STUDIA MATHEMATICA 176 (1) (2006)

Hankel forms and sums of random variables

by

HENRY HELSON (Berkeley)

Abstract. A well known theorem of Nehari asserts on the circle group that bilinear forms in H^2 can be lifted to linear functionals on H^1. We show that this result can be extended to Hankel forms in infinitely many variables of a certain type. As a corollary we find a new proof that all the L^p norms on the class of Steinhaus series are equivalent.

1. Hankel forms. A *Hankel form* in ℓ^2 is one of the form

$$\langle a, b\rangle = \sum_{j,k=0}^{\infty} a_j b_k \varrho_{j+k} \tag{1.1}$$

where (ϱ_n) $(n \geq 0)$ is a square-summable sequence. It can be written

$$\sum_{k=0}^{\infty}\sum_{j=0}^{k} a_j b_{k-j} \varrho_k. \tag{1.2}$$

From this it is easy to see that if

$$f(e^{ix}) = \sum a_k e^{ikx}, \qquad g(e^{ix}) = \sum b_k e^{ikx} \tag{1.3}$$

then the value of the sum (1.1) depends on the function fg, but not on f and g individually. Furthermore if (f_n), (g_n) is a finite collection of functions in $H^2(T)$ (T is the circle group) with coefficients respectively $(a_{nk}), (b_{nk})$ such that

$$\sum f_n g_n \tag{1.4}$$

is 0, then

$$\sum_n \sum_{k=0}^{\infty}\sum_{j=0}^{k} a_{nj} b_{n,k-j} \varrho_k = 0. \tag{1.5}$$

2000 *Mathematics Subject Classification*: 43A15, 30B50, 15A63.

Key words and phrases: Hankel form, Hilbert–Schmidt form, homogeneous Fourier series.

Thus the form defines a linear functional on the subspace of $H^1(T)$ spanned by such products fg. Actually every function of $H^1(T)$ is a product of two functions in $H^2(T)$, so the functional is defined on $H^1(T)$ itself. It follows that there is a function ϕ in $L^\infty(T)$ such that $\widehat{\phi}(k) = \varrho_k$ $(k \geq 0)$. This is a well known result of Nehari. The converse is easy: every bounded function ϕ leads to a bounded Hankel form by the reverse route.

The recent evolution of the theory of Dirichlet series leads to questions about analogous statements for forms in many variables. Let K be the infinite-dimensional torus whose dual Γ is realized as the subgroup of the line (in discrete topology) consisting of all real numbers $\log r$, where r is a positive rational number. If r has the prime factoring

$$r = \prod p_j^{n_j} \tag{1.6}$$

where $p_1 = 2$, $p_2 = 3, \ldots$ are the prime integers, then the character $\log r$ has values

$$\chi_{\log r}(e^{ix_1}, e^{ix_2}, \ldots) = \exp\Big(i \sum n_j x_j\Big). \tag{1.7}$$

Thus Γ can also be viewed as the group of sequences $(n_1, n_2, \ldots)$ of integers terminating in zeros.

The *narrow cone* Λ in Γ consists of all $\log r$ such that each n_j is non-negative; that is, r is a positive integer. $H^p(K)$ $(p \geq 1)$ consists of all functions f in $L^p(K)$ whose Fourier series are sums over the narrow cone:

$$f(e^{ix_1}, \ldots) \sim \sum_{\text{all } n_j \geq 0} a(n_1, \ldots) e^{i \sum n_j x_j}. \tag{1.8}$$

This is a power series in the variables $z_j = e^{ix_j}$ $(j = 1, 2, \ldots)$. It depends on infinitely many variables, but each term only contains a finite number of them.

Addition in Γ is addition of corresponding components of sequences $(n_1, \ldots)$. This mirrors multiplication of the corresponding rational numbers r given by (1.6). Thus it is natural to define a Hankel form in infinitely many variables to be a form

$$\langle a, b \rangle = \sum_{j,k \geq 1} a_j b_k \varrho_{jk} \tag{1.9}$$

where a, b are square-summable sequences indexed by the positive integers, the kernel ϱ is a square-summable sequence, and jk is a product of integers, not a double subscript. If ϕ is a bounded function on K and $\widehat{\phi}(\log n) = \varrho_n$ $(n \geq 1)$ then the form is bounded. We ask whether the converse, the analogue of Nehari's theorem, is true. It is no longer the case that every function of $H^1(K)$ is the product of functions in $H^2(K)$, so the argument given above does not apply. Nevertheless the form defines a linear functional at least in the part of $H^1(K)$ spanned by products gh, where g, h belong to $H^2(K)$, exactly as recounted for the circle group above, and if a bounded function ϕ realizes this functional, then ϕ is *lifted* from the form. If every bounded Hankel form of some class can be lifted we shall say the class has the *lifting property*.

In [5] the lifting property was connected to another question. Denote by $\mathcal{K}$ the linear set of finite sums (1.4), where the factors all belong to $H^2(K)$. In $\mathcal{K}$ we define the tensor norm

$$\|h\| = \inf \sum \|f_n\|_2 \|g_n\|_2 \tag{1.10}$$

where the infimum extends over all finite sums (1.4) equal to h. Thus $\mathcal{K}$ is a subspace of $H^1(K)$ with a larger norm. Perhaps the completion of $\mathcal{K}$, which we denote by $\mathcal{K}^*$, is all of $H^1(K)$. It is if and only if the tensor norm on $\mathcal{K}$ is equivalent to the norm in $H^1(K)$. We do not know whether this is the case, but

The class of all bounded Hankel forms has the lifting property if and only if $\mathcal{K}^* = H^1(K)$.

The equivalence, stated in [5], is easy to establish.

2. Hilbert–Schmidt forms. A Hankel form with kernel ϱ is of *Hilbert–Schmidt type* if

$$\sum_{j,k=1}^{\infty} |\varrho_{jk}|^2 < \infty. \tag{2.1}$$

Then the form is bounded, with bound at most the square root of the sum. The terms of the sum are the same for all pairs (j,k) such that the product jk has a given value n. Therefore the sum in (2.1) is the same as

$$\sum_{n=1}^{\infty} |\varrho_n|^2 d(n), \tag{2.2}$$

where $d(n)$ is the number of divisors of n. (For example, if n is prime then $d(n) = 2$.)

Our main result is this.

THEOREM. *The class of Hilbert–Schmidt Hankel forms in infinitely many variables has the lifting property.*

This answers Question 3 of [5, p. 54]. It does not settle the question raised above, but has its own consequences.

The proof rests on the following result about the circle.

LEMMA 1. *For f in $H^1(T)$ with Fourier coefficients a_n,*

$$\Big(\sum_{n=0}^{\infty} |a_n|^2/(n+1)\Big)^{1/2} \le \|f\|_1. \tag{2.3}$$

The convergence of the sum on the left is of course a weaker statement than the convergence of the sum in the better known inequality

$$\sum_{n=0}^{\infty} |a_n|/(n+1) \le \pi \|f\|_1 \tag{2.4}$$

of Hardy and Littlewood [4, p. 129] (the bound π was found by I. Schur); but for us the bound in (2.3) is essential, and it is due to Vukotić [8].

Throughout this paper, the Lebesgue spaces are constructed with normalized Lebesgue measure, which we denote generically by σ on various torus groups.

Here is a proof of (2.3). We may assume that f has norm 1. Factor f as gh with g, h in $H^2(T)$ and $|g| = |h|$, so that $\|g\|_2 = \|h\|_2 = 1$. Let g and h have Fourier coefficients (b_n) and (c_n), respectively. Then the left side of (2.3) is the supremum over sequences (e_n) of

$$\sum_{n=0}^{\infty} |a_n| e_n/\sqrt{n+1} = \sum_{n=0}^{\infty} \Big| \sum_{j=0}^{n} b_j c_{n-j} \Big| e_n/\sqrt{n+1} \tag{2.5}$$

where the e_n are non-negative, square-summable, with squared-sum equal to 1. If we replace the b_n and c_n by their moduli the expression on the right is increased, so we may take them to be non-negative. Setting $n = j + k$ transforms (2.5) to

$$\sum_{j,k=0}^{\infty} b_j c_k e_{j+k}/\sqrt{j+k+1}. \tag{2.6}$$

This is a Hankel form of Hilbert–Schmidt type, whose bound is at most the square root of

$$\sum_{j,k=0}^{\infty} e_{j+k}^2/(j+k+1). \tag{2.7}$$

For each non-negative integer m there are $m+1$ terms with $j+k = m$; therefore (2.7) is

$$\sum_{m=0}^{\infty} e_m^2 = 1. \tag{2.8}$$

This shows that the left side of (2.5) is at most 1, and the lemma is proved.

F. Bayart has proved this result [1, 2, 5]: *for f in $H^1(K)$, and $n = \prod p_j^{n_j}$,*

$$\sum |\widehat{f}(n_1, \ldots)|^2/n^{\varepsilon} < \infty \tag{2.9}$$

for every positive ε. We are going to prove that

$$\Big(\sum_{n=1}^{\infty} |\widehat{f}(n_1,\dots)|^2/d(n)\Big)^{1/2} \le \|f\|_1. \tag{2.10}$$

The statement of our theorem will follow easily by duality. This improves Bayart's theorem in two respects: $d(n) = O(n^{\varepsilon})$ for every positive ε, and indeed $d(n)$ is *much* smaller than n^{ε} if n is prime or has few factors; and secondly, the precise bound given by (2.10) has no analogue in (2.9). But our proof of (2.10) will be identical with the proof of Bayart, substituting Lemma 1 above for a different piece of information.

The proof will be given in the next section; first we state a needed inequality, introduced into this subject by A. Bonami [3]:

LEMMA 2. *Let ϱ be a positive function on the product of spaces X and Y carrying measures dx, dy. Then*

$$\Big(\int\Big(\int \varrho(x,y)\,dx\Big)^2 dy\Big)^{1/2} \le \int\Big(\int \varrho^2(x,y)\,dy\Big)^{1/2} dx. \tag{2.11}$$

This is an integral version of Minkowski's inequality. Note that the order of integration is reversed by the inequality! (A proof is given in [5].)

3. Proof of the theorem. Let n have the prime factoring (1.6). All the divisors of n are obtained by replacing each n_j by all the k_j satisfying $0 \le k_j \le n_j$. Hence the number of divisors of n is exactly

$$\prod_j (n_j + 1). \tag{3.1}$$

Of course only finitely many n_j are different from 0.

We repeat Bayart's argument. The statement to be proved is

$$\Big(\sum_{\text{all } n_j \ge 0} \frac{|\widehat{f}(n_1,n_2,\dots)|^2}{(n_1+1)(n_2+1)\dots}\Big)^{1/2} \le \|f\|_1. \tag{3.2}$$

It will suffice to prove this for all f that are analytic trigonometric polynomials. Lemma 1 is (3.2) for f depending on only one variable. Suppose that f depends on k variables. For $m = 1,\dots,k$ let T_m be the operator defined by

$$T_m \sum a(n_1,\dots)e^{i\sum n_j x_j} = \sum \frac{a(n_1,\dots)}{\sqrt{n_m+1}}\, e^{i\sum n_j x_j}. \tag{3.3}$$

Then (3.2) becomes

$$\|T_1 \cdots T_k f\|_2 \le \|f\|_1. \tag{3.4}$$

Lemma 1, applied to the first variable, gives us

$$(3.5)\qquad \Big(\int |T_1\cdots T_k f|^2\, d\sigma(x_1,\ldots,x_k)\Big)^{1/2} \le \Big(\int\Big(\int |T_2\cdots T_k f|\, d\sigma(x_1)\Big)^2 d\sigma(x_2,\ldots x_k)\Big)^{1/2}.$$

The next step uses Lemma 2; the right side is less than

$$(3.6)\qquad \int\Big(\int |T_2\cdots T_k f|^2\, d\sigma(x_2,\cdots,x_k)\Big)^{1/2} d\sigma(x_1).$$

Now we have one fewer T's, and one variable removed from the inside integral. We apply Lemma 1 to the second variable, and so forth. After k steps we have left $\|f\|_1$, and (2.10) is proved.

The dual of the operation that carries an analytic trigonometric polynomial f depending on k variables with the norm of $H^1(K)$ to $T_1\cdots T_k f$ in $H^2(K)$, which we have shown reduces norm, maps $H^2(K)$ into the dual of $H^1(K)$. This means that if ϱ satisfies

$$(3.7)\qquad \sum_{n=1}^{\infty} |\varrho_n|^2 d(n) < \infty$$

then there is a bounded function ϕ on K whose Fourier coefficients $\widehat{\phi}(n_1,\ldots)$ are ϱ_n. This is the statement of the theorem.

4. Homogeneous Fourier series. The theorem has an unexpected application to some results that are usually treated in probability theory. A *Steinhaus series* [6, p. 134] is a sum

$$(4.1)\qquad \sum_{n=1}^{\infty} a_n e^{ix_n}$$

where the x_n are independent real variables. These are the complex analogue of Rademacher series [6, p. 125], and we expect the same results about them. It is well known that all the p-norms are equivalent on such sums. This is a statement about Fourier series, and deserves a simple treatment in those terms. We shall show now that (2.10) contains this and other results.

THEOREM. *Let m be a positive integer, and*

$$(4.2)\qquad f(x) \sim \sum_{\sum n_j = m} a(n_1,\ldots) e^{i\sum n_j x_j}$$

the Fourier series of a function f belonging to $H^1(K)$ and homogeneous of degree m. Then f belongs to $H^q(K)$ for every finite q. For $q = 2$ we have

$$(4.3)\qquad \|f\|_2 \le 2^{m/2}\|f\|_1.$$

If $m = 2$ then $\exp|f|$ *is summable.*

The first statement was proved by Bayart [2]. The bound in the second statement is not the best possible; in a difficult paper, Sawa [7] shows for $m = 1$ (the case of Steinhaus series) that the best bound is $2/\pi^{1/2}$.

According to (2.10), for f in $H^1(K)$

$$\Big(\sum |\widehat{f}(n_1, \ldots)|^2/d(n)\Big)^{1/2} \leq \|f\|_1. \tag{4.4}$$

If n is the product of m distinct primes, then $d(n) = 2^m$. If some of the primes are repeated $d(n)$ is smaller. Therefore for f with Fourier series (4.2)

$$\Big(\sum |\widehat{f}(n_1, \ldots)|^2/2^m\Big)^{1/2} \leq \|f\|_1, \tag{4.5}$$

which is (4.3).

Now f^2 is summable, and its Fourier series is also homogeneous (of degree $2m$), so f^2 is also square-summable, and so on. Therefore f belongs to $H^q(K)$ for every finite q.

When we keep track of the bounds at each step, we find that

$$\|f\|_n \leq n^{m/2}\|f\|_1 \tag{4.6}$$

when n is a power of 2. For other values of n the next power of 2 is less than $2n$, so that

$$\|f\|_n \leq (2n)^{m/2}\|f\|_1 \tag{4.7}$$

for all positive integers n. The expansion

$$e^{\lambda|f|} = \sum_{n=0}^{\infty} \lambda^n |f|^n/n! \tag{4.8}$$

converges in the norm of $L^1(K)$ for positive λ such that

$$\sum_{n=0}^{\infty} \lambda^n \|f\|_n^n/n! < \infty. \tag{4.9}$$

Hadamard's formula relates the radius of convergence of this series to

$$\limsup \Big[\log \|f\|_n - \frac{1}{n}\sum_{k=1}^{n} \log k\Big]. \tag{4.10}$$

By comparing the sum with an integral we have

$$\frac{1}{n}\sum_{k=1}^{n} \log k > \log n - 1. \tag{4.11}$$

From this fact and (4.7), (4.10) is less than

$$\limsup \Big[\log \|f\|_1 + \frac{m}{2}(\log n + \log 2) - \log n + 1\Big]. \tag{4.12}$$

For $m = 2$ the bracket is the constant $\log \|f\|_1 + \log 2 + 1$. Thus the radius of convergence is large if $\|f\|_1$ is small.

The argument finishes in the conventional way. By omitting the first terms in (4.2) the norm of f in $L^2(K)$ can be made as small as we please. All the norms are equivalent, so the norm of f in $L^1(K)$ is small too, and (4.12) can be made negatively large, and (4.9) converges for large λ. The omitted terms are a bounded function, and the proof is finished.

The same proof (as pointed out by H. Queffélec) leads to the statement that for any positive integer m, $\exp |f|^{2/m}$ is summable if f is homogeneous of degree m.

Acknowledgements. I would like to express my appreciation to J. E. McCarthy and to H. Queffélec for correspondence about the subject of this paper.

References

[1] F. Bayart, *Hardy spaces of Dirichlet series and their composition operators*, Monatsh. Math. 136 (2002), 203–236.
[2] —, *Opérateurs de composition sur des espaces de séries de Dirichlet et problèmes d'hypercyclicité simultanée*, thèse, Université des Sciences et Technologies de Lille, 2002.
[3] A. Bonami, *Étude des coefficients de Fourier des fonctions de $L^p(G)$*, Ann. Inst. Fourier (Grenoble) 20 (1970), 335–402.
[4] H. Helson, *Harmonic Analysis*, 2nd ed., published by the author, 1995.
[5] —, *Dirichlet Series*, published by the author, 2005.
[6] S. Kaczmarz and H. Steinhaus, *Theorie der Orthogonalreihen*, Chelsea reprint, 1951.
[7] J. Sawa, *Best constant in the Khintchine inequality for complex Steinhaus variables, the case $p = 1$*, Studia Math. 81 (1985), 107–126.
[8] D. Vukotić, *The isoperimetric inequality and a theorem of Hardy and Littlewood*, Amer. Math. Monthly 110 (2003), 532–536.

Mathematics Department
University of California
Berkeley, CA 94720-3840, U.S.A.
E-mail: hhelson@aol.com

Received March 21, 2006
Revised version July 5, 2006 (5887)