STUDIA MATHEMATICA 169 (2) (2005)

Optimal L^p-properties of Green's functions for non-divergence elliptic equations in two dimensions

by

GIOCONDA MOSCARIELLO and CARLO SBORDONE (Napoli)

Abstract. A sharp integrability result for non-negative adjoint solutions to planar non-divergence elliptic equations is proved. A uniform estimate is also given for the Green's function.

1. Introduction. Given $K \geq 1$ and a smooth domain $\Omega \subset \mathbb{R}^2$, denote by $\mathcal{E}(K)$ the class of symmetric 2×2 matrix-valued functions $A = A(x)$ defined on Ω which satisfy the ellipticity bounds

$$\frac{|\xi|^2}{\sqrt{K}} \leq \langle A(x)\xi, \xi\rangle \leq \sqrt{K}\,|\xi|^2 \tag{1.1}$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^2$. For $w \in W^{2,2}_{\mathrm{loc}}(\Omega)$, set

$$\mathcal{M}[w] = \mathrm{Tr}(A(x)D^2 w)$$

and for $v \in L^2_{\mathrm{loc}}(\Omega)$,

$$\mathcal{N}[v] = \sum_{ij} \frac{\partial^2}{\partial y_i \partial y_j}(a_{ij}(y)v(y)), \qquad A = [a_{ij}].$$

This operator is nothing other than the formal adjoint of $\mathcal{M}$.

In this paper, following the ideas of [FS], we study the interior regularity of non-negative solutions $v \in L^2_{\mathrm{loc}}(\Omega)$ of the adjoint equation $\mathcal{N}[v] = 0$ (i.e. $v \in L^2_{\mathrm{loc}}(\Omega)$, $v \geq 0$, and $\int_\Omega v\mathcal{M}[\varphi]\,dx = 0$ for any $\varphi \in W^{2,2}(\Omega)$ with compact support). It is known [B] that such "adjoint solutions" need not be locally bounded, even if the a_{ij} are continuous. Here we determine the best integrability exponent of v, in terms of the ellipticity constant K.

Namely, we prove that for $2 \leq p < 2K/(K-1)$ the *reverse Hölder inequality*

$$\Big(\fint_B v(y)^p\,dy\Big)^{1/p} \leq c(K,p) \fint_B v(y)\,dy$$

holds for all balls $B = B(a,r) \subset B(a,2r) \subset \Omega$.

2000 *Mathematics Subject Classification*: 35J15, 35J70.

The same estimate holds for $v(y) = G(x, y)$ where $G(x, y)$ is the Green's function of $\mathcal{M}$ in Ω, with the constant $c = c(K, p)$ independent of x.

The aforesaid results are optimal.

The main tool for our proof is a generalization of the Aleksandrov–Bakelman–Pucci inequality (see [P], [FM]) recently obtained by Astala–Iwaniec–Martin [AIM].

2. The L^q-version of the Aleksandrov–Bakelman–Pucci inequality. Our discussion here is focused on the second order elliptic equation

$$\mathcal{M}[w] = \operatorname{Tr}(AD^2 w) = a_{11}(x)\,\frac{\partial^2 w}{\partial x_1^2} + 2a_{12}(x)\,\frac{\partial^2 w}{\partial x_1 \partial x_2} + a_{22}(x)\,\frac{\partial^2 w}{\partial x_2^2} = h$$

with given $h \in L^q(B)$, $q > 1$, defined on the ball $B = B(0, r)$. If $q = 2$ the Dirichlet problem

$$\begin{cases} \mathcal{M}[w] = h & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{2.1}$$

admits a unique solution $w \in W^{2,2}(B) \cap W_0^{1,2}(B)$ (see [C]).

Let us formulate the second order equations in terms of the complex derivatives

$$\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2}\right), \qquad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\,\frac{\partial}{\partial x_2}\right).$$

Upon a few elementary algebraic computations, we arrive at the formula

$$\operatorname{Tr}(AD^2 w) = (w_{z\bar z} - \mu w_{zz} - \overline{\mu}\,\overline{w}_{zz})\operatorname{Tr} A$$

where

$$\mu = \mu(z) = \frac{a_{22} - a_{11} - 2ia_{12}}{2(a_{11} + a_{22})}. \tag{2.2}$$

The ellipticity bounds at (1.1) imply

$$|\mu(z)| + |\overline{\mu}(z)| \le \frac{K-1}{K+1} < 1 \tag{2.3}$$

for a.e. $z \in B$.

Using the complex gradient

$$f(z) = w_z = \frac{1}{2}\,(w_{x_1} - i w_{x_2})$$

we are reduced to the Beltrami equation

$$f_{\bar z} - \mu(z) f_z - \overline{\mu(z)}\,\overline{f}_z = \frac{h(z)}{\operatorname{Tr} A}$$

in B. Optimal L^q-properties for its solutions have recently been established [AIS], [PV]. Precisely, given H defined on B, we set $H = 0$ for $z \in \mathbb{R}^2 \setminus B$ and

$\mu(z) = 0$ for $z \in \mathbb{R}^2 \setminus B$. Then the equation extends to the entire space $\mathbb{R}^2$,

$$F_{\bar{z}} - \mu(z)F_z - \overline{\mu(z)}\, \overline{F}_z = H.$$

It has a unique solution F such that

$$\|F_{\bar{z}}\|_{L^q(\mathbb{R}^2)} \le c(q,K)\|H\|_{L^q(\mathbb{R}^2)}$$

for $2K/(K+1) < q < 2K/(K-1)$. With the aid of this estimate the following result has been established in [AIM].

THEOREM 2.1. *Suppose* $2K/(K+1) < q \le 2$, *and* $w \in W^{2,q}_{\mathrm{loc}}(B_r)$ *satisfies*

$$\begin{cases} \mathcal{M}[w] = h & \text{a.e. in } B_r = B(0,r), \\ w = 0 & \text{on } \partial B_r. \end{cases}$$

Then

$$\|w\|_{L^\infty(B_r)} \le c(K,q) r^{2-2/q} \|h\|_{L^q(B_r)}. \tag{2.4}$$

The estimate no longer holds if $q \le 2K/(K+1)$.

3. A reverse Hölder inequality for non-negative adjoint solutions. In this section the letter c will denote a constant depending on K and p. It may vary at each occurrence.

We are now ready to prove the following

THEOREM 3.1. *Assume* $A = [a_{ij}]$ *satisfies* (1.1). *Let* $v \in L^2(\Omega)$, $v \ge 0$ *in* Ω, *satisfy the adjoint equation*

$$\mathcal{N}[v] = \sum_{i,j} \frac{\partial^2}{\partial y_i \partial y_j} (a_{ij}(y)v(y)) = 0.$$

Then, for all balls $B_r \subset B_{2r} \subset \Omega$, *we have*

$$\Big(\fint_{B_r} v(y)^p\, dy \Big)^{1/p} \le c(K,p) \fint_{B_r} v(y)\, dy, \tag{3.1}$$

where $2 \le p < 2K/(K-1)$.

Proof. We closely follow the arguments in [FS]. Note that here we dispense with the smoothness assumption on the coefficients. For $n = 2$ this assumption is redundant.

We make use of the dual expression of the L^p-norm,

$$\Big(\int_{B_r} v^p \Big)^{1/p} = \sup \Big\{ \int_{B_r} vh : h \ge 0,\ h \in C^1_0(B_r),\ \|h\|_{L^q(\mathbb{R}^2)} \le 1 \Big\}. \tag{3.2}$$

Fix $h \in C^1_0(B_r)$, $\|h\|_{L^q} \le 1$, $h \ge 0$. Applying (2.1) we solve the Dirichlet problem

$$\begin{cases} \mathcal{M}[w] = h & \text{in } B_{2r}, \\ w = 0 & \text{on } \partial B_{2r}. \end{cases}$$

Next, for $w \in W^{2,2}(B_{2r})$ fix $\varphi_r \in C_0^1(B_{3r/2})$ such that $\varphi_r = 1$ on B_r and $|\partial^\alpha \varphi_r / \partial x^\alpha| \le C_\alpha / r^{|\alpha|}$.

Then we have

$$\int_{B_r} vh \le \int_{B_{2r}} v\mathcal{M}[w]\varphi_r = -\int_{B_{2r}} vw\mathcal{M}[\varphi_r] - 2\int_{B_{2r}} v\langle A\nabla w\,,\nabla\varphi_r\rangle \\ \le \frac{c}{r^2}\,\|w\|_{L^\infty(B_{2r})}\int_{B_{3r/2}} v + \frac{c\sqrt{K}}{r}\int_{B_{3r/2}} v|\nabla w|. \tag{3.3}$$

By (2.4), $\|w\|_{L^\infty(B_{2r})} \le c(K,q)r^{2/p}$, hence (3.3) implies

$$\int_{B_r} vh \le \frac{c}{r^2}\,r^{2/p}\int_{B_{3r/2}} v + \frac{c}{r}\Big(\int_{B_{3r/2}} v\Big)^{1/2}\Big(\int_{B_{2r}} v|\nabla w|^2\Big)^{1/2}. \tag{3.4}$$

We now estimate the last integral in the right hand side, by using the Caccioppoli inequality. By (1.1) we have

$$\int_{B_{2r}} v|\nabla w|^2 \le \sqrt{K}\int_{B_{2r}} v\langle A\nabla w\,,\nabla w\rangle = \sqrt{K}\int_{B_{2r}} v[\mathcal{M}[w^2] - 2wh].$$

Since $w^2 = 0$ on ∂B_{2r}, and $\nabla(w^2) = 0$ on ∂B_{2r}, we deduce

$$\int_{B_{2r}} v\mathcal{M}[w^2] = 0 \quad \text{whenever} \quad \mathcal{N}[v] = 0.$$

Using again (2.4) yields

$$\int_{B_{2r}} v|\nabla w|^2 \le 2\sqrt{K}\int_{B_{2r}} v|w|h \le 2\sqrt{K}\,\|w\|_{L^\infty(B_{2r})}\int_{B_r} vh \\ \le 2\sqrt{K}\,cr^{2/p}\int_{B_r} vh. \tag{3.5}$$

By (3.4) and (3.5) it follows that

$$\int_{B_r} vh \le \frac{c}{r^{2(1-1/p)}}\int_{B_{3r/2}} v + \frac{c}{r^{1-1/p}}\Big(\int_{B_{3r/2}} v\Big)^{1/2}\Big(\int_{B_r} vh\Big)^{1/2}.$$

By the elementary inequality $\sqrt{a}\,\sqrt{b} \le a/2 + b/2$, we arrive at

$$\int_{B_r} vh \le \frac{c}{r^{2(1-1/p)}}\int_{B_{3r/2}} v + \frac{c}{r^{2(1-1/p)}}\int_{B_{3r/2}} v + \frac{1}{2}\int_{B_r} vh.$$

Rearranging yields

$$\int_{B_r} vh \le \frac{c}{r^{2(1-1/p)}}\int_{B_{3r/2}} v. \tag{3.6}$$

Since h is arbitrary, by (3.2), (3.6) we obtain

$$\Big(\fint_{B_r} v^p\Big)^{1/p} \le c \fint_{B_{3r/2}} v.$$

An application of the following lemma (Lemma 2.0 in [FS]) concludes the proof.

LEMMA 3.1. *There exists a constant c, depending only on K, such that for all non-negative weak solutions v of $\mathcal{N}[v] = 0$ and for all balls B_r with $B_{2r} \subset \Omega$ we have*

$$\int_{B_r} v(y)\,dy \le c \int_{B_{r/2}} v(y)\,dy. \ \blacksquare$$

4. A reverse Hölder inequality for the Green's function. Recall that the Green's function for $\mathcal{M}$ on a smooth domain Ω is non-negative and $G_\Omega(x,\cdot) \in L^1(\Omega)$ for every $x \in \Omega$. We have the identity

$$\varphi(x) = -\int_\Omega G_\Omega(x,y)\mathcal{M}\varphi(y)\,dy$$

for any $\varphi \in C^2(\overline{\Omega})$ such that $\varphi = 0$ on $\partial\Omega$.

THEOREM 4.1. *For every $2 \le p < 2K/(K-1)$ and for all balls $B_r \subset B_{4r} \subset \Omega$, we have*

$$\Big[\fint_{B_r} G_\Omega(x,y)^p\,dy\Big]^{1/p} \le c(K,p) \fint_{B_r} G_\Omega(x,y)\,dy \tag{4.1}$$

for $x \in \Omega$.

Let us first recall some well known properties of Green's functions. The Aleksandrov–Bakelman–Pucci theorem for $n = 2$ reads

THEOREM 4.2. *Let $w \in W^{2,2}(\Omega)$ satisfy*

$$\begin{cases} \mathcal{M}[w] = h & \text{with given } h \in L^2(\Omega), \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Then

$$\|w\|_{L^\infty(\Omega)} \le c(K)d(\Omega)\|h\|_{L^2(\Omega)}$$

with $d(\Omega) = \mathrm{diam}(\Omega)$.

The solution is unique. In what follows we write it as $w = w_h$ to indicate the dependence on $h \in L^2(\Omega)$. The following result is a well known consequence of Theorem 4.2.

Corollary 4.1. *There exists a unique function $G(x,\cdot)\in L^2(\Omega)$ such that $G(x,y)\geq 0$ in $\Omega\times\Omega$,*

$$w_h(x) = -\int_\Omega G(x,y)h(y)\,dy$$

and

$$\sup_{x\in\Omega}\|G(x,\cdot)\|_{L^2(\Omega)}\leq c(K)d(\Omega). \tag{4.3}$$

We need another preliminary fact:

Lemma 4.1 ([K, Lemma 3.3]). *Let $G_r(x,y)$ denote the Green's function for $\mathcal{M}$ in B_{3r}. Then there exist two positive constants $c_1(K)$, $c_2(K)$ such that*

$$c_1\leq \fint_{B_r} G_r(x,y)\,dy\leq c_2 \quad \textit{for } x\in B_{2r}.$$

Proof of Theorem 4.1. If $x\notin B_{2r}$ then $G(x,\cdot)$ is an adjoint solution of $\mathcal{M}$ in B_{2r} and then the estimate follows from Theorem 3.1.

Assume now that $x\in B_{2r}$. Let $G_r(x,y)$ be the Green's function for $\mathcal{M}$ in B_{3r}. By the maximum principle we know that $G(x,y)\geq G_r(x,y)$ and thus the function $v(y)=G(x,y)-G_r(x,y)$ is a non-negative solution to $\mathcal{N}[v]=0$ in B_{2r}. Hence, using Theorem 3.1, we have

$$\begin{aligned}\fint_{B_r} G(x,y)^p\,dy &\leq c\fint_{B_r}[G(x,y)-G_r(x,y)]^p\,dy + c\fint_{B_r} G_r(x,y)^p\,dy\\ &\leq c\Big\{\fint_{B_r}[G(x,y)-G_r(x,y)]\,dy\Big\}^p + c\fint_{B_r} G_r(x,y)^p\,dy.\end{aligned} \tag{4.4}$$

To estimate the last term we invoke the inequality

$$\Big[\int_{B_r} G_r(x,y)^p\,dy\Big]^{1/p}\leq c(K,p)r^{2/p}, \tag{4.5}$$

which comes from Theorem 3.1 in the following way. First observe that the solution w to the Dirichlet problem

$$\begin{cases}\mathcal{M}[w]=h & \text{in } B_{3r},\\ w=0 & \text{on } \partial B_{3r},\end{cases}$$

for $h\in L^q$ $(1/q+1/p=1)$ can be represented as

$$w(x) = -\int_{B_{3r}} G_r(x,y)h(y)\,dy.$$

Then (4.5) follows by duality arguments:

$$\Big[\int_{B_{3r}} G_r(x,y)^p\,dy\Big]^{1/p} = \sup_{\|h\|_{L^q(B_{3r})}\leq 1}|w(x)|\leq c(K,q)r^{2-2/q} = c(K,p)r^{2/p}.$$

In view of Lemma 4.1 inequality (4.5) implies

$$\Big[\fint_{B_r} G_r(x,y)^p \, dy \Big]^{1/p} \le c(K,p) \fint_{B_r} G_r(x,y) \, dy,$$

which, together with (4.4), concludes the proof. ∎

The following result parallels Corollary 2.4 in [FS] and can be proved in the same way.

COROLLARY 4.2. *Let $G(x,y)$ denote the Green's function corresponding to $\mathcal{M}$ on Ω. Then for every $2 \le p < 2K/(K-1)$ there exists a constant $A_p = A_p(K,d)$, $d = \operatorname{diam}(\Omega)$, such that*

$$\sup_{x\in\Omega} \int_{\Omega} G(x,y)^p \, dy \le A_p.$$

The optimality of the exponent p in Theorem 3.1 and in Theorem 4.1 follows again by duality arguments. Assume that inequality (4.1) holds for $p_0 = 2K/(K-1)$.

As in [AIM, Sect. 7], for $x \in B = B(0,1)$ let

$$\mathcal{M} = \operatorname{Tr}(A(x)D^2), \tag{4.6}$$

$$A(x) = \left(\sqrt{K} - \frac{1}{\sqrt{K}}\right)\frac{x\otimes x}{|x|^2} + \frac{I}{\sqrt{K}}, \qquad x\otimes x = \begin{bmatrix} x_1^2 & x_1x_2 \\ x_2x_1 & x_2^2 \end{bmatrix}, \tag{4.7}$$

$$w_N(x) = \varphi_N(|x|) \quad \text{for } N > 1, \tag{4.8}$$

where

$$\varphi_N(r) = \begin{cases} (\log r) r^{1-1/K} + \left(\log N - \dfrac{K}{K-1}\right)(r^{1-1/K} - 1) & \text{if } 1/N \le r, \\ -\log N + \dfrac{K}{K-1}\,(1 - N^{-1+1/K}) & \text{if } 0 \le r < 1/N, \end{cases} \tag{4.9}$$

and define

$$h_N(x) = \left(\sqrt{K} - \frac{1}{\sqrt{K}}\right)|x|^{-1-1/K}\chi_{1/N<|x|<1}(x).$$

It is easy to check that $w_N(x)$ is the solution to the Dirichlet problem

$$\begin{cases} \mathcal{M}[w_N] = h_N & \text{in } B, \\ w_N = 0 & \text{on } \partial B, \end{cases}$$

and therefore $w_N(x)$ can be represented as

$$w_N(x) = -\int_B G(x,y)h_N(y)\,dy, \qquad x \in B, \tag{4.10}$$

for G the Green's function of $\mathcal{M}$ with respect to B. An elementary calculation reveals that

$$\|h_N\|_{L^{q_0}(B)} = \left(\sqrt{K} - \frac{1}{\sqrt{K}}\right)(2\pi \log N)^{(2K+1)/2K} \tag{4.11}$$

where

$$q_0 = \frac{2K}{K+1} = \frac{p_0}{p_0-1}$$

and

$$\|w_N\|_{L^\infty(B)} \geq c(K)(\log N)^{1+(K+1)/2K}. \tag{4.12}$$

By (4.11) and (4.12) it follows that

$$\frac{\|w_N\|_{L^\infty}}{\|h_N\|_{L^{q_0}}} \to \infty \qquad \text{as } N \to \infty \tag{4.13}$$

An application of Hölder's inequality and Corollary 4.2 yield the estimates

$$|w_N(x)| \leq \|G(x,\cdot)\|_{L^{p_0}(B)} \cdot \|h_N\|_{L^{q_0}(B)} \leq A_{p_0}(K)\|h_N\|_{L^{q_0}(B)},$$

which are not consistent with (4.13).

Acknowledgments. The research of both authors has been supported by MIUR-PRIN 02 and GNAMPA-INdAM.

References

[AIM] K. Astala, T. Iwaniec and G. Martin, *Pucci's conjecture and the Aleksandrov inequality for elliptic PDEs in the plane*, to appear.

[AIS] K. Astala, T. Iwaniec and E. Saksman, *Beltrami operators in the plane*, Duke Math. J. 107 (2001), 27–56.

[B] P. Bauman, *Equivalence of Green's functions for diffusion operators in* $\mathbb{R}^n$*: a counterexample*, Proc. Amer. Math. Soc. 91 (1984), 64–68.

[C] S. Campanato, *Un risultato relativo ad equazioni ellittiche del secondo ordine di tipo non variazionale*, Ann. Scuola Norm. Sup. Pisa (3) 21 (1967), 701–707.

[FS] E. B. Fabes and D. W. Stroock, *The* L^p*-integrability of Green's functions and fundamental solutions for elliptic and parabolic equations*, Duke Math. J. 51 (1984), 997–1016.

[FM] M. Franciosi and G. Moscariello, *A note on the maximum principle for second order nonvariational linear elliptic equations*, Ricerche Mat. 35 (1986), 279–290.

[K] C. Kenig, *Potential theory of non-divergence form elliptic equations*, in: Dirichlet Forms (Varenna, 1992), Lecture Notes in Math. 1563, Springer, 1993, 89–128.

[PV] S. Petermichl and A. Volberg, *Heating of the Ahlfors–Beurling operator*; *weakly quasiregular maps on the plane are quasiregular*, Duke Math. J. 112 (2002), 281–305.

[P] C. Pucci, *Limitazioni per soluzioni di equazioni ellittiche*, Ann. Mat. Pura Appl. (4) 74 (1966), 15–30.

Dipartimento di Matematica "R. Caccioppoli"
Università di Napoli "Federico II"
Via Cintia, 80126 Napoli, Italy
E-mail: gmoscari@unina.it
sbordone@unina.it

Received August 2, 2004
Revised version February 8, 2005 (5470)