Dimensions of non-differentiability points of Cantor functions

by

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Abstract. For a probability vector (p_0, p_1) there exists a corresponding self-similar Borel probability measure μ supported on the Cantor set C (with the strong separation property) in $\mathbb{R}$ generated by a contractive similitude $h_i(x) = a_i x + b_i$, $i = 0, 1$. Let S denote the set of points of C at which the probability distribution function $F(x)$ of μ has no derivative, finite or infinite. The Hausdorff and packing dimensions of S have been found by several authors for the case that $p_i > a_i$, $i = 0, 1$. However, when $p_0 < a_0$ (or equivalently $p_1 < a_1$) the structure of S changes significantly and the previous approaches fail to be effective any more. The present paper is devoted to determining the Hausdorff and packing dimensions of S for the case $p_0 < a_0$.

1. Introduction. Let $C \subset \mathbb{R}$ be the unique nonempty compact set invariant under h_0, h_1:

$$C = h_0(C) \cup h_1(C),$$

where $h_j(x) = a_j x + b_j, j = 0, 1$, with $0 < a_j < 1$. The set C is also termed the *Cantor set*, or the self-similar set determined by h_0 and h_1. Without loss of generality we shall assume that $b_0 = 0$, $a_1 + b_1 = 1$ (so $h_0(0) = 0$ and $h_1(1) = 1$). We furthermore assume that the images $h_j([0,1])$, $j = 0, 1$, are pairwise disjoint (i.e. the h_j's satisfy the strong separation condition). Let

$$\beta = 1 - a_0 - a_1 > 0$$

be the length of the gap between $h_0([0,1])$ and $h_1([0,1])$. It is well known (cf. [4, 6]) that $\dim_{\mathrm{H}} C = \dim_{\mathrm{P}} C = \dim_{\mathrm{B}} C = \xi$ and $0 < \mathcal{H}^\xi(C) < \infty$ where ξ is given by

$$(1) \qquad\qquad a_0^\xi + a_1^\xi = 1.$$

The set C has a natural symbolic representation defined as follows. Let

$$\Sigma^* = \bigcup_{n=1}^{\infty} \{0,1\}^n \quad \text{and} \quad \Sigma^{\mathbb{N}} = \{0,1\}^{\mathbb{N}},$$

i.e. Σ^* is the family of all finite strings $j_1 \ldots j_n$ with entries j_i from $\{0,1\}$ and $\Sigma^{\mathbb{N}}$ denotes the family of all infinite strings $j_1 j_2 \ldots$ with entries j_i from $\{0,1\}$. For $\omega = j_1 j_2 \ldots \in \Sigma^{\mathbb{N}}$ and a positive integer n, let $\omega|n = j_1 \ldots j_n$ denote the truncation of ω to the nth place. Finally, we define $\pi : \Sigma^{\mathbb{N}} \to C$ by

$$\{\pi(\omega)\} = \bigcap_{n=1}^{\infty} h_{j_1} \circ \cdots \circ h_{j_n}([0,1])$$

for $\omega = j_1 j_2 \ldots \in \Sigma^{\mathbb{N}}$. It is easy to check that for $\omega = j_1 j_2 \ldots \in \Sigma^{\mathbb{N}}$,

$$\pi(\omega) = \lim_{n \to \infty} h_{\omega|n}(0) = b_{j_1} + \sum_{n=2}^{\infty} a_{j_1} \ldots a_{j_{n-1}} b_{j_n},$$

where $b_0 = 0$ and $b_1 = 1 - a_1$. It is well known that C can be symbolically represented as $C = \pi(\Sigma^{\mathbb{N}})$. Thus, each point of C can be encoded by a unique element from $\Sigma^{\mathbb{N}}$. Throughout this paper, by $\tilde{t} = t_1 t_2 \ldots$ we denote the code of $t \in C$, i.e., $\tilde{t} = t_1 t_2 \ldots \in \Sigma^{\mathbb{N}}$ with $\pi(\tilde{t}) = t$. Alternatively, we have

$$C = \bigcap_{n=1}^{\infty} \bigcup_{\omega \in \{0,1\}^n} h_{\omega}([0,1]),$$

where $h_{\omega} := h_{j_1} \circ \cdots \circ h_{j_n}$ for $\omega = j_1 \ldots j_n \in \{0,1\}^n$. The endpoints of $h_{\omega}([0,1])$ for $\omega \in \Sigma^*$ will be called the *endpoints of* C. Obviously any endpoint e of C lies in C and except for a finite number of terms, its coding $\tilde{e}$ consists of either only the digit 0 if e is the left endpoint of some $h_{\omega}([0,1])$, or only the digit 1 if e is the right endpoint of some $h_{\sigma}([0,1])$. Moreover, for each non-endpoint t of C its code $\tilde{t}$ consists of infinitely many 0s and infinitely many 1s.

Let μ be the *self-similar Borel measure* on C corresponding to the probability vector (p_0, p_1) where each $p_i > 0$ and $p_0 + p_1 = 1$, i.e., the measure satisfying

$$\mu(A) = p_0 \mu(h_0^{-1}(A)) + p_1 \mu(h_1^{-1}(A)) \quad \text{for any Borel set } A,$$

and so

$$\mu(h_{\omega}[0,1]) = \prod_{i=1}^{k} p_{j_i} =: p_{\omega} \quad \text{for any } \omega = j_1 \ldots j_k \in \{0,1\}^k, \ k \in \mathbb{N}.$$

Obviously, μ is atomless. Consider the distribution function of the probability measure μ, also called the *Cantor function* or a self-affine "devil's staircase" function:

$$F(x) = \mu([0,x]), \quad x \in [0,1].$$

Then $F(x)$ is a non-decreasing continuous function with $F(0) < F(1)$ that is constant off the support of μ. It is easy to check that the derivative of $F(x)$ is zero for all $x \in [0,1] \setminus C$. In particular, the set S of *points of non-differentiability* of $F(x)$, that is, those x where

$$(2) \qquad \lim_{\delta \to 0} \frac{F(x+\delta) - F(x)}{\delta}$$

does not exist either as a finite number or ∞, has Lebesgue measure 0. It has been shown that $F'(x) = \infty$ for μ-a.e. $x \in C$ and so $\mu(S) = 0$ (cf. [8]). Some further results on the size of S have also been obtained.

(I) When $(p_0, p_1) = (a_0^\xi, a_1^\xi)$, we have $\dim_{\mathrm{H}} S = \xi^2 = (\dim_{\mathrm{H}} C)^2$ and $\dim_{\mathrm{P}} S = \dim_{\mathrm{B}} S = \xi = \dim_{\mathrm{H}} C$ (cf. [1–3]). In this case, the corresponding μ is ξ-*Ahlfors regular*, which means that there exist $c_1, c_2 > 0$ such that $c_1 \delta^\xi \leq \mu([x-\delta, x+\delta]) \leq c_2 \delta^\xi$ for all $x \in C$ and $0 < \delta < 1$. This curious property is extended to a more general setting. K. J. Falconer [5] proved that for $0 < d < 1$ if μ is a positive finite Borel d-Ahlfors regular measure supported by a non-empty compact subset E of $\mathbb{R}$ (not necessarily the Cantor set), then $\dim_{\mathrm{H}} S = (\dim_{\mathrm{H}} E)^2$ and $\dim_{\mathrm{P}} S = \dim_{\mathrm{B}} S = \dim_{\mathrm{H}} E$.

(II) When (p_0, p_1) satisfies $p_0 > a_0$ and $p_1 > a_1$ (so μ is not Ahlfors regular except in a special case), $\dim_{\mathrm{P}} S = \dim_{\mathrm{B}} S = \xi = \dim_{\mathrm{H}} C$ still holds and another approach has to be taken to determine $\dim_{\mathrm{H}} S$ (cf. [7, 9]). In particular, when $p_0 = a_0/(a_0 + a_1)$ and $p_1 = a_1/(a_0 + a_1)$, $\dim_{\mathrm{H}} S$ was found by J. Morris [10].

Clearly, case (I) is included in case (II) since $p_i = a_i^\xi > a_i$, $i = 0, 1$. The condition $p_i > a_i$, $i = 0, 1$, implies that (cf. [9])

$$(3) \qquad \limsup_{\delta \to 0} \frac{F(x+\delta) - F(x)}{\delta} = \infty \quad \text{for all } x \in C.$$

Thus, we have

$$S = \left\{ x \in C : \liminf_{\delta \to 0} \frac{F(x+\delta) - F(x)}{\delta} < \infty \right\} \quad \text{if } p_0 > a_0, \, p_1 > a_1.$$

In particular,

$$(4) \quad \left\{ x \in C : F'(x) = \lim_{\delta \to 0} \frac{F(x+\delta) - F(x)}{\delta} = 0 \right\} = \emptyset \quad \text{if } p_0 > a_0, \, p_1 > a_1.$$

However, when $p_0 < a_0$ and $p_1 > a_1$ (or $p_0 > a_0$ and $p_1 < a_1$), the property (3) does not hold any more. This makes the structure of S more complicated, e.g., compared to (4); we will see in Theorem 1.1 that there are massive sets of points of C at which the derivative is zero in this case, and so the approaches used in the papers mentioned above are not valid any more.

The present paper is devoted to determining the fractal dimensions of S for the case $p_0 < a_0$ and $p_1 > a_1$. Let s be such that

$$(5) \qquad \left(\frac{p_0}{a_0}\right)\left(\frac{p_1}{a_1}\right)^s = 1, \quad \text{i.e.,} \quad s = \frac{\log(a_0/p_0)}{\log(p_1/a_1)} > 0.$$

We remark that s can take any positive value when p_0 varies in $(0, a_0)$. Let $\widetilde{C} = C \setminus \{\text{the endpoints of } C\}$. Since the Cantor set C has only countably many endpoints, removing the endpoints of C from S does not affect the size of S. In the following, we still use the notation S to denote the non-endpoints of C for which (2) does not exist either as a finite number or ∞. For $t \in \widetilde{C}$, let

$$N_n(t, 0) = \#\{1 \le l \le n : t_l = 0\} \quad \text{and} \quad N_n(t, 1) = \#\{1 \le l \le n : t_l = 1\}$$

where, as mentioned above, $\tilde{t} = t_1 t_2 \ldots \in \Sigma^{\mathbb{N}}$ is the code of t. We will first show that

$$\left\{ t \in \widetilde{C} : \liminf_{n \to \infty} \frac{N_n(t, 0)}{n} < \frac{1}{s+1} < \limsup_{n \to \infty} \frac{N_n(t, 0)}{n} \right\} \subseteq S$$

$$\subseteq \left\{ t \in \widetilde{C} : \liminf_{n \to \infty} \frac{N_n(t, 0)}{n} \le \frac{1}{s+1} \right\}.$$

By applying the results on divergence points in [11, 12] we obtain the following theorems.

THEOREM 1.1. *Let $p_0 < a_0$ and $p_1 > a_1$. Let s and ξ be defined as in* (5) *and* (1). *Let*

$$S_0 = \{t \in C : F'(t) = 0\} \quad and \quad S_\infty = \{t \in C : F'(t) = \infty\}.$$

Then $C = S \cup S_0 \cup S_\infty$, and

$$\dim_{\mathrm{H}} S_0 \ge \max_{p \in [(s+1)^{-1}, 1]} \frac{p \log p + (1-p) \log(1-p)}{p \log a_0 + (1-p) \log a_1},$$

$$\dim_{\mathrm{H}} S_\infty \ge \max_{p \in [0, (s+1)^{-1}]} \frac{p \log p + (1-p) \log(1-p)}{p \log a_0 + (1-p) \log a_1}.$$

In particular, $\dim_{\mathrm{H}} S_0 = \xi$ *if* $(s+1)^{-1} \le a_0^\xi$, *and* $\dim_{\mathrm{H}} S_\infty = \xi$ *if* $(s+1)^{-1} \ge a_0^\xi$.

THEOREM 1.2. *Let $p_0 < a_0$ and $p_1 > a_1$. Let s and ξ be defined as in* (5) *and* (1). *If* $1/(s+1) \le a_0^\xi$, *then*

$$\dim_{\mathrm{H}} S = \frac{s \log s - (s+1) \log(s+1)}{\log a_0 + s \log a_1}, \qquad \dim_{\mathrm{P}} S = \xi.$$

2. Proofs. In this section, we prove Theorems 1.1 and 1.2. We first characterize S by analyzing for each $t \in S$ the behavior of $N_n(t, 0)/n$, the frequency of the digit 0 occurring in the first n terms of the code of t. The desired results are then obtained by applying the results from [11, 12]. For

$t \in \widetilde{C}$, we denote by $z(t, n)$ the position of the nth occurrence of 0 in $\tilde{t}$ (recall we use $\tilde{t}$ to denote the code of t) and by $y(t, n)$ the position of the nth occurrence of 1 in $\tilde{t}$. Since $t \in \widetilde{C}$ (the non-endpoint of C), we have $n \le z(t, n), y(t, n) < \infty$ and $\lim_{n \to \infty} y(t, n) = \lim_{n \to \infty} z(t, n) = \infty$.

LEMMA 2.1. *For each* $t \in \widetilde{C}$ *we have*

$$\limsup_{n \to \infty} \frac{N_n(t, 0)}{n} = \limsup_{n \to \infty} \frac{n}{z(t, n)}, \qquad \liminf_{n \to \infty} \frac{N_n(t, 0)}{n} = \liminf_{n \to \infty} \frac{n}{z(t, n)},$$

$$\limsup_{n \to \infty} \frac{N_n(t, 1)}{n} = \limsup_{n \to \infty} \frac{n}{y(t, n)}, \qquad \liminf_{n \to \infty} \frac{N_n(t, 1)}{n} = \liminf_{n \to \infty} \frac{n}{y(t, n)}.$$

Proof. It is obvious that

$$(6) \qquad \limsup_{n \to \infty} \frac{N_n(t, 0)}{n} \ge \limsup_{n \to \infty} \frac{N_{z(t, n)}(t, 0)}{z(t, n)} = \limsup_{n \to \infty} \frac{n}{z(t, n)},$$

$$(7) \qquad \liminf_{n \to \infty} \frac{N_n(t, 0)}{n} \le \liminf_{n \to \infty} \frac{N_{z(t, n)}(t, 0)}{z(t, n)} = \liminf_{n \to \infty} \frac{n}{z(t, n)},$$

On the other hand, note that $z(t, N_n(t, 0)) \le n < z(t, N_n(t, 0) + 1)$ for all $n \in \mathbb{N}$. Thus, for all $n \in \mathbb{N}$,

$$\frac{N_n(t, 0)}{z(t, N_n(t, 0) + 1)} < \frac{N_n(t, 0)}{n} \le \frac{N_n(t, 0)}{z(t, N_n(t, 0))}.$$

Noting that $\lim_{n \to \infty} N_n(t, 0) = \infty$, this gives the opposite inequalities in (6) and (7). The other two equalities can be deduced in the same way. ∎

From Lemma 2.1 it follows that for each $t \in \widetilde{C}$,

$$(8) \qquad \limsup_{n \to \infty} \frac{z(t, n)}{n} > s + 1 \quad \text{if and only if} \quad \liminf_{n \to \infty} \frac{y(t, n)}{n} < \frac{s + 1}{s},$$

$$(9) \qquad \limsup_{n \to \infty} \frac{z(t, n)}{n} < s + 1 \quad \text{if and only if} \quad \liminf_{n \to \infty} \frac{y(t, n)}{n} > \frac{s + 1}{s}.$$

In fact, with the convention $0^{-1} = \infty$, $\infty^{-1} = 0$ and by Lemma 2.1 we have

$$\limsup_{n \to \infty} \frac{z(t, n)}{n} = \left(\liminf_{n \to \infty} \frac{n}{z(t, n)} \right)^{-1} = \left(\liminf_{n \to \infty} \frac{N_n(t, 0)}{n} \right)^{-1}$$

$$= \left(1 - \limsup_{n \to \infty} \frac{N_n(t, 1)}{n} \right)^{-1} = \left(1 - \limsup_{n \to \infty} \frac{n}{y(t, n)} \right)^{-1}$$

$$= \left(1 - \frac{1}{\liminf_{n \to \infty} y(t, n)/n} \right)^{-1},$$

which leads to (8) and (9).

LEMMA 2.2. *For each* $t \in \widetilde{C}$ *we have*

(I) $\qquad \liminf\limits_{n \to \infty} \dfrac{y(t,n)}{n} > \dfrac{s+1}{s} \;\Rightarrow\; \lim\limits_{x \to t} \dfrac{F(x) - F(t)}{x - t} = 0,$

(II) $\qquad \liminf\limits_{n \to \infty} \dfrac{y(t,n)}{n} < \dfrac{s+1}{s} \;\Rightarrow\; \limsup\limits_{x \to t} \dfrac{F(x) - F(t)}{x - t} = \infty,$

(III) $\qquad \limsup\limits_{n \to \infty} \dfrac{y(t,n)}{n} > \dfrac{s+1}{s} \;\Rightarrow\; \liminf\limits_{x \to t} \dfrac{F(x) - F(t)}{x - t} = 0.$

Proof. (I) If $\liminf_{n \to \infty} y(t,n)/n > (s+1)/s$, then there exist $\epsilon > 0$ and $k \in \mathbb{N}$ such that

(10) $\qquad\qquad y(t,n) > n\left(1 + \dfrac{1}{s}\right) + n\epsilon \quad \text{for } n \geq k.$

Let u be the distance between t and $[0,1] \backslash \bigcup_{\omega \in \{0,1\}^l} h_\omega([0,1])$ with $l = y(t,k)$. Then $u > 0$. Let x be a point in $(t - u, t)$. Then $t, x \in h_{\tilde{t}|(y(t,n)-1)}([0,1])$, but $x \notin h_{\tilde{t}|y(t,n)}([0,1])$ with some $n > k$. Therefore (recall that $\beta = 1 - a_0 - a_1$),

$$\frac{F(x) - F(t)}{x - t} \leq \frac{\prod_{i=1}^{y(t,n)-1} p_{t_i}}{\beta \prod_{i=1}^{y(t,n)-1} a_{t_i}} = \frac{a_1}{\beta p_1}\left(\frac{p_1}{a_1}\right)^n \left(\frac{p_0}{a_0}\right)^{y(t,n)-n} \leq \frac{a_1}{\beta p_1}\left(\frac{p_0}{a_0}\right)^{n\epsilon},$$

by (10). Hence $\lim_{x \uparrow t} (F(x) - F(t))/(x - t) = 0$.

On the other hand, by (9) there exist $\epsilon > 0$ and $k \in \mathbb{N}$ such that

(11) $\qquad\qquad z(t,n) < n(s+1) - n\epsilon \quad \text{for } n \geq k.$

Let u be the distance between t and $[0,1] \backslash \bigcup_{\omega \in \{0,1\}^l} h_\omega([0,1])$ with $l = z(t,k)$. Then $u > 0$. Let x be a point in $(t, t+u)$. Then $t, x \in h_{\tilde{t}|(z(t,n)-1)}([0,1])$, but $x \notin h_{\tilde{t}|z(t,n)}([0,1])$ for some $n > k$. Therefore,

$$\frac{F(x) - F(t)}{x - t} \leq \frac{\prod_{i=1}^{z(t,n)-1} p_{t_i}}{\beta \prod_{i=1}^{z(t,n)-1} a_{t_i}} = \frac{a_0}{\beta p_0}\left(\frac{p_0}{a_0}\right)^n \left(\frac{p_1}{a_1}\right)^{z(t,n)-n} \leq \frac{a_0}{\beta p_0}\left(\frac{p_1}{a_1}\right)^{-n\epsilon},$$

by (11). Hence $\lim_{x \downarrow t} (F(x) - F(t))/(x - t) = 0$.

(II) If $\liminf_{n \to \infty} y(t,n)/n < (s+1)/s$, then there exist $\epsilon > 0$ and a sequence $\{n_k\}_{k \geq 1}$ of integers with $n_k \uparrow \infty$ such that

(12) $\qquad\qquad y(t, n_k) < n_k\left(1 + \dfrac{1}{s}\right) - n_k\epsilon.$

Now take x_k as the left endpoint of the interval $h_{\tilde{t}|(y(t,n_k)-1)}([0,1])$. Then, by (12),

$$\frac{F(x_k) - F(t)}{x_k - t} \geq \frac{p_0 \prod_{i=1}^{y(t,n_k)-1} p_{t_i}}{\prod_{i=1}^{y(t,n_k)-1} a_{t_i}} = \frac{p_1^{n_k-1} p_0^{y(t,n_k)-n_k+1}}{a_1^{n_k-1} a_0^{y(t,n_k)-n_k}}$$

$$\geq a_1 p_0 p_1^{-1}\left(\frac{p_0}{a_0}\right)^{-n_k\epsilon},$$

which gives $\limsup_{\delta\uparrow 0} (F(t+\delta)-F(t))/\delta = \infty$. On the other hand, by (8) there exist $\epsilon > 0$ and a sequence $\{n_k\}_{k\geq 1}$ of integers with $n_k \uparrow \infty$ such that

$$z(t,n_k) > n_k(s+1) + n_k\epsilon.$$

By letting x_k be the right endpoint of the interval $h_{\tilde{t}|(z(t,n_k)-1)}([0,1])$ we have

$$\frac{F(x_k)-F(t)}{x_k-t} \geq \frac{p_1 \prod_{i=1}^{z(t,n_k)-1} p_{t_i}}{\prod_{i=1}^{z(t,n_k)-1} a_{t_i}} = \frac{p_0^{n_k-1} p_1^{z(t,n_k)-n_k+1}}{a_0^{n_k-1} a_1^{z(t,n_k)-n_k}}$$

$$\geq a_0 p_1 p_0^{-1} \left(\frac{p_1}{a_1}\right)^{n_k\epsilon},$$

yielding $\limsup_{\delta\downarrow 0} (F(t+\delta)-F(t))/\delta = \infty$.

(III) This can be proved in the same way as in (II). The details are left to the reader. ∎

From Lemma 2.2 it follows that

$$\left\{t \in \widetilde{C} : \liminf_{n\to\infty} \frac{y(t,n)}{n} < \frac{s+1}{s} < \limsup_{n\to\infty} \frac{y(t,n)}{n}\right\} \subseteq S$$

$$\subseteq \left\{t \in \widetilde{C} : \liminf_{n\to\infty} \frac{y(t,n)}{n} \leq \frac{s+1}{s}\right\},$$

and so

$$(13) \quad \left\{t \in \widetilde{C} : \liminf_{n\to\infty} \frac{N_n(t,0)}{n} < \frac{1}{s+1} < \limsup_{n\to\infty} \frac{N_n(t,0)}{n}\right\} \subseteq S$$

$$\subseteq \left\{t \in \widetilde{C} : \liminf_{n\to\infty} \frac{N_n(t,0)}{n} \leq \frac{1}{s+1}\right\},$$

by Lemma 2.1. For a sequence $\{x_n\}$ in a metric space X, we let $A(\{x_n\})$ denote the set of accumulation points of the sequence $\{x_n\}$, i.e.,

$$A(\{x_n\}) = \{x \in X : \text{there exists a subsequence } \{x_{n_k}\} \text{ such that } x_{n_k} \to x\}.$$

The following lemma is well known.

LEMMA 2.3. *If a sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty}(x_{n+1}-x_n) = 0$, then*

$$A(\{x_n\}) = [\liminf_{n\to\infty} x_n, \limsup_{n\to\infty} x_n].$$

Now we analyze the set of accumulation points of $\{N_n(t,0)/n\}_{n\geq 1}$ for each $t \in \widetilde{C}$. For a fixed $t \in \widetilde{C}$, note that either $N_{n+1}(t,0) = N_n(t,0)$ or $N_{n+1}(t,0) = N_n(t,0) + 1$. Hence, for any $n \in \mathbb{N}$, we have

$$-\frac{1}{n+1} \leq \frac{N_{n+1}(t,0)}{n+1} - \frac{N_n(t,0)}{n} \leq \frac{1}{n+1}.$$

From Lemma 2.3 it follows that

$$(14) \qquad A\left(\left\{\frac{N_n(t,0)}{n}\right\}\right) = \left[\liminf_{n\to\infty}\frac{N_n(t,0)}{n}, \limsup_{n\to\infty}\frac{N_n(t,0)}{n}\right].$$

The following lemma is a direct application of the results in [11, Theorem 2] and [12, Corollary 2.4] by Olsen.

LEMMA 2.4 (cf. [11, Theorem 2], [12, Corollary 2.4]). *For $s_i \in [0,1]$, $i = 1, 2, 3$, with $s_1 \leq s_2$ let*

$$A_{s_1,s_2} = \left\{t \in \widetilde{C} : \liminf_{n\to\infty}\frac{N_n(t,0)}{n} \leq s_1 \leq s_2 \leq \limsup_{n\to\infty}\frac{N_n(t,0)}{n}\right\},$$

$$B_{s_3} = \left\{t \in \widetilde{C} : \limsup_{n\to\infty}\frac{N_n(t,0)}{n} \leq s_3\right\}.$$

Then

$$\dim_{\mathrm{H}} A_{s_1,s_2} = \inf_{p\in[s_1,s_2]}\frac{p\log p + (1-p)\log(1-p)}{p\log a_0 + (1-p)\log a_1},$$

$$\dim_{\mathrm{P}} A_{s_1,s_2} = \xi,$$

$$\dim_{\mathrm{H}} B_{s_3} = \sup_{p\in[0,s_3]}\frac{p\log p + (1-p)\log(1-p)}{p\log a_0 + (1-p)\log a_1}.$$

Proof. Let $E_1 = \{(p, 1-p) : p \in [s_1, s_2]\}$, $F_1 = \{(p, 1-p) : p \in [0,1]\}$ and $F_2 = \{(p, 1-p) : p \in [0, s_3]\}$. Then, by (14) and Lemma 2.3, we have

$$A_{s_1,s_2} = \left\{t \in \widetilde{C} : E_1 \subset A\left(\left\{\frac{N_n(t,0)}{n}, \frac{N_n(t,1)}{n}\right\}\right) \subseteq F_1\right\} =: D(E_1, F_1),$$

and

$$B_{s_3} = \left\{t \in \widetilde{C} : A\left(\left\{\frac{N_n(t,0)}{n}, \frac{N_n(t,1)}{n}\right\}\right) \subseteq F_2\right\} =: D(\emptyset, F_2).$$

However, L. Olsen [11, Theorem 2(1), (2)] proved (in a more general setting) that

$$\dim_{\mathrm{H}} D(E_1, F_1) = \inf_{(p,1-p)\in E_1}\frac{p\log p + (1-p)\log(1-p)}{p\log a_0 + (1-p)\log a_1},$$

$$\dim_{\mathrm{H}} D(\emptyset, F_2) = \sup_{(p,1-p)\in F_2}\frac{p\log p + (1-p)\log(1-p)}{p\log a_0 + (1-p)\log a_1};$$

he also proved in [12, Corollary 2.4] (also in a more general setting) that

$$\dim_{\mathrm{P}} D(E_1, F_1) = \sup_{(p,1-p)\in F_1}\frac{p\log p + (1-p)\log(1-p)}{p\log a_0 + (1-p)\log a_1}.$$

Hence the desired results follow. ∎

Now we are ready to prove our theorems.

Proof of Theorem 1.1. STEP 1. First we prove that $C \setminus S = S_0 \cup S_\infty$. Suppose $t \in C \setminus S$. If t is an endpoint of C, then it is easy to check that

$$\liminf_{x \to t} \frac{F(x) - F(t)}{x - t} = 0, \quad \text{and so} \quad t \in S_0.$$

Note that for $x \in \widetilde{C}$ we have

$$x = (1 - a_1) \sum_{k=1}^{\infty} a_0^{y(x,k)-k} a_1^{k-1} \quad \text{and} \quad F(x) = \sum_{k=1}^{\infty} p_0^{y(x,k)-(k-1)} p_1^{k-1}.$$

Let $t \in \widetilde{C} \setminus S$. Take sequences $(x_n)_{n \geq 1}$ and $(x_n^*)_{n \geq 1}$ such that

$$y(x_n, k) = y(t, k) \quad \text{for } k < n, \quad y(x_n, k) = y(t, k) + 1 \quad \text{for } k \geq n,$$

$$y(x_n^*, k) = y(t, k) \quad \text{for } k < n, \quad y(x_n^*, k) = y(t, k) + 2 \quad \text{for } k \geq n.$$

Then $\lim_{n \to \infty} x_n = \lim_{n \to \infty} x_n^* = t$ and

$$c = \lim_{n \to \infty} \frac{F(x_n) - F(t)}{x_n - t} = \lim_{n \to \infty} \frac{a_1 p_0 (1 - p_0) \sum_{k=n}^{\infty} p_0^{y(t,k)-k} p_1^k}{p_1 (1 - a_1)(1 - a_0) \sum_{k=n}^{\infty} a_0^{y(t,k)-k} a_1^k}$$

$$= \lim_{n \to \infty} \frac{F(x_n^*) - F(t)}{x_n^* - t} = \lim_{n \to \infty} \frac{a_1 p_0 (1 - p_0^2) \sum_{k=n}^{\infty} p_0^{y(t,k)-k} p_1^k}{p_1 (1 - a_1)(1 - a_0^2) \sum_{k=n}^{\infty} a_0^{y(t,k)-k} a_1^k}$$

$$= \frac{1 + p_0}{1 + a_0} c,$$

implying $c \in \{0, \infty\}$, i.e., $t \in S_0 \cup S_\infty$.

STEP 2. For any $p \in ((s+1)^{-1}, 1]$, by Lemma 2.2(I) we have

$$\{x \in \widetilde{C} : F'(x) = 0\} \supseteq \left\{ t \in \widetilde{C} : \liminf_{n \to \infty} \frac{N_n(t, 0)}{n} > \frac{1}{s+1} \right\}$$

$$\supseteq \left\{ t \in \widetilde{C} : \lim_{n \to \infty} \frac{N_n(t, 0)}{n} = p \right\}$$

$$= \left\{ t \in \widetilde{C} : A \left(\left\{ \left(\frac{N_n(t, 0)}{n}, \frac{N_n(t, 1)}{n} \right) \right\} \right) = (p, 1 - p) \right\},$$

implying

$$\dim_H S_0 \geq \max_{p \in [(s+1)^{-1}, 1]} \frac{p \log p + (1 - p) \log(1 - p)}{p \log a_0 + (1 - p) \log a_1}.$$

STEP 3. We first show that

$$\text{(15)} \qquad S_\infty \supseteq \left\{ t \in \widetilde{C} : 1 < \lim_{n \to \infty} \frac{y(t, n)}{n} < \frac{s+1}{s} \right\} =: T.$$

It is easy to see (refer to the argument for (8))

$$\left\{ t \in \widetilde{C} : 1 < \lim_{n \to \infty} \frac{y(t,n)}{n} < \frac{s+1}{s} \right\} = \left\{ t \in \widetilde{C} : s+1 < \lim_{n \to \infty} \frac{z(t,n)}{n} < \infty \right\}.$$

Thus, for $t \in T$,

$$(16) \qquad \lim_{n \to \infty} \left(\frac{z(t,n+1)}{z(t,n)} - \frac{\log a_1}{\log p_1} + \left(\log \frac{p_0}{a_0} - \log \frac{p_1}{a_1} \right) \frac{n}{z(t,n) \log p_1} \right)$$

$$= \lim_{n \to \infty} \frac{\left(\frac{z(t,n+1)}{n} - s - 1 \right) \log p_1 + \left(\frac{z(t,n)}{n} - s - 1 \right) \log \frac{1}{a_1}}{\frac{z(t,n)}{n} \log p_1}$$

$$= \frac{\left(\lim_{n \to \infty} \frac{z(t,n)}{n} - s - 1 \right) \log \frac{p_1}{a_1}}{\log p_1 \lim_{n \to \infty} \frac{z(t,n)}{n}} < 0.$$

We claim that (16) implies that

$$(17) \qquad \lim_{x \downarrow t} \frac{F(x) - F(t)}{x - t} = \infty.$$

In fact, let k be a positive integer such that

$$(18) \qquad \frac{z(t,n+1)}{z(t,n)} - \frac{\log a_1}{\log p_1} + \left(\log \frac{p_0}{a_0} - \log \frac{p_1}{a_1} \right) \frac{n}{z(t,n) \log p_1} < q$$

for some negative real number q whenever $n \geq k$. Let u be the distance between t and $[0,1] \setminus \bigcup_{\omega \in \{0,1\}^l} h_\omega([0,1])$ with $l = z(t,k)$. Then $u > 0$. Let x be a point in the segment $(t, t+u)$. Then $t, x \in h_{\tilde{t} \mid (z(t,n)-1)}([0,1])$, but $x \notin h_{\tilde{t} \mid z(t,n)}([0,1])$ for some $n > k$. Therefore,

$$(19) \quad \frac{F(x) - F(t)}{x - t} \geq \frac{p_1 \prod_{i=1}^{z(t,n+1)-1} p_{t_i}}{\prod_{i=1}^{z(t,n)-1} a_{t_i}} = \frac{a_0 p_1 \prod_{i=1}^{z(t,n+1)} p_{t_i}}{p_0 \prod_{i=1}^{z(t,n)} a_{t_i}}$$

$$= \frac{a_0 p_0^{n+1} p_1^{z(t,n+1)-n}}{p_0 a_0^n a_1^{z(t,n)-n}}$$

$$= a_0 p_1^{z(t,n+1)-z(t,n)} \left(\frac{p_0}{a_0} \right)^n \left(\frac{p_1}{a_1} \right)^{z(t,n)-n}$$

$$= a_0 \left(p_1^{z(t,n+1)/z(t,n)-1} \left(\frac{p_0}{a_0} \right)^{n/z(t,n)} \left(\frac{p_1}{a_1} \right)^{1-n/z(t,n)} \right)^{z(t,n)}.$$

Let

$$Q = p_1^{z(t,n+1)/z(t,n)-1} \left(\frac{p_0}{a_0} \right)^{n/z(t,n)} \left(\frac{p_1}{a_1} \right)^{1-n/z(t,n)}$$

Taking logs, by (18) we have

(20)
$$\log Q = \left(\frac{z(t,n+1)}{z(t,n)} - \frac{\log a_1}{\log p_1} + \left(\log \frac{p_0}{a_0} - \log \frac{p_1}{a_1} \right) \frac{n}{z(t,n)\log p_1} \right) \log p_1$$
$$> q \log p_1 > 0.$$

Since t is a non-endpoint, $z(t,n) \to \infty$ as $n \to \infty$ and (17) holds by (19) and (20). On the other hand, for $t \in T$,

$$\lim_{n\to\infty} \left(\frac{y(t,n+1)}{y(t,n)} - \frac{\log a_0}{\log p_0} + \left(\log \frac{p_1}{a_1} - \log \frac{p_0}{a_0} \right) \frac{n}{y(t,n)\log p_0} \right)$$

$$= \lim_{n\to\infty} \frac{\left(\frac{y(t,n+1)}{n} - \frac{s+1}{s} \right) \log p_0 + \left(\frac{y(t,n)}{n} - \frac{s+1}{s} \right) \log \frac{1}{a_0}}{\frac{y(t,n)}{n} \log p_0}$$

$$= \frac{\left(\lim_{n\to\infty} \frac{y(t,n)}{n} - \frac{s+1}{s} \right) \log \frac{p_0}{a_0}}{\log p_0 \lim_{n\to\infty} \frac{y(t,n)}{n}} < 0.$$

By the same argument as above, this leads to

$$\lim_{x\uparrow t} \frac{F(x) - F(t)}{x - t} = \infty.$$

Finally, for each $p \in (0, (s+1)^{-1})$ it follows from (15) that

$$\dim_{\mathrm{H}} S_\infty \geq \dim_{\mathrm{H}} \left\{ t \in \widetilde{C} : \lim_{n\to\infty} \frac{N_n(t,0)}{n} = p \right\}$$

$$= \dim_{\mathrm{H}} \left\{ t \in \widetilde{C} : A\left(\left\{ \left(\frac{N_n(t,0)}{n}, \frac{N_n(t,1)}{n} \right) \right\} \right) = (p, 1-p) \right\}$$

$$= \frac{p \log p + (1-p) \log(1-p)}{p \log a_0 + (1-p) \log a_1},$$

implying the desired result. ∎

From the proof of Theorem 1.1 it follows that both S_0 and S_∞ are dense in C, and so $\dim_{\mathrm{B}} S_0 = \dim_{\mathrm{B}} S_\infty = \xi$.

Proof of Theorem 1.2. Let n be large enough that $\frac{1}{n} < \min\left\{ \frac{1}{s+1}, \frac{s}{s+1} \right\}$. For $k \geq n$ let

$$T_k = \left\{ t \in \widetilde{C} : \liminf_{n\to\infty} \frac{N_n(t,0)}{n} \leq \frac{1}{s+1} - \frac{1}{k} \leq \frac{1}{s+1} + \frac{1}{k} \leq \limsup_{n\to\infty} \frac{N_n(t,0)}{n} \right\},$$

$$L_k = \left\{ t \in \widetilde{C} : \liminf_{n\to\infty} \frac{N_n(t,0)}{n} \leq \frac{1}{s+1} \leq \frac{1}{s+1} + \frac{1}{k} \leq \limsup_{n\to\infty} \frac{N_n(t,0)}{n} \right\}.$$

Note that

$$
\left\{ t \in \widetilde{C} : \liminf_{n \to \infty} \frac{N_n(t,0)}{n} \leq \frac{1}{s+1} \right\} = \left\{ t \in \widetilde{C} : \limsup_{n \to \infty} \frac{N_n(t,0)}{n} \leq \frac{1}{s+1} \right\}
$$

$$
\cup \left\{ t \in \widetilde{C} : \liminf_{n \to \infty} \frac{N_n(t,0)}{n} \leq \frac{1}{s+1} < \limsup_{n \to \infty} \frac{N_n(t,0)}{n} \right\}
$$

$$
= \left\{ t \in \widetilde{C} : \limsup_{n \to \infty} \frac{N_n(t,0)}{n} \leq \frac{1}{s+1} \right\} \cup \bigcup_{k \geq n} L_k.
$$

Thus, by (13),

$$
\bigcup_{k \geq n} T_k \subseteq S \subseteq \left\{ t \in \widetilde{C} : \limsup_{n \to \infty} \frac{N_n(t,0)}{n} \leq \frac{1}{s+1} \right\} \cup \bigcup_{k \geq n} L_k.
$$

From Lemma 2.4 it follows that

$$
\sup_{k \geq n} \dim_{\mathrm{H}} T_k = \sup_{k \geq n} \dim_{\mathrm{H}} L_k = \frac{s \log s - (s+1) \log(s+1)}{\log a_0 + s \log a_1}, \quad \dim_{\mathrm{P}} T_k = \xi,
$$

and

$$
\dim_{\mathrm{H}} \left\{ t \in \widetilde{C} : \limsup_{n \to \infty} \frac{N_n(t,0)}{n} \leq \frac{1}{s+1} \right\} = \frac{s \log s - (s+1) \log(s+1)}{\log a_0 + s \log a_1},
$$

since $1/(s+1) \leq a_0^\xi$. ∎

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