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L^1 representation of Riesz spaces

by

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Abstract. Let E be a Riesz space. By defining the spaces L_E^1 and L_E^∞ of E, we prove that the center $Z(L_E^1)$ of L_E^1 is L_E^∞ and show that the injectivity of the Arens homomorphism $m : Z(E)'' \to Z(E^\sim)$ is equivalent to the equality $L_E^1 = Z(E)'$. Finally, we also give some representation of an order continuous Banach lattice E with a weak unit and of the order dual $E^\sim$ of E in L_E^1 which are different from the representations appearing in the literature.

1. Introduction. An ordered vector space E is called a *Riesz space* (or a *vector lattice*) if $\sup\{x,y\} = x \vee y$ (or $\inf\{x,y\} = x \wedge y$) exists in E for all $x, y \in E$. Sets of the form $[x,y] = \{z \in E : x \le z \le y\}$ are called *order intervals* or simply *intervals*. A subset A of E is said to be *order bounded* if A is included in some order interval.

A linear map T, between E and L, is said to be *order bounded* whenever T maps order bounded sets into order bounded sets. Order bounded linear maps between E and L will be denoted by $L_{\mathrm{b}}(E,L)$. We denote by $L_{\mathrm{b}}(E)$ the order bounded operators from E into itself and by $E^\sim$ the order bounded functionals on E. Furthermore, $E_{\mathrm{n}}^\sim$ will denote the order continuous members of $E^\sim$. The space $E^\sim$ is called the *order dual* of E. The norm dual of a Banach lattice E coincides with its order dual [3, p. 176].

A mapping $\pi \in L_{\mathrm{b}}(E)$ is called an *orthomorphism* if $x \perp y$ (i.e. $|x| \wedge |y| = 0$) implies $\pi x \perp y$. The set of orthomorphisms of E will be denoted by $\mathrm{Orth}(E)$. The principal order ideal generated by the identity operator I in $\mathrm{Orth}(E)$ is called the *ideal center* of E and is denoted by $Z(E)$ (i.e. $Z(E) = \{\pi \in L_{\mathrm{b}}(E) : |\pi| \le \lambda I \text{ for some } \lambda \in \mathbb{R}_+\}$).

If E is a uniformly complete Riesz space, then $Z(E)$ becomes a Banach lattice with respect to the I-uniform norm $\|\pi\| = \inf\{\lambda : |\pi| \le \lambda I, \lambda \in \mathbb{R}_+\}$. In particular, if E is a Banach lattice, the $Z(E)$ is a Banach lat-

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tice. Moreover, every operator norm coincides with the I-uniform norm on $Z(E)$. The space $Z(E)$ is an abstract AM-space and $Z(E)'$ is an abstract AL-space. Moreover, $Z(E)'$ has order continuous norm.

Let A be a Riesz algebra (lattice ordered algebra), i.e., A is a Riesz space which is simultaneously an associative algebra with the additional property that $a, b \in A_+$ implies that $ab \in A_+$. An *f-algebra* A is a Riesz algebra which satisfies the extra requirement that $a \perp b$ implies $ac \perp b = ca \perp b$ for all $c \in A_+$. Every Archimedean f-algebra is commutative. $\mathrm{Orth}(E)$ and $Z(E)$ are f-algebras under pointwise order and composition of operators.

Throughout, $E^\sim$ will be assumed to separate the points of E. This assumption implies that E is Archimedean.

Let us also recall that $(A^\sim)^\sim_{\mathrm{n}}$, the order continuous part of the order bidual $A^{\sim\sim}$, of an f-algebra A can be made an f-algebra, extending the product in A, whenever A has separating order dual. This is done as follows:

$$\begin{aligned}
&(1) && A \times A^\sim \to A^\sim, \\
& && (a, f) \mapsto f \cdot a: \quad f \cdot a(b) = f(a \cdot b) \quad \text{for } b \in A, \\
&(2) && (A^\sim)^\sim_{\mathrm{n}} \times A^\sim \to A^\sim, \\
& && (F, f) \mapsto F \cdot f: \quad F \cdot f(a) = F(f \cdot a) \quad \text{for } a \in A, \\
&(3) && (A^\sim)^\sim_{\mathrm{n}} \times (A^\sim)^\sim_{\mathrm{n}} \to (A^\sim)^\sim_{\mathrm{n}}, \\
& && (F, G) \mapsto F \cdot G: \quad F \cdot G(f) = F(G \cdot f) \quad \text{for } f \in A^\sim.
\end{aligned}$$

Then $(A^\sim)^\sim_{\mathrm{n}}$ is an f-algebra with the multiplication defined in step (3) (see [7]).

If $(A^\sim)^\sim_{\mathrm{n}}$ is an f-algebra with identity then the mapping $v : (A^\sim)^\sim_{\mathrm{n}} \to \mathrm{Orth}(A^\sim)$ defined as $F \mapsto v_F$ where $v_F(f) = F \cdot f$ for each $f \in A^\sim$ is an algebra isomorphism and a Riesz isomorphism [7].

Given the bilinear map $Z(E) \times E \to E$ defined by $(\pi, x) \mapsto \pi x$, consider its Arens extensions

$$\begin{aligned}
&(4) && E \times E^\sim \to Z(E)', \\
& && (x, f) \mapsto \mu_{x,f}: \quad \mu_{x,f}(\pi) = f(\pi x) \quad \text{for } \pi \in Z(E), \\
&(5) && Z(E)'' \times E^\sim \to E^\sim, \\
& && (F, f) \mapsto F \bullet f: \quad F \bullet f(x) = F(\mu_{x,f}) \quad \text{for } x \in E.
\end{aligned}$$

The maps defined in (4) and (5) are bipositive. (5) makes it possible to define a linear map $m : Z(E)'' \to Z(E^\sim)$ where $m(F)(f) = F \bullet f$; it is called *the Arens homomorphism*. If E is a Riesz space with topologically full center then the bilinear map $E \times E^\sim \to Z(E)'$ is a bilattice homomorphism and m is a unital algebra homomorphism and an order continuous surjective Riesz homomorphism [4, 5].

In all undefined terminology we will adhere to [3, 8, 9, 10, 12].

2. L^1 and L^∞ spaces. We define the L^1 and L^∞ spaces of a Riesz space E. Let A be an algebra and E be an A-module, and let A^*, E^* be their algebraic duals. Then a bilinear map can be defined by

$$\otimes : E \times E^* \to A^*,$$
$$(x, f) \mapsto x \otimes f : \quad x \otimes f(a) = f(a \cdot x) \quad \text{for } a \in A.$$

In general the image of this bilinear map may not be a linear space. For this reason, the linear space generated by the image of $\otimes$ has been taken as the A-tensor product of E and E^*. But the image of the bilinear map given in step (4) is a linear space as can be seen from the following lemma.

LEMMA 1. *Let E be a Riesz space and $B = \{\mu_{x,f} : x \in E,\ f \in E^\sim\}$. Then B is an order ideal in $Z(E)'$.*

Proof. Let $f, g \in E^\sim$, $x, y \in E$ and $\mu = \mu_{x,f} + \mu_{y,g}$. Then

$$0 \le \mu^+ \le |\mu_{x,f} + \mu_{y,g}| \le \mu_{|x|,|f|} + \mu_{|y|,|g|} \le \mu_{|x|+|y|,|f|+|g|}.$$

Similarly, $0 \le \mu^- \le \mu_{|x|+|y|,|f|+|g|}$. Set $h = |f| + |g|$ and $z = |x| + |y|$; then $0 \le \mu^+ \le \mu_{z,h}$ and $0 \le \mu^- \le \mu_{z,h}$. Since $0 \le \mu^+ \le \mu_{z,h}$ and $Z(E)'$ is Dedekind complete there exists $\pi \in Z(Z(E)')$ with $\pi(\mu_{z,h}) = \mu^+$ [3, Theorem 8.15]. By the algebra and Riesz isomorphism υ which is defined earlier, one has the equality $Z(E)'' = Z(Z(E)')$. It implies that $\upsilon_H = \pi$ for some $H \in Z(E)''$. For any two elements π_1, π_2 in $Z(E)$ we obtain

$$\mu_{x,f} \cdot \pi_1(\pi_2) = \mu_{x,f}(\pi_1 \cdot \pi_2) = f(\pi_2\pi_1 x) = \mu_{\pi_1 x,f}(\pi_2).$$

On the other hand, for this H one can easily calculate that $H \cdot \mu_{x,f} = \mu_{x,H\bullet f}$ and so

$$\pi(\mu_{z,h}) = \upsilon_H(\mu_{z,h}) = H \cdot \mu_{z,h} = \mu_{z,H\bullet h} = \mu^+.$$

Following the same argument, we see that $\mu^- = \mu_{z,S\bullet h}$ for $S \in Z(E)''$. By using these equalities, we have

$$\mu = \mu^+ - \mu^- = \mu_{z,H\bullet h} - \mu_{z,S\bullet h} = \mu_{z,(H-S)\bullet h},$$

which implies that the sum of two elements in B is again an element of B. As a consequence B is a subspace of $Z(E)'$. Using the same technique one can easily see that B is also an order ideal in $Z(E)'$.

Let E be a Riesz space. Then $Z(E)' = Z(E)^\sim$ is a Banach lattice with respect to operator norm. Since B is an order ideal in $Z(E)'$, it is well known that the $\sigma(Z(E)', Z(E)'')$ closure of B coincides with the norm closure of B. As $Z(E)'$ has order continuous norm, the closure of B is a band in $Z(E)'$ [12, Corollary 106.3]. The L^1 and L^∞ spaces of a $C(K)$-module E were defined and studied in [2]. Now we will give similar definitions for a Riesz space E.

DEFINITION 1. Let E be a Riesz space.

(a) The $\sigma(Z(E)', Z(E)'')$-closure of B is called the L^1 *space of* E and is denoted by L^1_E.
(b) The order dual of L^1_E is called the L^∞ *space of* E and is denoted by L^∞_E.

It follows from this definition that L^1_E is an abstract AL space and hence L^∞_E is an abstract AM space. Moreover, L^1_E has order continuous norm.

It is well known that if (X, Σ, μ) is a σ-finite measure space and $E = L^p(\mu)$ with $0 < p < \infty$ then $Z(E) = L^\infty(\mu)$. We now prove a similar result for L^1_E.

THEOREM 1. *If E is a Riesz space then $Z(L^1_E) = L^\infty_E$.*

Proof. We wish to define a map $u : L^\infty_E = (L^1_E)' \to Z(L^1_E)$ by $u_F(\mu) = F * \mu$ for each $F \in L^\infty_E$, $\mu \in L^1_E$ where $F * \mu(\pi) = F(\mu \cdot \pi)$, $\mu \cdot \pi(s) = \mu(\pi s)$ for all $\pi, s \in Z(E)$. We shall show that u is a Riesz isomorphism.

Let $\mu = \mu_{x,f} \in B$ for some $x \in E$, $f \in E^\sim$ and $\pi \in Z(E)$. We obtain

$$\mu_{x,f} \cdot \pi(s) = \mu_{x,f}(\pi s) = f(\pi s x) = f(s \pi x) = \mu_{\pi x, f}(s)$$

for each $s \in Z(E)$. Take a positive element μ in L^1_E. There exists a net $\{\mu_\alpha\} \subseteq B$ such that $0 \le \mu_\alpha \uparrow \mu$. Since $\mu_\alpha \cdot \pi \uparrow \mu \cdot \pi$ for each $\pi \in Z(E)_+$ and L^1_E is a band, it follows that $\mu \cdot \pi \in L^1_E$. It is easy to see that u_F is a positive operator for F positive in L^∞_E, and $F * (\lambda \mu_1 + \mu_2) = \lambda F * \mu_1 + F * \mu_2$ for all $\lambda \in \mathbb{R}$, $\mu_1, \mu_2 \in L^1_E$. This implies that $u_F \in L_{\mathrm{b}}(L^1_E)$ for each $F \in L^\infty_E$. Now let $F \in L^\infty_E$ and P be the band projection of L^1_E. Note that $F \circ P \in Z(E)''$ and $v_{F \circ P} \in Z(Z(E)')$. Since L^1_E is a band and $v_{F \circ P}|_{L^1_E} = u_F$, we see that $u_F \in Z(L^1_E)$. Thus the image of L^∞_E under u is included in $Z(L^1_E)$. It is routine to check that u is a positive operator.

If $u_F = 0$ for some $F \in L^\infty_E$, then

$$u_F(\mu_{x,f})(I) = F * \mu_{x,f}(I) = F(\mu_{x,f} \cdot I) = F(\mu_{x,f}) = 0$$

for each $\mu_{x,f} \in B$. By this fact and the order continuity of F, $F(\mu) = 0$ for each $\mu \in L^1_E$. Hence u is a one-to-one operator.

On the other hand, since $v : Z(E)'' \to Z(Z(E)')$ is surjective there exists $G \in Z(E)''$ such that $v_G = P$, the band projection considered above. Set $H = G|_{L^1_E}$. For each $\mu \in L^1_E$, we have $u_H(\mu) = H * \mu = P(\mu) = \mu$, which shows that $u_H = I$. Now let $s \in Z(L^1_E)$ and $\widetilde{s}$ be adjoint to s. Then $\widetilde{s}(H) \in L^\infty_E$. Observe that

$$\widetilde{s}(H) * \mu(\pi) = \widetilde{s}(H)(\mu \cdot \pi) = H(s(\mu \cdot \pi)) = H(s(\mu).\pi) = H * s(\mu)(\pi)$$

for all $\mu \in L^1_E$ and $\pi \in Z(E)$. Hence $u_{\widetilde{s}(H)}(\mu) = \widetilde{s}(H) * \mu = s(\mu)$. This shows that u is surjective.

Clearly, u^{-1} is a positive operator. Applying Theorem 7.3 of [3] we see that u is a Riesz isomorphism.

In the following theorem we characterize the injectivity of the Arens homomorphism in the case of $L_E^1 = Z(E)'$.

THEOREM 2. *Let E be a Riesz space. Then the Arens homomorphism $m : Z(E)'' \to Z(E^\sim)$ is injective if and only if $L_E^1 = Z(E)'$.*

Proof. Assume that $L_E^1 = Z(E)'$ and $m(F) = 0$. Then $0 = m(F)(f)(x) = F(\mu_{x,f})$ for each $x \in E$, $f \in E^\sim$. The order continuity of F implies that $F(\mu) = 0$ for each $\mu \in L_E^1 = Z(E)'$. This shows that m is injective.

Conversely, suppose that m is injective. Let $R : Z(E)' \to (L_E^1)^{\mathrm{d}}$ be the band projection. Since $Z(E)'' = Z(Z(E)')$, there exists $G \in Z(E)''$ such that $v_G = R$. For all $x \in E$, $f \in E^\sim$ we have

$$v_G(\mu_{x,f}) = G \cdot \mu_{x,f} = R(\mu_{x,f}) = 0$$

so

$$G \cdot \mu_{x,f}(I) = G(\mu_{x,f} \cdot I) = G(\mu_{x,f}) = m(G)(f)(x) = 0.$$

It follows that $G = 0$, as m is injective. Hence $R = 0$. Since $Z(E)' = L_E^1 \oplus (L_E^1)^{\mathrm{d}}$ and $R = 0$ one sees that $Z(E)' = L_E^1$, which completes the proof.

We now give two examples related to the characterization of L_E^1.

EXAMPLE 1.

(a) Let $E = l^1$, the absolutely summable sequences. Then $E^\sim = l^\infty$ (the bounded sequences) and $Z(E) = Z(E^\sim) = l^\infty$. On the other hand, $Z(E)'' = (l^\infty)''$ and m is the band projection of $(l^\infty)''$ onto l^∞. Then m is not one-to-one. Therefore $L_E^1 \neq Z(E)'$.

(b) Let K be a compact Hausdorff space and $E = C(K)$. Then $Z(E) = C(K)$ and $Z(E^\sim) = C(K)''$. Also m is the identity map of $C(K)''$. Hence $L_E^1 = Z(E)'$.

These examples show that, in general, $L_E^1 \neq Z(E)'$. We are now in a position to characterize L_E^1 in $Z(E)'$. First we introduce the weak operator topology on $Z(E)$, denoted by wo, corresponding to the dual pair $\langle E, E^\sim \rangle$. A net $\{\pi_\alpha\}$ converges to π with respect to the wo-topology if and only if $f(\pi_\alpha x) \to f(\pi x)$ for each $x \in E$ and $f \in E^\sim$. We are now ready to state the following theorem.

THEOREM 3. *Let E be a Banach lattice and $(Z(E), \mathrm{wo})'$ be the set of continuous functionals on $(Z(E), \mathrm{wo})$. Then $(Z(E), \mathrm{wo})' = B$.*

Proof. Clearly, B is a subset of $(Z(E),\text{wo})'$. Conversely, let μ be a functional on $Z(E)$ which is continuous in the wo-topology. By Theorem 4 in [6] there exist $x_1,\ldots,x_n \in E$ and $f_1,\ldots,f_n \in E'$ such that $\mu = \sum_{i=1}^n \mu_{x_i,f_i}$. This completes the proof.

From the above theorem and Lemma 1 one can deduce the following corollary.

COROLLARY 1. *Let E be a Banach lattice. Then $(Z(E),\text{wo})'$ is an order ideal in $Z(E)'$ and the closure of $(Z(E),\text{wo})'$ is equal to L_E^1.*

If E is an order continuous Banach lattice which has a weak unit, then there exists an AL space S such that E is an order dense Riesz subspace of S. In this case $E^\sim$ is also an order dense ideal in S [1, Theorem 2.1; 10, Theorem 2.7.8; 8, Theorem 1.b.14]. Now we shall give different representation theorems for order continuous Banach lattices with a weak unit. This representation clearly exhibits the relation between E, E' and $Z(E)'$.

THEOREM 4. *Let E be an order continuous Banach lattice which has a weak unit. Then $E^\sim$ is order isomorphic to an order dense ideal in L_E^1.*

Proof. Let $e > 0$ be a weak unit. Define a map $\Phi_e : E^\sim \to L_E^1$ such that $\Phi_e(f) = \mu_{e,f}$ for each $f \in E^\sim$. It is easy to see that Φ_e is a positive operator. Let $\Phi_e(f) = 0$. Then $\Phi_e(f)(\pi) = \mu_{e,f}(\pi) = f(\pi e) = 0$ for each $\pi \in Z(E)$. If $x \in I_e$ (where I_e is the principal ideal generated by e), then there exists $\pi \in Z(E)$ such that $\pi e = x$ by Lemma 2.7 in [11]. This implies that $f(\pi e) = f(x) = 0$. Take an arbitrary positive element in $B_e = E$ (where B_e is the principal band generated by e). Since I_e is order dense in B_e, there exists an upward directed net $\{x_\alpha\}$ in I_e such that $0 \le x_\alpha \uparrow x$. The order continuity of f implies that $f(x) = 0$. This shows that Φ_e is injective. Now let $f \in E^\sim$ and $\mu_{e,f} \ge 0$ in L_E^1. Using the above technique for each $0 \le x \in B_e = E$ we see that $f(x) \ge 0$. Thus $\Phi_e^{-1} : \Phi_e(E^\sim) \to E^\sim$ is positive. Applying Theorem 7.3 of [3] one can deduce that $\Phi_e : E^\sim \to \Phi_e(E^\sim)$ is a Riesz isomorphism. The order ideality of $\Phi_e(E^\sim)$ in L_E^1 follows from the technique used in Lemma 1.

Finally, we claim that $\Phi_e(E^\sim)$ is order dense in L_E^1. To see this, let D be the band generated by $\Phi_e(E^\sim)$ in L_E^1. If we show $B \subseteq D$, then the proof will be completed. For $0 \le x \in E$ and $0 \le f \in E^\sim$ take an element $\mu_{x,f}$ in B. If $x \in I_e$, then there exists $\pi \in Z(E)$ such that $\pi e = x$. For $s \in Z(E)$ the equality

$$\mu_{x,f}(s) = \mu_{\pi e,f}(s) = f(\pi s e) = \widetilde{\pi}(f)(se) = \mu_{e,\widetilde{\pi}(f)}(s)$$

shows that $\mu_{x,f}$ belongs to D. If we take an arbitrary $0 \le x \in B_e = E$, then there exists an upward directed net $\{x_\alpha\}$ in I_e such that $0 \le x_\alpha \uparrow x$. Since f is order continuous and positive, we have $f(\pi x_\alpha) \uparrow f(\pi x)$ for each positive

π in $Z(E)$ and so $\mu_{x_\alpha,f}(\pi) \uparrow \mu_{x,f}(\pi)$. As $\mu_{x_\alpha,f} \in D$ and D is a band in L^1_E we see that $\mu_{x,f} \in D$. It is routine to check that $\mu_{x,f} \in D$ for all x and f.

Theorem 5. *Let E be an order continuous Banach lattice which has a weak unit. Then E is order isomorphic to an order dense Riesz subspace of L^1_E.*

Proof. By Proposition 1.b.15 in [8] there exists $0 < h \in E^\sim$ such that $h(|x|) = 0$ implies that $x = 0$. Define $\Phi_h : E \to L^1_E$ by $\Phi_h(x) = \mu_{x,h}$. Clearly, Φ_h is a positive operator and since E has an order continuous norm, E is Dedekind complete. By Lemma 1 in [5], Φ_h is a Riesz homomorphism and hence $\Phi_h(E^\sim)$ is a Riesz subspace of L^1_E. To show the injectivity of Φ_h let $\Phi_h(x) = 0$. Then we have $\Phi_h(x)(\pi) = \mu_{x,h}(\pi) = h(\pi x) = 0$ for each $\pi \in Z(E)$. As $|x| \in I_x$, there exists $\pi \in Z(E)$ such that $\pi x = |x|$ and it follows that $h(|x|) = h(\pi x) = 0$. By the properties of h we deduce that Φ_h is injective.

Let D be the band generated by $\Phi_h(E)$ in L^1_E. It is sufficient to show that $B \subseteq D$. To do this for $0 \le x \in E$ and $0 \le f \in E^\sim$ take an element $\mu_{x,f}$ in B. If $f \in I_h$, then there exists $s \in Z(E^\sim)$ such that $s(h) = f$. On the other hand, Proposition 2 in [5] shows the equality $Z(E) = Z(E^\sim)$. Thus, there exists $\pi \in Z(E)$ such that $\widetilde{\pi} = s$. For each $t \in Z(E)$ we have

$$\mu_{x,f}(t) = \mu_{x,sh}(t) = \mu_{x,\widetilde{\pi}h}(t) = \widetilde{\pi}(h)(tx) = h(\pi t x) = \mu_{\pi x,h}(t),$$

which shows that $\mu_{x,f} \in D$. If we take $f \in B_h$, then there exists an upward directed net $\{f_\alpha\}$ in I_h such that $0 \le f_\alpha \uparrow f$. By a simple observation we find that $\mu_{x,f_\alpha} \uparrow \mu_{x,f}$. As D is a band and $\mu_{x,f_\alpha} \in D$, we have $\mu_{x,f} \in D$. By Theorem 2.4.9 in [10], h is a weak unit of $E^\sim$, and hence $\mu_{x,f} \in D$ for each $0 \le f \in E^\sim$. It is routine to calculate that $\mu_{x,f}$ belongs to D for all f and x. The proof of the theorem is now complete.

Corollary 2. *Under the hypothesis of Theorem* 5 *we have*

$$Z(E) = Z(E^\sim) = Z(L^1_E) = L^\infty_E.$$

Proof. The equalities $Z(E^\sim) = Z(E)$ and $Z(L^1_E) = L^\infty_E$ hold by Proposition 2 in [5] and Theorem 1. Now we show $Z(E^\sim) = Z(L^1_E)$. Since $E^\sim$ is an order ideal in L^1_E, for all $\pi \in Z(L^1_E)$ we have $\pi(E^\sim) \subseteq E^\sim$. Thus it makes sense to consider $\pi|_{E^\sim}$ for all $\pi \in Z(L^1_E)$, and one has $\pi|_{E^\sim} \in Z(E^\sim)$. Define $r : Z(L^1_E) \to Z(E^\sim)$ by $r(\pi) = \pi|_{E^\sim}$. It is clear that r is a positive operator. If $\pi_1, \pi_2 \in Z(L^1_E)$ and $\pi_1|_{E^\sim} = \pi_2|_{E^\sim}$, then by Corollary 140.6(ii) in [12], $\pi_1 = \pi_2$. Hence r is injective. Dedekind completeness of L^1_E ensures that each operator $0 \le \pi \in Z(E^\sim)$ has a unique extension $\widehat{\pi}(\mu) = \sup\{\pi(f) : 0 \le f \le \mu,\ f \in E^\sim\}$ in the ideal center of the band generated by $E^\sim$ in L^1_E. Thus r is surjective and r^{-1} is positive. This shows that r is a Riesz isomorphism.

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