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The Campanato, Morrey and Hölder spaces on spaces of homogeneous type

by

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Dedicated to Professor Sin-Ei Takahasi on his sixtieth birthday

Abstract. We investigate the relations between the Campanato, Morrey and Hölder spaces on spaces of homogeneous type and extend the results of Campanato, Mayers, and Macías and Segovia. The results are new even for the $\mathbb{R}^n$ case. Let (X, d, μ) be a space of homogeneous type and (X, δ, μ) its normalized space in the sense of Macías and Segovia. We also study the relations of these function spaces for (X, d, μ) and for (X, δ, μ). Using these relations, we can show that theorems for the Campanato, Morrey or Hölder spaces on the normal space are valid for the function spaces on any space of homogeneous type. As an application we obtain boundedness of some operators related to partial differential equations, boundedness of fractional differential and integral operators, and give characterizations of pointwise multipliers.

1. Introduction. The theory of function spaces on spaces of homogeneous type has been developed by many authors: Coifman and Weiss [4, 5], Macías and Segovia [14], Han and Sawyer [9], etc. In this paper we investigate the relations between the Campanato, Morrey and Hölder spaces on spaces of homogeneous type and extend the results of Campanato [2, 3], Mayers [15] and Macías and Segovia [14]. We also show the relations between these function spaces for (X, d, μ) and for (X, δ, μ), its normalized space in the sense of Macías and Segovia [14].

Let $X = (X, d, \mu)$ be a *space of homogeneous type*, i.e. X is a topological space endowed with a quasi-distance d and a nonnegative measure μ such that

$$d(x, y) \geq 0 \quad \text{and} \quad d(x, y) = 0 \text{ if and only if } x = y,$$

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$$d(x,y) = d(y,x),$$
$$d(x,y) \le K_1(d(x,z) + d(z,y)), \tag{1.1}$$

the balls (d-balls) $B(x,r) = B^d(x,r) = \{y \in X : d(x,y) < r\}$, $r > 0$, form a basis of neighborhoods of the point x, μ is defined on a σ-algebra of subsets of X which contains the balls, and

$$0 < \mu(B(x,2r)) \le K_2\,\mu(B(x,r)) < \infty, \tag{1.2}$$

where $K_i \ge 1$ $(i = 1, 2)$ are constants independent of $x, y, z \in X$ and $r > 0$.

We note that every open subset of X is expressible as a countable union of balls (see [4, p. 70]), and so it is measurable.

Let $1 \le p < \infty$ and $\phi : X \times \mathbb{R}_+ \to \mathbb{R}_+$. For a ball $B = B(x,r)$, we shall write $\phi(B)$ in place of $\phi(x,r)$. For a function $f \in L^1_{\mathrm{loc}}(X)$ and for a ball B, let $f_B = \mu(B)^{-1} \int_B f(x)\,d\mu(x)$. Then the *Campanato spaces* $\mathcal{L}_{p,\phi}(X)$, *Morrey spaces* $L_{p,\phi}(X)$ and *Hölder spaces* $\Lambda_\phi(X)$ are defined to be the sets of all f such that $\|f\|_{\mathcal{L}_{p,\phi}} < \infty$, $\|f\|_{L_{p,\phi}} < \infty$ and $\|f\|_{\Lambda_\phi} < \infty$, respectively, where

$$\|f\|_{\mathcal{L}_{p,\phi}} = \sup_B \frac{1}{\phi(B)} \left(\frac{1}{\mu(B)} \int_B |f(x) - f_B|^p \, d\mu(x) \right)^{1/p},$$

$$\|f\|_{L_{p,\phi}} = \sup_B \frac{1}{\phi(B)} \left(\frac{1}{\mu(B)} \int_B |f(x)|^p \, d\mu(x) \right)^{1/p},$$

$$\|f\|_{\Lambda_\phi} = \sup_{x,y\in X,\ x\ne y} \frac{2|f(x) - f(y)|}{\phi(x, d(x,y)) + \phi(y, d(y,x))}.$$

For $\phi(x,r) = r^\alpha$ $(\alpha > 0)$, we shall write $\mathrm{Lip}_\alpha(X)$ in place of $\Lambda_{r^\alpha}(X)$. If $p = 1$, then $\mathcal{L}_{1,\phi}(X) = \mathrm{BMO}_\phi(X)$. If $\phi \equiv 1$, then $\mathcal{L}_{1,\phi}(X) = \mathrm{BMO}(X)$ and $\Lambda_\phi(X) = L^\infty(X)$. If $\phi(B) = \mu(B)^{-1/p}$, then $L_{p,\phi}(X) = L^p(X)$.

If $X = \mathbb{R}^n$, $d(x,y) = |x - y|$, μ is Lebesgue measure and $\phi(x,r) = r^\alpha$, then the following are known (Campanato [2, 3], Mayers [15] and Peetre [22]):

$$\begin{aligned}
-n/p \le \alpha < 0 \ &\Rightarrow \mathcal{L}_{p,\phi}(\mathbb{R}^n)/\mathcal{C} = L_{p,\phi}(\mathbb{R}^n) \ (= L^p(\mathbb{R}^n) \text{ if } \alpha = -n/p),\\
\alpha = 0 \ &\Rightarrow \mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n) \supset L_{p,\phi}(\mathbb{R}^n) = \Lambda_\phi(\mathbb{R}^n) = L^\infty(\mathbb{R}^n),\\
0 < \alpha \le 1 \ &\Rightarrow \mathcal{L}_{p,\phi}(\mathbb{R}^n) = \Lambda_\phi(\mathbb{R}^n) = \mathrm{Lip}_\alpha(\mathbb{R}^n),
\end{aligned}$$

where $\mathcal{C}$ is the space of all constant functions (see also Spanne [23], Janson [10] and Nakai [18]). In this paper we give necessary and sufficient conditions on $\phi : X \times \mathbb{R}_+ \to \mathbb{R}_+$ for the relations

$$\mathcal{L}_{p,\phi}(X)/\mathcal{C} = L_{p,\phi}(X), \quad \mathcal{L}_{p,\phi}(X) = L_{p,\phi}(X), \quad \mathcal{L}_{p,\phi}(X) = \Lambda_\phi(X)$$

to hold.

If there are constants θ $(0 < \theta \le 1)$ and $K_3 \ge 1$ such that

$$(1.3)\quad |d(x,z) - d(y,z)| \le K_3\,(d(x,z) + d(y,z))^{1-\theta} d(x,y)^{\theta}, \qquad x,y,z \in X,$$

then the balls are open sets. Note that (1.1) for some $K_1 \ge 1$ follows from (1.3) (Lemarié [11]). Conversely, from (1.1) it follows that there exist $\theta > 0$, $K_3 \ge 1$ and a quasi-distance which is equivalent to the original d such that (1.3) holds (Macías and Segovia [14]).

Following Macías and Segovia [14], we shall say that a space of homogeneous type is *normal* if there are constants $K_4 > 0$ and $K_5 > 0$ such that

$$(1.4)\qquad K_4 r \le \mu(B(x,r)) \le K_5 r \quad \text{for } x \in X \text{ and } \mu(\{x\}) < r < \mu(X).$$

For a space (X,d,μ) of homogeneous type such that the balls are open sets, let

$$(1.5)\quad \delta(x,y) = \begin{cases} \inf\{\mu(B^d) : B^d \text{ is a } d\text{-ball containing } x \text{ and } y\} & \text{if } x \ne y, \\ 0 & \text{if } x = y. \end{cases}$$

Macías and Segovia [14] showed that (X,δ,μ) is a normal space of homogeneous type, that the topologies induced on X by d and δ coincide, and that

$$(1.6)\qquad \mathcal{L}_{p,\phi}(X,d,\mu) = \mathrm{Lip}_{\alpha}(X,\delta,\mu) \quad \text{for } \phi(x,r) = \mu(B^d(x,r))^{\alpha}.$$

In this paper we also study the relations of our function spaces for (X,d,μ) and (X,δ,μ). Our results include the equality (1.6) as a special case. Moreover, using these relations, we can show that theorems for the Campanato, Morrey or Hölder spaces on the normal space are valid for the function spaces on any space of homogeneous type. Conversely, results for these function spaces on any space of homogeneous type, for example spaces associated to vector fields (see [17]), adapt to the normal space. As an application we obtain boundedness of some operators related to partial differential equations, boundedness of fractional differential and integral operators, and characterizations of pointwise multipliers.

We regard $\mathcal{L}_{p,\phi}(X)$ and $L_{p,\phi}(X)$ as spaces of functions modulo null-functions, and $\Lambda_{\phi}(X)$ as a space of functions defined at all $x \in X$. Then $\mathcal{L}_{p,\phi}(X)/\mathcal{C}$, $L_{p,\phi}(X)$ and $\Lambda_{\phi}(X)/\mathcal{C}$ are Banach spaces with the norms $\|f\|_{\mathcal{L}_{p,\phi}}$, $\|f\|_{L_{p,\phi}}$ and $\|f\|_{\Lambda_{\phi}}$, respectively. For any fixed ball B_0 and for any fixed point x_0, $\|f\|_{\mathcal{L}_{p,\phi}} + |f_{B_0}|$ and $\|f\|_{\Lambda_{\phi}} + |f(x_0)|$ are norms on $\mathcal{L}_{p,\phi}(X)$ and $\Lambda_{\phi}(X)$, respectively. Thereby $\mathcal{L}_{p,\phi}(X)$ and $\Lambda_{\phi}(X)$ are Banach spaces. If $1 \in L_{p,\phi}(X)$, then $L_{p,\phi}(X)/\mathcal{C}$ is a Banach space with the norm $\|f - f_{B_0}\|_{L_{p,\phi}}$.

We note that for each ball B_1 and for each point $x_1 \in X$, $\|f\|_{\mathcal{L}_{p,\phi}} + |f_{B_0}| \sim \|f\|_{\mathcal{L}_{p,\phi}} + |f_{B_1}|$ and $\|f\|_{\Lambda_{\phi}} + |f(x_0)| \sim \|f\|_{\Lambda_{\phi}} + |f(x_1)|$. If $1 \in L_{p,\phi}(X)$, then $\|f - f_{B_0}\|_{L_{p,\phi}} \sim \|f - f_{B_1}\|_{L_{p,\phi}}$. If $\mu(X) < \infty$, then $\|f\|_{\mathcal{L}_{p,\phi}} + |f_{B_0}| \sim \|f\|_{\mathcal{L}_{p,\phi}} + \|f\|_{L^p}$. (See (3.3) and (2.7).)

We state our main results in the next section and prove them in the third section. We give applications in Section 4.

The letter C shall always denote a constant, not necessarily the same one.

2. Main results. Let (X, d, μ) be a space of homogeneous type satisfying (1.3), x_0 a fixed point in X, and $B_0 = B(x_0, 1)$.

We shall consider the following conditions on ϕ:

$$\frac{1}{A_1} \le \frac{\phi(a,s)}{\phi(a,r)} \le A_1, \qquad 1/2 \le s/r \le 2, \tag{2.1}$$

$$\frac{\phi(a,r)}{r^\theta} \le A_2 \frac{\phi(a,s)}{s^\theta}, \qquad 0 < s < r, \tag{2.2}$$

$$\int_0^r \mu(B(a,t))^{1/p} \frac{\phi(a,t)}{t}\, dt \le A_3\, \mu(B(a,r))^{1/p} \phi(a,r), \qquad r > 0, \tag{2.3}$$

$$\frac{1}{A_4} \le \frac{\phi(a,r)}{\phi(b,r)} \le A_4, \qquad d(a,b) \le r, \tag{2.4}$$

where $A_i > 0$ $(i = 1,2,3,4)$ are independent of $r, s > 0,\ a, b \in X$.

Let $r_0 \ge 0$. The following are equivalent (see [21, Lemma 5.2]):

$$B(x_0, K_6 r) \setminus B(x_0, r) \ne \emptyset, \qquad r > r_0, \qquad \text{for some } K_6 > 1, \tag{2.5}$$

$$\mu(B(x_0,r)) \le \tfrac{1}{2}\mu(B(x_0, K_6' r)), \qquad r > r_0, \qquad \text{for some } K_6' > 1, \tag{2.6}$$

where K_6 and K_6' are independent of $r > r_0$. We shall consider these conditions if $\mu(X) = \infty$.

Our first result is the following.

THEOREM 2.1. *Let $\mu(X) = \infty$ and X satisfy* (2.5) *for some $r_0 \ge 0$. If $1 \le p < \infty$ and ϕ satisfies* (2.1)–(2.4), *then the following are equivalent*:

(i) *There is a constant $C > 0$ such that*

$$\int_r^\infty \frac{\phi(a,t)}{t}\, dt \le C\phi(a,r), \qquad a \in X,\ r > 0.$$

(ii) *$\mathcal{L}_{p,\phi}(X)/\mathcal{C} = L_{p,\phi}(X)$ and $\|f\|_{\mathcal{L}_{p,\phi}} \sim \|f - \lim_{r\to\infty} f_{B(x_0,r)}\|_{L_{p,\phi}}$, i.e. for every $f \in \mathcal{L}_{p,\phi}(X)$, $f_{B(x_0,r)}$ converges as r tends to infinity, and the mapping $f \mapsto f - \lim_{r\to\infty} f_{B(x_0,r)}$ is bijective and bicontinuous from $\mathcal{L}_{p,\phi}(X)/\mathcal{C}$ to $L_{p,\phi}(X)$.*

In this case, $\lim_{r\to\infty} f_{B(a,r)} = \lim_{r\to\infty} f_{B(x_0,r)}$ *for all $a \in X$.*

REMARK 2.1. (i)⇒(ii) can be proved without (1.3), (2.1)–(2.5). In the case that $X = \mathbb{R}^n$, $d(x,y) = |x - y|$ and μ is Lebesgue measure, (i)⇒(ii) has been proved by Mizuhara [16].

For $\phi : X \times \mathbb{R}_+ \to \mathbb{R}_+$, we define

$$\Phi^*(a,r) = \int_1^{\max(2,d(x_0,a),r)} \frac{\phi(x_0,t)}{t}\,dt, \quad \Phi^{**}(a,r) = \int_r^{\max(2,d(x_0,a),r)} \frac{\phi(a,t)}{t}\,dt.$$

Then we have the following.

THEOREM 2.2. *If* $\mu(X) = \infty$, *then assume* (2.5) *for some* $r_0 \geq 0$. *If* $1 \leq p < \infty$ *and* ϕ *satisfies* (2.1)–(2.4), *then the following are equivalent*:

(i) *There is a constant* $C > 0$ *such that*

$$\Phi^*(a,r) + \Phi^{**}(a,r) \leq C\phi(a,r), \quad a \in X, r > 0.$$

(ii) $\mathcal{L}_{p,\phi}(X) = L_{p,\phi}(X)$ *and* $\|f\|_{\mathcal{L}_{p,\phi}} + |f_{B_0}| \sim \|f\|_{L_{p,\phi}}$.

(iii) $1 \in L_{p,\phi}(X)$, $\mathcal{L}_{p,\phi}(X)/\mathcal{C} = L_{p,\phi}(X)/\mathcal{C}$ *and* $\|f\|_{\mathcal{L}_{p,\phi}} \sim \|f - f_{B_0}\|_{L_{p,\phi}}$.

REMARK 2.2. (i)⇒(ii) can be proved without (1.3), (2.2)–(2.5). (ii)⇔(iii) can be proved without (1.3), (2.1)–(2.5).

REMARK 2.3. Let $\mu(B(a,r)) \sim r^\beta$ $(\beta > 0)$ and $\phi(a,r) = r^{-\alpha}\max(2, d(x_0,a),r)$, for $1 - \theta < \alpha < \min(1,\beta/p)$ and $1 \leq p < \infty$. Then ϕ satisfies (2.1)–(2.4) and (i).

If $\mu(X) < \infty$, then there is a constant $R_0 > 0$ such that

$$X = B(x,R_0) \quad \text{for all } x \in X \tag{2.7}$$

(see [21, Lemma 5.1]). For ϕ, we define

$$\Phi(a,r) = \int_r^{2R_0} \frac{\phi(a,t)}{t}\,dt, \qquad 0 < r \leq R_0.$$

Then we have the following.

COROLLARY 2.3. *Let* $\mu(X) < \infty$. *If* $1 \leq p < \infty$ *and* ϕ *satisfies* (2.1)–(2.4), *then the following are equivalent*:

(i) *There is a constant* $C > 0$ *such that*

$$\Phi(a,r) \leq C\phi(a,r), \quad a \in X, 0 < r \leq R_0.$$

(ii) $\mathcal{L}_{p,\phi}(X) = L_{p,\phi}(X)$ *and* $\|f\|_{\mathcal{L}_{p,\phi}} + |f_{B_0}| \sim \|f\|_{L_{p,\phi}}$.

(iii) $1 \in L_{p,\phi}(X)$, $\mathcal{L}_{p,\phi}(X)/\mathcal{C} = L_{p,\phi}(X)/\mathcal{C}$ *and* $\|f\|_{\mathcal{L}_{p,\phi}} \sim \|f - f_{B_0}\|_{L_{p,\phi}}$.

REMARK 2.4. On the assumption that $\Phi \geq C > 0$, (i)⇒(ii) can be proved without (1.3), (2.2)–(2.4). (ii)⇔(iii) can be proved without (1.3), (2.1)–(2.4).

To consider the Hölder spaces $\Lambda_\phi(X)$, we assume that there is a constant $A_5 > 0$ such that

$$\phi(a,r) \leq A_5\phi(b,s) \quad \text{for } B(a,r) \subset B(b,s). \tag{2.8}$$

THEOREM 2.4. *If* $1 \le p < \infty$ *and* ϕ *satisfies* (2.1)–(2.4) *and* (2.8), *then the following are equivalent*:

(i) *There is a constant* $C > 0$ *such that*

$$\int_0^{d(x,y)} \frac{\phi(x,t)}{t}\,dt \le C\phi(x, d(x,y)), \qquad x, y \in X.$$

(ii) $\mathcal{L}_{p,\phi}(X) = \Lambda_\phi(X)$ *and* $\|f\|_{\mathcal{L}_{p,\phi}} + |f_{B_0}| \sim \|f\|_{\Lambda_\phi} + |f(x_0)|$.

(iii) $\mathcal{L}_{p,\phi}(X)/\mathcal{C} = \Lambda_\phi(X)/\mathcal{C}$ *and* $\|f\|_{\mathcal{L}_{p,\phi}} \sim \|f\|_{\Lambda_\phi}$.

REMARK 2.5. (i)⇒(ii) can be proved without (1.3), (2.2)–(2.4). On the assumption that $\int_0^1 \phi(x_0,t)/t\,dt < \infty$, (ii)⇔(iii) can be proved without (1.3), (2.2)–(2.4).

Let δ be defined by (1.5). The relations of the function spaces for (X, d, μ) and (X, δ, μ) are as follows.

THEOREM 2.5. *Suppose that* $\psi : X \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* (2.1). *Let* $\phi(x,r) = \psi(x, \mu(B^d(x,r)))$ *and* $\psi_*(x,r) = \psi(x, \max(r, \mu(\{x\})))$. *Then*

$$\mathcal{L}_{p,\phi}(X, d, \mu) = \mathcal{L}_{p,\psi}(X, \delta, \mu),$$
$$\mathcal{L}_{p,\phi}(X, d, \mu)/\mathcal{C} = \mathcal{L}_{p,\psi}(X, \delta, \mu)/\mathcal{C},$$
$$L_{p,\phi}(X, d, \mu) = L_{p,\psi_*}(X, \delta, \mu),$$

with equivalent norms. If ψ *satisfies* (2.8) *also*, *then*

$$\Lambda_\phi(X, d, \mu) = \Lambda_\psi(X, \delta, \mu),$$
$$\Lambda_\phi(X, d, \mu)/\mathcal{C} = \Lambda_\psi(X, \delta, \mu)/\mathcal{C},$$

with equivalent norms.

Let $\psi(x,r) = r^\alpha$ $(\alpha > 0)$ in this theorem. Then ψ satisfies (2.1), (2.8) and (i) in Theorem 2.4. From Remark 2.5 it follows that $\mathcal{L}_{p,\psi}(X, \delta, \mu) = \mathrm{Lip}_\alpha(X, \delta, \mu)$. Therefore we have the following.

COROLLARY 2.6 (Macías and Segovia [14]). *Let* $\phi(x,r) = \mu(B^d(x,r))^\alpha$ *with* $\alpha > 0$. *Then*

$$\mathcal{L}_{p,\phi}(X, d, \mu) = \mathrm{Lip}_\alpha(X, \delta, \mu),$$
$$\mathcal{L}_{p,\phi}(X, d, \mu)/\mathcal{C} = \mathrm{Lip}_\alpha(X, \delta, \mu)/\mathcal{C},$$

with equivalent norms.

Let $M_{p,\lambda}(X)$ be the set of all f such that $\|f\|_{M_{p,\lambda}} < \infty$, where

$$\|f\|_{M_{p,\lambda}} = \sup_{B(a,r)} \left(\frac{1}{r^\lambda} \int_{B(a,r)} |f(x)|^p\,d\mu(x)\right)^{1/p}.$$

Let $\psi(x,r) = r^\alpha$ ($\alpha < 0$) in Theorem 2.5. Then ψ satisfies (2.1) and (i) in Theorem 2.1 and (i) in Corollary 2.3. If $\mu(\{x\}) = 0$ for all $x \in X$, then $\mu(B^\delta(x,r)) \sim r$ and $\psi_* = \psi$. Therefore we have the following.

COROLLARY 2.7. *Let $\mu(\{x\}) = 0$ for all $x \in X$, and let $\phi(x,r) = \mu(B^d(x,r))^\alpha$, $-1/p \le \alpha < 0$, $\lambda = 1 + p\alpha$. Then*

$$\mathcal{L}_{p,\phi}(X,d,\mu)/\mathcal{C} = M_{p,\lambda}(X,\delta,\mu) \qquad \textit{if } \mu(X) = \infty,$$

$$\begin{cases} \mathcal{L}_{p,\phi}(X,d,\mu) = M_{p,\lambda}(X,\delta,\mu) \\ \mathcal{L}_{p,\phi}(X,d,\mu)/\mathcal{C} = M_{p,\lambda}(X,\delta,\mu)/\mathcal{C} \end{cases} \qquad \textit{if } \mu(X) < \infty,$$

with equivalent norms.

3. Proofs. Let $1 \le p < \infty$ and

$$\mathrm{MO}_p(f,B) = \left(\frac{1}{\mu(B)} \int_B |f(x) - f_B|^p \, d\mu(x)\right)^{1/p}.$$

First, we state some simple inequalities (see for example [21]):

$$\left(\int_B |f(x) - f_B|^p \, d\mu(x)\right)^{1/p} \le 2 \inf_c \left(\int_B |f(x) - c|^p \, d\mu(x)\right)^{1/p}, \tag{3.1}$$

$$|F(z_1) - F(z_2)| \le C|z_1 - z_2| \;\Rightarrow\; \mathrm{MO}_p(F(f),B) \le 2C\,\mathrm{MO}_p(f,B), \tag{3.2}$$

$$|f_{B_1} - f_{B_2}| \le \frac{\mu(B_2)}{\mu(B_1)} \mathrm{MO}_p(f,B_2) \quad \text{for } B_1 \subset B_2, \tag{3.3}$$

$$\begin{aligned} &|f_{B(a,r)} - f_{B(a,s)}| \\ &\qquad \le 2K_2^2 (\log 2)^{-1} \int_r^{2s} \frac{\mathrm{MO}_p(f,B(a,t))}{t} \, dt \quad \text{for } 0 < r < s. \end{aligned} \tag{3.4}$$

If ϕ satisfies (2.1), then

$$\int_r^{2s} \frac{\phi(a,t)}{t} \, dt \le (1 + A_1) \int_r^s \frac{\phi(a,t)}{t} \, dt \quad \text{for } 0 < 2r \le s. \tag{3.5}$$

From (3.4) it follows that, if $f \in \mathcal{L}_{p,\phi}(X)$, then

$$|f_{B(a,r)} - f_{B(a,s)}| \le 2K_2^2 (\log 2)^{-1} \int_r^{2s} \frac{\phi(a,t)}{t} \, dt \, \|f\|_{\mathcal{L}_{p,\phi}} \quad \text{for } 0 < r < s. \tag{3.6}$$

3.1. *Proof of Theorem 2.1.* We state two lemmas.

LEMMA 3.1 ([21]). *If $1 \le p < \infty$ and ϕ satisfies (2.1)–(2.4), then*

$$f_a(x) = \int_{d(a,x)}^1 \frac{\phi(a,t)}{t} \, dt \tag{3.7}$$

is in $\mathcal{L}_{p,\phi}(X)$ for all $a \in X$, and there is a constant $C > 0$, independent of a, such that $\|f_a\|_{\mathcal{L}_{p,\phi}} \le C$.

LEMMA 3.2. *If $\int_1^\infty \phi(x_0,t)/t\,dt<\infty$, then for every $f\in\mathcal{L}_{p,\phi}(X)$ there exists a constant $\sigma(f)$ such that $\sigma(f)=\lim_{r\to 0} f_{B(a,r)}$ for all $a\in X$.*

Proof. Let $f\in\mathcal{L}_{p,\phi}(X)$. As r and s tend to infinity with $r<s$, the right-hand side of (3.6), with $a=x_0$, tends to zero. Thus

$$|f_{B(x_0,r)}-f_{B(x_0,s)}|\to 0 \quad \text{as } r,s\to\infty \text{ with } r<s.$$

Hence $f_{B(x_0,r)}$ converges as r tends to infinity. Let $\sigma(f)=\lim_{r\to\infty} f_{B(x_0,r)}$. If $d(x_0,a)\le r$, then $B(a,r)\subset B(x_0,2K_1r)$ and $\mu(B(a,r))\sim\mu(B(x_0,2K_1r))$. From (3.3) it follows that

$$\begin{aligned}(3.8)\qquad |f_{B(a,r)}-\sigma(f)| &\le |f_{B(a,r)}-f_{B(x_0,2K_1r)}|+|f_{B(x_0,2K_1r)}-\sigma(f)|\\ &\le \frac{\mu(B(x_0,2K_1r))}{\mu(B(a,r))}\|f\|_{\mathcal{L}_{p,\phi}}\phi(x_0,2K_1r)+|f_{B(x_0,2K_1r)}-\sigma(f)|\\ &\to 0 \quad \text{as } r\to\infty. \ \blacksquare\end{aligned}$$

Proof of Theorem 2.1. (i)⇒(ii): Let $f\in\mathcal{L}_{p,\phi}(X)$. Letting $s\to\infty$ in (3.6) and using Lemma 3.2, we have

$$|f_{B(a,r)}-\sigma(f)|\le C_0\int_r^\infty \frac{\phi(a,t)}{t}\,dt\,\|f\|_{\mathcal{L}_{p,\phi}}, \quad a\in X,\ r>0,$$

where $C_0=2K_2^2(\log 2)^{-1}$. Hence, by (i), we have

$$\begin{aligned}&\Big(\frac{1}{\mu(B(a,r))}\int_{B(a,r)}|f(x)-\sigma(f)|^p\,d\mu(x)\Big)^{1/p}\\ &\quad\le \Big(\frac{1}{\mu(B(a,r))}\int_{B(a,r)}|f(x)-f_{B(a,r)}|^p\,d\mu(x)\Big)^{1/p}+|f_{B(a,r)}-\sigma(f)|\\ &\quad\le (1+C_0C)\phi(a,r)\|f\|_{\mathcal{L}_{p,\phi}}, \quad a\in X,\ r>0.\end{aligned}$$

This shows that $f-\sigma(f)$ is in $L_{p,\phi}(X)$ and that

$$\|f-\sigma(f)\|_{L_{p,\phi}}\le(1+C_0C)\|f\|_{\mathcal{L}_{p,\phi}}.$$

From (3.1) it follows that

$$\|f\|_{\mathcal{L}_{p,\phi}}\le 2\|f-\sigma(f)\|_{L_{p,\phi}}.$$

Conversely, let $f\in L_{p,\phi}(X)$. Then f is in $\mathcal{L}_{p,\phi}(X)$ and $\sigma(f)=0$, since $|f_{B(x_0,r)}|\le\phi(x_0,r)\|f\|_{L_{p,\phi}}\to 0$ as $r\to\infty$. Therefore we have (ii).

(ii)⇒(i): For $f\in\mathcal{L}_{p,\phi}(X)$, let $\sigma(f)=\lim_{r\to\infty} f_{B(x_0,r)}$. First we show that $\phi(x_0,t)\to 0$ $(t\to\infty)$. Then, by (3.8), we have $\sigma(f)=\lim_{r\to\infty} f_{B(a,r)}$ for all $a\in X$. Let

$$g(x)=\int_1^{\max(1,d(x_0,x))}\frac{\phi(x_0,t)}{t}\,dt=\max(0,-f_{x_0}(x)),$$

where f_{x_0} is defined by (3.7). By Lemma 3.1 and (3.2), g is in $\mathcal{L}_{p,\phi}(X)$. From (2.6) it follows that there is a constant $C > 0$ such that

$$\mu(B(x_0, K_6' r)) \le 2\mu(B(x_0, K_6' r) \setminus B(x_0, r)), \quad r > r_0.$$

Since $g_{B(x_0,r)}$ is increasing with respect to r, we have

$$\sigma(g) \ge g_{B(x_0,K_6' r)} \ge \frac{1}{\mu(B(x_0, K_6' r))} \int_{B(x_0,K_6' r)\setminus B(x_0,r)} g(x)\, d\mu(x)$$
$$\ge \frac{1}{2}\int_1^r \frac{\phi(x_0,t)}{t}\, dt \quad \text{for all } r > r_0.$$

Hence $\phi(x_0,t) \to 0$ $(t \to \infty)$. Now, for each ball $B(a,r)$, let

$$h(x) = \int_r^{\max(r,d(a,x))} \frac{\phi(a,t)}{t}\, dt = \max\left(0, -\left(f_a(x) + \int_1^r \frac{\phi(a,t)}{t}\, dt\right)\right),$$

where f_a is defined by (3.7). By Lemma 3.1 and (3.2), h is in $\mathcal{L}_{p,\phi}(X)$ and there is a constant $C > 0$, independent of $B(a,r)$, such that $\|h\|_{\mathcal{L}_{p,\phi}} \le C$. From (2.6) it follows that there are constants $r_a \ge 0$ and $K_a > 1$ such that

$$\mu(B(a, K_a s)) \le 2\mu(B(a, K_a s) \setminus B(a, s)), \quad s > r_a.$$

Since $h_{B(a,s)}$ is increasing with respect to s, we have

$$\sigma(h) \ge h_{B(a,K_a s)} \ge \frac{1}{\mu(B(a, K_a s))} \int_{B(a,K_a s)\setminus B(a,s)} h(x)\, d\mu(x) \tag{3.9}$$
$$\ge \frac{1}{2}\int_r^s \frac{\phi(a,t)}{t}\, dt \quad \text{for all } s > \max(r, r_a).$$

Since $h(x) = 0$ on $B(a,r)$, we have

$$\sigma(h) = \frac{1}{\mu(B(a,r))} \int_{B(a,r)} |h(x) - \sigma(h)|\, d\mu(x) \tag{3.10}$$
$$\le \|h - \sigma(h)\|_{L_{p,\phi}} \phi(a,r) \sim \|h\|_{\mathcal{L}_{p,\phi}} \phi(a,r) \le C\phi(a,r).$$

By (3.9) and (3.10), we have (i). ■

3.2. *Proof of Theorem 2.2.* We need the following lemmas.

LEMMA 3.3 ([19]). *If* $1 \le p < \infty$ *and* ϕ *satisfies* (2.1), *then*

$$\mathcal{L}_{p,\phi}(X) \subset L_{p,\Phi^*+\Phi^{**}}(X) \quad \text{and} \quad \|f\|_{L_{p,\Phi^*+\Phi^{**}}} \le C(\|f\|_{\mathcal{L}_{p,\phi}} + |f_{B_0}|).$$

LEMMA 3.4 ([21]). *If* $1 \le p < \infty$ *and* ϕ *satisfies* (2.1)–(2.4), *then, for any ball* $B(a,r)$, *there is a function* $f \in \mathcal{L}_{p,\phi}(X)$ *such that*

$$\|f\|_{\mathcal{L}_{p,\phi}} + |f_{B_0}| \le C_1 \quad \text{and} \quad f_{B(a,r)} \ge C_2(\Phi^*(a,r) + \Phi^{**}(a,r)),$$

where $C_1 > 0$ *and* $C_2 > 0$ *are independent of* f *and* $B(a,r)$.

Proof of Theorem 2.2. (i)⇒(ii): In general $\mathcal{L}_{p,\phi}(X) \supset L_{p,\phi}(X)$ and

$$\|f\|_{\mathcal{L}_{p,\phi}} + |f_{B_0}| \le (2 + \phi(B_0))\|f\|_{L_{p,\phi}}.$$

From (i) and Lemma 3.3 it follows that

$$\mathcal{L}_{p,\phi}(X) \subset L_{p,\phi}(X) \quad \text{and} \quad \|f\|_{L_{p,\phi}} \le C(\|f\|_{\mathcal{L}_{p,\phi}} + |f_{B_0}|).$$

(ii)⇒(i): For any ball $B(a,r)$, let f be as in Lemma 3.4. Since

$$|f_{B(a,r)}| \le \phi(a,r)\|f\|_{L_{p,\phi}} \sim \phi(a,r)(\|f\|_{\mathcal{L}_{p,\phi}} + |f_{B_0}|),$$

we have (i).

(ii)⇒(iii): Since $\mathcal{L}_{p,\phi}(X) = L_{p,\phi}(X)$, all constant functions are in $L_{p,\phi}(X)$. Putting $f - f_{B_0}$ instead of f in (ii), we obtain (iii).

(iii)⇒(ii): Since $1 \in L_{p,\phi}(X)$, it turns out that $\sup_B 1/\phi(B) < \infty$ and that $\|f_{B_0}\|_{L_{p,\phi}} = (\sup_B 1/\phi(B))|f_{B_0}|$. Hence,

$$\|f\|_{L_{p,\phi}} \le \|f - f_{B_0}\|_{L_{p,\phi}} + \|f_{B_0}\|_{L_{p,\phi}} \sim \|f\|_{\mathcal{L}_{p,\phi}} + |f_{B_0}|. \blacksquare$$

3.3. *Proof of Corollary 2.3.* From (2.4) it follows that $\inf_{a\in X} \phi(a, R_0) > 0$ and that $\Phi \ge C > 0$. Hence, if ϕ satisfies (2.1), then Φ is comparable to $\Phi^* + \Phi^{**}$. Therefore this corollary follows from Theorem 2.2.

3.4. *Proof of Theorem 2.4.* We need some lemmas.

Lemma 3.5. *If $\int_0^1 \phi(x,t)/t\,dt < \infty$, then for every $f \in \mathcal{L}_{p,\phi}(X)$, $f_{B(x,r)}$ converges as r tends to zero.*

Proof. As r and s tend to zero with $0 < r < s$, the right-hand side of (3.6) tends to zero. ■

Lemma 3.6. *Suppose that ϕ satisfies* (2.1). *Let*

$$X^* = \left\{x \in X : \int_0^1 \frac{\phi(x,t)}{t}\,dt < \infty\right\}.$$

For $f \in \mathcal{L}_{p,\phi}(X)$ and for $x \in X^*$, let $g(x) = \lim_{r\to 0} f_{B(x,r)}$. Then

$$|g(x) - g(y)| \le C \int_0^{d(x,y)} \frac{\phi(x,t) + \phi(y,t)}{t}\,dt\,\|f\|_{\mathcal{L}_{p,\phi}} \quad \textit{for all } x, y \in X^*, \tag{3.11}$$

where the constant $C > 0$ is independent of f.

Proof. Let $f \in \mathcal{L}_{p,\phi}(X)$, $x, y \in X^*$ and $2r < d(x,y) = s$. By (3.6) and (3.3), we have

$$
\begin{aligned}
&|f_{B(x,r)} - f_{B(y,r)}| \\
&\quad \le |f_{B(x,r)} - f_{B(x,s)}| + |f_{B(x,s)} - f_{B(y,2K_1 s)}| + |f_{B(y,2K_1 s)} - f_{B(y,r)}| \\
&\quad \le \Bigg(C_0 \int_r^{2s} \frac{\phi(x,t)}{t}\,dt + \frac{\mu(B(y,2K_1 s))}{\mu(B(x,s))}\,\phi(y,2K_1 s) + C_0 \int_r^{4K_1 s} \frac{\phi(y,t)}{t}\,dt \Bigg) \\
&\qquad \times \|f\|_{\mathcal{L}_{p,\phi}},
\end{aligned}
$$

where $C_0 = 2K_2^2(\log 2)^{-1}$. Since $\mu(B(y,2K_1 s)) \sim \mu(B(x,s))$ and

$$
\phi(y,2K_1 s) \sim \int_{2K_1 s}^{4K_1 s} \frac{\phi(y,t)}{t}\,dt,
$$

using (3.5), we deduce that

$$
|f_{B(x,r)} - f_{B(y,r)}| \le C \int_r^{d(x,y)} \frac{\phi(x,t)+\phi(y,t)}{t}\,dt\,\|f\|_{\mathcal{L}_{p,\phi}}.
$$

Letting $r \to 0$, we obtain (3.11). ■

LEMMA 3.7. *Suppose that ϕ satisfies* (2.1). *Let*

$$
X^{**} = \Big\{ x \in X : \int_0^{d(x,y)} \frac{\phi(x,t)+\phi(y,t)}{t}\,dt \to 0 \text{ as } y \to x \Big\}.
$$

Then for every $f \in \mathcal{L}_{p,\phi}(X)$, the function $g(x) = \lim_{r\to 0} f_{B(x,r)}$ is equal to $f(x)$ for a.e. $x \in X^{**}$.

Proof. As in the proof of Theorem 4 in [14], using (3.11) instead of [14, (3.7)], we find that for every $x \in X^{**}$ and $\varepsilon > 0$ there exists a ball $B(x,r(x))$ such that $0 < r(x) < 1$ and

$$
\int_{B(x,r(x))} |f(y) - g(y)|^p\,d\mu(y) \le \varepsilon \mu(B(x,r(x))),
$$

which shows that $f(x) = g(x)$ for a.e. $x \in X^{**}$. ■

Proof of Theorem 2.4. (i)⇒(ii): Let $f \in \mathcal{L}_{p,\phi}(X)$. We regard $\mathcal{L}_{p,\phi}(X)$ as a set of functions modulo null-functions. By Lemma 3.7, we may assume that $f(x) = \lim_{r\to 0} f_{B(x,r)}$ for all $x \in X$. By Lemma 3.6, we have

$$
\begin{aligned}
|f(x) - f(y)| &\le C \int_0^{d(x,y)} \frac{\phi(x,t)+\phi(y,t)}{t}\,dt\,\|f\|_{\mathcal{L}_{p,\phi}} \\
&\le C'(\phi(x,d(x,y)) + \phi(y,d(x,y)))\|f\|_{\mathcal{L}_{p,\phi}}, \quad x,y \in X.
\end{aligned}
$$

Hence $f \in \Lambda_\phi(X)$ and $\|f\|_{\Lambda_\phi} \le 2C'\|f\|_{\mathcal{L}_{p,\phi}}$.

Conversely, let $f \in \Lambda_\phi(X)$. If $x, y \in B = B(z,r)$, then

$$
B(x,d(x,y)), B(y,d(x,y)) \subset B(z,4K_1^2 r).
$$

By (2.8), we have

$$\phi(x,d(x,y)),\phi(y,d(x,y))\le A_5\phi(z,4K_1^2r).$$

Hence,

$$|f(x)-f(y)|\le C\phi(B)\|f\|_{\Lambda_\phi}.$$

It follows that

$$\begin{aligned}\left(\frac{1}{\mu(B)}\int_B|f(x)-f_B|^p\,d\mu(x)\right)^{1/p}&\\ &\le\left(\frac{1}{\mu(B)}\int_B\left(\frac{1}{\mu(B)}\int_B|f(x)-f(y)|\,d\mu(y)\right)^p d\mu(x)\right)^{1/p}\\ &\le C\phi(B)\|f\|_{\Lambda_\phi}.\end{aligned}$$

Hence $f\in\mathcal{L}_{p,\phi}(X)$ and $\|f\|_{\mathcal{L}_{p,\phi}}\le C\|f\|_{\Lambda_\phi}$.

(ii)⇒(i): Let f_a be defined by (3.7). By Lemma 3.1 and (ii), f_a is in $\Lambda_\phi(X)$ and there is a constant $C>0$, independent of $a\in X$, such that $\|f_a\|_{\Lambda_\phi}\le C$. Since $\Lambda_\phi(X)\subset L^\infty_{\mathrm{loc}}(X)$, we have

$$f_a(a)=\int_0^1\frac{\phi(a,t)}{t}\,dt<\infty\quad\text{ for every } a\in X.$$

Hence,

$$\begin{aligned}\int_0^{d(x,y)}\frac{\phi(x,t)}{t}\,dt&=|f_x(x)-f_x(y)|\\ &\le\tfrac12(\phi(x,d(x,y))+\phi(y,d(x,y)))\|f_x\|_{\Lambda_\phi}\le\tfrac12(1+A_4)C\phi(x,d(x,y)).\end{aligned}$$

(ii)⇔(iii): Let $f\in\Lambda_\phi(X)$. By (2.1) and (2.8), we have

$$|f_{B_0}-f(x_0)|\le\frac{1}{\mu(B_0)}\int_{B_0}|f(x)-f(x_0)|\,d\mu(x)\le C\phi(B_0)\|f\|_{\Lambda_\phi}.$$

Let $f\in\mathcal{L}_{p,\phi}(X)$. By (3.6), we have

$$|f_{B_0}-f(x_0)|=\lim_{r\to0}|f_{B(x_0,1)}-f_{B(x_0,r)}|\le C\int_0^2\frac{\phi(x_0,t)}{t}\,dt\,\|f\|_{\mathcal{L}_{p,\phi}}.\ \blacksquare$$

3.5. *Proof of Theorem 2.5*

LEMMA 3.8. *Let* (X,d_i,μ) $(i=1,2)$ *be spaces of homogeneous type. Suppose that* $\psi:X\times\mathbb{R}_+\to\mathbb{R}_+$ *satisfies* (2.1). *Let* $\phi_i(x,r)=\psi(x,\mu(B^{d_i}(x,r)))$ $(i=1,2)$. *If for every* d_1*-ball* B_1 *there is a* d_2*-ball* B_2 *such that*

(3.12) $$B_1\subset B_2\quad\textit{and}\quad\mu(B_2)\le C\mu(B_1)$$

for some constant $C > 0$ independent of B_1 and B_2, then

$$\mathcal{L}_{p,\phi_1}(X, d_1, \mu) \supset \mathcal{L}_{p,\phi_2}(X, d_2, \mu),$$
$$\mathcal{L}_{p,\phi_1}(X, d_1, \mu)/\mathcal{C} \supset \mathcal{L}_{p,\phi_2}(X, d_2, \mu)/\mathcal{C},$$
$$L_{p,\phi_1}(X, d_1, \mu) \supset L_{p,\phi_2}(X, d_2, \mu),$$

and the embeddings are continuous. If ψ also satisfies (2.8), *then*

$$\Lambda_{\phi_1}(X, d_1, \mu) \supset \Lambda_{\phi_2}(X, d_2, \mu),$$
$$\Lambda_{\phi_1}(X, d_1, \mu)/\mathcal{C} \supset \Lambda_{\phi_2}(X, d_2, \mu)/\mathcal{C},$$

and the embeddings are continuous.

Proof. First we show $\|f\|_{\mathcal{L}_{p,\phi_1}(X,d_1,\mu)} \le C\|f\|_{\mathcal{L}_{p,\phi_2}(X,d_2,\mu)}$. From (3.12) it follows that $\mu(B_1) \sim \mu(B_2)$ and $\phi_1(B_1) \sim \phi_2(B_2)$. Hence, by (3.1) we have

$$\begin{aligned}\frac{1}{\phi_1(B_1)}&\biggl(\frac{1}{\mu(B_1)}\int_{B_1}|f(x)-f_{B_1}|^p\,d\mu(x)\biggr)^{1/p}\\ &\le \frac{2}{\phi_1(B_1)}\biggl(\frac{1}{\mu(B_1)}\int_{B_1}|f(x)-f_{B_2}|^p\,d\mu(x)\biggr)^{1/p}\\ &\le C\frac{1}{\phi_2(B_2)}\biggl(\frac{1}{\mu(B_2)}\int_{B_2}|f(x)-f_{B_2}|^p\,d\mu(x)\biggr)^{1/p}.\end{aligned}$$

In the same way, we have $\|f\|_{L_{p,\phi_1}(X,d_1,\mu)} \le C\|f\|_{L_{p,\phi_2}(X,d_2,\mu)}$.

Let $B_1 = B^{d_1}(x_0, 1)$ in (3.12). Then by (3.3) we have

$$|f_{B^{d_1}(x_0,1)}| \le C|f_{B_2}| \le C'(|f_{B^{d_2}(x_0,1)}| + \|f\|_{\mathcal{L}_{p,\phi_2}(X,d_2,\mu)}).$$

Next we show

$$\phi_2(x, d_2(x,y)) \le C\phi_1(x, d_1(x,y)), \quad x, y \in X, \tag{3.13}$$

which implies that $\|f\|_{\Lambda_{\phi_1}(X,d_1,\mu)} \le C\|f\|_{\Lambda_{\phi_2}(X,d_2,\mu)}$. For $B^{d_1}(x, 2d_1(x,y))$, there is $B^{d_2}(x,r)$ such that

$$B^{d_1}(x, 2d_1(x,y)) \subset B^{d_2}(x,r) \quad \text{and} \quad \mu(B^{d_2}(x,r)) \le C\mu(B^{d_1}(x, 2d_1(x,y))).$$

Since $y \in B^{d_1}(x, 2d_1(x,y)) \subset B^{d_2}(x,r)$, it turns out that $d_2(x,y) < r$. Hence,

$$\begin{aligned}\mu(B^{d_2}(x, d_2(x,y))) &\le \mu(B^{d_2}(x,r)) \le C\mu(B^{d_1}(x, 2d_1(x,y)))\\ &\le C'\mu(B^{d_1}(x, d_1(x,y))).\end{aligned}$$

Using (2.1) and (2.8) for ψ, we obtain (3.13). ■

By the definition of δ, we have

$$\begin{aligned}&B^d(x,r) \subset B^\delta(x, \mu(B^d(x,r))),\\ &\mu(B^\delta(x, \mu(B^d(x,r)))) \le K_5\mu(B^d(x,r)).\end{aligned} \tag{3.14}$$

Conversely, we have the following.

LEMMA 3.9. *For any δ-ball $B^\delta(x,r)$, there is a constant $\widetilde{r}>0$ such that*

$$B^\delta(x,r)\subset B^d(x,\widetilde{r}) \quad \text{and} \quad \mu(B^d(x,\widetilde{r}))\le C\mu(B^\delta(x,r))$$

for some constant $C>0$ independent of x, r and $\widetilde{r}$.

Proof. CASE 1: $\mu(\{x\})<r<\mu(X)$. Choose a constant $M>0$ such that

$$\mu(B^d(z,(2K_1)^2 s))\le M\mu(B^d(z,s)), \quad z\in X,\ s>0.$$

If $y\in B^\delta(x,r)$, then $\delta(x,y)<r$. By the definition of δ, there is a d-ball $B^d(z,s)$ such that $x,y\in B^d(z,s)$ and $\mu(B^d(z,s))<r$. Since $B^d(z,s)\subset B^d(x,2K_1s)\subset B^d(z,(2K_1)^2s)$ and $\mu(B^d(x,2K_1s))\le M\mu(B^d(z,s))<Mr$, we have

$$B^\delta(x,r)\subset\bigcup\{B^d(x,t):\mu(B^d(x,t))<Mr\}.$$

Let

$$\widetilde{r}=\begin{cases}\sup\{0<t\le R_0:\mu(B^d(x,t))<Mr\} & \text{if } \mu(X)<\infty,\\ \sup\{t>0:\mu(B^d(x,t))<Mr\} & \text{if } \mu(X)=\infty,\end{cases}$$

where R_0 is the constant in (2.7). Then $B^\delta(x,r)\subset B^d(x,\widetilde{r})$ and

$$\mu(B^d(x,\widetilde{r}))\le K_2\mu(B^d(x,\widetilde{r}/2))\le K_2Mr\le (K_2M/K_4)\mu(B^\delta(x,r)).$$

CASE 2: $\mu(\{x\})>0$ and $0<r\le\mu(\{x\})$. By [14, Theorem 1], we have

$$B^\delta(x,r)=\{x\}=B^d(x,r_x) \quad \text{for some } r_x>0.$$

CASE 3: $\mu(X)<\infty$ and $\mu(X)\le r$. We have

$$B^\delta(x,r)\subset X=B^d(x,R_0)=B^\delta(x,2r). \ \blacksquare$$

Proof of Theorem 2.5. Since $\psi_*(x,r)\sim\psi(x,\mu(B^\delta(x,r)))$, using Lemma 3.8, (3.14) and Lemma 3.9, we obtain

$$\begin{gathered}\mathcal{L}_{p,\phi}(X,d,\mu)=\mathcal{L}_{p,\psi_*}(X,\delta,\mu),\\ \mathcal{L}_{p,\phi}(X,d,\mu)/\mathcal{C}=\mathcal{L}_{p,\psi_*}(X,\delta,\mu)/\mathcal{C},\\ L_{p,\phi}(X,d,\mu)=L_{p,\psi_*}(X,\delta,\mu),\\ \Lambda_\phi(X,d,\mu)=\Lambda_{\psi_*}(X,\delta,\mu),\\ \Lambda_\phi(X,d,\mu)/\mathcal{C}=\Lambda_{\psi_*}(X,\delta,\mu)/\mathcal{C},\end{gathered}$$

with equivalent norms. If $\mu(\{x\})>0$ and $0<r<\mu(\{x\})$, then $\{x\}=B^\delta(x,r)$ and

$$\mathrm{MO}_p(f,B^\delta(x,r))=0.$$

Hence,

$$\begin{gathered}\mathcal{L}_{p,\psi_*}(X,\delta,\mu)=\mathcal{L}_{p,\psi}(X,\delta,\mu),\\ \mathcal{L}_{p,\psi_*}(X,\delta,\mu)/\mathcal{C}=\mathcal{L}_{p,\psi}(X,\delta,\mu)/\mathcal{C},\end{gathered}$$

with equivalent norms. If $x\ne y$, then $\mu(\{x\}),\mu(\{y\})\le\delta(x,y)$, and then

$$\psi_*(x,\delta(x,y))=\psi(x,\delta(x,y)), \quad \psi_*(y,\delta(x,y))=\psi(y,\delta(x,y)).$$

Hence,

$$\Lambda_{\psi_*}(X,\delta,\mu)=\Lambda_{\psi}(X,\delta,\mu),$$
$$\Lambda_{\psi_*}(X,\delta,\mu)/\mathcal{C}=\Lambda_{\psi}(X,\delta,\mu)/\mathcal{C},$$

with equivalent norms. We have completed the proof. ■

4. Applications. We now state some applications of the main results.

By using Theorem 2.5, we can show that theorems for the Campanato, Morrey or Hölder spaces on the normal space are valid for the function spaces on any space of homogeneous type with the relation $\phi(x,r)=\psi(x,\mu(B(x,r)))$. Conversely, results for these function spaces on any space of homogeneous type also adapt to the normal space.

It was proved by Macías and Segovia [14] that the set of points of X which have positive measure is countable, and that for every such point there exists a constant $r>0$ such that $B(x,r)=\{x\}$. We define

$$r_x=\begin{cases}0 & \text{if } \mu(\{x\})=0,\\ \sup\{r>0: B(x,r)=\{x\}\} & \text{if } \mu(\{x\})>0.\end{cases}$$

Similarly, we set (see (2.7))

$$R_x=\begin{cases}\infty & \text{if } \mu(X)=\infty,\\ \inf\{r>0: B(x,r)=X\} & \text{if } \mu(X)<\infty.\end{cases}$$

If (X,d,μ) has the property that there exists a constant $K_6''>1$ such that

$$\mu(B^d(x,r))\le \tfrac{1}{2}\mu(B^d(x,K_6''r)) \quad \text{for all } x\in X,\ r_x<r<K_6''r<R_x, \tag{4.1}$$

then, for every $\phi: X\times\mathbb{R}_+\to\mathbb{R}_+$ with (2.1), there exists $\psi: X\times\mathbb{R}_+\to\mathbb{R}_+$ with (2.1) such that $\phi(x,r)\sim\psi(x,\mu(B^d(x,r)))$ for $x\in X$ and $r_x<r<R_x$. Actually, for $x\in X$, there exists a continuous, increasing and bijective function $b_x:\mathbb{R}_+\to\mathbb{R}_+$ such that $b_x(r)\sim\mu(B^d(x,r))$ and $b_x^{-1}(r)$ satisfies (2.1). Let $\psi(x,r)=\phi(x,b_x^{-1}(r)))$. Then $\phi(x,r)\sim\psi(x,\mu(B^d(x,r)))$.

If there exist constants $C>0$ and $\beta>0$ such that

$$\frac{\phi(x,t)}{\mu(B^d(x,t))^\beta}\le C\frac{\phi(x,r)}{\mu(B^d(x,r))^\beta}, \quad x\in X,\ r_x<r<t<R_x,$$

then (2.3) implies (4.1) (see [21, Lemma 5.4]).

4.1. *Boundedness of operators.* Let M_α^d $(0<\alpha\le 1)$ be the fractional maximal operator with respect to a quasi-distance d, i.e.,

$$M_\alpha^d f(x)=\sup_{B^d\ni x}\frac{1}{\mu(B^d)^\alpha}\int_{B^d}|f(y)|\,d\mu(y),$$

where the supremum is taken over all d-balls B^d containing x. If $\alpha=1$, then M_α^d is the Hardy–Littlewood maximal operator. Then, by (3.14) and Lemma 3.9, we have the following:

Proposition 4.1. *Let (X, d, μ) be a space of homogeneous type satisfying* (1.3) *and δ defined by* (1.5). *Let $0 < \alpha \le 1$. Then there exists a constant $C > 0$ such that, for all $f \in L^1_{\mathrm{loc}}(X)$ and for all $x \in X$,*

$$C^{-1} M^d_\alpha f(x) \le M^\delta_\alpha f(x) \le C M^d_\alpha f(x).$$

Let (X, δ, μ) be a normal space of homogeneous type, μ a Borel measure or its completion and $\mu(\{x\}) = 0$ for all $x \in X$.

Arai and Mizuhara [1, Theorem 3.1] gave a sufficient condition for an operator to be bounded on Morrey spaces on (X, δ, μ). They used the Muckenhoupt A_1-weights defined by using the Hardy–Littlewood maximal operator. By Proposition 4.1 and Theorem 2.5, we can extend their result to any space of homogeneous type. Moreover, by using Theorem 2.2 or Corollary 2.3, we have results for the Campanato spaces. Arai and Mizuhara proved the boundedness of several operators on the Morrey spaces as corollaries of [1, Theorem 3.1]: Lusin area integral for harmonic functions on $\mathbb{R}^{n+1}_+$, Hardy–Littlewood maximal function, maximal Calderón–Zygmund singular integral operators, Cauchy–Szegő projection on the Heisenberg group and the Kohn Laplacian $\Box_b$.

Lu [12, 13] proved embedding theorems on Campanato–Morrey spaces for vector fields. He considered spaces of homogeneous type associated to vector fields of homogeneous degree Q. His proofs rely on the fractional maximal functions. We can get his results also on the normal space.

4.2. *Fractional differentiation and integration.* Following Macías and Segovia [14], we shall say that a space of homogeneous type is of order θ if the condition (1.3) holds.

Gatto, Segovia and Vági [7] investigated fractional differentiation and integration of functions in Lipschitz spaces on normal spaces of homogeneous type.

Let (X, δ, μ) be a normal space of order θ and $\mu(\{x\}) = 0$ for all $x \in X$. It was proved in [7] that, for each α, $-\infty < \alpha < 1$, there exists a quasi-distance δ_α equivalent to δ such that (X, δ_α, μ) is a normal space of order θ and δ_α has the cancellation property for $0 < \alpha < \theta$, i.e.,

$$\int_X \left(\frac{1}{\delta_\alpha(x, y)^{1-\alpha}} - \frac{1}{\delta_\alpha(x', y)^{1-\alpha}} \right) d\mu(y) = 0 \quad \text{for any } x, x' \in X.$$

For $0 < \alpha < \theta$, the *fractional derivative* of order α of f in $\mathrm{Lip}_\beta(X, \delta, \mu) \cap L^\infty(X, \mu)$, $\alpha < \beta < \theta$, is defined by

$$D_\alpha f(x) = \int_X \frac{f(y) - f(x)}{\delta_{-\alpha}(x, y)^{\alpha+1}}\, d\mu(y).$$

For $0 < \alpha < 1$, the *fractional integral* of order α of f in $\mathrm{Lip}_\beta(X,\delta,\mu) \cap L^1(X,\mu)$ is defined by

$$I_\alpha f(x) = \int_X \frac{f(y)}{\delta_\alpha(x,y)^{1-\alpha}}\, d\mu(y). \tag{4.2}$$

The definitions of D_α and I_α can be extended to Lip_β for β as above. This requires the following modification:

$$\widetilde{D}_\alpha f(x) = \int_X \left(\frac{f(y)-f(x)}{\delta_{-\alpha}(x,y)^{\alpha+1}} - \frac{f(y)-f(x_0)}{\delta_{-\alpha}(x_0,y)^{\alpha+1}} \right) d\mu(y),$$

and

$$\widetilde{I}_\alpha f(x) = \int_X f(y) \left(\frac{1}{\delta_\alpha(x,y)^{1-\alpha}} - \frac{1}{\delta_\alpha(x_0,y)^{1-\alpha}} \right) d\mu(y), \tag{4.3}$$

where x_0 is a fixed but arbitrary point of X.

It was proved in [7] that, for $0 < \alpha < \beta \le \theta$, $\widetilde{D}_\alpha$ is bounded from $\mathrm{Lip}_{\alpha+\beta}/\mathcal{C}$ to $\mathrm{Lip}_\beta/\mathcal{C}$ and $\widetilde{I}_\alpha$ is bounded from $\mathrm{Lip}_{\beta-\alpha}/\mathcal{C}$ to $\mathrm{Lip}_\beta/\mathcal{C}$. The operator $\widetilde{I}_\alpha$ is well defined because of the cancellation property.

It follows from the results in [8] that $\widetilde{I}_\alpha$, $0 < \alpha < 1$, is bounded from $L^p(X,\mu)$ to $\mathrm{BMO}(X,\delta,\mu)/\mathcal{C}$ when $\alpha = 1/p$, from $L^p(X,\mu)$ to $\mathrm{Lip}_\beta(X,\delta,\mu)/\mathcal{C}$ when $0 < \alpha - 1/p = \beta < \theta$, and from $\mathrm{BMO}(X,\delta,\mu)/\mathcal{C}$ to $\mathrm{Lip}_\alpha(X,\delta,\mu)/\mathcal{C}$ when $0 < \alpha < \theta$.

These results of [7] and [8] can be extended to any space of homogeneous type (X,d,μ). Let δ be defined by (1.5) and δ_α as above. By Theorem 2.5 we have the relations $\mathrm{BMO}(X,\delta,\mu) = \mathrm{BMO}(X,d,\mu)$ and $\mathrm{Lip}_\beta(X,\delta,\mu) = \Lambda_{\phi_\beta}(X,d,\mu)$ with $\phi_\beta(x,r) = \mu(B^d(x,r))^\beta$. Therefore $\widetilde{D}_\alpha$ is bounded from $\Lambda_{\phi_{\alpha+\beta}}(X,d,\mu)/\mathcal{C}$ to $\Lambda_{\phi_\beta}(X,d,\mu)/\mathcal{C}$, and $\widetilde{I}_\alpha$ is bounded between $L^p(X,\mu)$, $\mathrm{BMO}(X,d,\mu)/\mathcal{C}$, $\Lambda_{\phi_\beta}(X,d,\mu)/\mathcal{C}$ and $\Lambda_{\phi_{\alpha+\beta}}(X,d,\mu)/\mathcal{C}$ for suitable p,α,β.

In the definitions (4.2) and (4.3), we can replace $\delta_\alpha(x,y)$ by $\mu(B^d(x, d(x,y)))$ for the boundedness from $L^p(X,\mu)$ to $\mathrm{BMO}(X,d,\mu)/\mathcal{C}$ or to $\Lambda_{\phi_\beta}(X,d,\mu)/\mathcal{C}$, since $\delta_\alpha(x,y) \sim \delta(x,y) \sim \mu(B^d(x,d(x,y)))$ and the cancellation property is not needed. See also Genebashvili, Gogatishvili, Kokilashvili and Krbec [6].

4.3. *Pointwise multipliers.* Let E and F be spaces of real- or complex-valued functions defined on a set X. A function g defined on X is called a *pointwise multiplier* from E to F if the pointwise product fg belongs to F for each $f \in E$. We denote by $\mathrm{PWM}(E,F)$ the set of all pointwise multipliers from E to F.

The author studied pointwise multipliers on Morrey spaces in [20]. Combining the results in [20], Theorem 2.5, Corollary 2.3 and Theorem 2.1 we deduce the following:

THEOREM 4.2. *Let* (X,d,μ) *be a space of homogeneous type and* $\mu(\{x\}) = 0$ *for all* $x \in X$. *If* $1 \le p_2 \le p_1 < \infty$, $1/p_1 + 1/p_3 = 1/p_2$, $0 < \alpha_1 \le \alpha_2 \le 1$, $0 < p_2\alpha_2 \le p_1\alpha_1 \le 1$, $\alpha_3 = \alpha_2 - \alpha_1$ *and* $\phi_i(x,r) = \mu(B^d(x,r))^{-\alpha_i}$ $(i = 1,2,3)$, *then*

$$\mathrm{PWM}(\mathcal{L}_{p_1,\phi_1}(X), \mathcal{L}_{p_2,\phi_2}(X)) = \begin{cases} \mathcal{L}_{p_3,\phi_3}(X), & p_1 \ne p_2, \\ L^\infty(X), & p_1 = p_2, \end{cases} \qquad \text{if } \mu(X) < \infty,$$

$$\mathrm{PWM}(\mathcal{L}_{p_1,\phi_1}(X), \mathcal{L}_{p_2,\phi_2}(X)) = \begin{cases} \mathcal{L}_{p_3,\phi_3}(X) \cap L_{p_2,\phi_2}(X), & p_1 \ne p_2, \\ L^\infty(X) \cap L_{p_2,\phi_2}(X), & p_1 = p_2, \end{cases} \qquad \text{if } \mu(X) = \infty.$$

Moreover, *the operator norm of* $g \in \mathrm{PWM}(\mathcal{L}_{p_1,\phi_1}(X), \mathcal{L}_{p_2,\phi_2}(X))$ *is comparable to the norm in the function space of the right-hand side.*

The author studied pointwise multipliers on BMO_ϕ in [19]. Combining [19, Examples 2.6 and 2.9] and Theorem 2.4 we deduce the following:

THEOREM 4.3. *Let* (X,d,μ) *be a space of homogeneous type with* (1.3), *and* μ *a Borel measure or its completion with the following property*:

$$\frac{\mu(B^d(x,t))}{\mu(B^d(x,r))} \le C\left(\frac{t}{r}\right)^c, \quad x \in X,\ 0 < t < r, \tag{4.4}$$

for some $C, c > 0$. *If* $0 < \beta \le \alpha \le \theta$, *then*

$$\mathrm{PWM}(\mathrm{Lip}_\alpha(X), \mathrm{Lip}_\beta(X)) = \mathrm{Lip}_\beta(X), \qquad \mu(X) < \infty,$$
$$\mathrm{PWM}(\mathrm{Lip}_\alpha(X), \mathrm{Lip}_\beta(X)) = \mathrm{BMO}_\phi(X) \cap L_{r^{\beta-\alpha}}(X), \qquad \mu(X) = \infty,$$

where

$$\phi(x,r) = \frac{r^\beta}{(2 + d(x_0,x) + r)^\alpha}.$$

Moreover, *the operator norm of* $g \in \mathrm{PWM}(\mathrm{Lip}_\alpha(X), \mathrm{Lip}_\beta(X))$ *is comparable to the norm in the function space of the right-hand side.*

REMARK 4.1. For example, the Muckenhoupt A_p-weights on $\mathbb{R}^n$ satisfy (4.4).

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References

[1] H. Arai and T. Mizuhara, *Morrey spaces on spaces of homogeneous type and estimates for* $\Box_b$ *and the Cauchy–Szegő projection*, Math. Nachr. 185 (1997), 5–20.

[2] S. Campanato, *Proprietà di hölderianità di alcune classi di funzioni*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 17 (1963), 175–188.

[3] —, *Proprietà di una famiglia di spazi funzionali*, ibid. 18 (1964), 137–160.

[4] R. R. Coifman et G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Math. 242, Springer, Berlin, 1971.

[5] —, —, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. 83 (1977), 569–645.

[6] I. Genebashvili, A. Gogatishvili, V. Kokilashvili and M. Krbec, *Weight Theory for Integral Transforms on Spaces of Homogeneous Type*, Longman, Harlow, 1998.

[7] A. E. Gatto, C. Segovia and S. Vági, *On fractional differentiation and integration on spaces of homogeneous type*, Rev. Mat. Iberoamericana 12 (1996), 111–145.

[8] A. E. Gatto and S. Vági, *Fractional integrals on spaces of homogeneous type*, in: Analysis and Partial Differential Equations, C. Sadosky (ed.), Dekker, New York, 1990, 171–216.

[9] Y. Han and E. Sawyer, *Littlewood–Paley theory on spaces of homogeneous type and the classical function spaces*, Mem. Amer. Math. Soc. 110 (1994), no. 530.

[10] S. Janson, *On functions with conditions on the mean oscillation*, Ark. Mat. 14 (1976), 189–196.

[11] P. G. Lemarié, *Algèbres d'opérateurs et semi-groupes de Poisson sur un espace de nature homogène*, Publ. Math. Orsay 84–3 (1984).

[12] G. Lu, *Embedding theorems on Campanato–Morrey spaces for vector fields and applications*, C. R. Acad. Sci. Paris Sér. I Math. 320 (1995), 429–434.

[13] —, *Embedding theorems on Campanato–Morrey spaces for vector fields on Hörmander type*, Approx. Theory Appl. (N.S.) 14 (1998), 69–80.

[14] R. A. Macías and C. Segovia, *Lipschitz functions on spaces of homogeneous type*, Adv. Math. 33 (1979), 257–270.

[15] N. G. Mayers, *Mean oscillation over cubes and Hölder continuity*, Proc. Amer. Math. Soc. 15 (1964), 717–721.

[16] T. Mizuhara, *Relations between Morrey and Campanato spaces with some growth functions, II*, in: Proceedings of Harmonic Analysis Seminar 11 (1995), 67–74 (in Japanese).

[17] A. Nagel, E. M. Stein and S. Wainger, *Balls and metrics defined by vector fields. I. Basic properties*, Acta Math. 155 (1985), 103–147.

[18] E. Nakai, *On the restriction of functions of bounded mean oscillation to the lower dimensional space*, Arch. Math. (Basel) 43 (1984), 519–529.

[19] —, *Pointwise multipliers on weighted BMO spaces*, Studia Math. 125 (1997), 35–56.

[20] —, *Pointwise multipliers on the Morrey spaces*, Mem. Osaka Kyoiku Univ. III Natur. Sci. Appl. Sci., 46 (1997), 1–11.

[21] E. Nakai and K. Yabuta, *Pointwise multipliers for functions of weighted bounded mean oscillation on spaces of homogeneous type*, Math. Japon. 46 (1997), 15–28.

[22] J. Peetre, *On the theory of $\mathcal{L}_{p,\lambda}$ spaces*, J. Funct. Anal. 4 (1969), 71–87.

[23] S. Spanne, *Some function spaces defined using the mean oscillation over cubes*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 19 (1965), 593–608.

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