STUDIA MATHEMATICA 169 (2) (2005)

Distinctness of spaces of Lorentz–Zygmund multipliers

by

KATHRYN E. HARE and PARASAR MOHANTY (Waterloo, ON)

Abstract. We study the spaces of Lorentz–Zygmund multipliers on compact abelian groups and show that many of these spaces are distinct. This generalizes earlier work on the non-equality of spaces of Lorentz multipliers.

1. Introduction. Let G be a compact abelian group. The *Lorentz–Zygmund space*, denoted by $L^{p,q}(\log L)^A(G)$ for $0 < p, q \le \infty$ and $A \in \mathbb{R}$, is the class of measurable functions f on G for which the quasi-norm, $\|f\|_{p,q,A}$, given by

$$(1.1)\qquad \|f\|_{p,q,A} = \begin{cases} \left(\int_0^1 (t^{1/p}(1-\log t)^A f^*(t))^q \,\frac{dt}{t}\right)^{1/q} & \text{if } 0 < q < \infty, \\ \sup\limits_{0<t<1} t^{1/p}(1-\log t)^A f^*(t) & \text{if } q = \infty, \end{cases}$$

is finite, where f^* denotes the decreasing rearrangement of f. This quasi-norm is equivalent to a norm when $1 < p \le \infty,\ 1 \le q \le \infty,\ A \in \mathbb{R}$ or $p = q = 1,\ A \ge 0$. For $A = 0$ these spaces are the usual Lorentz spaces, and for $p = q$ they are known as *Zygmund spaces*, particular examples of Orlicz spaces. Of course, if $p = q$ and $A = 0$, then the Lorentz–Zygmund space $L^{p,q}(\log L)^A$ is simply the classical Banach space L^p.

For $1 \le p, r \le \infty$, $1 \le q, s \le \infty$ and $A, B \in \mathbb{R}$, we let $M(p, q, A; r, s, B)$ be the space of all bounded linear operators T from $L^{p,q}(\log L)^A$ to $L^{r,s}(\log L)^B$ which commute with translations. We call this a *space of Lorentz–Zygmund multipliers*. We call the operator norm of T the *multiplier norm* of T, and denote it by $\|T\|_{M(p,q,A;r,s,B)}$ (or $\|T\|_{M(p,q,A)}$ if $r, s, B = p, q, A$.)

Lorentz–Zymund multipliers have arisen in a number of recent papers. For example, in [6] Grafakos and Mastyło derived bilinear interpolation theorems for operators on Lorentz–Zygmund spaces, while in [12] Tao and Wright

2000 *Mathematics Subject Classification*: Primary 43A22; Secondary 42A45, 46E30.
Key words and phrases: multipliers, Lorentz–Zygmund space, convolution operators.
This research was supported in part by NSERC.

showed that the (usual difficult) end point result of the Marcinkiewicz multiplier theorem is from $L(\log L)^A$ (the Lorentz–Zygmund space $L^{1,1}(\log L)^A$) to $L^{1,q}$, for $A \geq 1/2+1/q$. Bak *et al.*, in [1], showed that measures which are compactly supported on the flat curves in $\mathbb{R}^2$ map L^2 to the smaller Orlicz space $L^{2,2}(\log L)^A$ for $A < 0$, and so are examples of Orlicz-improving measures. The dimension of these Orlicz improving measures was studied in [8]. Lorentz-improving measures were investigated in [7].

The problem of inclusion and non-inclusion of different spaces of multipliers for Lorentz spaces has been extensively studied, see for example [5], [13], [4] and [9]. Cowling and Fournier [4] improved the result $M(p;p) \subsetneq M(p;p,\infty)$ for $1 < p < 2$ of Zafran [13] to general locally compact groups. In the same paper they showed that $M(p,q;p,r) \subsetneq M(p,q;p,t)$ for $p \neq 1,2,\infty$ and $1 < q \leq r < t \leq q'$, and that $M(p,s;p,r) \subsetneq M(p,q;p,r)$ if $r' \leq q < s \leq r$.

In [9], Hare and Sato studied the Lorentz multiplier spaces $M(p,q;p,r)$ and showed that for infinite, compact abelian groups, $M(p,q;p,r) \subsetneq M(p,t;p,s)$ if $0 < 1/r - 1/q < 1/s - 1/t$ and $M(p,t;p,s) \neq M(r,v;r,u)$ if $1 < p,\ r < \infty$ and $r \neq p,p'$.

In this paper we generalize the results of [9] to Lorentz–Zygmund spaces. Our proofs depend heavily on their techniques. Our main new contribution is to find an upper bound for the Lorentz–Zygmund multiplier norms of trigonometric polynomials which depend (logarithmically) on the cardinality of the support of their Fourier transforms. This allows us to show, for example, that if $1 \leq p \leq \infty$, then the multiplier spaces $M(p,s,A;p,t,B)$ and $M(p,q,C;p,r,D)$ are distinct if $t \leq s$, $B \geq A$, and $1/t - 1/s + B - A < 1/r - 1/q + D - C$ (where if $p = 1,\infty$ we understand all the second indices to be p and all third indices to be non-negative if $p = 1$, non-positive if $p = \infty$). Explicit constructions of multipliers that are combinations of Fejér kernels are given to show the distinctness of these spaces.

By combining these multipliers using a Rudin–Shapiro-like construction, we also show that if $r \neq p,p'$, the spaces $M(p,t,A;p,s,B)$ and $M(r,v,C;r,u,D)$ are distinct for any $1 \leq s,t,u,v \leq \infty$ and A,B,C,D (with the same caveat as above if the first index is 1 or ∞).

2. Lorentz–Zygmund spaces

2.1. *Notation and basic facts.* In this section we summarize basic properties of the Lorentz–Zygmund spaces. For results stated here without proofs we refer the reader to Bennett and Rudnick [2] where these spaces are studied extensively.

For $1 < p \leq \infty$, $1 \leq q \leq \infty$, $A \in \mathbb{R}$, or $p = q = 1$ and $A \geq 0$, the Lorentz–Zygmund space $L^{p,q}(\log L)^A$ is a Banach space and its dual is identified with $L^{p',q'}(\log L)^{-A}$ when $p<\infty$, where p' and q' are the conjugate indices to p and q respectively.

Lorentz spaces arise as interpolation spaces of Lorentz spaces. Indeed, in [11], Merucci showed that by considering the classical K functional of J. Peetre and the function $f(t)=t^{\theta}(1+|\log t|)^{-A}$ one can obtain

$$L^{p,q}(\log L)^A = (L^{p_0,q_0}, L^{p_1,q_1})_{f,q;K}$$

with $1/p=(1-\theta)/p_0+\theta/p_1$.

As with Lorentz spaces, the first index dominates the inclusion relation: for any $q,s>0$ and $A,B\in\mathbb{R}$,

$$L^{p,q}(\log L)^A \subseteq L^{r,s}(\log L)^B \quad \text{if } 0<r<p\le\infty. \tag{2.1}$$

When we vary the second and third indices then, for any $0<p\le\infty$, if either

$$\text{(i) } q\le r \text{ and } A\ge B \quad \text{or} \quad \text{(ii) } q>r \text{ and } A+1/q>B+1/r,$$

we have

$$L^{p,q}(\log L)^A \subseteq L^{p,r}(\log L)^B.$$

With this inclusion relation one can easily see that if either

$$\text{(i) } s<r\le q \text{ and } B+1/r>C+1/s \quad \text{or} \quad \text{(ii) } r\le s\le q \text{ and } B\ge C,$$

then

$$M(p,q,A;p,r,B) \subseteq M(p,q,A;p,s,C). \tag{2.2}$$

Similarly, if either $r\le q\le v$ and $C\ge A$ or $r\le v<q$ and $C+1/q>A+1/v$, then

$$M(p,v,A;p,r,B) \subseteq M(p,q,C;p,r,B). \tag{2.3}$$

Our interest lies in asking where these containments are strict.

There is a Marcinkiewicz interpolation theorem for these operators (see [3, p. 253]). Taking $A=0$ gives the interpolation theorem for operators on Lorentz spaces (cf. [10], [3, p. 225]).

THEOREM 2.1. *Suppose T is a quasilinear operator and is of weak types (p_0,q_0) and (p_1,q_1), with respective weak norms M_0 and M_1 and $0<\theta<1$. Then there is a constant $M=M(M_0,M_1,\theta)$ such that*

$$\|Tf\|_{p_\theta,r,A} \le M(M_0,M_1,\theta)\|f\|_{q_\theta,r,A}$$

for any $r\ge 1$ and

$$\frac{1}{p_\theta}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}, \qquad \frac{1}{q_\theta}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}.$$

Trigonometric polynomials are dense in $L(\log L)^A$ for $A\ge 0$ as these spaces are homogeneous Banach spaces. Hence using the dominated convergence theorem it follows that the $L(\log L)^A$ multiplier norm of a polynomial can be approximated by the $L^{p_n,1}(\log L)^A$ multiplier norms where $p_n \downarrow 1$. As a consequence we have the following lemma.

LEMMA 2.2. *Let g be a trigonometric polynomial. Then* $\|g\|_{M(1,1,A)} \le \|g\|_1$.

Throughout this paper by $f(x) \simeq g(x)$ we will mean that there exist constants C_1, C_2, independent of x, such that $C_1 g(x) \le f(x) \le C_2 g(x)$.

In our calculations, generic constants which appear may vary from line to line.

It will be convenient to know the $L^{p,q}(\log L)^A$ norm for the characteristic function, χ_I, of an interval I. In [8] the case $p = q = 2$, $A < 0$ was addressed. For the general case the proof is similar.

LEMMA 2.3. *Let I be an interval of length $|I|$. For any of the Banach spaces $L^{p,q}(\log L)^A$ we have*

$$\|\chi_I\|_{p,q,A} \simeq |I|^{1/p} |\log |I|\,|^A$$

where the equivalence constants C_1, C_2 depend only on p.

2.2. *Inequalities relating Lorentz–Zymund norms.* It will also be helpful to derive some inequalities relating Lorentz–Zygmund norms with different indices. The proofs follow from Hölder's inequality and are omitted.

PROPOSITION 2.4. *Suppose* $1 < b < \infty$, $1 \le p, q < \infty$ *and* $\alpha \in \mathbb{R}$.

(i) *If* $\alpha q(1-\delta)b' < -1$ *then*

$$\|f\|_{p,q,\alpha} \le \left(\frac{-1}{1 + \alpha q(1-\delta)b'} \right)^{1/qb'} \|f\|_{p,qb,\alpha\delta}.$$

(ii) *If* $1 < \delta < \infty$, *then*

$$\|f\|_{p,q,\alpha} \le \left(\frac{pb'}{q\delta'} \right)^{(1/q\delta')'} \|f\|_{pb,q\delta,\alpha}.$$

3. Upper bounds for multiplier norms. Throughout the remainder of the paper we will restrict our attention to certain Lorentz–Zymund spaces which are Banach spaces, namely the spaces $L^{p,q}(\log L)^A$ where

$$1 < p < \infty,\ 1 \le q \le \infty,\ A \in \mathbb{R},\ \text{or}$$
$$p = q = 1,\ A \ge 0,\ \text{or}$$
$$p = q = \infty,\ A \le 0.$$

These spaces are all contained in L^1 and contain L^∞.

We first derive upper bounds for multiplier norms of trigonometric polynomials (acting by convolution) when the first two indices of the Lorentz–Zygmund spaces are the same. This uses the following observation which can be proved by elementary calculus.

LEMMA 3.1. $\sup_{0<t<1}(t^x(1-\log t)^y) = e^{x-y}\left(\frac{y}{x}\right)^y$.

PROPOSITION 3.2. *Let G be a compact abelian group, $1<p<\infty$ and $A\le B$. Suppose P is a trigonometric polynomial on G with $\|P\|_{M(r,r)}\le C_0$ for all $1<r<\infty$. Then there is a constant $C=C(p,q,A,B,C_0)$ such that*

$$\|P\|_{M(p,q,A;p,q,B)} \le C(\log|\operatorname{supp}\widehat{P}|)^{B-A}.$$

Proof. Assume first $q<\infty$. Proposition 2.4 shows that if $1<b<\infty$ and δ is chosen such that $Bq(1-\delta)b'<-1$, then

$$\|P*f\|_{p,q,B} \le \left(\frac{-1}{1+Bq(1-\delta)b'}\right)^{1/qb'} \|P*f\|_{p,qb,B\delta}. \tag{3.1}$$

The definition of the Lorentz–Zygmund norm gives

$$\begin{aligned}\|P*f\|_{p,qb,B\delta} \\ \le \|P*f\|_{p,q,A}^{1/b}\sup_t(t^{1/pb'}(1-\log t)^{B\delta-A/b})\sup_t|P*f(t)|^{1/b'}\end{aligned} \tag{3.2}$$

and according to the elementary lemma, the middle term is bounded by

$$e^{1/pb'}e^{-(B\delta-A/b)}((B\delta-A/b)pb')^{B\delta-A/b}.$$

Without loss of generality we can assume $N\equiv\log|\operatorname{supp}\widehat{P}|>p$. Let $b=b_N\equiv N/(N-p)$ (so $b'_N=N/p$) and choose $\delta=\delta_N$ such that for some fixed $\varepsilon_0>0$,

$$Bq(1-\delta_N)b'_N = -(1+\varepsilon_0).$$

Then $\delta_N\to 1$ as $N\to\infty$ and

$$\left(\frac{-1}{1+Bq(1-\delta_N)b'_N}\right)^{1/qb'_N} = \left(\frac{1}{\varepsilon_0}\right)^{p/qN} \le \left(\frac{1}{\varepsilon_0}\right)^{p/N}$$

is clearly bounded. Also, since $B\delta_N - A/b_N \to B-A$,

$$e^{-(B\delta_N-A/b_N)}((B\delta_N-A/b_N)pb'_N)^{B\delta_N-A/b_N}\to N^{B-A}.$$

Together with equations (3.1) and (3.2) this establishes

$$\|P*f\|_{p,q,B}\le CN^{B-A}\|P*f\|_{p,q,A}^{1/b_N}\sup_t|P*f(t)|^{1/b'_N}. \tag{3.3}$$

The boundedness of the $M(r,r)$ multiplier norms and the Interpolation Theorem ensure that the $M(p,q,A)$ norm of P is also bounded by the same constant. Thus

$$\|P * f\|_{p,q,B} \le C C_0^{1/b_N} N^{B-A} \|f\|_{p,q,A}^{1/b_N} \sup_t |P * f(t)|^{1/b'_N}. \tag{3.4}$$

Moreover, as the $M(r,r)$ norm dominates the supremum of the Fourier coefficients of P, the inclusions of the Lorentz–Zygmund spaces with different first indices yield

$$\|P\|_{p',q',-A} \le C(p,q,A)\|P\|_\infty \le C_0 C(p,q,A)|\operatorname{supp} \widehat{P}|.$$

Together with Hölder's inequality and the translation invariance of these norms, we obtain the bound

$$\sup_t |P * f(t)|^{1/b'_N} \le (\|P\|_{p',q',-A}\|f\|_{p,q,A})^{1/b'_N} \le C\|f\|_{p,q,A}^{1/b'_N}. \tag{3.5}$$

The desired result follows by combining inequalities (3.3)–(3.5).

For the case $q=\infty$, a duality argument shows

$$\|P\|_{M(p,\infty,A;p,\infty,B)} = \|P\|_{M(p',1,-B;p',1,-A)} \le C(\log|\operatorname{supp} \widehat{P}|)^{B-A}. \ ∎$$

Corollary 3.3. *Suppose $B \ge A \ge 0$ and P is a trigonometric polynomial on G satisfying $\|P\|_1 \le C_0$. Then*

$$\|P\|_{M(1,1,A;1,1,B)} = \|P\|_{M(\infty,\infty,-B;\infty,\infty,-A)} \le C(\log|\operatorname{supp} \widehat{P}|)^{B-A}.$$

Proof. For $p=1$, the proof is similar to the proposition above noting that the L^1 boundedness of P gives $\|P\|_{M(1,1,A)} \le C_0$. The case $p=\infty$ follows by duality. ∎

This proposition is used in the next result where we vary both the second and third indices.

Proposition 3.4. *Let G be a compact abelian group and suppose $p<\infty$, $1 \le r \le q \le \infty$ and $B \ge A$. Let P be a trigonometric polynomial on G with $\|P\|_{M(s,s)} \le C_0$ for all $1<s<\infty$ if $p \ne 1$, and $\|P\|_1 \le C_0$ if $p=1$. Then*

$$\|P\|_{M(p,q,A;p,r,B)} \le C(\log|\operatorname{supp} \widehat{P}|)^{1/r-1/q+B-A}$$

where the constant C depends only on the indices p,q,r,A and the constant C_0.

Proof. First, suppose $q<\infty$. For any $1<b,\delta<\infty$ we have, from Proposition 2.4 and the definition of the Lorentz–Zygmund norm,

$$\begin{aligned}\|P * f\|_{p,r,B} &\le \left(\frac{pb'}{r\delta'}\right)^{1/r\delta'} \|P * f\|_{pb,r\delta,B} \\ &= \left(\frac{pb'}{r\delta'}\right)^{1/r\delta'} \left(\int_0^1 t^{r\delta/pb}(1-\log t)^{Br\delta}(P * f)^{*r\delta}\,\frac{dt}{t}\right)^{1/r\delta}.\end{aligned} \tag{3.6}$$

Suppose we choose b and δ such that $bq = \delta r$. Then, of course, $t^{r\delta/pb} = t^{q/p}$ and $(1-\log t)^{r\delta B} = (1-\log t)^{qbB}$, hence it easily follows that

$$\|P*f\|_{pb,r\delta,B}^{r\delta} \le \|P*f\|_{p,q,Bb}^{q} \sup_t |P*f(t)|^{bq/b'}. \tag{3.7}$$

As in the previous proposition, we can assume $N \equiv \log|\operatorname{supp}\widehat{P}| > p$ and we put $b = b_N \equiv N/(N-p)$. Take $\delta = \delta_N = qb_N/r$ and note that the assumption $q \ge r$ ensures $\delta > 1$. For this choice of b and δ Proposition 3.2 yields

$$\|P*f\|_{p,q,Bb} \le CN^{Bb_N - A}\|f\|_{p,q,A} \tag{3.8}$$

(where C depends on C_0). The inequalities (3.6)–(3.8), along with Hölder's inequality, give

$$\begin{aligned}\|P*f\|_{p,r,B} &\le \left(\frac{N}{r\delta'}\right)^{1/r\delta'} N^{B-A+Ap/N}\|P\|_{p',q',-A}^{p/N}\|f\|_{p,q,A}\\ &\le \left(\frac{N}{r\delta'}\right)^{1/r\delta'} N^{B-A+Ap/N}C(p,C_0)\|f\|_{p,q,A}.\end{aligned}$$

But

$$\frac{1}{r\delta'} = \frac{1}{r} - \frac{1}{q} + \frac{p}{rN},$$

so upon simplifying we obtain

$$\|P*f\|_{p,r,B} \le C_1C_2(p,q,r,A)N^{1/r-1/q+B-A}\|f\|_{p,q,A}$$

where $C_2 = \max((1/r - 1/q + p/r)^{1/r-1/q+p/r}, 1)$ and C_1 is independent of r (and B).

Now we will deal with the case $q = \infty$. First consider $1 \ne r < \infty$. Then by duality we have

$$\|P\|_{M(p,\infty,A;p,r,B)} = \|P\|_{M(p',r',-B;p',1,-A)} \le C(\log|\operatorname{supp}\widehat{P}|)^{1/r+B-A}$$

where $C = C_1(1/r + p')^{p'+1/r}$.

The care we have taken with the constants will be helpful in considering the final case, $q = \infty$ and $r = 1$ ($q = r = \infty$ was done in the previous corollary). Choose a sequence $\{s_n\}$ which decreases to 1. Then $\|f\|_{p,1,B} = \lim_{n\to\infty}\|f\|_{p,s_n,B}$. (This can be proved in the same way as shown in [7] for Lorentz spaces.) Thus,

$$\begin{aligned}\|P*f\|_{p,1,B} &= \lim_{n\to\infty}\|P*f\|_{p,s_n,B}\\ &\le \lim_{n\to\infty} C_1\left(\frac{1}{s_n} + p'\right)^{p'+1/s_n}(\log|\operatorname{supp}\widehat{P}|)^{B-A+1/s_n}\|f\|_{p,\infty,A}\end{aligned}$$

where C_1 is independent of s_n and B. Hence

$$\|P*f\|_{p,1,B} \le C(\log|\operatorname{supp}\widehat{P}|)^{B-A+1}\|f\|_{p,\infty,A}. \blacksquare$$

REMARK 3.5. This generalizes [9, Prop. 3.1].

COROLLARY 3.6. *Suppose $r \le q$ and $A \le B$. If P is any trigonometric polynomial then*

$$\|P\|_{M(p,q,A;p,r,B)} \le C(\log|\operatorname{supp}\widehat{P}|)^{1/r-1/q+B-A}$$

where the constant C depends on p,q,r,A,B and the L^1 norm of P.

Proof. We always have $\|P\|_{M(s,s)} \le \|P\|_1$. ∎

4. Non-equality of multiplier spaces for the circle group. To show the non-equality of certain Lorentz–Zygmund multiplier spaces we will exhibit polynomials for which the upper bounds on the multiplier norms given in Proposition 3.4 are sharp. In this section, we consider the circle group, and in the next section, groups with elements of finite order and arbitrary compact abelian groups.

4.1. *Multiplier norms of Fejér and Dirichlet kernels*

PROPOSITION 4.1. *Let $1 \le r, q \le \infty$ and $A, B \in \mathbb{R}$. If P is the Fejér or de la Vallée Poussin kernel of degree N, then there is a constant C, independent of N, such that*

$$\|P\|_{M(p,q,A;p,r,B)} \le C(\log N)^{\alpha}$$

where $\alpha = \max(1/r - 1/q + B - A, 1/r - 1/q, B - A, 0)$. The same bound holds for P the Dirichlet kernel of degree N if $p \ne 1, \infty$.

Proof. When $r \le q$ and $A \le B$ we clearly have

$$\|P\|_{M(p,q,A;p,r,B)} \le C(\log N)^{1/r-1/q+B-A}$$

from Proposition 3.4. If, instead, $r \ge q$ and $A \le B$, then the inclusions of the Lorentz–Zygmund spaces, together with this bound (applied with $r = q$), give

$$\|P * f\|_{p,r,B} \le C\|P * f\|_{p,q,B} \le C(\log N)^{B-A}\|f\|_{p,q,A}.$$

The other cases are similar. ∎

One can explicitly calculate the Lorentz–Zygmund norms of these kernels and this allows us to prove the sharpness of the upper bounds in certain cases. For this calculation it is convenient to identify the circle group with $[-1/2, 1/2]$.

PROPOSITION 4.2. *Let P denote either the Dirichlet kernel of degree N, the Fejér kernel of degree N, or the de la Vallée Poussin kernel of degree $2N+1$. Then*

$$\|P\|_{p,q,A} \simeq \begin{cases} (\log N)^{A+1} & \text{if } P = \text{Dirichlet kernel and } p = q = 1, \\ N^{1/p'}(\log N)^{A} & \text{otherwise}, \end{cases}$$

where the equivalence constants depend only on p.

Proof. We will prove the result for the Dirichlet kernel, d_N, as the proof of the estimates of the other kernels is similar in nature.

It is routine to check that there are positive constants A, B such that for $t \in [0, 1/2]$,

$$AN\chi_{[0,1/(4N+2)]} \le |d_N(t)| \le B\left(N\chi_{[0,1/(N+1/2)]} + \frac{1}{t}\,\chi_{[1/(N+1/2),1/2]}\right).$$

Notice that

$$\frac{1}{x}\,\chi^*_{[1/(N+1/2),1/2)} = \frac{\chi_{[0,1/2-1/(N+1/2)]}}{x+1/(N+1/2)}.$$

As usual, suppose first that $q < \infty$ and $p \ne 1$. In view of Lemma 2.3 it suffices to prove that

$$J = \int_0^{1/2} \frac{(1-\log t)^{qA}}{(t+1/(N+1/2))^q} t^{q/p-1}\,dt \le CN^{q/p'}(\log N)^{qA}.$$

Choose m such that $2^m \le N + 1/2 < 2^{m+1}$. By dividing the integral into subintervals of width 2^{-k} we see that

$$\begin{aligned}(4.1)\qquad J &\le \Big(\sum_{k=1}^{m} + \sum_{k=m+1}^{\infty}\Big)\frac{(1-\log 2^{-k})^{qA}}{(2^{-k}+1/(N+1/2))^q}\,(2^{-k})^{q/p-1}\\ &\le C\sum_{k=1}^{m} k^{qA}(2^{-k})^{q/p-q} + \sum_{k=m+1}^{\infty} N^q k^{qA}(2^{-k+1})^{q/p}.\end{aligned}$$

Summing gives the desired result.

Suppose now $q = \infty$ (and $p \ne 1$). We need to bound

$$\sup\left\{\frac{t^{1/p}(1-\log t)^A}{t+a_N} : t \in \left[0, \frac{1}{2} - a_N\right]\right\},$$

where $a_N = 1/(N+1/2)$. Observe that the function $t^{1/p}(t+a_N)^{-1}$ increases until $t = a_N/(p-1)$ and then decreases. Thus if $A \ge 0$ and $t \in [a_N, 1/2]$, then since $(1-\log t)^A$ decreases,

$$\frac{t^{1/p}(1-\log t)^A}{t+a_N} \le C\left(\frac{a_N}{p-1}\right)^{-1/p'}(\log a_N)^A \le CN^{1/p'}(\log N)^A.$$

Notice that $t^{1/p}(1-\log t)^A$ increases for small t and then decreases. Thus for N sufficiently large and $t \in [0, a_N]$,

$$\frac{t^{1/p}(1-\log t)^A}{t+a_N} \le C\,\frac{a_N^{1/p}(\log a_N)^A}{a_N} \le CN^{1/p'}(\log N)^A.$$

Suppose $A < 0$ and N is large enough that $\sqrt{a_N} \ge a_N/(p-1)$. As $(1-\log t)^A$

is an increasing function, if $t \in [0, \sqrt{a_N}]$, then

$$\frac{t^{1/p}(1-\log t)^A}{t+a_N} \le \begin{cases} Ca_N^{-1/p'}(\log \sqrt{a_N})^A & \text{if } t \in [0, \sqrt{a_N}], \\ Ca_N^{-1/2p'} & \text{if } t \in [\sqrt{a_N}, 1/4]. \end{cases}$$

In either case, it is a trivial calculation to obtain the desired bound.

Finally, we consider the case $p = q = 1$, $A \ge 0$. The estimates of (4.1) still give the correct upper bound. To obtain the same order of magnitude for the lower bound we observe that $|D_N(t)| \ge c/t$ (for a suitable constant c) provided

$$t \in \bigcup_{k=0}^{N-1} \left[\frac{1/4+2k}{2N+1}, \frac{3/4+2k}{2N+1}\right].$$

Thus $m\{|D_N| \ge x\} \ge c/x$ for $x < N$. Hence

$$\begin{aligned} \|D_N\|_{1,1,A} &\ge C \sum_{k=0}^{[\log N]} \int_{2^{-k}}^{2^{-k+1}} \frac{(1-\log t)^A}{t}\, dt \ge C \sum_{k=0}^{[\log N]} (\log 2^k)^A \\ &\ge C(\log N)^{A+1}. \ \blacksquare \end{aligned}$$

This calculation enables us to verify that the upper bounds already determined for the common kernels are sharp when $r \ge q$, $A \le B$. In particular, note that the following corollary shows that it is not true that $\|P_N\|_{M(p,q,A;p,r,B)} \simeq (\log N)^{1/r-1/q+B-A}$ when $r > q$ and $1/r - 1/q + B - A > 0$ (although the Lorentz–Zygmund spaces satisfy $L^{p,q}(\log L)^A \subseteq L^{p,r}(\log L)^B$).

Corollary 4.3. *Suppose $r \ge q$ and $A \le B$. If P_N is the Fejér or de la Vallée Poussin kernel of degree N (or Dirichlet kernel of degree N if $p \ne 1, \infty$), then*

$$\|P_N\|_{M(p,q,A;p,r,B)} \simeq (\log N)^{B-A}.$$

Proof. Let d_N denote the Dirichlet kernel of degree N. Then for any $1 \le r \le \infty$,

$$\|P_N * d_N\|_{p,r,B} = \|P_N\|_{p,r,B} \ge C(\log N)^{B-A}\|d_N\|_{p,q,A}.$$

Together with the earlier work, this completes the proof. ■

Unfortunately, the same easy estimates will not suffice to give the lower bound for these multipliers when $q > r$. Instead, we consider the test function F_N defined in [9, 3.2]. For convenience we repeat the definition here. For a large integer λ and $M_N = 2\lambda^N + 1$, set $x_j = 2(j-1)/\sqrt{M_N}$ for $j = 1, \ldots, 2^N$, $z_k = 3^N k/\sqrt{M_N}$ for $k = 1, \ldots, N$, and let D_N denote the Dirichlet kernel of degree λ^N. Define

$$D_{j,k}(x) = D_N(x - (x_j + z_k)) \quad \text{and} \quad \widetilde{D_{j,k}(x)} = D_{j,k}(x)\chi_{[-2/M_N, 2/M_N] + x_j + z_k}.$$

Notice that if N is sufficiently large then the functions $\widetilde{D_{j,k}(x)}$ are disjointly supported.

Our test function for determining sharp lower bounds on the operator norms will be given by

$$F_N(x) = \frac{1}{M_N} \sum_{k=1}^{N} 2^{-k/p} \sum_{j=1}^{2^k} \widetilde{D_{j,k}(x)}. \tag{4.2}$$

PROPOSITION 4.4. *There is a constant C (independent of N) such that*

$$\|F_N\|_{p,q,A} \le C N^{1/q+A} M_N^{-1/p}.$$

Proof. In [9] the following estimates for F_N^* were calculated:

(1) $F_N^*(0) \le 2^{-1/p}$,
(2) $F_N^*(2^{n+3}/M_N) \le 2^{-n/p}$ for $n = 1, \ldots, N$,
(3) $F_N^*(y) = 0$ for $y > 2^{N+3}/M_N$.

From the definition of the Lorentz–Zygmund norm, for $q < \infty$ we have

$$\begin{aligned} \|F_N\|_{p,q,A}^q &\le \int_0^{16/M_N} t^{q/p} F_N^{*q} (1-\log t)^{Aq} \frac{dt}{t} \\ &\quad + \sum_1^{N-1} \int_{2^{n+3}/M_N}^{2^{n+4}/M_N} t^{q/p} F_N^{*q} (1-\log t)^{Aq} \frac{dt}{t} \\ &= I_1 + I_2. \end{aligned}$$

Let $2^{-k_0} \simeq 16/M_N$. As $F_N^* \le 2^{-1/p}$ and $k_0 \simeq N$, we have

$$\begin{aligned} I_1 &\le \sum_{k=k_0}^{\infty} \int_{2^{-k}}^{2^{-k+1}} 2^{-q/p} t^{q/p-1} (1-\log t)^{Aq}\, dt \\ &\le c \sum_{k=k_0}^{\infty} 2^{-q/p} 2^{-kq/p} k^{Aq} \simeq \left(\frac{16}{M_N}\right)^{q/p} (\log M_N)^{Aq}. \end{aligned}$$

Since $M_N \gg 2^N$, property (2) similarly implies that for I_2 we have

$$I_2 \le CN \frac{(\log M_N)^{Aq}}{M_N^{q/p}},$$

and as $(\log M_N)^{Aq} \simeq N^{Aq}$ this gives $\|F_N\|_{p,q,A} \le C N^{1/q+A} M_N^{-1/p}$.

The case $q = \infty$ is a routine exercise. ∎

Now we will calculate the multiplier norm of K_N, the Fejér kernel of degree λ^{8N}. For this purpose we need the following lemma from [9, 3.5].

LEMMA 4.5. *Let $p < \infty$. There is a constant $C > 0$ such that for any $n = 2, \ldots, N$, $N \in \mathbb{N}$,*

$$m\{x : |F_N * K_N(x)| \geq 2^{-n/p}\} \geq C2^n/M_N.$$

REMARK 4.6. This is stated for $p > 1$ in [9], but remains true when $p = 1$.

THEOREM 4.7. *Let $1 \leq r \leq q \leq \infty$ and $A \leq B$. If K_N is the Fejér kernel of degree λ^{8N}, then*

$$\|K_N\|_{M(p,q,A;p,r,B)} \simeq N^{B-A+1/r-1/q}.$$

Proof. Since a duality argument gives the case $p = \infty$, we can assume $p < \infty$.

To estimate the lower bound we consider the test function F_N of (4.2). Applying Lemma 4.5 and simplifying gives

$$\begin{aligned}\|F_N * K_N\|_{p,r,B}^r &\geq \sum_{n=1}^{N-1} \int_{\alpha 2^n/M_N}^{\alpha 2^{n+1}/M_N} t^{r/p} (F_N * K_N)^{*r}(t)(1 - \log t)^{Ar} \, \frac{dt}{t} \\ &\geq C \, \frac{N^{Br+1}}{M_N^{r/p}}.\end{aligned}$$

Using the bound of F_N obtained in the previous proposition we have

$$\|K_N\|_{M(p,q,A;p,r,B)} \geq \frac{\|F_N * K_N\|_{p,r,B}}{\|F_N\|_{p,q,A}} \geq CN^{B-A+1/r-1/q}.$$

The upper bound has already been noted in Proposition 4.1. ■

REMARK 4.8. The same arguments show that for any r, q, A, B,

$$\|K_N\|_{M(p,q,A;p,r,B)} \geq CN^{B-A+1/r-1/q}.$$

4.2. *Comparing multiplier spaces with differing second or third indices.* The previous theorem shows that the norms of many Lorentz–Zygmund multiplier spaces are not comparable. Indeed, using a similar technique to [9, 3.7] we can construct examples of operators which belong to certain multiplier spaces and not to others.

THEOREM 4.9. *Let $1 \leq t \leq s \leq \infty$, $1 \leq r, q \leq \infty$, $B \geq A$ and $C, D \in \mathbb{R}$. Let $\varepsilon > 0$. There is an $F \in L^1(\mathbb{T})$ such that*

$$F \in \bigcap_{1/t-1/s+B-A=\varepsilon} M(p, s, A; p, t, B)$$

but

$$F \notin \bigcup_{1/r-1/q+D-C>\varepsilon} M(p, q, C; p, r, D).$$

Proof. Let $K'_{2^n}(x) = K_{2^n}(x)e^{iL_n x}$ for $n = 1, 2, \ldots$ where, as before, K_N denotes the Fejér kernel of degree λ^{8N} and the integers L_n are chosen in such a way that the $\widehat{K'_{2^n}}$ have disjoint support. Put

$$F = \sum_{n=1}^{\infty} \frac{2^{-\varepsilon n}}{n^2} K'_{2^n}.$$

Then $F \in L^1(\mathbb{T})$ and

$$\|F\|_{M(p,s,A;p,t,B)} \le C_0 \sum_n \frac{2^{-\varepsilon n}}{n^2} 2^{n(1/t-1/s+B-A)} = C_0 \sum_n \frac{1}{n^2} < \infty.$$

On the other hand, we saw in the previous proof that there were functions F_{2^n} satisfying

$$\|F_{2^n} * K_{2^n}\|_{p,r,D} \ge C_0 2^{n(1/r-1/q+D-C)} \|F_{2^n}\|_{p,q,C}.$$

If we let f_{2^n} denote F_{2^n} convolved with the Dirichlet kernel of degree $\lambda^{2^n 8}$, then since the functions $\widehat{K'_{2^n}}$ have disjoint support it follows that $|F * f_{2^n}(x)e^{iL_n x}| = |K_{2^n} * F_{2^n}(x)| 2^{-\varepsilon n}/n^2$. As the Dirichlet kernels are uniformly bounded as multipliers on $L^{p,q}(\operatorname{Log} L)^C$, it follows that $\|f_{2^n}\|_{p,q,C} \le C_0 \|F_{2^n}\|_{p,q,C}$. Thus if $F \in M(p,q,C;p,r,D)$ for $1/r - 1/q + D - C = \delta > \varepsilon$, then

$$\infty > \|F\|_{M(p,q,C;p,r,D)} \ge C_0 \frac{2^{-\varepsilon n}\|F_{2^n} * K_{2^n}\|_{p,r,D}}{n^2 \|F_{2^n}\|_{p,q,C}} \ge C_0 \sup_n \frac{2^{-\varepsilon n}}{n^2} 2^{n\delta}$$

and this is a contradiction as $\delta > \varepsilon$. ■

REMARK 4.10. More generally, these arguments can be used to prove that

$$M(p,s,A;p,t,B) \ne M(p,q,C;p,r,D)$$

if $\max(1/t - 1/s + B - A, 1/t - 1/s, B - A, 0) < 1/r - 1/q + D - C$.

4.3. *Comparing multiplier spaces with differing first indices.* Lorentz–Zymund multiplier spaces can also be shown to be distinct when the first indices are different. For this we will adopt the usual technique of constructing multipliers using Rudin–Shapiro type polynomials.

Choose $y_1, \ldots, y_N$ such that the intervals

$$\sum_{j=1}^{N} \epsilon_j y_j + \left[\frac{-4}{\lambda^{N/3}}, \frac{4}{\lambda^{N/3}}\right]$$

are disjoint for $\epsilon_j = 0, 1$. Let $L(y_j)$ denote translation by y_j. Set $\varrho_0 = \sigma_0 = K_N$ and inductively define Rudin–Shapiro polynomials ϱ_{n+1} and σ_{n+1} by

$$\varrho_{n+1} = \varrho_n - L(y_{n+1})\sigma_n, \qquad \sigma_{n+1} = \varrho_n + L(y_{n+1})\sigma_n.$$

Proposition 4.11. *Let* $1 \le p < 2$, $1 \le q, r \le \infty$ *and* $A, B \in \mathbb{R}$. *There is a constant* C, *independent of* N, *such that*

$$\|\varrho_N\|_{M(p,q,A;p,r,B)} \le C2^{N/p} N^{\max(1/r-1/q+B-A,1/r-1/q,B-A,0)}.$$

Proof. Let V_N denote the de la Vallée Poussin kernel of degree $3\lambda^{8N}$. As ϱ_N is a trigonometric polynomial of degree λ^{8N}, $\widehat{V}_N = 1$ on $\operatorname{supp} \widehat{\varrho}_N$ and thus $\varrho_N * V_N = \varrho_N$.

The polynomial ϱ_N is a linear combination of 2^N translates of Fejér kernels, with coefficients ± 1, thus $\|\varrho_N\|_1 = 2^N$. The usual Rudin–Shapiro arguments show that $\|\varrho_N\|_{M(2,2)} \le c2^{N/2}$. Interpolating (or using Lemma 2.2 if $p = 1$) gives the estimate $\|\varrho_N\|_{M(p,r,B)} \le C2^{N/p}$ for $p > 1$. Thus if $r \le q$ and $B \ge A$ we have

$$\begin{aligned}\|\varrho_N * f\|_{p,r,B} &= \|\varrho_N * V_N * f\|_{p,r,B} \le C2^{N/p}\|V_N * f\|_{p,r,B} \\ &\le C2^{N/p} N^{1/r-1/q+B-A}\|f\|_{p,q,A}.\end{aligned}$$

The other cases are easy exercises. ∎

For $p = 2$ the arguments are more delicate.

Proposition 4.12. *For* $1 \le r \le q \le \infty$ *and* $B \ge A$,

$$\|\varrho_N\|_{M(2,q,A;2,r,B)} \le C2^{N/2} N^{\max(1/r-1/q,1/r-1/2,1/2-1/q)} N^{\max(B-A,B,-A)}.$$

Proof. First, suppose $A \le 0$ and $q \ge 2$. Using the fact that $\|\varrho_N\|_{M(2,2)} \le C2^{N/2}$ and the known multiplier norm of V_N, similar arguments to those given above show

$$\|\varrho_N\|_{M(2,q,A;2,2,0)} = \|\varrho_N\|_{M(2,2,0;2,q',-A)} \le C2^{N/2} N^{1/2-1/q-A}.$$

By factoring through L^2, it follows that if $r \le 2 \le q$ and $B \ge 0 \ge A$, then

$$\begin{aligned}\|\varrho_N * f\|_{M(2,q,A;2,r,B)} &\le \|\varrho_N\|_{M(2,2,0;2,r,B)}\|V_N\|_{M(2,q,A;2,2,0)} \\ &\le C_1 2^{N/2} N^{1/r-1/q+B-A}.\end{aligned}$$

Taking into account the fact that $L^{2,q}(\log L)^A \subset L^{2,q}$ if $A \ge 0$, one can easily see that if A and B are both non-negative, then

$$\frac{\|\varrho_N * f\|_{2,r,B}}{\|f\|_{2,q,A}} \le C\,\frac{\|\varrho_N * f\|_{2,r,B}}{\|f\|_{2,q,0}} \le C2^{N/2} N^{1/r-1/q+B}.$$

The case when A, B are both non-positive is similar.

If $2 \le r \le q$ we note that $\|\varrho_N\|_{M(2,q,A;2,r,B)} \le C\|\varrho_N\|_{M(2,q,A;2,2,B)}$ and use the previous estimates. The case $r \le q \le 2$ is dual. ∎

Remark 4.13. There is a similar bound if $r > q$ and/or $B < A$.

Next we will find the lower bounds for the multiplier norms of ϱ_N.

PROPOSITION 4.14. *For* $1 \le p \le 2$,

$$\|\varrho_N\|_{M(p,q,A;p,r,B)} \ge C 2^{N/p} N^{1/r-1/q+B-A}.$$

Proof. Again, we will use the test function F_N introduced in (4.2). The calculations that show

$$\|\varrho_N * F_N\|_{p,r,B} \ge C 2^{N/p} N^{1/r+B} M_N^{-1/p}$$

are similar to those given for $K_N * F_N$ in Theorem 4.7 (see [9, 3.11] for details). The extra factor of 2^N in the computation of $m\{x : |\varrho_N * F_N(x)| \ge 2^{-N/p}\}$ gives rise to $2^{N/p}$. ∎

These two results show that the Lorentz–Zygmund multiplier spaces with different first indices, $r < p \le 2$, are distinct.

THEOREM 4.15. *Let* $1 < p \le \infty$, $1 \le s, t \le \infty$ *and* $A, B \in \mathbb{R}$. *There exists* $F \in M(p, s, A; p, t, B)$ *such that* $F \notin M(r, u, C; r, v, D)$ *for any* $1 \le r < p$, $1 \le u, v \le \infty$ *and* $C, D \in \mathbb{R}$.

Proof. By translating ϱ_N suitably construct ϱ'_N whose Fourier coefficients have disjoint support. Consider

$$F = \sum_N \varrho'_N N^{-\log N} 2^{-N/p}.$$

According to Propositions 4.11 and 4.12,

$$\|F\|_{M(p,s,A;p,t,B)} \le \sum_N 2^{N/p} N^{\alpha} N^{-2\log N} 2^{-N/p}$$

where α depends only on s, t, A, B. As $\alpha \le \log N$ for N sufficiently large, $F \in M(p, s, A; p, t, B)$.

Now, take any $r < p$ and fixed u, v, C, D. Similar arguments to those given in Theorem 4.9, but based on the previous proposition, show

$$\begin{aligned}\|F\|_{M(r,u,C;r,v,D)} &\ge \sup_N \|\varrho_N\|_{M(r,u,C;r,v,D)} 2^{-N/p} N^{-\log N} \\ &\ge C_0 \sup_N 2^{N(1/r-1/p)} N^{-2\log N}.\end{aligned}$$

But $2^{N(1/r-1/p)} N^{-2\log N} \to \infty$ as $N \to \infty$, hence $F \notin M(r, u, C; r, v, D)$. ∎

5. Non-equality of multiplier spaces for arbitrary groups

5.1. *Groups with finite subgroups.* In this section we show the non-equality of certain Lorentz–Zygmund multiplier spaces when G is a compact, abelian group whose dual contains a (large) finite subgroup X. The arguments are similar to the circle case; we will sketch the main ideas.

Let H be the annihilator of X and let $D_H = \chi_H/m(H)$ where m denotes the Haar measure on G. As $\widehat{D}_H = \chi_X$, D_H is a trigonometric polynomial of L^1-norm one.

It is shown in [9, Section 4] that if $|X| \geq 100^N$ then there is a trigonometric polynomial F_N, defined on G, whose Fourier transform is supported on X and whose rearrangement is given by

$$F_N^*(u) = \begin{cases} 2^{-1/p} & \text{if } u < 2m(H), \\ 2^{-n/p} & \text{if } \sum_{k=1}^{n-1} 2^k m(H) \leq u < \sum_{k=1}^{n} 2^k m(H), \\ 0 & \text{if } u \geq \sum_{k=1}^{N} 2^k m(H). \end{cases}$$

Using this test function F_N and our previous work we can estimate the multiplier norms of D_H.

Proposition 5.1. *Let $1 \leq r \leq q \leq \infty$ and $B \geq A$. Then*

$$\|D_H\|_{M(p,q,A;p,r,B)} \simeq (\log|X|)^{1/r-1/q+B-A}.$$

Proof. Since $m(H) = |X|^{-1}$, Corollary 3.6 gives

$$\|D_H\|_{M(p,q,A;p,r,B)} \leq C(\log|X|)^{1/r-1/q+B-A}.$$

For the lower bound choose $N = \left[\frac{1}{7}\log|X|\right]$ and observe that since $\operatorname{supp}\widehat{F}_N \subseteq X$, we have $D_H * F_N = F_N$. It is straightforward to calculate that $\|F_N\|_{p,q,A} \simeq m(H)^{1/p} N^{A+1/q}$ since F_N^* is a sum of characteristic functions of intervals. Hence,

$$\|D_H\|_{M(p,q,A;p,r,B)} \geq \frac{\|D_H * F_N\|_{p,r,B}}{\|F_N\|_{p,q,A}} \geq CN^{1/r-1/q+B-A}. \ \blacksquare$$

Remark 5.2. Similarly, for any q, r, A, B,

$$C_0(\log|X|)^{1/r-1/q+B-A} \leq \|D_H\|_{M(p,q,A;p,r,B)} \leq C(\log|X|)^{|1/r-1/q|+|B-A|}.$$

Corollary 5.3. *Let G be an infinite, compact abelian group and suppose $\widehat{G}$ contains infinitely many elements of finite order. Let $1 \leq t \leq s \leq \infty$ and $B \geq A$. Suppose $\varepsilon > 0$. There is an $F \in L^1(\mathbb{T})$ such that*

$$F \in \bigcap_{1/t-1/s+B-A=\varepsilon} M(p,s,A;p,t,B)$$

but

$$F \notin \bigcup_{1/r-1/q+D-C>\varepsilon} M(p,q,C;p,r,D).$$

Proof. This is similar to the proof of [9, 4.3]; we take F to be a suitable weighted sum of translates of functions D_{H_n}, where H_n are the annihilators of a sequence of finite subgroups X_n whose cardinalities tend to infinity. $\blacksquare$

Remark 5.4. This corollary is the analogue of Theorem 4.9.

We can also show the non-comparability of norms of multiplier spaces with different first indices.

PROPOSITION 5.5. *Let G be an infinite, compact abelian group, X_N be a finite subgroup of $\widehat{G}$ with $|X_N| \simeq 100^N$, $1 \le p < 2$, $1 \le q, r \le \infty$, and $A, B \in \mathbb{R}$. There is a trigonometric polynomial ϱ_N such that*

$$C_0 2^{N/p} N^{1/r-1/q+B-A} \le \|\varrho_N\|_{M(p,q,A;p,r,B)} \le C 2^{N/p} N^{|1/r-1/q|+|B-A|}.$$

Proof. We use the Rudin–Shapiro polynomials ϱ_N constructed in [9, 4.5]. Since $\operatorname{supp} \widehat{\varrho}_N \subseteq X_N$, similar arguments to those given in Proposition 4.11, but factoring through D_{H_N}, where H_N is the annihilator of X_N (rather than V_N), show that

$$\|\varrho_N\|_{M(p,q,A;p,r,B)} \le C 2^{N/p} N^{|1/r-1/q|+|B-A|}.$$

These functions are known to satisfy $(\varrho_N * F_N)^*(u) = F_N^*(u2^{-N})$ (see proof of [9, 4.5]), thus an easy calculation gives

$$\begin{aligned} \|\varrho_N * F_N\|_{p,r,B} &\ge C m(H_N)^{1/p} 2^{N/p} N^{B+1/r} \\ &\ge C 2^{N/p} N^{1/r-1/q+B-A} \|F_N\|_{p,q,A}. \ \blacksquare \end{aligned}$$

Here is the analogue of Theorem 4.15.

COROLLARY 5.6. *Let G be an infinite, compact abelian group and suppose $\widehat{G}$ contains infinitely many elements of finite order. Suppose $1 \le s, t, u, v \le \infty$, and $A, B, C, D \in \mathbb{R}$. If $r \ne p, p'$ then*

$$M(p, s, A; p, t, B) \ne M(r, u, C; r, v, D).$$

5.2. *Arbitrary compact abelian groups.* To obtain the analogous results for arbitrary compact abelian groups we use the following lemma, which can be proved by a change of variables argument.

LEMMA 5.7. *Let G and H be compact abelian groups and $\pi : G \to H$ a continuous, onto homomorphism. Let $F \in L^1(H)$ and define a function $\widetilde{F}$ on G by $\widetilde{F} = F \circ \pi$. Then $\widetilde{F} \in L^1(G)$ and for any $1 \le p, q, r, s \le \infty$ and $A, B \in \mathbb{R}$,*

$$\|\widetilde{F}\|_{L^{p,q}(\log L)^A} = \|F\|_{L^{p,q}(\log L)^A}, \qquad \|F\|_{M(p,q,A;p,r,B)} \le \|\widetilde{F}\|_{M(p,q,A;p,r,B)}.$$

THEOREM 5.8. *Suppose that G is an infinite, compact abelian group and $1 \le t, s, r, q \le \infty$, and $A, B, C, D \in \mathbb{R}$.*

(i) *If $t \le s$, $A \le B$ and $\varepsilon > 0$, then there is a function $F \in L^1(G)$ such that*

$$F \in \bigcap_{1/t-1/s+B-A=\varepsilon} M(p, s, A; p, t, B)$$

but

$$F \notin \bigcup_{1/r-1/q+D-C>\varepsilon} M(p,q,C;p,r,D).$$

(ii) *Suppose that* $w \neq p, p'$. *Then*

$$M(p,s,A;p,t,B) \neq M(w,q,C;w,r,D).$$

Proof. (i) If $\widehat{G}$ contains an element of infinite order, then $\mathbb{T}$ is a homomorphic image of G and we appeal to Theorem 4.9. Otherwise all elements are of finite order and the result follows from Corollary 5.3.

(ii) is similar. ∎

Our results, combined with the known inclusions for multiplier spaces (2.2), (2.3), imply:

COROLLARY 5.9. *Suppose G is an infinite, compact abelian group.*

(i) *If either* (a) $s < r \leq q$ *and* $B + 1/r > C + 1/s$, *or* (b) $r < s \leq q$ *and* $B \geq C$, *then* $M(p,q,A;p,r,B) \subsetneq M(p,q,A;p,s,C)$.

(ii) *If either* (a) $r \leq q < v$ *and* $C \geq A$, *or* (b) $r \leq v < q$ *and* $C + 1/q > A + 1/v$, *then* $M(p,v,A;p,r,B) \subsetneq M(p,q,C;p,r,B)$.

REMARK 5.10. To study the non-equality of Lorentz–Zygmund multiplier spaces with other indices a version of Zafran's multilinear interpolation theorem [14] may be needed.

References

[1] J.-G. Bak, J. D. McMichael and D. M. Oberlin, *Convolution estimates for some measures on flat curves*, J. Funct. Anal. 101 (1991), 81–96.

[2] C. Bennett and K. Rudnick, *On Lorentz–Zygmund spaces*, Dissertationes Math. 175 (1980).

[3] C. Bennett and R. Sharpley, *Interpolation of Operators*, Pure Appl. Math. 129, Academic Press, Boston, MA, 1988.

[4] M. Cowling and J. Fournier, *Inclusions and noninclusion of spaces of convolution operators*, Trans. Amer. Math. Soc. 221 (1976), 59–95.

[5] G. I. Gaudry, *Bad behavior and inclusion results for multipliers of the type* (p,q), Pacific J. Math. 35 (1970), 83–94.

[6] L. Grafakos and M. Mastyło, *Interpolation of bilinear operators between quasi-Banach spaces*, preprint.

[7] R. Grinnell and K. E. Hare, *Lorentz improving measures*, Illinois J. Math. 38 (1994), 366–389.

[8] K. E. Hare, P. Mohanty and M. Roginskaya, *A general energy formula*, preprint.

[9] K. E. Hare and E. Sato, *Spaces of Lorentz multipliers*, Canad. J. Math. 53 (2001), 565–591.

[10] R. Hunt, *On* $L(p,q)$ *spaces*, Enseign. Math. 12 (1966), 249–276.

[11] C. Merucci, *Applications of interpolation with a function parameter to Lorentz, Sobolev and Besov spaces*, in: Interpolation Spaces and Allied Topics in Analysis (Lund, 1983), Lecture Notes in Math. 1070, Springer, Berlin, 1984, 183–201.

[12] T. Tao and J. Wright, *Endpoint multiplier theorems of Marcinkiewicz type*, Rev. Mat. Iberoamericana 17 (2001), 521–558.
[13] M. Zafran, *Multiplier transformations of weak type*, Ann. of Math. 101 (1975), 34–44.
[14] —, *A multilinear interpolation theorem*, Studia Math. 62 (1978), 107–124.

Department of Pure Mathematics
University of Waterloo
Waterloo, ON, N2L 3G1, Canada
E-mail: kehare@uwaterloo.ca
mparasar@yahoo.com

Current address of Parasar Mohanty:
Department of Mathematics
University of Delhi
Delhi, India 110007

Received August 30, 2004
Revised version February 11, 2005 (5488)