Optimal $L^p$-properties of Green’s functions for non-divergence elliptic equations in two dimensions

by

GIOCONDA MOSCARIELLO and CARLO SBORDONE (Napoli)

Abstract. A sharp integrability result for non-negative adjoint solutions to planar non-divergence elliptic equations is proved. A uniform estimate is also given for the Green’s function.

1. Introduction. Given $K \geq 1$ and a smooth domain $\Omega \subset \mathbb{R}^2$, denote by $E(K)$ the class of symmetric $2 \times 2$ matrix-valued functions $A = A(x)$ defined on $\Omega$ which satisfy the ellipticity bounds

$$\frac{|\xi|^2}{\sqrt{K}} \leq \langle A(x)\xi, \xi \rangle \leq \sqrt{K} |\xi|^2$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^2$. For $w \in W^{2,2}_\text{loc}(\Omega)$, set

$$\mathcal{M}[w] = \text{Tr}(A(x)D^2w)$$

and for $v \in L^2_\text{loc}(\Omega)$,

$$\mathcal{N}[v] = \sum_{ij} \frac{\partial^2}{\partial y_i \partial y_j} (a_{ij}(y)v(y)), \quad A = [a_{ij}].$$

This operator is nothing other than the formal adjoint of $\mathcal{M}$.

In this paper, following the ideas of [FS], we study the interior regularity of non-negative solutions $v \in L^2_\text{loc}(\Omega)$ of the adjoint equation $\mathcal{N}[v] = 0$ (i.e. $v \in L^2_\text{loc}(\Omega)$, $v \geq 0$, and $\int_{\Omega} v\mathcal{M}[\varphi] \, dx = 0$ for any $\varphi \in W^{2,2}(\Omega)$ with compact support). It is known [B] that such “adjoint solutions” need not be locally bounded, even if the $a_{ij}$ are continuous. Here we determine the best integrability exponent of $v$, in terms of the ellipticity constant $K$.

Namely, we prove that for $2 \leq p < 2K/(K - 1)$ the reverse Hölder inequality

$$\left( \frac{1}{B} \int_B v(y)^p \, dy \right)^{1/p} \leq c(K, p) \frac{1}{B} \int_B v(y) \, dy$$

holds for all balls $B = B(a, r) \subset B(a, 2r) \subset \Omega$.

2000 Mathematics Subject Classification: 35J15, 35J70.
The same estimate holds for $v(y) = G(x, y)$ where $G(x, y)$ is the Green’s function of $\mathcal{M}$ in $\Omega$, with the constant $c = c(K, p)$ independent of $x$.

The aforesaid results are optimal.

The main tool for our proof is a generalization of the Aleksandrov–Bakelman–Pucci inequality (see [P], [FM]) recently obtained by Astala–Iwaniec–Martin [AIM].

2. The $L^q$-version of the Aleksandrov–Bakelman–Pucci inequality. Our discussion here is focused on the second order elliptic equation

$$\mathcal{M}[w] = \text{Tr}(AD^2w) = a_{11}(x) \frac{\partial^2 w}{\partial x_1^2} + 2a_{12}(x) \frac{\partial^2 w}{\partial x_1 \partial x_2} + a_{22}(x) \frac{\partial^2 w}{\partial x_2^2} = h$$

with given $h \in L^q(B)$, $q > 1$, defined on the ball $B = B(0, r)$. If $q = 2$ the Dirichlet problem

$$\begin{cases}
\mathcal{M}[w] = h & \text{in } B, \\
u = 0 & \text{on } \partial B,
\end{cases}$$

admits a unique solution $w \in W^{2,2}(B) \cap W^{1,2}_0(B)$ (see [C]).

Let us formulate the second order equations in terms of the complex derivatives

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right), \quad \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right).$$

Upon a few elementary algebraic computations, we arrive at the formula

$$\text{Tr}(AD^2w) = (w_{zz} - \mu w_{zz} - \overline{\mu} \overline{w}_{zz}) \text{Tr} A$$

where

$$\mu = \mu(z) = \frac{a_{22} - a_{11} - 2ia_{12}}{2(a_{11} + a_{22})}.$$ 

The ellipticity bounds at (1.1) imply

$$|\mu(z)| + |\overline{\mu}(z)| \leq \frac{K - 1}{K + 1} < 1$$

for a.e. $z \in B$.

Using the complex gradient

$$f(z) = w_z = \frac{1}{2} \left( w_{x_1} - iw_{x_2} \right)$$

we are reduced to the Beltrami equation

$$f_z - \mu(z) f_{\overline{z}} - \overline{\mu(z)} \overline{f}_{\overline{z}} = \frac{h(z)}{\text{Tr} A}$$

in $B$. Optimal $L^q$-properties for its solutions have recently been established [AIS], [PV]. Precisely, given $H$ defined on $B$, we set $H = 0$ for $z \in \mathbb{R}^2 \setminus B$ and
\[ \mu(z) = 0 \text{ for } z \in \mathbb{R}^2 \setminus B. \] Then the equation extends to the entire space \( \mathbb{R}^2 \),

\[ F_z - \mu(z)F_z - \mu(z) \overline{F}_z = H. \]

It has a unique solution \( F \) such that

\[ \|F\|_{L^q(\mathbb{R}^2)} \leq c(q, K)\|H\|_{L^q(\mathbb{R}^2)} \]

for \( 2K/(K + 1) < q < 2K/(K - 1) \). With the aid of this estimate the following result has been established in [AIM].

**Theorem 2.1.** Suppose \( 2K/(K + 1) < q \leq 2 \), and \( w \in W^{2,q}_{\text{loc}}(B_r) \) satisfies

\[
\begin{cases}
\mathcal{M}[w] = h & \text{a.e. in } B_r = B(0, r), \\
w = 0 & \text{on } \partial B_r.
\end{cases}
\]

Then

\[ \|w\|_{L^\infty(B_r)} \leq c(K, q)r^{2-2/q}\|h\|_{L^q(B_r)}. \]

The estimate no longer holds if \( q \leq 2K/(K + 1) \).

3. **A reverse Hölder inequality for non-negative adjoint solutions.** In this section the letter \( c \) will denote a constant depending on \( K \) and \( p \). It may vary at each occurrence.

We are now ready to prove the following

**Theorem 3.1.** Assume \( A = [a_{ij}] \) satisfies (1.1). Let \( v \in L^2(\Omega), v \geq 0 \) in \( \Omega \), satisfy the adjoint equation

\[ \mathcal{N}[v] = \sum_{i,j} \frac{\partial^2}{\partial y_i \partial y_j} (a_{ij}(y)v(y)) = 0. \]

Then, for all balls \( B_r \subset B_{2r} \subset \Omega \), we have

\[ \left( \int_{B_r} v(y)^p \, dy \right)^{1/p} \leq c(K, p) \| v \|_{B_r}, \]

where \( 2 \leq p < 2K/(K - 1) \).

**Proof.** We closely follow the arguments in [FS]. Note that here we dispense with the smoothness assumption on the coefficients. For \( n = 2 \) this assumption is redundant.

We make use of the dual expression of the \( L^p \)-norm,

\[ \left( \int_{B_r} v \right)^{1/p} = \sup \left\{ \int_{B_r} vh : h \geq 0, h \in C^1_0(B_r), \|h\|_{L^q(\mathbb{R}^2)} \leq 1 \right\}. \]

Fix \( h \in C^1_0(B_r), \|h\|_{L^q} \leq 1, h \geq 0 \). Applying (2.1) we solve the Dirichlet problem

\[
\begin{cases}
\mathcal{M}[w] = h & \text{in } B_{2r}, \\
w = 0 & \text{on } \partial B_{2r}.
\end{cases}
\]
Next, for $w \in W^{2,2}(B_{2r})$ fix $\varphi_r \in C^1_0(B_{3r/2})$ such that $\varphi_r = 1$ on $B_r$ and $|\partial^{\alpha} \varphi_r / \partial x^\alpha| \leq C_\alpha / r^{\alpha}$. Then we have

$$(3.3) \quad \int_{B_r} vh \leq \int_{B_{2r}} vM[w] \varphi_r = -\int_{B_{2r}} vwM[\varphi_r] - 2 \int_{B_{2r}} v\langle A\nabla w, \nabla \varphi_r \rangle$$

$$\leq \frac{c}{r^2} \|w\|_{L^\infty(B_{2r})} \int_{B_{3r/2}} v + \frac{c\sqrt{K}}{r} \int_{B_{3r/2}} v|\nabla w|.$$ 

By (2.4), $\|w\|_{L^\infty(B_{2r})} \leq c(K, q)r^{2/p}$, hence (3.3) implies

$$(3.4) \quad \int_{B_r} vh \leq \frac{c}{r^2} r^{2/p} \int_{B_{3r/2}} v + \frac{c}{r} \left( \int_{B_{3r/2}} v \right)^{1/2} \left( \int_{B_{2r}} v|\nabla w|^2 \right)^{1/2}.$$ 

We now estimate the last integral in the right hand side, by using the Caccioppoli inequality. By (1.1) we have

$$\int_{B_{2r}} v|\nabla w|^2 \leq \sqrt{K} \int_{B_{2r}} v\langle A\nabla w, \nabla w \rangle = \sqrt{K} \int_{B_{2r}} v[M[w^2] - 2wh].$$

Since $w^2 = 0$ on $\partial B_{2r}$, and $\nabla (w^2) = 0$ on $\partial B_{2r}$, we deduce

$$\int_{B_{2r}} vM[w^2] = 0 \quad \text{whenever} \quad N[v] = 0.$$ 

Using again (2.4) yields

$$(3.5) \quad \int_{B_{2r}} v|\nabla w|^2 \leq 2\sqrt{K} \int_{B_{2r}} v|w|h \leq 2\sqrt{K} \|w\|_{L^\infty(B_{2r})} \int_{B_r} vh$$

$$\leq 2\sqrt{K} cr^{2/p} \int_{B_r} vh.$$ 

By (3.4) and (3.5) it follows that

$$\int_{B_r} vh \leq \frac{c}{r^{2(1-1/p)}} \int_{B_{3r/2}} v + \frac{c}{r^{1-1/p}} \left( \int_{B_{3r/2}} v \right)^{1/2} \left( \int_{B_r} vh \right)^{1/2}.$$ 

By the elementary inequality $\sqrt{a} \sqrt{b} \leq a/2 + b/2$, we arrive at

$$\int_{B_r} vh \leq \frac{c}{r^{2(1-1/p)}} \int_{B_{3r/2}} v + \frac{c}{r^{2(1-1/p)}} \int_{B_{3r/2}} v + \frac{1}{2} \int_{B_r} vh.$$ 

Rearranging yields

$$(3.6) \quad \int_{B_r} vh \leq \frac{c}{r^{2(1-1/p)}} \int_{B_{3r/2}} v.$$
Since \( h \) is arbitrary, by (3.2), (3.6) we obtain
\[
\left( \int_{B_r} v^p \right)^{1/p} \leq c \int_{B_{3r/2}} v.
\]
An application of the following lemma (Lemma 2.0 in [FS]) concludes the proof.

**Lemma 3.1.** There exists a constant \( c \), depending only on \( K \), such that for all non-negative weak solutions \( v \) of \( \mathcal{N}[v] = 0 \) and for all balls \( B_r \) with \( B_{2r} \subset \Omega \) we have
\[
\int_{B_r} v(y) \, dy \leq c \int_{B_{r/2}} v(y) \, dy.
\]

**4. A reverse Hölder inequality for the Green’s function.** Recall that the Green’s function for \( \mathcal{M} \) on a smooth domain \( \Omega \) is non-negative and \( G_{\Omega}(x, \cdot) \in L^1(\Omega) \) for every \( x \in \Omega \). We have the identity
\[
\varphi(x) = -\int_{\Omega} G_{\Omega}(x, y) \mathcal{M}\varphi(y) \, dy
\]
for any \( \varphi \in C^2(\mathbb{R}^n) \) such that \( \varphi = 0 \) on \( \partial \Omega \).

**Theorem 4.1.** For every \( 2 \leq p < 2K/(K - 1) \) and for all balls \( B_r \subset B_{4r} \subset \Omega \), we have
\[
(4.1) \quad \left[ \int_{B_r} G_{\Omega}(x, y)^p \, dy \right]^{1/p} \leq c(K, p) \int_{B_r} G_{\Omega}(x, y) \, dy
\]
for \( x \in \Omega \).

Let us first recall some well known properties of Green’s functions. The Aleksandrov–Bakelman–Pucci theorem for \( n = 2 \) reads

**Theorem 4.2.** Let \( w \in W^{2,2}(\Omega) \) satisfy
\[
(4.2) \quad \begin{cases}
\mathcal{M}[w] = h & \text{with given } h \in L^2(\Omega), \\
w = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Then
\[
\|w\|_{L^\infty(\Omega)} \leq c(K)d(\Omega)\|h\|_{L^2(\Omega)}
\]
with \( d(\Omega) = \text{diam}(\Omega) \).

The solution is unique. In what follows we write it as \( w = w_h \) to indicate the dependence on \( h \in L^2(\Omega) \). The following result is a well known consequence of Theorem 4.2.
Corollary 4.1. There exists a unique function $G(x, \cdot) \in L^2(\Omega)$ such that $G(x, y) \geq 0$ in $\Omega \times \Omega$,

$$w_h(x) = - \int_\Omega G(x, y) h(y) \, dy$$

and

$$\sup_{x \in \Omega} \|G(x, \cdot)\|_{L^2(\Omega)} \leq c(K)d(\Omega).$$

We need another preliminary fact:

Lemma 4.1 ([K, Lemma 3.3]). Let $G_r(x, y)$ denote the Green’s function for $\mathcal{M}$ in $B_{3r}$. Then there exist two positive constants $c_1(K)$, $c_2(K)$ such that

$$c_1 \leq \int_{B_r} G_r(x, y) \, dy \leq c_2$$

for $x \in B_{2r}$.

Proof of Theorem 4.1. If $x \notin B_{2r}$ then $G(x, \cdot)$ is an adjoint solution of $\mathcal{M}$ in $B_{2r}$, and then the estimate follows from Theorem 3.1.

Assume now that $x \in B_{2r}$. Let $G_r(x, y)$ be the Green’s function for $\mathcal{M}$ in $B_{3r}$. By the maximum principle we know that $G(x, y) \geq G_r(x, y)$ and thus the function $v(y) = G(x, y) - G_r(x, y)$ is a non-negative solution to $\mathcal{N}[v] = 0$ in $B_{2r}$. Hence, using Theorem 3.1, we have

$$\int_{B_r} G(x, y)^p \, dy \leq c \int_{B_r} [G(x, y) - G_r(x, y)]^p \, dy + c \int_{B_r} G_r(x, y)^p \, dy$$

$$\leq c \left\{ \int_{B_r} [G(x, y) - G_r(x, y)] \, dy \right\}^p + c \int_{B_r} G_r(x, y)^p \, dy.$$

To estimate the last term we invoke the inequality

$$\left[ \int_{B_r} G_r(x, y)^p \, dy \right]^{1/p} \leq c(K, p)r^{2/p},$$

which comes from Theorem 3.1 in the following way. First observe that the solution $w$ to the Dirichlet problem

$$\begin{cases}
\mathcal{M}[w] = h & \text{in } B_{3r}, \\
w = 0 & \text{on } \partial B_{3r},
\end{cases}$$

for $h \in L^q$ $(1/q + 1/p = 1)$ can be represented as

$$w(x) = - \int_{B_{3r}} G_r(x, y) h(y) \, dy.$$

Then (4.5) follows by duality arguments:

$$\left[ \int_{B_{3r}} G_r(x, y)^p \, dy \right]^{1/p} = \sup_{\|h\|_{L^q(B_{3r})} \leq 1} |w(x)| \leq c(K, q)r^{2-2/q} = c(K, p)r^{2/p}. $$
In view of Lemma 4.1 inequality (4.5) implies
\[
\left( \int_{B_r} G_r(x, y)^p \, dy \right)^{1/p} \leq c(K, p) \int_{B_r} G_r(x, y) \, dy,
\]
which, together with (4.4), concludes the proof. ■

The following result parallels Corollary 2.4 in [FS] and can be proved in the same way.

**Corollary 4.2.** Let \( G(x, y) \) denote the Green’s function corresponding to \( \mathcal{M} \) on \( \Omega \). Then for every \( 2 \leq p < 2K/(K - 1) \) there exists a constant \( A_p = A_p(K, d), d = \text{diam}(\Omega) \), such that
\[
\sup_{x \in \Omega} \int_{\Omega} G(x, y)^p \, dy \leq A_p.
\]

The optimality of the exponent \( p \) in Theorem 3.1 and in Theorem 4.1 follows again by duality arguments. Assume that inequality (4.1) holds for \( p_0 = 2K/(K - 1) \).

As in [AIM, Sect. 7], for \( x \in B = B(0, 1) \) let
\begin{align}
\mathcal{M} &= \text{Tr}(A(x)D^2), \\
A(x) &= \left( \sqrt{K} - \frac{1}{\sqrt{K}} \right) \frac{x \otimes x}{|x|^2} + \frac{I}{\sqrt{K}}, \quad x \otimes x = \begin{bmatrix} x_1^2 & x_1x_2 \\ x_2x_1 & x_2^2 \end{bmatrix}, \\
(4.7) \quad w_N(x) &= \varphi_N(|x|) \quad \text{for } N > 1,
\end{align}
where
\begin{align}
(4.9) \quad \varphi_N(r) &= \begin{cases} 
(\log r)r^{1-1/K} + \left( \log N - \frac{K}{K - 1} \right)(r^{1-1/K} - 1) & \text{if } 1/N \leq r, \\
- \log N + \frac{K}{K - 1} (1 - N^{-1 + 1/K}) & \text{if } 0 \leq r < 1/N,
\end{cases}
\end{align}
and define
\[
h_N(x) = \left( \sqrt{K} - \frac{1}{\sqrt{K}} \right)|x|^{-1-1/K} \chi_{1/N < |x| < 1}(x).
\]
It is easy to check that \( w_N(x) \) is the solution to the Dirichlet problem
\[
\begin{cases}
\mathcal{M}[w_N] = h_N & \text{in } B, \\
w_N = 0 & \text{on } \partial B,
\end{cases}
\]
and therefore $w_N(x)$ can be represented as
\begin{equation}
(4.10) \quad w_N(x) = -\int_B G(x,y)h_N(y)\,dy, \quad x \in B,
\end{equation}
for $G$ the Green’s function of $\mathcal{M}$ with respect to $B$. An elementary calculation reveals that
\begin{equation}
(4.11) \quad \|h_N\|_{L^{q_0}(B)} = \left(\frac{\sqrt{K} - 1}{\sqrt{K}}\right)(2\pi \log N)^{(2K+1)/2K}
\end{equation}
where
\begin{equation}
q_0 = \frac{2K}{K+1} = \frac{p_0}{p_0-1}
\end{equation}
and
\begin{equation}
(4.12) \quad \|w_N\|_{L^\infty(B)} \geq c(K)(\log N)^{1+(K+1)/2K}.
\end{equation}
By (4.11) and (4.12) it follows that
\begin{equation}
(4.13) \quad \frac{\|w_N\|_{L^\infty}}{\|h_N\|_{L^{q_0}}} \to \infty \quad \text{as } N \to \infty
\end{equation}
An application of Hölder’s inequality and Corollary 4.2 yield the estimates
\begin{equation}
|w_N(x)| \leq \|G(x,\cdot)\|_{L^{p_0}(B)} \cdot \|h_N\|_{L^{q_0}(B)} \leq A_{p_0}(K)\|h_N\|_{L^{q_0}(B)},
\end{equation}
which are not consistent with (4.13).

Acknowledgments. The research of both authors has been supported by MIUR-PRIN 02 and GNAMPA-INdAM.

References


Dipartimento di Matematica “R. Caccioppoli”
Università di Napoli “Federico II”
Via Cintia, 80126 Napoli, Italy
E-mail: gmoscari@unina.it
sbordone@unina.it

*Received August 2, 2004*
*Revised version February 8, 2005*