operatorname{Alg}\mathcal{L}$. *Suppose* $\dim K \geq 2$ *for every $K \in \mathcal{J}(\mathcal{L})$. Then every multiplicative isomorphism ϕ of $\mathcal{A}$ onto an arbitrary ring is additive.*

Proof. Consider the family $S = \{x \otimes f : x \in K,\ f \in K_-^\perp,\ f(x) = 1,\ K \in \mathcal{J}(\mathcal{L})\}$ of idempotent rank one operators. Now it suffices to verify that conditions (1)–(3) in Theorem M are satisfied.

(1) Suppose $T \in \mathcal{A}$ is such that $T\mathcal{A} = 0$. For each $K \in \mathcal{J}(\mathcal{L})$, fix a nonzero functional $f_K \in K_-^\perp$. Then for every $x \in K$, $x \otimes f_K \in \mathcal{A}$. So $Tx \otimes f_K = 0$. This implies that $Tx = 0$ for every $x \in K$. Hence $Tx = 0$ for every $x \in \langle \mathcal{J}(\mathcal{L}) \rangle$. Since T is linear and continuous, and $\langle \mathcal{J}(\mathcal{L}) \rangle$ is dense in X, it follows that $T = 0$.

(2) Suppose that $T \in \mathcal{A}$ is such that $(x \otimes f)AT = 0$ for every $x \otimes f \in S$ and $A \in \mathcal{A}$. In particular, setting $A = x \otimes f$, we have

$$(x \otimes f)T = 0 \tag{2.2}$$

for every $x \otimes f \in S$. Let $K \in \mathcal{J}(\mathcal{L})$ and $y \in K$. For every nonzero functional $f \in K_-^\perp$, since $K \vee K_- = X$, there exists $x \in K$ such that $f(x) = 1$. Thus by (2.2), $f(Ty) = 0$ for every $f \in K_-^\perp$. This implies that $Ty \in K_-$. But clearly $Ty \in K$. Therefore $Ty = 0$ for every $y \in \langle \mathcal{J}(\mathcal{L}) \rangle$. Consequently, $T = 0$.

(3) Let $x \otimes f \in S$ and $T \in \mathcal{A}$. Suppose that

$$(x \otimes f)T(x \otimes f)A(1 - x \otimes f) = 0 \tag{2.3}$$

for every $A \in \mathcal{A}$. Now $x \in K$ and $f \in K_-^\perp$ for some $K \in \mathcal{J}(\mathcal{L})$. Since $\dim K \geq 2$ and $K \vee K_- = X$ and $K \cap K_- = 0$, we have $\langle x \rangle \vee K_- \neq X$. Thus there is a nonzero functional $g \in K_-^\perp$ such that $g(x) = 0$. Putting $A = x \otimes g$ in (2.3), we get $(x \otimes f)T(x \otimes g) = 0$. This implies that $f(Tx) = 0$. Therefore $(x \otimes f)T(x \otimes f) = f(Tx)(x \otimes f) = 0$.

For a pentagon subspace lattice, the condition that $\dim K \geq 2$ for every $K \in \mathcal{J}(\mathcal{L})$ in Theorem 2.10 is automatically satisfied. However, for a general $\mathcal{J}$-subspace lattice, this condition cannot be removed. The simplest example is $\phi(\lambda) = \lambda|\lambda|$ for $\lambda \in \mathbb{F}$.

3. Jordan derivations. We begin with the continuity of linear derivations of $\mathcal{J}$-subspace lattice algebras.

THEOREM 3.1. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X. Let δ be a linear derivation from* $\operatorname{Alg}\mathcal{L}$ *to $B(X)$. Then δ is automatically* (*norm*) *continuous.*

Proof. By the closed graph theorem, it suffices to prove that δ is a closed operator from $\operatorname{Alg}\mathcal{L}$ into $B(X)$. Let T_n, T in $\operatorname{Alg}\mathcal{L}$ and S in $B(X)$ be such that $T_n \to T$ and $\delta(T_n) \to S$.

Let F be in $\mathcal{J}(\mathcal{L})$ and x in F. We want to prove that $\delta(T)x = Sx$. Take a nonzero functional f in $F_-^\perp$. Let E be an arbitrary element in $\mathcal{J}(\mathcal{L})$, and take nonzero vectors $y \in E$ and $g \in E_-^\perp$. Then $x \otimes f$ and $y \otimes g$ are both in $\operatorname{Alg}\mathcal{L}$. From the fact that for every $A \in \operatorname{Alg}\mathcal{L}$,

$$\begin{aligned}
&\delta((y\otimes g)A(x\otimes f))\\
&\qquad = \delta(y\otimes g)A(x\otimes f) + (y\otimes g)\delta(A)(x\otimes f) + (y\otimes g)A\delta(x\otimes f),
\end{aligned}$$

we have

$$\begin{aligned}
(y\otimes g)S(x\otimes f) &= \lim_{n\to\infty}(y\otimes g)\delta(T_n)(x\otimes f)\\
&= \lim_{n\to\infty}\delta((y\otimes g)T_n(x\otimes f))\\
&\quad - \lim_{n\to\infty}\delta(y\otimes g)T_n(x\otimes f) - \lim_{n\to\infty}(y\otimes g)T_n\delta(x\otimes f))\\
&= \delta((y\otimes g)T(x\otimes f))\\
&\quad - \delta(y\otimes g)T(x\otimes f) - (y\otimes g)T\delta(x\otimes f)\\
&= (y\otimes g)\delta(T)(x\otimes f).
\end{aligned}$$

It follows that $g(\delta(T)x) = g(Sx)$ for every $E \in \mathcal{J}(\mathcal{L})$ and every $g \in E_-^\perp$. Since $\operatorname{span}\{K_-^\perp : K \in \mathcal{J}(\mathcal{L})\}$ is weak* dense in X^*, we conclude that $\delta(T)x = Sx$. Hence $\delta(T) = S$ since $\langle \mathcal{J}(\mathcal{L})\rangle$ is dense in X.

As we have seen above and will see below, finite rank operators and rank one operators play an important role in the study of $\mathcal{J}$-subspace lattice algebras. The question of whether a finite rank operator in an operator algebra can be written as a finite sum of rank one operators in that algebra has been studied by many authors (see [15] and its references). It has been shown that finite rank operators in nest algebras, in finite width CSL algebras, and in atomic Boolean subspace lattice algebras have this property. However, Hopenwasser and Moore [5] produced an example of a commutative, completely distributive subspace lattice algebra in which there is a rank two operator which cannot be written as a finite sum of rank one operators. So the following result has independent interest. It was first proved in [17, Proposition 3.3.1], and we include a proof here for the convenience of the reader.

PROPOSITION 3.2. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X and suppose that A is an operator of rank n in* $\operatorname{Alg}\mathcal{L}$. *Then A can be written as a sum of n rank one operators in* $\operatorname{Alg}\mathcal{L}$.

Proof. Let

$$\{K_\alpha \in \mathcal{J}(\mathcal{L}) : \alpha \in \Lambda\} = \{K \in \mathcal{J}(\mathcal{L}) : Ax \neq 0 \text{ for some } x \in K\}.$$

For each $\alpha \in \Lambda$, let $\{x_1^\alpha, \ldots, x_{n_\alpha}^\alpha\}$ be a Hamel basis for $\{Ax : x \in K_\alpha\}$. Then $n_\alpha < \infty$ for each $\alpha \in \Lambda$ and $\{x_1^\alpha, \ldots, x_{n_\alpha}^\alpha\}_{\alpha\in\Lambda}$ is a linearly independent set of vectors. It is clear that the linear span of $\{x_1^\alpha, \ldots, x_{n_\alpha}^\alpha\}_{\alpha\in\Lambda}$ is contained in the range of A. It follows that the set $\{x_1^\alpha, \ldots, x_{n_\alpha}^\alpha\}_{\alpha\in\Lambda}$ is finite and Λ has at most n elements. Hence since $\bigvee\{K : K \in \mathcal{J}(\mathcal{L})\} = X$, the range of A is equal to the linear span of $\{x_1^\alpha, \ldots, x_{n_\alpha}^\alpha\}_{\alpha\in\Lambda}$. Suppose that $\Lambda = \{1, \ldots, m\}$ $(m \le n)$. Then $n_1 + \ldots + n_m = n$ and

$$A = x_1^1 \otimes f_1^1 + \ldots + x_{n_1}^1 \otimes f_{n_1}^1 + \ldots + x_1^m \otimes f_1^m + \ldots + x_{n_m}^m \otimes f_{n_m}^m, \tag{3.1}$$

where $f_i^j \in X^*$. Now it suffices to prove that each f_i^j, $1 \le i \le n_j$, is in $(K_j)_-^\perp$. For simplicity, we only prove that $f_1^1, \ldots, f_{n_1}^1 \in (K_1)_-^\perp$. For every $x \in (K_1)_-$, $Ax \in (K_1)_-$. Since $K_j \ne K_1$, $j = 2, \ldots, m$, we have $K_j \subseteq (K_1)_-$. It follows that $\{x_i^j : 2 \le j \le m,\ 1 \le i \le n_j\} \subset (K_1)_-$. Thus, from (3.1), $f_1^1(x)x_1^1 + \ldots + f_{n_1}^1(x)x_{n_1}^1$ belongs to $(K_1)_-$. But this vector also belongs to K_1. It follows from $K_1 \cap (K_1)_- = (0)$ that $f_1^1(x)x_1^1 + \ldots + f_{n_1}^1(x)x_{n_1}^1 = 0$. Hence since $x_1^1, \ldots, x_{n_1}^1$ are linearly independent, $f_1^1(x) = \ldots = f_{n_1}^1(x) = 0$. Since x is an abitrary vector in $(K_1)_-$, we conclude that $f_1^1, \ldots, f_{n_1}^1 \in (K_1)_-^\perp$.

The following is the main result in this section.

THEOREM 3.3. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X and $\mathcal{A}$ be a standard subalgebra of* $\operatorname{Alg}\mathcal{L}$. *Then every Jordan derivation δ of $\mathcal{A}$ to $B(X)$ is an additive derivation.*

For the proof of Theorem 3.3, we need some lemmas. The first can be found in [4].

LEMMA 3.4. *For $A, B, C \in \mathcal{A}$, we have*

(i) $\delta(AB + BA) = A\delta(B) + \delta(A)B + B\delta(A) + \delta(B)A$.
(ii) $\delta(ABA) = \delta(A)BA + A\delta(B)A + AB\delta(A)$.

In what follows, for $K \in \mathcal{J}(\mathcal{L})$, write $\mathcal{F}(K) = \langle\{x \otimes f : x \in K,\ f \in K_-^\perp\}\rangle$. Then $\mathcal{F}(K)$ is an ideal of $\mathcal{A}$.

LEMMA 3.5. *Let $K \in \mathcal{J}(\mathcal{L})$ and suppose that* $\dim K \ge 2$. *Suppose that φ is a ring homomorphism from $\mathcal{F}(K)$ to $B(X)$ and ψ is a ring anti-homomorphism from $\mathcal{F}(K)$ to $B(X)$. If, for every $A \in \mathcal{F}(K)$,*

$$\varphi(A) + \psi(A) = A, \tag{3.2}$$

then $\psi = 0$.

Proof. Since $\dim K \ge 2$, we can choose nonzero vectors $x_1, x_2 \in K$ and $f \in K_-^\perp$ satisfying $f(x_1) = 1$ and $f(x_2) = 0$. Then both $x_1 \otimes f$ and $x_2 \otimes f$ are in $\mathcal{F}(K)$. Moreover, by (3.2), $x_1 \otimes f$ is the sum of two idempotents $\varphi(x_1 \otimes f)$ and $\psi(x_1 \otimes f)$. It follows that one of $\varphi(x_1 \otimes f)$ and $\psi(x_1 \otimes f)$ is zero. We will show that $\psi(x_1 \otimes f) = 0$. Otherwise, $\varphi(x_1 \otimes f) = 0$. Then

$\varphi(x_2 \otimes f) = \varphi((x_2 \otimes f)(x_1 \otimes f)) = 0$. By (3.2), we have $\psi(x_1 \otimes f) = x_1 \otimes f$ and $\psi(x_2 \otimes f) = x_2 \otimes f$. Thus we would have $x_2 \otimes f = \psi(x_2 \otimes f) = \psi((x_2 \otimes f)(x_1 \otimes f)) = \psi(x_1 \otimes f)\psi(x_2 \otimes f) = (x_1 \otimes f)(x_2 \otimes f) = 0$. This is impossible. So $\psi(x_1 \otimes f) = 0$. For every $x \otimes g \in \mathcal{F}(K)$, $\psi(x \otimes g) = \psi((x \otimes f)(x_1 \otimes f)(x_1 \otimes g)) = 0$. Consequently, $\psi = 0$.

Recall that an algebra $\mathcal{A}$ is called a *matrix algebra of rank* n if there exists a system $\{e_{ij} : 1 \leq i, j \leq n\} \subseteq \mathcal{A}$ satisfying $e_{ij}e_{kl} = \delta_{jk}e_{il}$ such that $x = \sum_{i,j} t_{ij}e_{ij}$ for each $x \in \mathcal{A}$, where $t_{ij} \in \mathbb{F}$. An algebra $\mathcal{A}$ is called a *locally matrix algebra* if for every finite set of elements $A_1, \ldots, A_n$ in $\mathcal{A}$ there is a subalgebra $\mathcal{B}$ of $\mathcal{A}$, which is a matrix algebra of rank ≥ 2, such that all $A_1, \ldots, A_n$ are in $\mathcal{B}$. The following lemma ensures that Theorem 8 in [6] can be applied.

LEMMA 3.6. *Let $K \in \mathcal{J}(\mathcal{L})$ and suppose that* $\dim K \geq 2$. *Then $\mathcal{F}(K)$ is a locally matrix algebra.*

Proof. We first establish the following claim.

CLAIM. *For every finite set of rank one operators $x_1 \otimes f_1, \ldots, x_n \otimes f_n \in \mathcal{F}(K)$, there is an idempotent operator P of* rank ≥ 2 *in $\mathcal{F}(K)$ such that $Px_i \otimes f_iP = x_i \otimes f_i$, $i = 1, \ldots, n$.*

Proceed by induction. Consider a rank one operator $x_1 \otimes f_1 \in \mathcal{F}(K)$. If $f_1(x_1) = \lambda \neq 0$, it follows from $\dim K \geq 2$ that there are $y \in K$ and $g \in K_-^\perp$ such that $g(y) = 1$ and $g(x_1) = f_1(y) = 0$. Set $P = \lambda^{-1}x_1 \otimes f_1 + y \otimes g$. It is easy to verify that this is the desired P. If $f_1(x_1) = 0$, it follows from $\dim K \geq 2$ that there are $y \in K$ and $g \in K_-^\perp$ such that $g(x_1) = f_1(y) = 1$ and $g(y) = 0$. The desired P is $P = x_1 \otimes g + y \otimes f_1$.

Now suppose that the claim is valid for $n-1$, that is, there is an idempotent operator Q of rank ≥ 2 in $\mathcal{F}(K)$ such that $Qx_i \otimes f_iQ = x_i \otimes f_i$, $i = 1, \ldots, n-1$. We want to prove that the claim is also valid for n. We distinguish some cases.

CASE 1: $(I-Q)x_n \neq 0$ and $(I-Q)^*f_n \neq 0$. Note that $(I-Q)x_n \in K$ and $(I-Q)^*f_n \in K_-^\perp$.

If $f_n((I-Q)x_n) = \lambda \neq 0$, we set $P = Q + \lambda^{-1}(I-Q)x_n \otimes (I-Q)^*f_n$.

If $f_n((I-Q)x_n) = 0$, then there is $y \in K$ such that $f_n((I-Q)y) = 1$. Let $g \in K_-^\perp$ be such that $g((I-Q)y) = 0$ and $g((I-Q)x_n) = 1$. Set

$$P = Q + (I-Q)x_n \otimes (I-Q)^*g + (I-Q)y \otimes (I-Q)^*f_n.$$

CASE 2: $(I-Q)x_n \neq 0$ and $(I-Q)^*f_n = 0$. Then $f_n((I-Q)x_n)) = 0$. Let $g \in K_-^\perp$ be such that $g((I-Q)x_n) = 1$. Set

$$P = Q + (I-Q)x_n \otimes (I-Q)^*g.$$

CASE 3: $(I-Q)x_n = 0$ and $(I-Q)^* f_n \neq 0$. Pick $y \in K$ such that $f_n((I-Q)y) = 1$. Set

$$P = Q + (I-Q)y \otimes (I-Q)^* f_n.$$

CASE 4: $(I-Q)x_n = 0$ and $(I-Q)^* f_n = 0$. Set $P = Q$.

This establishes the claim.

Let $A_1, \dots, A_n$ be in $\mathcal{F}(K)$. By the claim, there is an idempotent operator P of rank ≥ 2 in $\mathcal{F}(K)$ such that $PA_iP = A_i$, $i = 1, \dots, n$. Let $\{x_1, \dots, x_m\}$ be a basis of the range of P. Let $f_i \in X^*$, $i = 1, \dots, m$, be such that

$$f_i(x_j) = \delta_{ij}, \quad f_i(z) = 0 \quad \text{for every } z \in \ker P.$$

Then $f_i \in K_-^\perp$ and $P = x_1 \otimes f_1 + \ldots + x_n \otimes f_n$. Define $\mathcal{B} = \langle \{x_k \otimes f_j : 1 \leq k, j \leq m\} \rangle$. Then $\mathcal{B}$ is a matrix algebra and all $A_1, \dots, A_n$ belong to $\mathcal{B}$. Moreover, for each $1 \leq i \leq n$,

$$A_i = \sum_{k,j} t_{kj}^i x_k \otimes f_j,$$

where $t_{kj}^i = f_k(A_i x_j)$.

LEMMA 3.7. *Let $K \in \mathcal{J}(\mathcal{L})$ and δ_K be the restriction of δ to $\mathcal{F}(K)$. Then δ_K is an additive derivation.*

Proof. We distinguish two cases.

CASE 1: $\dim K = 1$. Then $K + K_- = X$, and hence $\dim K_-^\perp = 1$. Thus $\mathcal{F}(K)$ is of dimension one. (Though in this case $\mathcal{F}(K)$ and $B(X)$ are prime, it seems that Herstein's result in [4] cannot be directly used. We also believe that a complete description of all Jordan derivations from a prime subalgebra to $B(X)$ has not yet been published. Here we give an elementary proof.)

Let $x_0 \in K$ and $f_0 \in K_-^\perp$ be such that $f_0(x_0) = 1$, and set $P = x_0 \otimes f_0$. Then $\mathcal{F}(K) = \{\lambda P : \lambda \in \mathbb{F}\}$. Multiplying by P the equation $\delta_K(P) = \delta_K(P^2) = \delta_K(P)P + P\delta_K(P)$ from the left, we get $P\delta_K(P)P = 0$. Let $x = \delta_K(P)x_0$ and $f = \delta_K(P)^* f_0$. Then $f_0(x) = f(x_0) = 0$. For $\lambda \in \mathbb{F}$, let $h(\lambda) = f_0(\delta_K(\lambda P)x_0)$. Then $P\delta_K(\lambda P)P = h(\lambda)P$. Moreover, by Lemma 3.4, we have

$$\begin{aligned}\delta_K(\lambda P) &= \delta_K(P(\lambda P)P) = \delta_K(P)(\lambda P)P + P\delta_K(\lambda P)P + P(\lambda P)\delta_K(P) \\ &= \lambda x_0 \otimes f + \lambda x \otimes f_0 + h(\lambda)P.\end{aligned}$$

Thus

$$\delta_K(\lambda P)P = \lambda x \otimes f_0 + h(\lambda)P, \quad P\delta_K(\lambda P) = \lambda x_0 \otimes f + h(\lambda)P.$$

Therefore, for $\lambda, \mu \in \mathbb{F}$,

$$\begin{aligned}
&2(\lambda\mu x_0 \otimes f + \lambda\mu x \otimes f_0 + h(\lambda\mu)P) \\
&\qquad = 2\delta_K(\lambda\mu P) = \delta_K(2\lambda\mu P) = \delta_K((\lambda P)(\mu P) + (\mu P)(\lambda P)) \\
&\qquad = \delta_K(\lambda P)\mu P + \lambda P\delta_K(\mu P) + \delta_K(\mu P)\lambda P + \mu P\delta_K(\lambda P) \\
&\qquad = \mu(\lambda x \otimes f_0 + h(\lambda)P) + \lambda(\mu x_0 \otimes f + h(\mu)P) \\
&\qquad\quad + \lambda(\mu x \otimes f_0 + h(\mu)P) + \mu(\lambda x_0 \otimes f + h(\lambda)P) \\
&\qquad = 2(\lambda\mu x_0 \otimes f + \lambda\mu x \otimes f_0 + (\lambda h(\mu) + \mu h(\lambda))P).
\end{aligned}$$

It follows that $h(\lambda\mu) = \lambda h(\mu) + \mu h(\lambda)$. Further, for $\lambda, \mu \in \mathbb{F}$,

$$\begin{aligned}
&\delta_K(\lambda P)\mu P + \lambda P\delta_K(\mu P) \\
&\qquad = \mu(\lambda x \otimes f_0 + h(\lambda)P) + \lambda(\mu x_0 \otimes f + h(\mu)P) \\
&\qquad = \lambda\mu x_0 \otimes f + \lambda\mu x \otimes f_0 + (\lambda h(\mu) + \mu h(\lambda))P \\
&\qquad = \lambda\mu x_0 \otimes f + \lambda\mu x \otimes f_0 + h(\lambda\mu)P = \delta_K((\lambda P)(\mu P)).
\end{aligned}$$

CASE 2: $\dim K \geq 2$. Define a mapping $\phi : \mathcal{F}(K) \to B(X \oplus X)$ by

$$\phi(A) = \begin{bmatrix} A & \delta_K(A) \\ 0 & A \end{bmatrix}.$$

Then ϕ is an additive Jordan homomorphism. Since $\mathcal{F}(K)$ is a locally matrix algebra by Lemma 3.6, by Theorem 8 in [6], $\phi = \varphi + \psi$, where φ is a ring homomorphism from $\mathcal{F}(K)$ to $B(X \oplus X)$ and ψ is a ring anti-homomorphism from $\mathcal{F}(K)$ to $B(X \oplus X)$. Furthermore, φ and ψ are of the form

$$\varphi(A) = \begin{bmatrix} \varphi_1(A) & \varphi_2(A) \\ 0 & \varphi_3(A) \end{bmatrix}, \qquad \psi(A) = \begin{bmatrix} \psi_1(A) & \psi_2(A) \\ 0 & \psi_3(A) \end{bmatrix}, \tag{3.3}$$

where φ_1 and φ_3 are ring homomorphisms from $\mathcal{F}(K)$ to $B(X)$, and ψ_1 and ψ_3 are ring anti-homomorphisms from $\mathcal{F}(K)$ to $B(X)$. Thus the equations $\varphi_1(A) + \psi_1(A) = A$ and $\varphi_3(A) + \psi_3(A) = A$ hold for every $A \in \mathcal{F}(K)$. By Lemma 3.5, $\psi_1 = \psi_3 = 0$. Thus relation (3.3) becomes

$$\varphi(A) = \begin{bmatrix} A & \varphi_2(A) \\ 0 & A \end{bmatrix}, \qquad \psi(A) = \begin{bmatrix} 0 & \psi_2(A) \\ 0 & 0 \end{bmatrix}. \tag{3.4}$$

It follows from (3.4) that φ_2 is an additive derivation and $\psi(AB) = \psi(B)\psi(A) = 0$. Hence for all A, B in $\mathcal{F}(K)$, we have $\psi_2(AB) = 0$. For every rank one operator $x \otimes f \in \mathcal{F}(K)$, we take $y \in K$ such that $f(y) = 1$. Thus

$$\psi_2(x \otimes f) = \psi_2((x \otimes f)(y \otimes f)) = 0.$$

Hence $\psi_2 = 0$. Thus $\delta_K = \varphi_2$ is an additive derivation.

Proof of Theorem 3.3. Let K be an arbitrary element in $\mathcal{J}(\mathcal{L})$ and δ_K be the restriction of δ to $\mathcal{F}(K)$. Fix $f_K \in K_-^\perp$ and $x_K \in K$ such that

$f_K(x_K) = 1$. Then $x \otimes f_K \in \mathcal{F}(K)$ for any $x \in K$. Define a map T_K from K to X by

$$T_K x = \delta_K(x \otimes f_K)x_K, \qquad x \in K.$$

For every $C \in \mathcal{F}(K)$, by Lemma 3.7,

$$\delta_K(Cx \otimes f_K) = \delta_K(C)x \otimes f_K + C\delta_K(x \otimes f_K), \qquad x \in K.$$

Applying the two sides of the above equation to x_K, we get

$$\delta_K(C)x = (T_K C - CT_K)x, \qquad x \in K.$$

Let $A \in \mathcal{A}$ be arbitrary. For every $C \in \mathcal{F}(K)$, since AC and CA are both in $\mathcal{F}(K)$, we have, for any $x \in K$,

$$\begin{aligned}\delta(AC + CA)x &= \delta_K(AC)x + \delta_K(CA)x\\ &= (T_K AC - ACT_K)x + (T_K CA - CAT_K)x\\ &= ((T_K A - AT_K)C + A(T_K C - CT_K))x\\ &\quad + ((T_K C - CT_K)A + C(T_K A - AT_K))x\\ &= (T_K A - AT_K)Cx + A\delta(C)x + \delta(C)Ax + C(T_K A - AT_K)x.\end{aligned}$$

On the other hand, since δ is a Jordan derivation, by Lemma 3.4,

$$\delta(AC + CA)x = \delta(A)Cx + A\delta(C)x + \delta(C)Ax + C\delta(A)x.$$

So we have

$$(T_K A - AT_K - \delta(A))Cx_K = -C(T_K A - AT_K - \delta(A))x_K, \qquad C \in \mathcal{F}(K).$$

In particular,

$$(T_K A - AT_K - \delta(A))(x \otimes f_K)x_K = -(x \otimes f_K)(T_K A - AT_K - \delta(A))x_K, \quad x \in K.$$

It follows that $(T_K A - AT_K - \delta(A))x = \lambda x$ for some $\lambda \in \mathbb{F}$ (where λ is independent of x). Thus the above equation becomes

$$\lambda x = -\lambda x, \qquad x \in K,$$

from which $\lambda = 0$ and then

$$\delta(A)x = (T_K A - AT_K)x$$

for every $x \in K$.

Now let A and B be in $\mathcal{A}$. Let K be an arbitrary element in $\mathcal{J}(\mathcal{L})$. For every $x \in K$, since $Bx \in K$ we have

$$\begin{aligned}\delta(AB)x &= (T_K AB - ABT_K)x = (T_K A - AT_K)Bx + A(T_K B - BT_K)x\\ &= \delta(A)Bx + A\delta(B)x.\end{aligned}$$

Consequently, δ is an additive derivation.

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