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Bergman coordinates

by

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Abstract. Various incarnations of Stefan Bergman's notion of representative coordinates will be given that are useful in a variety of contexts. Bergman wanted his coordinates to map to canonical regions, but they fail to do this for multiply connected regions. We show, however, that it is possible to define generalized Bergman coordinates that map multiply connected domains to quadrature domains which satisfy a long list of desirable properties, making them excellent candidates to be called Bergman representative domains. We also construct a kind of Bergman coordinate that maps a domain to an algebraic variety in $\mathbb{C}^2$ in a natural way, and thereby show that Bergman-style coordinates can be used to convert problems in conformal mapping to problems in algebraic geometry. Many of these results generalize routinely to finite Riemann surfaces.

1. Introduction. Stefan Bergman discovered that the Riemann map associated to a simply connected domain $\Omega \neq \mathbb{C}$ in the complex plane can be expressed very simply in terms of his kernel function. Indeed, suppose that f is the one-to-one analytic map of Ω onto the unit disc which maps a point $a \in \Omega$ to zero with $f'(a)$ real and positive. Let $F = f^{-1}$ and let $K(z,w)$ denote the Bergman kernel associated to Ω. The Bergman kernel $K_1(z,w)$ associated to the unit disc is $\pi^{-1}(1-\overline{w}z)^{-2}$. The transformation formula for the Bergman kernels under conformal mappings yields

$$(1.1)\quad K(z,F(w))\overline{F'(w)} = f'(z)K_1(f(z),w) = f'(z)\,\frac{1}{\pi(1-\overline{w}f(z))^2}.$$

Let $w=0$ in this expression to obtain

$$(1.2)\qquad K(z,a)\overline{F'(0)} = \pi^{-1}f'(z).$$

Next, let $K'(z,w) = (\partial/\partial\overline{w})K(z,w)$ and differentiate (1.1) with respect to $\overline{w}$ to obtain

$$(1.3)\quad K'(z,F(w))\overline{F'(w)^2} + K(z,F(w))\overline{F''(w)} = 2f'(z)f(z)\,\frac{1}{\pi(1-\overline{w}f(z))^3}.$$

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Let $w = 0$ in this expression to obtain

$$K'(z,a))\overline{F'(0)^2} + K(z,a)\overline{F''(0)} = 2\pi^{-1} f'(z) f(z). \tag{1.4}$$

Finally, divide (1.4) by (1.2) to get

$$2f(z) = c_1 \frac{K'(z,a)}{K(z,a)} + c_2,$$

where $c_1 = \overline{F'(0)}$ and $c_2 = \overline{F''(0)}/\overline{F'(0)}$. This shows that the mapping

$$\frac{K'(z,a)}{K(z,a)}$$

is a one-to-one analytic map of Ω onto a disc. This quotient is what Bergman hoped would map any domain in the plane onto a "representative" domain. He showed that it had invariant properties that would make representative domains well defined if they existed. It all worked splendidly in the simply connected case, but he could not show that the maps were one-to-one or even well defined in multiply connected domains. Later, Suita and Yamada [15] showed that the Bergman kernel must vanish at certain points in multiply connected domains, dashing all hopes that the quotients would do what Bergman wanted them to do in multiply connected domains.

Bergman coordinates might have been a disappointment at first in one complex variable, but the idea proved to be very useful in several complex variables. Sidney Webster started a thread of research in his paper [16] that inspired Ewa Ligocka to revive and revise a sort of Bergman coordinates in [13], and this allowed Bergman-style coordinates to be applied directly in [9] to prove that regularity estimates for the $\overline{\partial}$-problem imply boundary regularity of biholomorphic mappings. I have been wrestling with the idea of Bergman coordinates for many years now, and I believe I am just beginning to understand how they should be described. My understanding of Bergman coordinates evolved as I went back and forth between mapping problems in one and several variables. This paper will dwell mostly on the one variable case where the theory is particularly beautiful. The last section mentions some problems in several complex variables. The high point of the paper will be to show that there is a new way to view Bergman coordinates in such a light that they effect a morphism between problems in conformal mapping and problems in algebraic geometry.

Roughly speaking, the term "Bergman coordinates" in this paper will mean any change of variables $H(z)$ which is a quotient of finite linear combinations of the Bergman kernel $K(z,a)$ or its derivatives $(\partial^m/\partial\overline{a}^m)K(z,a)$ in the second variable as a ranges over a finite set in the domain.

2. Local Bergman coordinates. In this section, we show that very simple Bergman coordinates can be defined which linearize biholomorphic mappings near a boundary point and which transform boundaries into real-algebraic sets. To begin, suppose that Ω is an n-connected bounded domain in the plane bounded by n non-intersecting C^∞ smooth simple closed curves. (It will be easy to drop this strong assumption later because of the way the Bergman kernel transforms under conformal mappings.) Let $K(z,w)$ denote the Bergman kernel associated to Ω (which is the kernel function for the orthogonal projection of $L^2(\Omega)$ onto the subspace consisting of holomorphic functions in $L^2(\Omega)$). Let $A^\infty(\Omega)$ denote the subspace of $C^\infty(\overline{\Omega})$ consisting of holomorphic functions that are smooth up to the boundary. It is well known that $K(z,w)$ is in $A^\infty(\Omega)$ as a function of z for each fixed w in Ω. (In fact, $K(z,w)$ is in $C^\infty(\overline{\Omega}\times\overline{\Omega}-\{(z,z):z\in b\Omega\})$.)

The following lemma proved in [3] will be one of two main tools we need to set up various species of Bergman coordinates with a host of interesting properties.

LEMMA 2.1. *Suppose that Ω is an n-connected bounded domain in the plane bounded by n non-intersecting C^∞ smooth simple closed curves. Let $\mathcal{S}$ denote the set of functions $H(z)$ of the form $H(z)=K(z,a)$ where a is a point in Ω. The complex linear span of $\mathcal{S}$ is dense in $A^\infty(\Omega)$, i.e., given a function h in $A^\infty(\Omega)$, an $\varepsilon>0$, and a positive integer M, there is a function of the form $\mathcal{K}(z)=\sum_{j=1}^N c_jK(z,a_j)$ such that $h-\mathcal{K}$ and all its derivatives up to order M are uniformly bounded by ε on $\overline{\Omega}$.*

Pick a point p_0 in the boundary of Ω. Lemma 2.1 will allow us to find points a_0 and a_1 in Ω such that the quotient $K(z,a_1)/K(z,a_0)$ is a change of coordinates on Ω near p_0 that is C^∞ up to the boundary, meaning that there is a disc $D_r(p_0)$ such that the quotient is one-to-one and C^∞ smooth with non-vanishing derivative on $\overline{\Omega}\cap D_r(p_0)$. In particular, the quotient maps the boundary curve of Ω near p_0 one-to-one onto a smooth curve segment.

Indeed, we first choose a_0 such that $K(p_0,a_0)\neq 0$. Such a point a_0 exists because if $K(p_0,a)=0$ for all a in Ω, then the function $G(z)\equiv 1$ could not be approximated in $A^\infty(\Omega)$ by linear combinations of functions in $\mathcal{S}$ because all such linear combinations would vanish at p_0, and this violates Lemma 2.1. Next, pick a point a_1 so that the function $\Phi(z)=K(z,a_1)/K(z,a_0)$ has non-vanishing derivative at p_0. Such a point exists because if $\Phi'(p_0)$ were equal to zero for all choices of a_1, then it would not be possible to approximate the function $zK(z,a_0)$ in $A^\infty(\Omega)$ by linear combinations of functions in $\mathcal{S}$.

The transformation $w=\Phi(z)$ is a local change of variables on Ω near p_0 which maps the boundary of Ω to a C^∞ smooth curve. We shall show in a moment that this curve is a real-algebraic curve. However, first we shall show how these Bergman coordinates behave under conformal mappings, and how this transformation property controls the boundary behavior of the mappings.

Suppose that $f : \Omega_1 \to \Omega_2$ is a biholomorphic mapping between n-connected bounded domains with C^∞ smooth boundaries. Let p_1 be a boundary point in Ω_1 and let z_j be a sequence of points in Ω_1 that converges to p_1. By taking a subsequence of z_j, we may also assume that $f(z_j)$ converges to a point p_2 in the boundary of Ω_2. We may define a local Bergman coordinate Φ_1 near p_1 using two points a_0 and a_1 as above. However, this time, we also want $A_0 = f(a_0)$ and $A_1 = f(a_1)$ to serve as the parameters in a local Bergman coordinate at p_2. This can easily be accomplished because the set of suitable points a_0 such that $K_1(p_1, a_0) \neq 0$ is a dense open subset of Ω_1. (In fact, it is all of Ω_1 minus finitely many points by the result of Suita and Yamada [15].) Hence, the set of a_0 such that $K_1(p_1, a_0) \neq 0$ and $K_2(p_2, A_0) \neq 0$, where $A_0 = f(a_0)$, is dense and open in Ω_1. Once a_0 has been chosen, then, similarly, the set of suitable a_1 is a dense open subset of Ω_1. Let $\Phi_1(z) = K_1(z, a_1)/K_1(z, a_0)$ and let $\Phi_2(z)$ denote the Bergman coordinate near p_2 given by $K_2(z, A_1)/K_2(z, A_0)$. By choosing j large enough that z_j is inside the special coordinate chart near p_1 and $f(z_j)$ is inside the chart near p_2, and by shrinking the charts, if necessary, it can be seen that f is a *linear* map in the new coordinates. Indeed, the transformation formula

$$K_1(z, w) = f'(z) K_2(f(z), f(w)) \overline{f'(w)}$$

for the Bergman kernels under biholomorphic mappings yields

$$\frac{K_1(z, a_1)}{K_1(z, a_0)} = \frac{K_2(f(z), A_1)}{K_2(f(z), A_0)} \, \frac{\overline{f'(a_1)}}{\overline{f'(a_0)}},$$

i.e., $\Phi_1(z) = c\Phi_2(f(z))$ where c is a non-zero constant, and this is exactly what it means for f to be linear in the new Bergman coordinates. Thus, the changes of coordinates linearize the mapping.

Notice that the formulas above also reveal that the mapping f is as differentiable up to the boundary near p_1 as the Φ's are, and the Φ's are as differentiable as quotients of Bergman kernels. Thus, we obtain a rather elementary proof that conformal mappings between domains with C^∞ smooth boundaries are C^∞ smooth up to the boundary. This is a very old and well known fact in one complex variable, but it was new in several in the 1970's. The argument above is easy to generalize to several variables and this is essentially what was done in [9] and [13]. (In a moment, we shall see that quotients of Bergman kernels extend to the double of a domain. This will show that quotients of Bergman kernels extend nicely to the boundary even when it might not be known that the kernels themselves do.)

We shall next show that these Bergman coordinates transform the boundaries into real-algebraic sets.

THEOREM 2.2. *Suppose that Ω_1 and Ω_2 are n-connected bounded domains in the plane, each bounded by n non-intersecting C^∞ smooth simple* *closed curves, and suppose $f : \Omega_1 \to \Omega_2$ is a biholomorphic mapping between the domains. Given boundary points $p_j \in b\Omega_j$ for $j = 1, 2$, it is possible to construct Bergman coordinates which are given as simple quotients $K_j(z, a_1^j)/K_j(z, a_0^j)$ such that the boundaries of each Ω_j near p_j get mapped to smooth real-algebraic sets, and such that the mapping f becomes linear in the new coordinates.*

To prove this, we shall show that a Bergman coordinate $\Phi(z)$ associated to a point p in the boundary of a C^∞ smooth bounded n-connected domain Ω extends meromorphically to the double of Ω. Let $\widehat{\Omega}$ denote the double of Ω (see Ahlfors and Sario [2] and/or Farkas and Kra [11] for the relevant definitions and facts) and let $\widetilde{\Omega}$ denote the reflected copy of Ω in the double. Let $R(z)$ denote the anti-holomorphic reflection function on $\widehat{\Omega}$ which maps Ω one-to-one onto $\widetilde{\Omega}$ in the double. Note that $R \circ R$ is the identity map and $R(z) = z$ for all points $z \in b\Omega$. Assume for the moment that the boundary of Ω consists of C^∞ smooth *real-analytic* curves. We shall need to use the simple and well known fact that if H and G are meromorphic functions on Ω which extend meromorphically past the boundary, and if $H(z) = \overline{G(z)}$ for points z in the boundary that are not poles of H, then H extends to the double of Ω as a meromorphic function, and for $z \in \widetilde{\Omega}$, the extension of H is given as $\overline{G(R(z))}$.

The Bergman kernel is related to the classical Green's function via

$$K(z, w) = -\frac{2}{\pi}\frac{\partial^2 G(z, w)}{\partial z \partial \overline{w}},$$

and the complementary kernel

$$\Lambda(z, w) = -\frac{2}{\pi}\frac{\partial^2 G(z, w)}{\partial z \partial w}$$

is holomorphic in z and w and is in $C^\infty(\overline{\Omega} \times \overline{\Omega} - \{(z, z) : z \in \overline{\Omega}\})$. If $a \in \Omega$, then $\Lambda(z, a)$ has a double pole at $z = a$ as a function of z and $\Lambda(z, a) = \Lambda(a, z)$ for $z \neq a$ (see [10] or [4, p. 134]). The Bergman kernel $K(z, w)$ and the kernel $\Lambda(z, w)$ extend holomorphically past the boundary in z when w is held fixed in Ω. They also satisfy the identity

$$(2.1) \qquad \Lambda(w, z)T(z) = -K(w, z)\overline{T(z)} \quad \text{for } w \in \Omega \text{ and } z \in b\Omega,$$

where $T(z)$ denotes the complex number of unit modulus pointing in the direction of the tangent vector at the boundary point z pointing in the direction of the standard sense of the boundary (see [10] or [4, p. 135] for a proof).

Recall that $\Phi(z) = K(z, a_1)/K(z, a_0)$ is a Bergman coordinate near a boundary point p. Let $\Psi(z) = \Lambda(z, a_1)/\Lambda(z, a_0)$. Identity (2.1) implies that $\Phi(z) = \overline{\Psi(z)}$ when $z \in b\Omega$. This shows that $\Phi(z)$ extends to the double of Ω as a meromorphic function. Let $\widehat{\Phi}$ denote the meromorphic function on $\widehat{\Omega}$ which is the extension of Φ. Now the function $G(z)$ which is the complex conjugate of $\widehat{\Phi}(R(z))$ is also meromorphic on $\widehat{\Omega}$. It is well known (see Farkas and Kra [11]) that any two meromorphic functions on a compact Riemann surface are algebraically dependent. Hence, there exists an irreducible polynomial $P(z,w)$ of two complex variables such that $P(\Phi(z),G(z))\equiv 0$ on Ω.

Now let z be a point in the boundary of Ω to see that $P(\Phi(z),\overline{\Phi(z)})=0$. This shows that Φ maps the boundary of Ω to the real-algebraic curve $\{z: P(z,\bar z)=0\}$. If the boundary of Ω is C^∞ smooth, but not real-analytic, we may use a biholomorphic mapping F from Ω to a domain Ω_{r} which does have real-analytic boundary to come to the same conclusions. Indeed, we know that F extends C^∞ smoothly up to the boundaries, and the argument we used above to show that Bergman coordinates linearize biholomorphic mappings yields $\Phi(z)=c\Phi_{\mathrm{r}}(F(z))$ for a quotient of Bergman kernels Φ_{r} on Ω_{r}. We have shown that functions of the form Φ_{r} extend to the double of Ω_{r}. This property is preserved under the biholomorphic mapping F. Thus, Φ extends meromorphically to the double of Ω and we can repeat the argument above to see that Φ maps the boundary of Ω near p to a real-algebraic curve. When this fact is added to the argument preceding the statement of Theorem 2.2, the proof is complete.

We shall now relax the boundary smoothness hypotheses in Theorem 2.2 to the condition that the domains are bounded with boundaries consisting of finitely many non-intersecting Jordan curves. Indeed, in this case, the domains are finitely connected bordered Riemann surfaces, and the doubles are well defined. Suppose that $f:\Omega_1\to\Omega_2$ is a biholomorphic mapping between two such domains. Then f extends continuously to the boundary and maps each boundary component one-to-one onto a boundary component of the image domain.

It is well known that there exist biholomorphic mappings $G_j:\Omega_j\to\widetilde{\Omega}_j$, $j=1,2$, where $\widetilde{\Omega}_j$ are bounded domains with C^∞ smooth real-analytic boundaries. Let $\widetilde{f}=G_2\circ f\circ G_1^{-1}$. Suppose p_1 is a boundary point of Ω_1, and let $p_2=f(p_1)$. Then $G_1(p_1)$ is a boundary point of $\widetilde{\Omega}_1$ and we may create local Bergman coordinates near $G_1(p_1)$ and $G_2(p_2)$ that linearize $\widetilde{f}$ in a neighborhood. Let $\widetilde{K}_j$ denote the Bergman kernels of $\widetilde{\Omega}_j$, $j=1,2$, and let K_j denote the kernels of Ω_j, $j=1,2$. Given Bergman coordinates $\widetilde{K}_j(z,\widetilde{a}_1^j)/\widetilde{K}_j(z,\widetilde{a}_0^j)$ on the real-analytically bounded domains that are related as in the previous result, the transformation formula for the Bergman kernel yields

$$\widetilde{K}_j(z,\widetilde{a}_1^j)/\widetilde{K}_j(z,\widetilde{a}_0^j)=c_jK_j(G_j^{-1}(z),a_1^j)/K_j(G_j^{-1}(z),a_0^j),$$

where $\widetilde{a}_i^j=G_j(a_i^j)$, and where c_j is a non-zero constant. Since the Bergman coordinates on the $\widetilde{\Omega}_j$ extend meromorphically to the double of $\widetilde{\Omega}_j$ without any poles on the boundary near $G(p_1)$ or $G(p_2)$, it follows that the quotients $\widetilde{K}_j(z,\widetilde{a}_1^j)/\widetilde{K}_j(z,\widetilde{a}_0^j)$ extend meromorphically to the double of Ω_j without any poles near p_1 or p_2. Hence, the quotients extend continuously up to the boundary near p_1 and p_2. The transformation formulas and the properties of the Bergman coordinates for the smooth domains now show that these Bergman coordinates for Ω_j, $j=1,2$, linearize the map f and locally map the boundaries to smooth real-algebraic curves. Thus, we have proved the following theorem.

THEOREM 2.3. *Suppose that Ω_1 and Ω_2 are n-connected bounded domains in the plane, each bounded by n non-intersecting Jordan curves, and suppose $f:\Omega_1\to\Omega_2$ is a biholomorphic mapping between the domains. Given boundary points $p_j\in b\Omega_j$ for $j=1,2$, there exist Bergman coordinates which are given as quotients $K_j(z,a_1^j)/K_j(z,a_0^j)$ which extend continuously to the boundary in such a way that the boundaries of each Ω_j near p_j get mapped to smooth real-algebraic sets, and such that the mapping f becomes linear in the new coordinates.*

It is interesting that our quotients of Bergman kernels can be seen to extend continuously up to boundary curves even though it may happen that the kernels by themselves do not.

All that is required in the localization argument in the proof of Theorem 2.3 is that a neighborhood of a boundary point be realizable as a bordered Riemann surface, and so there is room for further relaxation of the requirements on the boundary. We shall not pursue this point further here.

It is natural to revisit the proof above that Bergman coordinates linearize conformal maps $f:\Omega_1\to\Omega_2$ without making special choices for the points A_0 and A_1 in the target domain. This time, let Φ_1 be constructed as a local coordinate near p_1 and let the points A_0 and A_1 be chosen so that Φ_2 is a local coordinate near p_2 (without requiring that $A_0=f(z_0)$ and $A_1=f(a_1)$). We observed above that Φ_1 extends to the double of Ω_1 and Φ_2 extends to the double of Ω_2. The property of extending to the double is invariant under conformal mappings between bordered Riemann surfaces, and hence, $\Phi_2\circ f$ also extends meromorphically to the double of Ω_1. Thus, Φ_1 and $\Phi_2\circ f$ are algebraically dependent, i.e., $\Phi_2\circ f$ is an algebraic function of Φ_1. This means that the conformal map becomes an algebraic mapping in the new Bergman coordinates and that the boundaries become real-algebraic curves. Isn't that fascinating? The same technique of proof reveals that this fact extends to be true in case f is only assumed to be a *proper* holomorphic mapping from Ω_1 to Ω_2.

We have called the Bergman coordinates we have studied in this section *local*. However, they clearly depend on the global domain. They are local algebraic objects attached to the domain that satisfy invariance properties under global conformal and proper holomorphic transformations.

3. Global Bergman coordinates. Suppose that Ω is an n-connected bounded domain in the plane bounded by n non-intersecting C^∞ smooth simple closed curves. Let $K(z,w)$ and $K_0(z,w)$ denote the Bergman kernel associated to Ω and let $K_m(z,w)$ denote the function $(\partial/\partial\overline{w})^m K(z,w)$. We shall now study quotients of functions $\mathcal{K}$ which are finite sums of the form

$$\mathcal{K}(z) = \sum_{k=1}^{N} \sum_{m=0}^{M_k} c_{km} K_m(z, a_k).$$

Let $\mathcal{K}_j(z)$ for $j = 0, 1$ denote two such functions. If $\mathcal{K}_0$ is non-vanishing on $\overline{\Omega}$ and if the quotient $Q(z) = \mathcal{K}_1(z)/\mathcal{K}_0(z)$ is one-to-one on $\overline{\Omega}$ with non-vanishing derivative on $\overline{\Omega}$, then we call it a *global Bergman coordinate.* This type of Bergman coordinate was first defined in [7–8]. We shall show here that such coordinates have a number of strong properties and that they are extremely flexible, in that they can be used to approximate any smooth conformal change of variables, and that they improve upon any change of variables in a substantial way.

We shall call an n-connected bounded domain Ω in the plane such that no boundary component is a point a *quadrature domain* if there exist finitely many points $\{w_j\}_{j=1}^{N}$ in the domain and non-negative integers n_j such that complex numbers c_{jk} exist satisfying

$$\int_{\Omega} f \, dA = \sum_{j=1}^{N} \sum_{k=0}^{n_j} c_{jk} f^{(k)}(w_j) \tag{3.1}$$

for every function f in the Bergman space of square integrable holomorphic functions on Ω. Here, dA denotes Lebesgue area measure. Recall that Stefan Bergman's original coordinates mapped simply connected domains to discs, which are famous for being one-point quadrature domains. The papers [1, 12, 14] and the volume in which [7] appears give an excellent compendium of the special properties of quadrature domains.

We shall see that the global Bergman coordinates mentioned above have the remarkable property that they map domains to quadrature domains which have smooth real-algebraic boundaries, and that biholomorphic and proper holomorphic mappings become algebraic in the new coordinates. I think this result would have made Stefan Bergman happy. Multiply connected quadrature domains can be thought of as being rather analogous to a disc in the simply connected case, which are the only one-point quadrature domains in the plane. One could look upon a quadrature domain as a kind of "representative domain" if one were willing to drop the uniqueness requirement for such a concept.

Suppose that Ω is an n-connected domain bounded by C^∞ smooth curves, and suppose $F:\Omega\to\widetilde{\Omega}$ is a conformal change of variables where $\widetilde{\Omega}$ is another domain bounded by C^∞ smooth curves. Then F extends C^∞ smoothly up to the boundary to $\overline{\Omega}$. Lemma 2.1 allows us to find a Bergman sum $\mathcal{K}_1$ that is as close in $C^\infty(\overline{\Omega})$ as desired to the function $F(z)$, and a sum $\mathcal{K}_0$ that is as close to the function 1 as desired. In this way, the quotient Q can be made as close in $C^\infty(\overline{\Omega})$ to $F(z)$ as desired. In particular, we may construct Q sufficiently close to F that Q maps Ω to another domain $\mathcal{Q}$ that is C^∞ close to $\widetilde{\Omega}$. We shall now show that $\mathcal{Q}$ is a quadrature domain with a real-algebraic boundary.

Identity (2.1) can be differentiated with respect to w to yield

$$\frac{\partial^m}{\partial w^m}\Lambda(w,z)T(z)=-\frac{\partial^m}{\partial w^m}K_m(w,z)\overline{T(z)}\quad \text{for } w\in\Omega \text{ and } z\in b\Omega, \tag{3.2}$$

where $T(z)$ denotes the complex number of unit modulus pointing in the direction of the tangent vector at the boundary point z pointing in the direction of the standard sense of the boundary. These identities can be used exactly as we used (2.1) to show that local Bergman coordinates extend to the double to show that global Bergman coordinates extend meromorphically to the double.

Björn Gustafsson [12] proved that the property of being the image of a smooth domain Ω under a biholomorphic mapping f that extends meromorphically to the double of Ω is the defining feature of a quadrature domain. Thus, the domain $\mathcal{Q}$ that we constructed above is a quadrature domain. Aharonov and Shapiro [1] (see also Gustafsson [12]) showed that the boundaries of quadrature domains are real-algebraic sets. They also proved that Ahlfors mappings associated to quadrature domains are algebraic. Multiply connected quadrature domains have many properties that make them rather analogous to a disc in the simply connected case. For example, the Bergman and Szegő kernel functions associated to a quadrature domain are algebraic functions. In fact $K(z,w)$ is a rational combination of z and $\overline{w}$, and the Schwarz function $S(z)$ and $\overline{S(w)}$. Furthermore, the Schwarz function itself is an algebraic function. Hence, it is rather like having nice formulas for the classical functions that are so handy in the unit disc. (See [7, 8] for proofs of these facts about the kernel functions.)

We have proved the following theorem.

THEOREM 3.1. *Suppose that Ω is an n-connected domain bounded by C^∞ smooth curves, and suppose $F:\Omega\to\widetilde{\Omega}$ is a conformal change of variables where $\widetilde{\Omega}$ is another domain bounded by C^∞ smooth curves. There exist global Bergman coordinates that are as close as desired to F in $C^\infty(\overline{\Omega})$*

which map Ω to a quadrature domain Q which is C^∞ close to $\widetilde{\Omega}$. Such a quadrature domain has smooth real-algebraic boundary curves.

I like to think of this last result as meaning that, given a smooth change of coordinates, they can be improved upon via global Bergman coordinates so that the image is a quadrature domain. This line of reasoning can be applied to the identity change of coordinates $F(z) = z$, to see that any smooth domain can be approximated by a smooth quadrature domain. This allows us to see that any domain in the plane bounded by finitely many non-intersecting Jordan curves can be turned into a quadrature domain by "filing down" the edges ever so slightly. Indeed, such a domain can be approximated from the inside by a domain bounded by non-intersecting C^∞ smooth simple closed curves, and this second domain can be approximated by a quadrature domain that stays within the original domain. Similarly it is possible to approximate via domains that contain the original.

Global Bergman coordinates are easy to generalize to non-smooth domains. Suppose Ω is an n-connected domain in the plane such that no boundary component is a point. There is a biholomorphic map F of Ω onto a domain Ω_2 that does have C^∞ smooth boundary. Let Q_2 be global Bergman coordinates for Ω_2 as constructed above. The transformation formula for the Bergman kernel under biholomorphic mappings allows us to easily see that there are global Bergman coordinates Q on Ω that are given as $\mathcal{K}_1/\mathcal{K}_0$ where $\mathcal{K}_0$ does not vanish on Ω and Q is one-to-one on Ω. The coordinates transform via $Q(z) = Q_2(F(z))$.

Now suppose $f : \Omega_1 \to \Omega_2$ is a biholomorphic mapping between two n-connected domains in the plane such that no boundary component is a point in the Riemann sphere. The global Bergman coordinates defined above transform the domains into quadrature domains with smooth real-algebraic boundaries. Furthermore, the mapping f becomes complex-algebraic in the new coordinates. Indeed, since the Bergman kernels associated to quadrature domains are algebraic, results of [6] show that proper holomorphic mappings between them must be algebraic.

Björn Gustafsson has recently shown (see [7, §4]) that Bergman coordinates are actually characterized by the property that they effect a change of variables that transforms a region into a quadrature domain. To be more precise, Gustafsson has shown that if Φ is a one-to-one holomorphic mapping on a finitely connected domain with smooth boundary that maps the domain to a quadrature domain, then Φ is equal to a Bergman coordinate. (In this context, Gustafsson has also shown [12] that the property of mapping to a quadrature domain is equivalent to the condition that Φ extends to the double of the domain as a meromorphic function with no poles on the boundary.)

4. The Bergman map from conformal mapping to algebraic geometry. Suppose that Ω is a finitely connected bounded domain in the plane bounded by n non-intersecting smooth real-analytic curves. Let $\mathbb{P}$ denote the extended complex plane, i.e., the Riemann sphere. We will define a mapping $\Phi(z)$ from Ω into $\mathbb{P}^2$ which maps Ω into an algebraic curve in $\mathbb{P}^2$ which is given as the zero set $\{(z,w) : P(z,w) = 0\}$ of a single irreducible polynomial $P(z,w)$. The mapping Φ will have the form

$$\Phi(z) = \left(\frac{K(z,a_1)}{K(z,a_0)}, \frac{K(z,a_2)}{K(z,a_0)}\right)$$

where the points a_0, a_1, and a_2 can be chosen from a dense open set in Ω^3. The two quotients $K(z,a_j)/K(z,a_0)$, $j = 1,2$, will be meromorphic functions on Ω that extend meromorphically to the double of Ω and the extensions will form a primitive pair for the field of meromorphic functions on the double.

If $f : \Omega_1 \to \Omega_2$ is a biholomorphic mapping between two such domains, we shall see that there is a rational function $\mathcal{R}$ from $\mathbb{C}^2$ to itself such that Bergman maps of the two domains satisfy $\Phi_2(f(z)) = \mathcal{R}(\Phi_1(z))$, where Φ_1 and Φ_2 are Bergman maps defined as above corresponding to Ω_1 and Ω_2, respectively. Thus, this rational map sends the algebraic variety corresponding to Ω_1 mentioned above to an algebraic variety corresponding to Ω_2. Applying this fact to the inverse map reveals that the algebraic varieties mapped to by the Bergman maps are *birationally equivalent*. Although choices will be made in the construction of the algebraic varieties, applying this reasoning to the identity map $f(z) = z$ also reveals that the algebraic varieties that we construct are unique up to birational equivalence. Thus, the Bergman map will be seen to convert problems in conformal mapping to problems in algebraic geometry. Since an n-connected domain such that no boundary component is a point is conformally equivalent to a bounded domain with real-analytic boundaries, and since quotients of Bergman kernels transform under biholomorphic mappings, these concepts will extend routinely to this more general context. The details of this process are routine and so similar to arguments given in Sections 2 and 3 that we omit them here.

We now turn to the details of the construction. As in §2, let $R(z)$ denote the anti-holomorphic reflection function on the double $\widehat{\Omega}$ of Ω which maps Ω to its reflected copy $\widetilde{\Omega}$ in $\widehat{\Omega}$. Suita and Yamada [15] proved that $K(z,w) \neq 0$ and $\Lambda(z,w) \neq 0$ if z and w are boundary points of Ω with $z \neq w$. Therefore, it is possible to choose a point a_0 in Ω near the boundary such that $K(z,a_0)$ and $\Lambda(z,a_0)$ have no zeroes on the boundary in the z variable. Let a_1 be any other point in Ω with $a_1 \neq a_0$. It was shown in §2 that quotients of the form $K(z,a_1)/K(z,a_0)$ extend meromorphically to the double of Ω. Let Q_1 denote the meromorphic extension. To find an appropriate point a_2, we shall need to show that, given any meromorphic function G on the double of Ω, there is a point a in Ω such that the meromorphic extension $Q_a(z)$ of $K(z,a)/K(z,a_0)$ to the double of Ω is such that G and Q_a form a primitive pair for the double. It will then follow that any meromorphic function on the double is given as a rational combination of G and Q_a.

Let $\widehat{\mathbb{C}}$ denote the extended complex plane. Suppose that the mapping degree of G as a map from $\widehat{\Omega}$ to $\widehat{\mathbb{C}}$ is m and choose a point $w_0 \in \mathbb{C}$ such that the set $W = G^{-1}(w_0)$ consists of m distinct points in $\widehat{\Omega}$. If necessary, we may move w_0 slightly so that none of the points in W fall on the boundary of Ω in $\widehat{\Omega}$ and so that no point in W is a zero of $K(z,a_0)$ in Ω or a zero of $\Lambda(R(z),a_0)$ in $\widetilde{\Omega}$. To prove that G and Q_a form a primitive pair, it will be enough to show that there exists a point a in Ω such that Q_a separates the points of W (see Farkas and Kra [11]). The only tool we shall need to show this is the simple fact that, for fixed distinct points w_1 and w_2 in Ω, it can never happen that the function of z given by $K(z,w_1) - cK(z,w_2)$ is identically zero in z on Ω. Indeed, if this function were the zero function, then the reproducing property of the Bergman kernel would yield $h(w_1) = \bar{c}h(w_2)$ for all functions h in the Bergman space, and this is false.

Let w_i and w_j be two distinct points from W. For the moment, assume that w_i and w_j are both in the subset of $\widehat{\Omega}$ identified with Ω. The set E_{ij} of points a in Ω such that Q_a does not separate w_i and w_j is given by $\{a \in \Omega : K(w_i,a) - cK(w_j,a) = 0\}$ where $c = K(w_i,a_0)/K(w_j,a_0)$ is a non-zero constant. Since the anti-holomorphic function of a which defines this set extends anti-holomorphically past the boundary and cannot be identically zero on Ω, it follows that E_{ij} is a finite subset of Ω.

Next, suppose that w_i and w_j are both in $\widetilde{\Omega}$, the reflected copy of Ω in $\widehat{\Omega}$. The function $Q_a(z)$ is given by $K(z,a)/K(z,a_0)$ when z is in $\overline{\Omega}$ and it is given by the conjugate of $\Lambda(R(z),a)/\Lambda(R(z),a_0)$ when z is in the reflected copy of Ω in $\widehat{\Omega}$. Let w_i^* and w_j^* denote the points $R(w_i)$ and $R(w_j)$ in Ω, respectively. The set E_{ij} of points a in Ω such that Q_a does not separate w_i and w_j is given by $\{a \in \Omega : \Lambda(w_i^*,a) - c\Lambda(w_j^*,a) = 0\}$ where $c = \Lambda(w_i^*,a_0)/\Lambda(w_j^*,a_0)$ is a non-zero constant. It is clear that the holomorphic function of a that defines this set cannot be identically zero in a because it has double poles at w_i^* and w_j^*. Hence, again, we may conclude that E_{ij} is a finite subset of Ω.

Finally, suppose that w_i belongs to Ω and w_j belongs to $\widetilde{\Omega}$. The set E_{ij} of points a in Ω such that Q_a does not separate w_i and w_j is given by $\{a \in \Omega : K(w_i,a) - c\overline{\Lambda(w_j^*,a)} = 0\}$ where $c = K(w_i,a_0)/\overline{\Lambda(w_j^*,a_0)}$ is a non-zero constant. Since the function of a which defines this set is anti-holomorphic in a and since $\Lambda(w_j^*,a)$ has a double pole at w_j^*, it is clear that there are only finitely many points a in E_{ij}. We may now deduce that the set of a in Ω such that Q_a separates the points in W is equal to Ω minus the finite set of points consisting of the union of all the E_{ij}'s. Pick one such a and call it a_2.

We now let $Q_j(z)$ denote the meromorphic extension of $K(z,a_j)/K(z,a_0)$ to the double of Ω, $j=1,2$. There is an irreducible polynomial $P(z,w)$ of two complex variables such that $P(Q_1(z),Q_2(z))\equiv 0$ on the double. This shows that the corresponding Bergman map sends Ω into an algebraic curve in $\mathbb{P}^2$.

Now suppose that $f:\Omega_1\to\Omega_2$ is a biholomorphic mapping between bounded n-connected domains with real-analytic boundaries. Let Q_1 and Q_2 denote the functions constructed above for Ω_2, and let $q_1(z)=K_1(z,A_1)/K_1(z,A_0)$ and $q_2(z)=K_1(z,A_2)/K_1(z,A_0)$ be components of a Bergman map associated to Ω_1. The property of being meromorphically extendible to the double is preserved by f and so $Q_j\circ f$ extends meromorphically to the double of Ω_1, $j=1,2$. Since q_1 and q_2 extend to form a primitive pair for the double of Ω_1, it follows that $Q_1\circ f$ and $Q_2\circ f$ are both rational functions of q_1 and q_2, i.e., that there is a rational function from $\mathbb{C}^2$ to itself that maps the algebraic curve associated to Ω_1 to the algebraic curve associated to Ω_2. Applying this same reasoning to the inverse of f yields a rational map going in the other direction, and therefore, we may conclude that the two curves are *birationally equivalent.*

An avenue of further research would be to investigate if anything is gained by adding more components of quotients to the Bergman map to get mappings into algebraic varieties in higher dimensions.

The same procedure we have carried out for domains can be done on general finite Riemann surfaces. Indeed, in this setting, the Bergman kernel is viewed as a differential form via

$$K(z,w)=\frac{\partial^2}{\partial z\partial\overline{w}}G(z,w)dzd\overline{w}$$

where $G(z,w)$ denotes the classical Green's function. If G_1 and G_2 are any two meromorphic functions on the surface that extend meromorphically to the double and form a primitive pair for the double, and if f is any proper holomorphic mapping of the surface onto the unit disc (and there are many), it was proved in [5] that the Bergman kernel can be expressed via

$$K(z,w)=R(G_1(z),G_2(z),\overline{G_1(w)},\overline{G_2(w)})df(z)\overline{df(w)},$$

where R is a rational function of four complex variables. Hence quotients of the form $K(z,a_1)/K(z,a_2)$ can be viewed as meromorphic functions that extend meromorphically to the double, and the same reasoning above reveals that Bergman maps send the surface to an algebraic variety in $\mathbb{P}^2$. Furthermore, it is a safe bet that Bergman maps convert biholomorphic maps between finite Riemann surfaces to birational mappings between algebraic varieties.

5. Several complex variables. Since Lemma 2.1 is true for bounded domains Ω in $\mathbb{C}^n$ with C^∞ smooth boundaries such that their Bergman projections preserve the space $C^\infty(\overline{\Omega})$, and since the Bergman kernel transforms under biholomorphic mappings between domains in $\mathbb{C}^n$ in a similar manner to the one-dimensional case, many of the applications of Bergman coordinates given above carry over routinely to several variables. For example, it is possible to use the same reasoning as in §2 to choose points a_0 and $a_1,\dots,a_n$ in such a domain so that the Bergman coordinates $w=(w_1,\dots,w_n)$ where $w_j=K(z,a_j)/K(z,a_0)$ effect a change of variables (with non-vanishing jacobian) which linearize biholomorphic mappings near given boundary points. (This is very similar to what was done in [9, 13] to prove regularity of biholomorphic mappings in several complex variables.)

Global Bergman coordinates can be defined in several variables, but there utility is something that will need to be studied. If $F(z)$ is a biholomorphic mapping of a domain Ω onto another domain Ω_0 where each coordinate function $F_j(z)=\mathcal{K}_j(z)/\mathcal{K}_0(z)$ is a quotient of linear combinations of the Bergman kernel and where each F_j is C^∞ close to the coordinate variable z_j as in §3, I have serious doubts that Ω_0 is a quadrature domain, or that the boundary of Ω_0 is real-algebraic or real-analytic, or that proper holomorphic mappings become algebraic in the new coordinates. The only thing I have been able to deduce about Ω_0 is that its Bergman kernel satisfies a rather special identity. Indeed, let $K_0(z,w)$ denote the Bergman kernel for Ω_0. Let $f=F^{-1}$ and define $u=\det f'$ and $U=\det F'$ to be the holomorphic jacobian determinants. The transformation formula for the Bergman kernels is

$$u(z)K(f(z),w)=K_0(z,F(w))\overline{U(w)}.$$

Hence, if we write

$$F_j(z)=\frac{\sum_{j=1}^N A_jK(z,a_j)}{\sum_{j=1}^M B_jK(z,b_j)},$$

then the coordinate function $z_j=F_j(f(z))$ on Ω_0 can be expressed via

$$z_j=\frac{u(z)\sum_{j=1}^N A_jK(f(z),a_j)}{u(z)\sum_{j=1}^M B_jK(f(z),b_j)},$$

and the transformation formula for the kernels yields

$$z_j=\frac{\sum_{j=1}^N A'_jK_0(z,F(a_j))}{\sum_{j=1}^M B'_jK_0(z,F(b_j))},$$

where $A'_j=A_j\overline{U(a_j)}$ and $B'_j=B_j\overline{U(b_j)}$. Thus, we see that the coordinate functions are given as Bergman quotients akin to Bergman coordinates. This is similar to being a quadrature domain because such domains are characterized by the fact that the function which is identically one is equal to a linear combination of Bergman kernel functions, and in one variable, the coordinate function z on a quadrature domain is also equal to a linear combination of Bergman kernels (see [7]). We must leave it for future research to uncover further special properties of these coordinates.

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