Some weighted norm inequalities for a one-sided version of $g^*_\lambda$

by

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Abstract. We study the boundedness of the one-sided operator $g^+_\lambda$ between the weighted spaces $L^p(M^{-w})$ and $L^p(w)$ for every weight $w$. If $\lambda = 2/p$ whenever $1 < p < 2$, and in the case $p = 1$ for $\lambda > 2$, we prove the weak type of $g^+_\lambda$. For every $\lambda > 1$ and $p = 2$, or $\lambda > 2/p$ and $1 < p < 2$, the boundedness of this operator is obtained. For $p > 2$ and $\lambda > 1$, we obtain the boundedness of $g^+_\lambda$ from $L^p((M^{-})^{[p/2]+1}w)$ to $L^p(w)$, where $(M^{-})^k$ denotes the operator $M^-$ iterated $k$ times.

1. Notations and definitions. As usual, $S$ denotes the class of all those $C^\infty$-functions defined on $\mathbb{R}$ such that
$$\sup_{x \in \mathbb{R}} |x^m(D^n \varphi)(x)| < \infty$$
for all non-negative integers $m$ and $n$. We also consider the space $C^\infty_0$ of all $C^\infty$-functions defined on $\mathbb{R}$ with compact support.

If $E \subset \mathbb{R}$ is a Lebesgue measurable set, we denote its Lebesgue measure by $|E|$, and the characteristic function of $E$ by $\chi_E(x)$.

Let $f$ be a measurable function defined on $\mathbb{R}$. The one-sided Hardy-Littlewood maximal functions $M^-f$ and $M^+f$ are given by
$$M^-f(x) = \sup_{h > 0} \frac{1}{h} \int_{x-h}^{x} |f(t)| \, dt, \quad M^+f(x) = \sup_{h > 0} \frac{1}{h} \int_{x}^{x+h} |f(t)| \, dt.$$

A weight $w$ is a measurable and non-negative function defined on $\mathbb{R}$. If $E \subset \mathbb{R}$ is a measurable set, we denote its $w$-measure by $w(E) = \int_E w(t) \, dt$. Given $p \geq 1$, $L^p(w)$ is the space of all measurable functions $f$ such that
$$\|f\|_{L^p(w)} = \left( \int_{-\infty}^{\infty} |f(x)|^p w(x) \, dx \right)^{1/p} < \infty.$$
If $w = 1$, we simply write $L^p$ and $\|f\|_{L^p}$.

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We shall say that a function $B : [0, \infty) \to [0, \infty)$ is a Young function if it is continuous, convex, increasing and satisfies $\lim_{t \to \infty} B(t) = \infty$. The Luxemburg norm of a function $f$ is given by
\[
\|f\|_B = \inf \left\{ \lambda > 0 : \int B(|f|/\lambda) \leq 1 \right\},
\]
and the average over an interval $I$ is:
\[
\|f\|_{B,I} = \inf \left\{ \lambda > 0 : \frac{1}{|I|} \int I B(|f|/\lambda) \leq 1 \right\}.
\]
The one-sided maximal operators associated to $B$ are defined as
\[
M^+_B(f)(x) = \sup_{h > 0} \|f\|_{B,[x,x+h]}, \quad M^-_B(f)(x) = \sup_{h > 0} \|f\|_{B,[x-h,x]}.
\]

Let $\varphi$ belong to $S$ and be supported on $(-\infty, 0]$ with $\int \varphi(x) \, dx = 0$. For every $\lambda > 1$, the one-sided operator $g^+_{\lambda,\varphi}$ was defined in [RoSe] as
\[
g^+_{\lambda,\varphi}(f)(x) = \left( \int_0^\infty \int_x^{\infty} \left( \frac{t}{t + y - x} \right)^\lambda |f * \varphi| \frac{dy \, dt}{t^2} \right)^{1/2}.
\]

Throughout this paper the letter $C$ will always mean a positive constant not necessarily the same at each occurrence. If $1 < p < \infty$ then $p'$ denotes its conjugate exponent: $p + p' = pp$.  

2. Statement of the results. In [CW], S. Chanillo and R. Wheeden obtained the boundedness of the area integral between the spaces $L^p(Mw)$ and $L^p(w)$ when $1 < p \leq 2$. For $p = 2$ and $\lambda > 1$, if the support of $\varphi$ is compact, they showed in [CW, Lemma (1.1)] that the operator $g^+_{\lambda,\varphi}$ maps $L^2(Mw)$ into $L^2(w)$. We shall give, in Theorem A, a one sided-version of this result without the restriction on the support of $\varphi$. For $1 < p < 2$ and $\lambda = 2/p$, in order to prove Theorem B below, we use some arguments due to C. Fefferman (see [F]). As a consequence of Theorems A and B, for $1 < p \leq 2$ and $\lambda > 2/p$, we obtain, in Theorem C, the boundedness of $g^+_{\lambda,\varphi}$ between $L^p(M^{-w})$ and $L^p(w)$. For $p > 2$, the known techniques (see [P]) allow us to prove Theorem D.

Next, we state the already mentioned Theorems A–D.

**THEOREM A.** Let $\varphi \in S$ with $\text{supp}(\varphi) \subset (-\infty, 0]$ and $\int \varphi(x) \, dx = 0$. Then, for every $\lambda > 1$,
\[
\left( \int_{-\infty}^{\infty} g^+_{\lambda,\varphi}(f)(x)^2 w(x) \, dx \right)^{1/2} \leq C_{\lambda,\varphi} \left( \int_{-\infty}^{\infty} |f(x)|^2 M^{-w}(x) \, dx \right)^{1/2},
\]
with a constant $C_{\lambda,\varphi}$ not depending on $f$.

**THEOREM B.** Let $\varphi \in S$ with $\text{supp}(\varphi) \subset (-\infty, 0]$ and $\int \varphi(x) \, dx = 0$. Let $\lambda > 2$ if $p = 1$, and $\lambda = 2/p$ whenever $1 < p < 2$. Then there exists a
constant $C_{p,\lambda,w,\varphi}$ such that
\[
w(\{x : g_{\lambda,\varphi}^+(f)(x) > \mu\}) \leq \frac{C_{p,\lambda,w,\varphi}}{\mu^p} \int_{-\infty}^{\infty} |f(x)|^p M^{-w}(x) \, dx\]
for every function $f$ and $\mu > 0$.

**Theorem C.** Let $\varphi \in \mathcal{S}$ with $\text{supp}(\varphi) \subset (-\infty, 0]$ and $\int \varphi(x) \, dx = 0$. Let $1 < p \leq 2$. If $\lambda > 2/p$, then there exists a constant $C_{p,\lambda,w,\varphi}$ such that
\[
\int_{-\infty}^{\infty} g_{\lambda,\varphi}^+(f)(x)^pw(x) \, dx \leq C_{p,\lambda,w,\varphi} \int_{-\infty}^{\infty} |f(x)|^p M^{-w}(x) \, dx
\]
for every function $f$.

**Theorem D.** Let $\varphi \in \mathcal{S}$ with $\text{supp}(\varphi) \subset (-\infty, 0]$ and $\int \varphi(x) \, dx = 0$. Let $\lambda > 1$ and $p > 2$. Then there exists a constant $C_{p,\lambda,w,\varphi}$ such that
\[
\int_{-\infty}^{\infty} g_{\lambda,\varphi}^+(f)(x)^pw(x) \, dx \leq C_{p,\lambda,w,\varphi} \int_{-\infty}^{\infty} |f(x)|^p (M^{-p/2+1}(w))(x) \, dx
\]
for every function $f$.

**3. Proof of the results.** The following lemma and remark will be used in the proof of Theorem A.

**Lemma 1.** Let $\varphi \in C_0^\infty$ with $\text{supp}(\varphi) \subset [-2^s, 0]$, $s \geq 0$, and $\int \varphi(x) \, dx = 0$. Then
\[
\int_{-\infty}^{\infty} g_{\lambda,\varphi}^+(f)(x)^2w(x) \, dx \leq C_{\lambda}2^{s\lambda} \left( \int_{-\infty}^{\infty} |\tilde{\varphi}(t)|^2 \frac{dt}{|t|} \right) \int_{-\infty}^{\infty} |f(x)|^2 M^{-w}(x) \, dx,
\]
with a constant $C_{\lambda}$ depending neither on $f$ nor on $\varphi$.

**Proof.** By Fubini’s theorem, we have
\[
\int_{-\infty}^{\infty} g_{\lambda,\varphi}^+(f)(x)^2w(x) \, dx
\]
\[
= \int_{-\infty}^{\infty} \int_0^\infty \int_x^{t+y-x} \frac{t}{t+y-x} \lambda |f \ast \varphi_t(y)|^2 \frac{dy \, dt}{t^2} w(x) \, dx
\]
\[
= \int_{-\infty}^{\infty} \int_0^\infty |f \ast \varphi_t(y)|^2 \frac{1}{t} \int_{-\infty}^y \frac{t}{t+y-x} \lambda w(x) \, dx \frac{dy \, dt}{t}.
\]
For each integer $k$, we consider the set
\[
A_k = \left\{ (y, t) : 2^{k-1} < \frac{1}{t} \int_{-\infty}^y \frac{t}{t+y-x} \lambda w(x) \, dx \leq 2^k \right\}.
\]
Then
\begin{equation}
\int_{-\infty}^{\infty} \frac{g_{x,\varphi}^+(f)(x)^2 w(x)}{dx} \leq \sum_{k \in \mathbb{Z}} 2^k \int_{0}^{\infty} \int_{-\infty}^{\infty} |f * \varphi_t(y)|^2 \chi_{A_k(y, t)} \frac{dy dt}{t}.
\end{equation}

For every \((y, t)\) belonging to \(A_k\) and \(y \leq z \leq y + 2^s t\), we have
\[ \frac{1}{t} \int_{-\infty}^{z} \left( \frac{t}{t + z - x} \right)^{\lambda} w(x) dx \geq \frac{1}{2(s+1)\lambda} \frac{1}{t} \int_{-\infty}^{y} \left( \frac{t}{t + y - x} \right)^{\lambda} w(x) dx \]
\[ > \frac{2^{k-1}}{2(s+1)\lambda}. \]

On the other hand, since \(\lambda > 1\), there exists a constant \(C_{\lambda}\) such that for every \(z\),
\[ \frac{1}{t} \int_{-\infty}^{z} \left( \frac{t}{t + z - x} \right)^{\lambda} w(x) dx \leq C_{\lambda} M^{-w}(z). \]

Therefore, if \((y, t) \in A_k\) and \(y \leq z \leq y + 2^s t\) then \(z\) belongs to \(E_k = \{z : M^{-w}(z) \geq (C_{\lambda}/2(s+1)\lambda)^2k^{-1}\}\). Taking into account that \(\text{supp}(\varphi) \subset [-2^s, 0]\), we get
\[ f * \varphi_t(y) = \int f(z) \chi_{E_k}(z) \varphi_t(y - z) \, dz = (f \chi_{E_k} * \varphi_t)(y). \]

Then, by Plancherel’s and Fubini’s theorems, (2) is majorized by
\[ \sum_{k \in \mathbb{Z}} 2^k \int_{0}^{\infty} \int_{-\infty}^{\infty} |f \chi_{E_k} * \varphi_t(y)|^2 \frac{dy dt}{t} = \sum_{k \in \mathbb{Z}} 2^k \int_{-\infty}^{\infty} |\hat{\chi}_{E_k}(y)|^2 \int_{0}^{\infty} |\hat{\varphi}(ty)|^2 \frac{dt}{t} \, dy. \]

The inner integral is bounded by \(C_{\varphi} = \int_{-\infty}^{\infty} (|\hat{\varphi}(t)|/|t|) \, dt\). Thus, applying Plancherel’s theorem again, we get
\[ \int_{-\infty}^{\infty} g_{x,\varphi}^+(f)(x)^2 w(x) dx \leq C_{\varphi} \int_{-\infty}^{\infty} |f(y)|^2 \sum_{k \in \mathbb{Z}} 2^k \chi_{E_k}(y) dy. \]

Finally, we observe that by the definition of \(E_k\),
\[ \sum_{k \in \mathbb{Z}} 2^k \chi_{E_k}(y) \leq C_{\lambda} 2^{s\lambda} M^{-w}(y) \]
for almost every \(y\), ending the proof of the lemma. \(\blacksquare\)

**Remark.** We observe that if \(\varphi \in S\) and \(\int \varphi(x) \, dx = 0\), then
\begin{equation}
\int_{-\infty}^{\infty} |\hat{\varphi}(s)|^2 \frac{ds}{|s|} \leq 4\pi^2 \left( \int_{-\infty}^{\infty} |s||\varphi(s)| \, ds \right)^2 + \int_{-\infty}^{\infty} |\varphi(s)|^2 \, ds.
\end{equation}

In fact, since \(\int \varphi(x) \, dx = 0\), we have
\[ |\hat{\varphi}(s)| = \left| \int_{-\infty}^{\infty} \varphi(t)(e^{-2\pi ist} - 1) \, dt \right| \leq 2\pi |s| \int_{-\infty}^{\infty} |t| |\varphi(t)| \, dt. \]
Consequently,
\[
\int_{|s| \leq 1} |\hat{\varphi}(s)|^2 \frac{ds}{|s|} \leq 4\pi^2 \left( \int_{-\infty}^{\infty} |s| |\varphi(s)| \, ds \right)^2.
\]

On the other hand, in view of Plancherel’s theorem
\[
\int_{|s| \geq 1} |\hat{\varphi}(s)|^2 \frac{ds}{|s|} \leq \int_{-\infty}^{\infty} |\hat{\varphi}(s)|^2 \, ds \leq \int_{-\infty}^{\infty} |\varphi(s)|^2 \, ds,
\]
which shows that (3) holds.

Let \( \eta \) be a non-negative and \( C^\infty_0 \)-function with support contained in \([-2, -1]\) and \( \int \eta(x) \, dx = 1 \). For every non-negative integer \( k \), let \( \eta_k(x) = 2^{-k} \eta(2^{-k} x) \). We define
\[
\theta(x) = \int_{|x|/2 \leq |t| \leq |x|} \eta(t) \, dt.
\]
Then \( \theta \in C^\infty_0 \) and \( \text{supp}(\theta) \subset [-4, -1] \cup [1, 4] \). For every positive integer \( k \), let
\[
\theta_k(x) = \theta(2^{-k+1} x),
\]
and for \( k = 0 \), let
\[
\theta_0(x) = 1 - \int_{|y| \leq |x|} \eta(y) \, dy.
\]
Then \( \sum_{k=0}^{\infty} \theta_k(x) = 1 \) for every \( x \). Given \( \varphi \in S \) with \( \text{supp}(\varphi) \subset (-\infty, 0] \) and \( \int \varphi(x) \, dx = 0 \), we define
\[
a_k = \sum_{h=0}^{k} \int \theta_h(y) \varphi(y) \, dy, \quad k \geq 0, \quad a_{-1} = 0.
\]
For every non-negative integer \( k \), let \( \varphi_k \) be given by
\[
(4) \quad \varphi_k(x) = \theta_k(x) \varphi(x) + a_{k-1} \eta_{k-1}(x) - a_k \eta_k(x).
\]
It is easy to check that \( \text{supp}(\varphi_k) \subset [-2^{k+1}, -2^{-k-1}] \) for \( k \geq 1 \), and \( \text{supp}(\varphi_0) \subset [-2, 0] \). Moreover, \( \int \varphi_k(x) \, dx = 0 \) for every \( k \geq 0 \), and \( \sum_{k=0}^{\infty} \varphi_k = \varphi \). We shall show that for every \( N > 2 \),
\[
C_{\varphi_k} = \int_{-\infty}^{\infty} |\hat{\varphi}_k(s)|^2 \frac{ds}{|s|} \leq C_{N, \varphi} 2^{-2k(N-2)}.
\]

By definition of \( \varphi_k \),
\[
(5) \quad \left( \int_{-\infty}^{\infty} |\varphi_k(x)|^2 \, dx \right)^{1/2} \leq \left( \int_{-\infty}^{\infty} |\theta_k(x) \varphi(x)|^2 \, dx \right)^{1/2}
\]
\[+ |a_{k-1}| \left( \int_{-\infty}^{\infty} |\eta_{k-1}(x)|^2 \, dx \right)^{1/2} + |a_k| \left( \int_{-\infty}^{\infty} |\eta_k(x)|^2 \, dx \right)^{1/2}.
\]
Since $0 \leq \theta_k(x) \leq 1$ and $\text{supp}(\theta_k \varphi) \subset [-2^{k+1}, -2^{k-1}]$ for $k \geq 1$, and $\text{supp}(\theta_0 \varphi) \subset [-2, 0]$, we have

\[
(\int_{-\infty}^{\infty} |\theta_k(x) \varphi(x)|^2 \, dx)^{1/2} \leq \left( \int_{\text{supp}(\theta_k \varphi)} \frac{C_{N, \varphi}}{(1 + |x|)^N} \, dx \right)^{1/2} \leq C_{N, \varphi} 2^{-k(N-1/2)}.
\]

By definition of $a_k$, and taking into account that $\int \phi(x) \, dx = 0$, we get

\[
|a_k| = -\int \sum_{h=k+1}^{\infty} \theta_h(y) \varphi(y) \, dy \leq \int |\varphi(y)| \, dy \leq C_{N, \varphi} \int \frac{dy}{(1 + |y|)^N} \leq C_{N, \varphi} 2^{-k(N-1)}.
\]

Thus,

\[
(8) \quad |a_k| \left( \int_{-\infty}^{\infty} |\eta_k(x)|^2 \, dx \right)^{1/2} = \frac{|a_k|}{2k/2} \left( \int_{-\infty}^{\infty} |\eta(x)|^2 \, dx \right)^{1/2} \leq C_{N, \varphi} 2^{-k(N-1/2)}.
\]

Then, by (6)–(8),

\[
\int_{-\infty}^{\infty} |g_k(x)|^2 \, dx \leq C_{N, \varphi} 2^{-2k(N-1/2)}.
\]

Simple calculations show that

\[
\int_{-\infty}^{\infty} |x| \, |g_k(x)|^2 \, dx \leq C_{N, \varphi} 2^{-2k(N-2)}.
\]

Now, using (3) we obtain (5).

**Proof of Theorem A.** We consider the sequence of functions $\{g_k, k \geq 0\}$ defined in (4). Since $\sum_{k=0}^{\infty} g_k = \varphi$ and $\sum_{k=0}^{\infty} \chi_{\text{supp}(g_k)}(x) \leq 3$, we have

\[
f * \varphi_t(y) = \sum_{k=0}^{\infty} f * (g_k)_t(y)
\]

for every $y$. Then

\[
(9) \quad \left( \int_{-\infty}^{\infty} g_{\lambda, \varphi}^+(f)(x)^2 w(x) \, dx \right)^{1/2} \leq \sum_{k=0}^{\infty} \left( \int_{-\infty}^{\infty} \int_{0}^{\infty} \left( \frac{t}{t + y - x} \right)^\lambda |f * (g_k)_t(y)|^2 \frac{dy \, dt}{t^2} \, w(x) \, dx \right)^{1/2} = \sum_{k=0}^{\infty} \left( \int_{-\infty}^{\infty} g_{\lambda, \varphi}^+(f)(x)^2 w(x) \, dx \right)^{1/2}.
\]
Keeping in mind that \( \text{supp}(g_k) \subset [-2^{k+1}, 0] \) and \( \int g_k(x) \, dx = 0 \), we can apply Lemma 1. Then, by the estimate (5) with \( N > \lambda + 2 \), we find that (9) is bounded by a constant times

\[
\sum_{k=0}^{\infty} 2^{(k+1)\lambda/2} \left( \int_{-\infty}^{\infty} |\hat{g}_k(t)|^2 \frac{dt}{|t|} \right)^{1/2} \left( \int_{-\infty}^{\infty} |f(x)|^2 M^- w(x) \, dx \right)^{1/2} \leq C_{\lambda, \varphi} \int_{-\infty}^{\infty} |f(x)|^2 M^- w(x) \, dx \right)^{1/2}.
\]

In order to prove Theorem B, we shall need the following one-sided Fefferman–Stein type inequality and Lemma 11.

**Lemma 10.** There exists a positive constant \( C \), such that

\[
w(\{ x : M^+(f)(x) > \mu \}) \leq \frac{C}{\mu} \int_{-\infty}^{\infty} |f(x)| M^- w(x) \, dx
\]

for every function \( f \), and \( \mu > 0 \).

**Proof.** The proof is similar to the proof of Theorem 1 in [M, p. 693], and it shall not be given.

**Lemma 11.** Let \( I = (\alpha, \beta) \), a bounded interval, \( 1 < \lambda < 2 \), and \( k \geq 4 \). Then there exists a constant \( C_{\lambda, k} \) such that for every \( x < \alpha - 2|I| \),

\[
\int_{0}^{\alpha - 2|I|} \int_{x}^{\alpha - y} \left( \frac{t}{t+y-x} \right)^{\lambda} \left( \frac{t}{t+\alpha-y} \right)^{k} \frac{dy \, dt}{t^4} \leq C_{\lambda, k} \frac{|I|^{\lambda-2}}{(\alpha-x)^\lambda}.
\]

**Proof.** Changing the variables \((y, t)\) to

\[
z = (\alpha - y)/t \quad \text{and} \quad u = (\alpha - x)/t,
\]

we obtain

\[
\int_{0}^{\infty} \int_{\alpha - x \geq \alpha - y \geq 2|I|} \left( \frac{1}{1 + \frac{y-z}{t}} \right)^{\lambda} \left( \frac{1}{1 + \frac{\alpha-y}{t}} \right)^{k} \frac{dy \, dt}{t^4} = \frac{1}{(\alpha-x)^2} \int_{0}^{\infty} \int_{u \geq 2|I|u/(\alpha-x)} \frac{1}{(1 + u - z)^\lambda} \frac{1}{(1 + z)^k} \, u \, du \, dz.
\]

We set \( A = 2|I|/(\alpha-x) \). Applying Fubini’s theorem, it is enough to show that

\[
\int_{0}^{\infty} \frac{1}{(1 + z)^k} \int_{z \leq u \leq z/A} \frac{u}{(1 + u - z)^\lambda} \, du \, dz \leq C_{\lambda, k} A^{\lambda-2}.
\]
Recalling that $1 < \lambda < 2$, we have
\begin{align*}
\int_0^\infty \frac{1}{(1+z)^k} \int_{z \leq u \leq z/A, u-z > u/2} \frac{u}{(1+u-z)^\lambda} du \, dz \\
\leq \int_0^\infty \frac{1}{(1+z)^k} \int_0^{z/A} \frac{2^\lambda}{u} u du \, dz = C_\lambda \int_0^\infty \frac{1}{(1+z)^k} \left( \frac{z}{A} \right)^{2-\lambda} \, dz = A^{\lambda-2}.
\end{align*}
Since $k \geq 4$, $A < 1$ and $\lambda < 2$, it follows that
\begin{align*}
\int_0^\infty \frac{1}{(1+z)^k} \int_0^{z/A} \frac{2^\lambda}{u} u du \, dz \\
\leq \int_0^\infty \frac{1}{(1+z)^k} \int_0^{2z} u du \, dz = 2 \int_0^\infty \frac{z^2}{(1+z)^k} \, dz \leq C_k A^{\lambda-2},
\end{align*}
which ends the proof of the lemma.

**Proof of Theorem B.** By a density argument it is enough to consider $f \in L^p(M^-w) \cap L^p$. It is well known that the set $\Omega = \{ x : M^+(|f|^p)(x)^{1/p} > \mu \}$ is open. Let $\{I_j\}_{j \geq 1}$ be its connected components. Since $f \in L^p$, each $I_j$ is a bounded interval, and it is well known (see [HSt, pp. 421–424]) that
\begin{equation}
\frac{1}{|I_j|} \int_{I_j} |f(x)|^p \, dx = \mu^p.
\end{equation}
Given $I_j = (\alpha_j, \beta_j)$, we write $I_j^- = (\alpha_j - 4|I_j|, \alpha_j)$. By (12), we have
\begin{align*}
w(I_j^-) &= \frac{1}{\mu^p} \int_{I_j} |f(x)|^p \frac{w(I_j^-)}{|I_j|} \, dx \leq \frac{5}{\mu^p} \int_{I_j} |f(x)|^p M^-w(x) \, dx.
\end{align*}
Therefore, if we define $\tilde{\Omega} = \bigcup_{j \geq 1} I_j \cup I_j^-$, applying Lemma 10 we obtain
\begin{align*}
w(\tilde{\Omega}) &\leq w(\Omega) + \sum_{j \geq 1} w(I_j^-) \\
&\leq \frac{C}{\mu^p} \int_{-\infty}^\infty |f(x)|^p M^-w(x) \, dx + \frac{5}{\mu^p} \sum_{j \geq 1} \int_{I_j} |f(x)|^p M^-w(x) \, dx \\
&\leq \frac{C}{\mu^p} \int_{-\infty}^\infty |f(x)|^p M^-w(x) \, dx.
\end{align*}
Consequently, it is enough to prove that
\begin{equation}
w(\{ x \notin \tilde{\Omega} : g^+_{\lambda, \phi}(f)(x) > \mu \}) \leq \frac{C}{\mu^p} \int_{-\infty}^\infty |f(x)|^p M^-w(x) \, dx.
\end{equation}
We define
\[ g(x) = f(x)\chi_{\Omega_c}(x) + \sum_{j \geq 1} \left( \frac{1}{|I_j|} \int_{I_j} f \right) \chi_{I_j}(x), \]
\[ b_j(x) = \left( f(x) - \frac{1}{|I_j|} \int_{I_j} f \right) \chi_{I_j}(x), \quad j \geq 1. \]

Then \( f = g + b \) where \( b = \sum_{j \geq 1} b_j \).

By Chebyshev’s inequality and applying Theorem A, we get
\[ w(\{ x \notin \tilde{\Omega} : g^+(f)(x) > \mu \}) \leq \frac{1}{\mu^p} \int_{\tilde{\Omega}c} g^+(g)(x)^p w(x) \, dx \]
\[ \leq \frac{C}{\mu^p} \int_{-\infty}^{\infty} |g(x)|^2 M^-(w\chi_{\tilde{\Omega}c})(x) \, dx \]
\[ = \frac{C}{\mu^p} \int_{-\infty}^{\infty} |g(x)|^{2-p} |g(x)|^p M^-(w\chi_{\tilde{\Omega}c})(x) \, dx. \]

We observe that \( |g(x)| \leq \mu \) almost everywhere. Then, by the definition of \( g \) and Hölder’s inequality, (14) is bounded by
\[ \frac{C}{\mu^p} \left[ \int_{\Omega_c} |f(x)|^p M^-(w\chi_{\tilde{\Omega}c})(x) \, dx + \sum_{j \geq 1} \left( \frac{1}{|I_j|} \int_{I_j} |f(z)|^p \, dz \right) M^-(w\chi_{\tilde{\Omega}c})(x) \, dx \right]. \]

It is easy to see that \( M^-(w\chi_{\tilde{\Omega}c})(x) \leq CM^-w(z) \) for every \( x, z \in I_j \). Thus,
\[ w(\{ x \notin \tilde{\Omega} : g^+(g)(x) > \mu \}) \leq \frac{C}{\mu^p} \int_{-\infty}^{\infty} |f(x)|^p M^-w(x) \, dx. \]

We define \( I_j^* = (\alpha_j - 2|I_j|, \beta_j) \) for every \( j \geq 1 \). We can write
\[ g^+_\lambda,\varphi(b)(x) \leq g^1(x) + g^2(x), \]
where
\[ g^1(x) = \left( \int_0^\infty \int_x^\infty \left( \frac{t}{t + y - x} \right)^\lambda \sum_{i \notin I_j^*} b_i * \varphi_t(y) \, \frac{dy \, dt}{t^2} \right)^{1/2}, \]
\[ g^2(x) = \left( \int_0^\infty \int_x^\infty \left( \frac{t}{t + y - x} \right)^\lambda \sum_{i \in I_j^*} b_i * \varphi_t(y) \, \frac{dy \, dt}{t^2} \right)^{1/2}. \]

Let us consider \( g^1(x) \). Taking into account that \( b_i * \varphi_t(y) = 0 \) if \( y > \beta_i \), and \( \int |b_i(z)| \, dz \leq 2|I_i| \mu \), it follows that
\[ \left| \sum_{i : y \notin I_j^*} b_i * \varphi_t(y) \right| \leq \frac{2\mu}{t} \sum_{i : y \notin I_j^*, y < \beta_i} |I_i| \sup_{z \in I_i} \varphi \left( \frac{y - z}{t} \right). \]
Since $\varphi \in \mathcal{S}$, and $\text{supp}(\varphi) \subset (-\infty, 0]$, we deduce that
\[
\left| \varphi \left( \frac{y - z}{t} \right) \right| \leq \frac{C}{(1 + \frac{w-y}{t})^2} \text{ for } y \notin I_i^* \text{ and } z, w \in I_i.
\]
Then
\[
\left| \sum_{i: y \notin I_i^*} b_i * \varphi_t(y) \right| \leq \frac{C\mu}{t} \sum_{i: y \notin I_i^*, y < \beta} \int \frac{dw}{(1 + \frac{w-y}{t})^2} \leq c\mu.
\]
Therefore,
\[
g^1(x)^2 \leq C\mu \int \int \left( \frac{t}{t + y - x} \right)^\lambda \sum_{i: y \notin I_i^*} |b_i * \varphi_t(y)| \frac{dy dt}{t^2} = C\mu F(x),
\]
and by Chebyshev’s inequality we get
\[
(17) \quad w(\{x \notin \tilde{\Omega} : g^1(x) > \mu\}) \leq \frac{C}{\mu} \int F(x)w(x) \, dx.
\]
Since $\int b_i(z) \, dz = 0$, applying the mean value theorem, for every $y \leq \alpha_i - 2|I_i|$ we obtain the estimate
\[
|b_i * \varphi_t(y)| \leq \frac{1}{t} \int |b_i(z)| \left| \varphi \left( \frac{y - z}{t} \right) - \varphi \left( \frac{y - \alpha_i}{t} \right) \right| \, dz
\]
\[
\leq \frac{C}{t} \int |b_i(z)| \left| z - \alpha_i \right| \left( \frac{t}{t + \alpha_i - y} \right)^4 \, dz
\]
\[
\leq C|I_i| \frac{t^2}{(t + \alpha_i - y)^4} \int f(z) \, dz.
\]
Then, by the definition of $F(x)$, (17) is majorized by
\[
(18) \quad \frac{C}{\mu} \sum_{i \geq 1} \int |f(z)| \, dz \int_{\tilde{\Omega}^c} \int_{0}^{\infty} \int_{\tilde{\Omega}^c} \left( \frac{t}{t + y - x} \right)^{\lambda'}
\]
\[
\times \frac{1}{(t + \alpha_i - y)^4} \, dy \, dt \, w(x) \, dx,
\]
where $1 < \lambda' < \inf(\lambda, 2)$. Now, applying Lemma 11 with $k = 4$, we find that
(18) is bounded by
\[
\frac{C}{\mu} \sum_{i \geq 1} \int |f(z)| \, dz \int_{-\infty}^{\alpha_i - 4|I_i|} \frac{|I_i|^{\lambda' - 1}}{(\alpha_i - x)^{\lambda'}} \, w(x) \chi_{\tilde{\Omega}^c}(x) \, dx.
\]
The inner integral is bounded by $CM^- (w \chi_{\tilde{\Omega}^c})(\alpha_i)$. It is easy to verify that, by Hölder’s inequality and (12),
\[
\frac{1}{\mu} \int_{I_i} |f| \leq \frac{1}{\mu^p} \int_{I_i} |f|^p.
\]

Thus, we obtain

\[w(\{x \notin \tilde{\Omega} : g^1(x) > \mu\}) \leq \frac{C}{\mu^p} \sum_i \left( \int_{I_i} |f(z)|^p \, dz \right) M^{-}(w \chi_{\tilde{\Omega}^c})(\alpha_i) \]

\[\leq \frac{C}{\mu^p} \int_{-\infty}^{\infty} |f(z)|^p M^{-}w(z) \, dz.\]

Now, let us consider \(g^2(x)\). By (12), there exists an integer \(k_0\) such that \(|I_j| \leq \|f\|_p^p \mu^{-p} \leq 2^{k_0}\) for every \(j \geq 1\). Let \(A_k = \{j : 2^{k-1} < |I_j| \leq 2^k\}\), \(k \leq k_0\). We can write

\[\bigcup_{j \geq 1} I^*_j = \bigcup_{k \leq k_0} \bigcup_{j \in A_k} E^*_j,\]

where \(E^*_j = I^*_j \setminus \bigcup_{l > k} \bigcup_{s \in A_l} I^*_s\) for each \(j \in A_k\). We observe that if \(I^*_j \cap E^*_j\) is not empty then \(I^*_j \subset I^*_j\), where \(I^*_j\) is the interval with the same center of \(I_j\) and with measure \(20|I_j|\). For each \(x \notin \tilde{\Omega}\), we have

\[g^2(x)^2 = \sum_{k \leq k_0} \sum_{j \in A_k} \sum_{0 < y, y \in E^*_j} \left( \int_{0}^{\infty} \lambda | \sum_{i : y \in I^*_j} b_i \ast \varphi_t(y) |^{\lambda} \, dy \right) \left( \frac{t}{t+y-x} \right)^{\lambda} \sum_{i : y \in I^*_j} b_i \ast \varphi_t(y) \right)^2 \frac{dy \, dt}{t^2}.
\]

We observe that if \(x \notin \tilde{\Omega}^c\), \(x < y\) and \(y \in E^*_j\) then \(x < \alpha_j - 4|I_j|\) and \(t + y - x \geq (\alpha_j - x) - (\alpha_j - y) \geq (\alpha_j - x)/2\). Then

\[g^2(x)^2 \leq C \sum_{k \leq k_0} \sum_{j \in A_k, x < \alpha_j} \frac{1}{(\alpha_j - x)^{\lambda}} \times \int_{0}^{\infty} \int_{0}^{\infty} t^{\lambda-2} \left( \sum_{i : y \in I^*_j} b_i \ast \varphi_t(y) \right)^2 \, dy \, dt.
\]

If we define \(D_j = \bigcup_{i : E^*_j \cap I^*_i \neq \emptyset} I_i\) and \(b^j(x) = |b(x)| \chi_{D_j}(x)\) then, for every \(y \in E^*_j\), we obtain

\[\left| \sum_{i : y \in I^*_j} b_i \ast \varphi_t(y) \right| \leq \sum_{i : y \in I^*_j} \int_{I_i} |b(z)| |\varphi_t(y - z)| \, dz \leq \int_{D_j} |b(z)| |\varphi_t(y - z)| \, dz = (b^j \ast |\varphi_t|)(y).
\]
Consequently, by (20), we have

\[(21) \quad g^2(x)^2 \leq C \sum_{k \leq k_0} \sum_{j \in A_k, x < \alpha_j} \frac{1}{(\alpha_j - x)^\lambda} \times \int_0^\infty \int_{x < y, y \in E_j^*} t^{\lambda - 2} |(b^j * |\varphi|_t)(y)|^2 \, dy \, dt.\]

We claim that

\[(22) \quad \int_0^\infty \int_{E_j^*} t^{\lambda - 2} |(b^j * |\varphi|_t)(y)|^2 \, dy \, dt \leq C |E_j^*|^{\lambda - 2/p} \|b^j\|_{p}^2.\]

In fact, by Fubini's theorem, we have

\[
\int_0^\infty \int_{E_j^*} t^{\lambda - 2} |(b^j * |\varphi|_t)(y)|^2 \, dy \, dt = \int_y b^j(z) \int_y b^j(w) \int_0^\infty t^{\lambda - 4} |\varphi| \left(\frac{y - z}{t}\right) |\varphi| \left(\frac{y - w}{t}\right) \, dt \, dw \, dz.
\]

Since \(\varphi \in \mathcal{S}\), and \(\lambda < 3\),

\[
\int_0^\infty t^{\lambda - 4} |\varphi| \left(\frac{y - z}{t}\right) |\varphi| \left(\frac{y - w}{t}\right) \, dt \leq C \int_0^\infty \frac{1}{(1 + \frac{z - y}{t})^2} \left(\frac{1}{(1 + \frac{w - y}{t})^2}\right) \, dt \leq C \int_0^\infty \frac{t^{\lambda - 4}}{(1 + \frac{z + w - 2y}{t})^2} \, dt = C_\lambda (z + w - 2y)^{\lambda - 3}.
\]

Then the left hand side of (22) is bounded by

\[
C \int_{E_j^*} \int_y^\infty \int_y^\infty b^j(z) b^j(w) \frac{1}{(z + w - 2y)^{3-\lambda}} \, dw \, dz \, dy \leq C' \int_{E_j^*} \int_y^\infty \frac{b^j(z)}{(z - y)^{(3-\lambda)/2}} \, dz \int_y^\infty \frac{b^j(w)}{(w - y)^{(3-\lambda)/2}} \, dw \, dy \leq C' \int_{E_j^*} |I_{(\lambda-1)/2}^+ (b^j)(y)|^2 \, dy,
\]

where \(I_{(\lambda-1)/2}^+\) denotes the one-sided fractional integral operator of order \((\lambda - 1)/2\). In the case \(1 < p < 2\) and \(\lambda = 2/p\), since, as is well known, \(I_{(\lambda-1)/2}^+\) is a bounded operator from \(L^p\) to \(L^2\), it follows that (22) holds.
For $2 < \lambda < 3$, the operator $I_{(\lambda-1)/2}^+$ maps $L^1$ into weak-$L^{2/(3-\lambda)}$. Then, by Kolmogorov’s condition (see [GRu, p. 485]), we obtain (22).

On the other hand, since $\int |b_i(y)|^p \, dy \leq (2\mu)^p |I_i|$, we have

$$\|b_j\|_p \leq \left( \sum_{i : E_i^c \cap I_i^c \neq \emptyset} (2\mu)^p |I_i| \right)^{1/p} \leq 2\mu |I_j|^1/p = C\mu |I_j|^1/p.$$ 

Therefore, by (21) and (22) we get

$$g^2(x)^2 \leq C' \mu^2 \sum_{k=k_0}^{\infty} \sum_{j \in A_k, x < \alpha_j} |I_j|^{\lambda} \frac{(\alpha_j - x)^{\lambda}}{\lambda}.$$ 

Consequently,

$$w(\{x \notin \tilde{\Omega} : g^2(x) > \mu\}) \leq C \sum_j |I_j|^{\lambda} \int_{-\infty}^{\alpha_j-4|I_j|} w(x) \chi_{\tilde{\Omega}_e}(x) \frac{w(x) \chi_{\tilde{\Omega}_e}(x)}{(\alpha_j - x)^{\lambda}} \, dx$$

$$\leq \frac{C}{\mu^p} \sum_j \int_{I_j} |f(z)|^p \, dz M^- (w \chi_{\tilde{\Omega}_e})(\alpha_j)$$

$$\leq \frac{C}{\mu^p} \int_{-\infty}^{\infty} |f(z)|^p M^- w(z) \, dz.$$ 

From (15), (16), (19) and (23) we deduce that (13) holds for $\lambda = 2/p$ if $1 < p < 2$ and for $2 < \lambda < 3$ if $p = 1$. Taking into account that if $\lambda_1 \leq \lambda_2$ then $g_{\lambda_2,\varphi}^+(f)(x) \leq g_{\lambda_1,\varphi}^+(f)(x)$, the proof of the theorem is complete.

We now deduce Theorem C from Theorems A and B.

**Proof of Theorem C.** The case $p = 2$ and $\lambda > 1$ was considered in Theorem A. Let $1 < p < 2$ and $2/p < \lambda < 2$. We have $\lambda = 2/q$ with $1 < q < p$. Then, by Theorem B, $g_{\lambda,\varphi}^+$ maps $L^q(M^- w)$ into weak-$L^q(w)$. Since $g_{\lambda,\varphi}^+$ is bounded from $L^2(M^- w)$ to $L^2(w)$, by interpolation, we get the assertion for $\lambda < 2$. The case $\lambda \geq 2$ follows by simple arguments.

The following remark shows that for $\lambda = 2$ and $p = 1$, a weak type inequality as in Theorem B cannot be valid.

**Remark.** Let $\varphi \neq 0$ belong to $\mathcal{S}$ with $\text{supp}(\varphi) \subset [-1,0]$ and $\int \varphi(x) \, dx = 0$. There exists $f \in L^1$ such that $g_{2,\varphi}^+(f)(x) = \infty$ for every $x$ belonging to an unbounded set.

In fact, we consider

$$f(t) = \left( \frac{1}{|t| \ln^{3/2}(1/|t|)} - c \right) \chi_{[-1/2,0]}(t),$$
where \( c \) is the unique constant such that \( \int f(t) \, dt = 0 \). For every \( x < -4 \), we have

\[
(24) \quad g^+_{2,\varphi}(f)(x)^2 \geq \frac{1}{(1-x)^2} \int_0^1 |f * \varphi_t(y)|^2 \, dy \, dt.
\]

The support of \( f * \varphi_t \) is contained in \((-\infty, 0]\) and the fractional integral \( I_{1/2}(f) \notin L^2 \) (see [Z, p. 232]). Then Plancherel’s theorem yields

\[
A := \int_0^\infty \int_0^\infty |f * \varphi_t(y)|^2 \, dy \, dt = \int_0^\infty \int_{-\infty}^\infty |\hat{f}(ty)|^2 |\hat{\varphi}(y)|^2 \, dy \, dt \\
\geq C_{\varphi} \int_{-\infty}^\infty \frac{|\hat{f}(y)|^2}{|y|} \, dy \\
= C_{\varphi} \int_{-\infty}^\infty |I_{1/2}(f)(y)|^2 \, dy = \infty.
\]

Applying the mean value theorem, for every \( y \leq -2 \) we obtain

\[
|f * \varphi_t(y)| \leq \frac{1}{t} \int_{-1/2}^0 |f(z)| \left| \varphi\left( \frac{y-z}{t} \right) - \varphi\left( \frac{y}{t} \right) \right| \, dz \\
\leq \frac{1}{t} \int_{-1/2}^0 |f(z)| \left| \frac{z}{t} \right| C_{\varphi} \left( \frac{t}{t+|y|} \right)^2 \, dz \leq C_{\varphi} \frac{1}{(t+|y|)^2}.
\]

Using these inequalities we get

\[
A_1 := \int_0^\infty \int_{-\infty}^{-2} |f * \varphi_t(y)|^2 \, dy \, dt \leq C \int_{-\infty}^{-2} \int_0^\infty \frac{1}{(t+|y|)^4} \, dy \, dt < \infty.
\]

Since \( |f * \varphi_t(y)| \leq \frac{1}{t} \| \varphi \|_\infty \| f \|_1 \), we have

\[
A_2 := \int_0^\infty \int_{-2}^0 |f * \varphi_t(y)|^2 \, dy \, dt \leq C \int_{-2}^0 \int_0^\infty \frac{1}{t^2} \, dy \, dt < \infty.
\]

By (24) and the estimates obtained for \( A, A_1, \) and \( A_2 \) it follows that \( g^+_{2,\varphi}(f)(x) = \infty \) for every \( x < -4 \).

To prove Theorem D, we proceed as in Theorem 1.10 of [P, p. 150].

**Proof of Theorem D.** More generally, we shall prove that

\[
\int_{-\infty}^{\infty} g^+_{\lambda,\varphi}(f)(x)^p M^{-}_B(w^{2/p})(x)^{p/2} \, dx \\
\geq \int_{-\infty}^{\infty} f(x)^p M^{-}_B(w^{2/p})(x)^{p/2} \, dx,
\]

where \( \lambda \) is the unique constant such that \( \int f(t) \, dt = 0 \). For every \( x < -4 \), we have

\[
(24) \quad g^+_{2,\varphi}(f)(x)^2 \geq \frac{1}{(1-x)^2} \int_0^1 |f * \varphi_t(y)|^2 \, dy \, dt.
\]
where $B$ is a Young function that satisfies
\begin{equation}
\int_c^\infty \left( \frac{tp/2}{B(t)} \right)^{(p/2)'} - 1 \frac{dt}{t} < \infty.
\end{equation}

In the case $B(t) \approx t^{p/2}(1 + \ln t)^{[p/2]}$, we get Theorem D.

Let $r = p/2$. We have

$$I = \| g^+_{\lambda,\varphi}(f) \|_{L^p(w)}^2 = \| g^+_{\lambda,\varphi}(f) \|^2 \| w^{1/r} \|_{L^r} = \int_{-\infty}^\infty g^+_{\lambda,\varphi}(f)(x)^2 w(x)^{1/r} g(x) \, dx,$$

for some $g \in L^{r'}$ with unit norm. We recall that

$$M^-((g_1 g_2)(x)) \leq M^-_B(g_1)(x) M^-_B(g_2)(x),$$

where $B$ is the complementary function to $B$. Then Theorem A and Hölder’s inequality yield

$$I \leq C \int_{-\infty}^\infty |f(x)|^2 M^-((w^{1/r}g)(x)) \, dx
\leq C \int_{-\infty}^\infty |f(x)|^2 M^-_B(w^{1/r})(x) M^-_B(g)(x) \, dx
\leq C \left( \int_{-\infty}^\infty |f(x)|^p M^-_B(w^{1/r})(x)^{p/2} \, dx \right)^{2/p} \left( \int_{-\infty}^\infty M^-_B(g)(x)^{r'} \, dx \right)^{1/r'}$$

$$= C \| f \|^2_{L^p(v)} \| M^-_B(g) \|_{L^{r'}},$$

where $v = M^-_B(w^{1/r})(x)^{r'}$. By Theorem 2.6 in [RiRoT], if $B$ satisfies (25), then

$$I \leq C \| f \|^2_{LP(v)} \| g \|_{L^{r'}} \leq C \| f \|^2_{LP(v)}.$$

It is easy to check that $M^-_B(w^{1/r})(x)^r = M^-_B(w)(x)$, where $\tilde{B}(t) = B(t^{1/r})$. If $\tilde{B}(t) = t(1 + \ln t)^{[r]}$ then $B$ satisfies (25), and by Proposition 2.15 in [RiRoT] there exist two constants $C_1$ and $C_2$ such that

$$C_1 M^-_B(w)(x) \leq (M^-)^{[r]+1} w(x) \leq C_2 M^-_B(w)(x),$$

which completes the proof.  

References


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