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Functions of bounded variation on compact subsets of the plane

by

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Abstract. A major obstacle in extending the theory of well-bounded operators to cover operators whose spectrum is not necessarily real has been the lack of a suitable variation norm applicable to functions defined on an arbitrary nonempty compact subset σ of the plane. In this paper we define a new Banach algebra $\mathrm{BV}(\sigma)$ of functions of bounded variation on such a set and show that the function-theoretic properties of this algebra make it better suited to applications in spectral theory than those used previously.

1. Introduction. The motivation for this work lies in the spectral theory of linear operators on Banach spaces. It has long been known that the possession of a suitable functional calculus for an operator T on a Banach space X is often enough to ensure that T has some sort of integral or sum representation with respect to a family of projections on X.

In 1960, Smart [15] introduced the class of well-bounded operators in order to give a representation theory for operators whose integral representations were of a conditional, rather than unconditional, nature. A bounded operator was said to be well-bounded if it has an $\mathrm{AC}([a,b])$ functional calculus (where $\mathrm{AC}([a,b])$ denotes the absolutely continuous functions on the compact interval $[a,b]$). On reflexive spaces, all well-bounded operators have an integral representation with respect to a family of projections known as a spectral family. An account of the theory of well-bounded operators can be found in [10].

A serious restriction of this theory is that it only handles operators whose spectrum is a subset of the real line. Attempts to address this problem were made in even the earliest papers on well-bounded operators (see [13]). Over

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the past 40 years a number of authors have examined classes of operators which generalize the well-bounded theory to operators with complex spectrum ([5], [7], [8], [17]). Although these theories have proved rather important in applications (especially the theory of trigonometrically well-bounded operators developed in [8]), each has contained either restrictions on the allowable spectrum, or else an unsatisfactory matching between the function algebras used and the spectrum of the operator.

A first step in trying to develop a suitable theory is to find an appropriate analogue for the functions of bounded variation on an interval for functions whose domain is now a subset of $\mathbb{C}$. There have been, of course, many definitions of the variation of a function of two or more variables. As early as 1933 Clarkson and Adams [9] had collected 7 variants. New definitions continue to be introduced for various applications (see, for example, [1] for a more recent definition from the theory of partial differential equations). Berkson and Gillespie [7] used a definition ascribed to Hardy and Krause to define a Banach algebra $\mathrm{BV}_{\mathrm{HK}}(R)$ where R is a rectangle in the plane. (Here and throughout the paper, rectangles will be assumed to have sides parallel to the coordinate axes.) The closure of the polynomials in two variables in this algebra is denoted $\mathrm{AC}_{\mathrm{HK}}(R)$. They defined an operator $T \in B(X)$ to be an AC operator if T admits an $\mathrm{AC}_{\mathrm{HK}}(R)$ functional calculus for some rectangle R.

The theory of AC operators has some appealing features. For example, T is an AC operator if and only if there exist commuting well-bounded operators A and B such that $T = A + iB$. Since their introduction however, a number of less desirable properties have become apparent.

As was shown in [6], the class of AC operators is not closed under scalar multiplication. From an operator theorist's point of view this is unsatisfactory since if one's theory provides a structure theorem for T, then it should also provide a structure theorem for $\alpha T + \beta I$ for any $\alpha, \beta \in \mathbb{C}$. In any case, a more natural domain for the functions for which a functional calculus for an operator T might be defined is usually the spectrum of T (or at least some small neighbourhood of $\sigma(T)$) rather than a rectangle.

We shall show in this paper that it is possible to define functions of bounded variation on arbitrary nonempty compact subsets of the plane in a way which is much better suited for spectral-theoretic purposes. Even for well-bounded operators it would actually be more natural to write the theory in terms of functions defined on $\sigma(T)$ rather than an interval $[a, b]$. Defining $\mathrm{BV}(\sigma)$ and $\mathrm{AC}(\sigma)$ for a compact subset $\sigma \subset \mathbb{R}$ is of course a relatively straightforward extension of the usual interval definitions, but as these definitions will be important when we extend to complex domains, we quickly summarize the main results in Section 2.

From a spectral-theoretic point of view, a new definition for $\mathrm{BV}(\sigma)$, the Banach algebra of functions of bounded variation on a nonempty compact set $\sigma \subset \mathbb{C}$ (or $\sigma \subset \mathbb{R}^2$), should have at least the following properties:

(i) it should agree with the "usual definition" if $\sigma \subset \mathbb{R}$;
(ii) it should contain all sufficiently well-behaved functions (polynomials, C^∞ functions, characteristic functions of polygons and so forth);
(iii) for all $\alpha, \beta \in \mathbb{C}$, $\alpha \neq 0$, we should have $\mathrm{BV}(\alpha\sigma + \beta) \cong \mathrm{BV}(\sigma)$.

The main part of this paper (Section 3) goes to giving a definition which satisfies these properties.

Our new definition agrees with the standard one when $\sigma \subset \mathbb{R}$, and, up to an equivalent norm, with the natural definition given in [8] for the case that σ is the unit circle. We show in [4] that if σ is a rectangle, the new definition gives a strictly larger algebra of functions than the one that arises from the Hardy–Krause definition used by Berkson and Gillespie.

For the applications to operator theory, one is interested in working with a smaller algebra of "absolutely continuous" functions. In Section 4 we define a subalgebra $\mathrm{AC}(\sigma) \subset \mathrm{BV}(\sigma)$ and examine its properties. An $\mathrm{AC}(\sigma)$ operator is then defined to be one which admits an $\mathrm{AC}(\sigma)$ functional calculus.

As was shown in [2], one can develop generalizations of the well-bounded theory to cover these $\mathrm{AC}(\sigma)$ operators. For example, whereas well-bounded operators admit projection-valued decompositions for projections associated with half-lines, $\mathrm{AC}(\sigma)$ operators have decompositions involving projections associated to half-planes. The main direction of this paper however is to develop an appropriate function theory and so, although we shall comment on the operator theory throughout, most of the details will appear in [3].

2. $\mathrm{BV}(\sigma)$ **for** $\sigma \subset \mathbb{R}$ **compact.** Let σ be a nonempty compact subset of $\mathbb{R}$. Since σ inherits an order from $\mathbb{R}$, one may define the variation of a function $f : \sigma \to \mathbb{C}$ in exactly the same way as one does for functions defined on intervals. This concept of variation will be important when we go on to consider functions of bounded variation in two real variables so we shall give here a summary of the important similarities and differences between $\mathrm{BV}(\sigma)$ and $\mathrm{BV}([a, b])$. Since most of the proofs in this section are exact analogs of the more classical situation we shall generally refer the reader to references such as [14] for the details.

Let $J = [a, b]$ be the smallest interval which contains σ. We say $\{s_i\}_{i=1}^n$ is a *partition* of σ if $s_1 \le \cdots \le s_n$ and $s_i \in \sigma$ for all i. The set of partitions of σ is denoted by $\Lambda(\sigma)$. Let $S = \{s_i\}_{i=1}^n$, $T = \{t_i\}_{i=1}^m \in \Lambda(\sigma)$. The set T is said to be a *refinement* of S if $S \subset T$. Then $\Lambda(\sigma)$ is a lattice using refinement as a partial ordering.

For $f : \sigma \to \mathbb{C}$ we define the *variation* of f by

$$\operatorname{var}(f,\sigma) = \sup_{\{s_i\}_{i=1}^n \in \Lambda(\sigma)} \sum_{i=1}^{n-1} |f(s_{i+1}) - f(s_i)|.$$

Since $\Lambda(\sigma)$ is a lattice and because of the triangle inequality one deduces that $\operatorname{var}(f,\sigma)$ can equivalently be defined by replacing the supremum in the above expression with a limit. Set

$$\|f\|_{\mathrm{BV}(\sigma)} = \|f\|_\infty + \operatorname{var}(f,\sigma).$$

The set of functions of bounded variation is

$$\mathrm{BV}(\sigma) = \{f : \sigma \to \mathbb{C} : \|f\|_{\mathrm{BV}(\sigma)} < \infty\}.$$

We shall show below that $\mathrm{BV}(\sigma)$ is a Banach algebra.

Many of the following properties of variation will be generalized to the two-variable situation.

PROPOSITION 2.1. *Let $f, g \in \mathrm{BV}(\sigma)$, $k \in \mathbb{C}$ and $\sigma = \sigma_1 \cup \sigma_2$ where σ_1, σ_2 are nonempty compact subsets of $\mathbb{R}$. Then*

(i) $\operatorname{var}(f+g,\sigma) \le \operatorname{var}(f,\sigma) + \operatorname{var}(g,\sigma)$,
(ii) $\operatorname{var}(kf,\sigma) = |k| \operatorname{var}(f,\sigma)$,
(iii) $\operatorname{var}(fg,\sigma) \le \|f\|_\infty \operatorname{var}(g,\sigma) + \|g\|_\infty \operatorname{var}(f,\sigma)$,
(iv) $\operatorname{var}(f,\sigma) \ge |f(b) - f(a)|$,
(v) *if f is nondecreasing or nonincreasing then* $\operatorname{var}(f,\sigma) = |f(b) - f(a)|$,
(vi) $\operatorname{var}(f,\sigma_1) \le \operatorname{var}(f,\sigma)$,
(vii) *if $\sigma_1 \subset [a,c]$, $\sigma_2 \subset [c,b]$ and $\sigma_1 \cap \sigma_2 = \{c\}$ then*

$$\operatorname{var}(f,\sigma) = \operatorname{var}(f,\sigma_1) + \operatorname{var}(f,\sigma_2).$$

Proof. The proofs of (i) through (v) are the same as in the case $\sigma = [a,b]$. Since $\Lambda(\sigma_1) \subset \Lambda(\sigma)$, (vi) follows. We now prove (vii). Let $\{s_i\}_{i=1}^n \in \Lambda(\sigma)$. By refining if necessary we may assume that $c = s_j$ for some j. Then $\{s_i\}_{i=1}^j \in \Lambda(\sigma_1)$ and $\{s_i\}_{i=j}^n \in \Lambda(\sigma_2)$. Hence

$$\begin{aligned}\sum_{i=1}^{n-1} |f(s_{i+1}) - f(s_i)| &= \sum_{i=1}^{j-1} |f(s_{i+1}) - f(s_i)| + \sum_{i=j}^{n-1} |f(s_{i+1}) - f(s_i)| \\ &\le \operatorname{var}(f,\sigma_1) + \operatorname{var}(f,\sigma_2).\end{aligned}$$

Taking the supremum over partitions shows that $\operatorname{var}(f,\sigma) \le \operatorname{var}(f,\sigma_1) + \operatorname{var}(f,\sigma_2)$. The reverse inequality follows from noting that any partitions of σ_1 and σ_2 generate a partition of σ. ∎

It is easy to use Proposition 2.1 to show that $\|\cdot\|_{\mathrm{BV}(\sigma)}$ is an algebra norm on $\mathrm{BV}(\sigma)$.

For many of the properties of $\mathrm{BV}(\sigma)$ it is easier to embed $\mathrm{BV}(\sigma)$ into $\mathrm{BV}(J)$ and then use the classical theory. For $t \in J \setminus \sigma$ define

$$\alpha(t) = \sup\{x : [t, x] \subset J \setminus \sigma\}, \qquad \beta(t) = \inf\{x : [x, t] \subset J \setminus \sigma\}.$$

Given $f : \sigma \to \mathbb{C}$ define the function $\iota(f) : J \to \mathbb{C}$ by

$$\iota(f)(t) = \begin{cases} f(t) & \text{if } t \in \sigma, \\ \left(\dfrac{f(\alpha(t)) - f(\beta(t))}{\alpha(t) - \beta(t)}\right)(t - \beta(t)) + f(\beta(t)) & \text{if } t \in J \setminus \sigma. \end{cases} \tag{1}$$

In other words, $\iota(f)$ is defined so that it is linear on the gaps in σ. The following results are readily verified.

PROPOSITION 2.2. *Let $\sigma_1 \subset \sigma_2$ be compact subsets of $\mathbb{R}$ and let $f \in \mathrm{BV}(\sigma_2)$. Then $\|f|\sigma_1\|_{\mathrm{BV}(\sigma_1)} \le \|f\|_{\mathrm{BV}(\sigma_2)}$ and so $f|\sigma_1 \in \mathrm{BV}(\sigma_1)$.*

PROPOSITION 2.3. *Let $f \in \mathrm{BV}(\sigma)$. Then* $\mathrm{var}(f, \sigma) = \mathrm{var}(\iota(f), J)$.

PROPOSITION 2.4. *Let $f : \sigma \to \mathbb{C}$. Then $f \in \mathrm{BV}(\sigma)$ if and only if $\iota(f) \in \mathrm{BV}(J)$.*

PROPOSITION 2.5. *The map $\iota : \mathrm{BV}(\sigma) \to \mathrm{BV}(J)$ is a linear isometry.*

Note that $\mathrm{BV}(J) \to \mathrm{BV}(\sigma) : F \mapsto F|\sigma$ is a left inverse of ι. That is, if $f : \sigma \to \mathbb{C}$ then $\iota(f)|\sigma = f$.

LEMMA 2.6. *Suppose that $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathrm{BV}(\sigma)$. Then $F = \lim_{n\to\infty} \iota(f_n) \in \mathrm{BV}(J)$ exists and $F = \iota(F|\sigma)$.*

Proof. From Proposition 2.3, $\{\iota(f_n)\}_{n=1}^\infty$ is a Cauchy sequence in $\mathrm{BV}(J)$ and so converges as claimed to some $F \in \mathrm{BV}(J)$. To complete the proof we need to show that if $t \in J \setminus \sigma$ then

$$F(t) = \left(\frac{F(\alpha(t)) - F(\beta(t))}{\alpha(t) - \beta(t)}\right)(t - \beta(t)) + F(\beta(t)).$$

First we notice that we must have pointwise convergence of both $\{f_n\}_{n=1}^\infty$ and $\{\iota(f_n)\}_{n=1}^\infty$. Hence

$$\begin{aligned} F(t) &= \lim_{n\to\infty} \iota(f_n(t)) \\ &= \lim_{n\to\infty} \left(\left(\frac{f_n(\alpha(t)) - f_n(\beta(t))}{\alpha(t) - \beta(t)}\right)(t - \beta(t)) + f_n(\beta(t)) \right) \\ &= \left(\frac{\lim_{n\to\infty} f_n(\alpha(t)) - \lim_{n\to\infty} f_n(\beta(t))}{\alpha(t) - \beta(t)}\right)(t - \beta(t)) + \lim_{n\to\infty} f_n(\beta(t)) \\ &= \left(\frac{F(\alpha(t)) - F(\beta(t))}{\alpha(t) - \beta(t)}\right)(t - \beta(t)) + F(\beta(t)). \ \blacksquare \end{aligned}$$

THEOREM 2.7. $(\mathrm{BV}(\sigma), \|\cdot\|_{\mathrm{BV}(\sigma)})$ *is a Banach algebra.*

Proof. The only thing to show is completeness. Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathrm{BV}(\sigma)$. Then by Proposition 2.5, $\{\iota(f_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathrm{BV}(J)$, and so converges say to F. By Proposition 2.2, $f = F|\sigma \in \mathrm{BV}(\sigma)$ and by Lemma 2.6, $F = \iota(f)$. Finally, we note that

$$\begin{aligned}\lim_{n\to\infty} \|f_n - f\|_{\mathrm{BV}(\sigma)} &= \lim_{n\to\infty} \|\iota(f_n - f)\|_{\mathrm{BV}(J)} \\ &= \lim_{n\to\infty} \|\iota(f_n) - F\|_{\mathrm{BV}(J)} = 0. \ \blacksquare\end{aligned}$$

It is easy to check that (the restrictions of) any C^∞ functions (in particular polynomials), or any Lipschitz functions sit inside $\mathrm{BV}(\sigma)$, as do piecewise polynomial functions.

In the theory of well-bounded operators, the most important subalgebra of $\mathrm{BV}([a,b])$ is the algebra of absolutely continuous functions on $[a,b]$. In dealing with more general domain sets, one has to decide which of the characterizations of absolute continuity one wishes to work with.

DEFINITION 2.8. Let $f : \sigma \to \mathbb{C}$. We say that f is *absolutely continuous* if given $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite number of non-overlapping intervals $\{[s_i, t_i]\}_{i=1}^n$ with $s_i, t_i \in \sigma$ for all i and $\sum_{i=1}^n |t_i - s_i| < \delta$ we have $\sum_{i=1}^n |f(t_i) - f(s_i)| < \varepsilon$. We let the set of absolutely continuous functions with domain σ be denoted $\mathrm{AC}(\sigma)$.

If $\sigma = [a,b]$ then this is the usual definition of absolute continuity. See [11] and [12] for more information on $\mathrm{AC}(J)$. An equivalent definition of $\mathrm{AC}(\sigma)$ is the following.

PROPOSITION 2.9. *Let $f : \sigma \to \mathbb{C}$. Then $f \in \mathrm{AC}(\sigma)$ if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every finite sequence of non-overlapping intervals $\{[s_i, t_i]\}_{i=1}^n$ with $\sum_{i=1}^n |t_i - s_i| < \delta$ we have*

$$\sum_{i=1}^n \mathrm{var}(f, [s_i, t_i] \cap \sigma) < \varepsilon.$$

Proof. We first show the "if" part of the statement. Suppose $f : \sigma \to \mathbb{C}$ has the properties on the right hand side of the "if and only if" statement above. Fix $\varepsilon > 0$ and choose δ accordingly. Let $\{[s_i, t_i]\}_{i=1}^n$ be a set of nonoverlapping intervals with $s_i, t_i \in \sigma$ for all i and $\sum_{i=1}^n |t_i - s_i| < \delta$. Then

$$\sum_{i=1}^n |f(t_i) - f(s_i)| \le \sum_{i=1}^n \mathrm{var}(f, [s_i, t_i] \cap \sigma) < \varepsilon.$$

Hence $f \in \mathrm{AC}(\sigma)$.

Suppose $f \in \mathrm{AC}(\sigma)$. Fix $\varepsilon > 0$ and choose δ as in Definition 2.8 using $\varepsilon/2$ instead of ε. Let $\{[s_i, t_i]\}_{i=1}^n$ be a set of nonoverlapping intervals with $\sum_{i=1}^n |t_i - s_i| < \delta$. For each i there exists a sequence of nonoverlapping intervals $\{[u_{i,j}, v_{i,j}]\}_{j=1}^{m_i}$ such that $u_{i,j}, v_{i,j} \in \sigma \cap [s_i, t_i]$ for all j and

$$\operatorname{var}(f, [s_i, t_i] \cap \sigma) \le \sum_{j=1}^{m_i} |f(v_{i,j}) - f(u_{i,j})| + \frac{\varepsilon}{2n}.$$

Then $\{[u_{i,j}, v_{i,j}]\}_{i,j=1}^{j=m_i, i=n}$ is a set of nonoverlapping intervals satisfying $\sum_{i=1}^{n}\sum_{j=1}^{m_i} |v_{i,j} - u_{i,j}| < \delta$, and so $\sum_{i=1}^{n}\sum_{j=1}^{m_i} |f(v_{i,j}) - f(u_{i,j})| < \varepsilon/2$. Hence

$$\sum_{i=1}^{n} \operatorname{var}(f, [s_i, t_i] \cap \sigma) \le \sum_{i=1}^{n} \left(\sum_{j=1}^{m_i} |f(v_{i,j}) - f(u_{i,j})| + \frac{\varepsilon}{2n} \right) \le \varepsilon.$$

This shows the "only if" portion of the statement. ∎

LEMMA 2.10. *Let $\sigma_1 \subset \sigma_2$ both be compact and let $f \in \mathrm{AC}(\sigma_2)$. Then $f|\sigma_1 \in \mathrm{AC}(\sigma_1)$.*

Proof. Fix $\varepsilon > 0$ and choose $\delta > 0$ as in the definition of $f \in \mathrm{AC}(\sigma_2)$. Then for any sequence of intervals $\{[s_i, t_i]\}_{i=1}^{n}$ with $\sum_{i=1}^{n} |t_i - s_i| < \delta$, we have by Proposition 2.1(vi),

$$\sum_{i=1}^{n} \operatorname{var}(f, [s_i, t_i] \cap \sigma_1) \le \sum_{i=1}^{n} \operatorname{var}(f, [s_i, t_i] \cap \sigma_2) < \varepsilon. \ ∎$$

The following is an easy consequence of the characterization of AC functions on intervals as the integrals of L^1 functions (see [14, Corollary 5.4.14]).

LEMMA 2.11. *Let $a = s_1 \le \cdots \le s_n = b$ and let $f \in C([a, b])$ where $f|[s_i, s_{i+1}] \in \mathrm{AC}([s_i, s_{i+1}])$ for all i. Then $f \in \mathrm{AC}([a, b])$.*

COROLLARY 2.12. *Let $a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_n < b_n$. Suppose $\sigma = \bigcup_{i=1}^{n} [a_i, b_i]$, $f : \sigma \to \mathbb{C}$ is continuous and $f|[a_i, b_i] \in \mathrm{AC}([a_i, b_i])$ for each i. Then $f \in \mathrm{AC}(\sigma)$.*

Proof. Let $[a, b]$ be the smallest interval containing σ. Then $\iota(f) \in C([a, b])$ and $\iota(f)|[a_i, b_i] = f|[a_i, b_i] \in \mathrm{AC}([a_i, b_i])$ for each i. Now $\iota(f)$ is linear on $[b_i, a_{i+1}]$ for each i and so $\iota(f)|[b_i, a_{i+1}] \in \mathrm{AC}([b_i, a_{i+1}])$. We now apply Lemma 2.11 to conclude $\iota(f) \in \mathrm{AC}([a, b])$. Finally, we conclude from Lemma 2.10 that $f = \iota(f)|\sigma \in \mathrm{AC}(\sigma)$. ∎

We now have a version of Proposition 2.4 for $\mathrm{AC}(\sigma)$.

THEOREM 2.13. *Let $f : \sigma \to \mathbb{C}$. Then $f \in \mathrm{AC}(\sigma)$ if and only if $\iota(f) \in \mathrm{AC}(J)$.*

Proof. If $\iota(f) \in \mathrm{AC}(J)$ then by Lemma 2.10, $f \in \mathrm{AC}(\sigma)$.

Suppose then that $f \in \mathrm{AC}(\sigma)$. Since σ is compact, $J \setminus \sigma$ can be written as a countable union of disjoint open intervals $\bigcup O_n$. For each n let I_n denote the largest closed interval satisfying $O_n \subset I_n \subset O_n \cup \sigma$. Let $\sigma_n = I_1 \cup \cdots \cup I_n$. Clearly σ_n can be written as a finite union of disjoint closed intervals. Let

J' be one of these intervals. If we set $V_1 = \sigma \cap J'$ and $V_2 = \overline{J' \setminus \sigma}$, then both V_1 and V_2 are disjoint unions of closed intervals. Now $\iota(f)|V_1 = f|V_1$, so by Lemma 2.10, $\iota(f)|V_1 \in \mathrm{AC}(V_1)$. On the other hand, $\iota(f)$ is linear on each of the components of V_2, so $\iota(f)|V_2 \in \mathrm{AC}(V_2)$. It follows from Lemma 2.11 and Corollary 2.12 that $\iota(f)|\sigma_n \in \mathrm{AC}(\sigma_n)$.

For each n, let $\tau_n = \overline{J \setminus \sigma_n}$. Again τ_n is a finite union of disjoint closed intervals. Since $J = \bigcup \sigma_n$, for any $\delta > 0$, there exists N such that for all $n \geq N$, the measure of τ_n is less than δ.

Fix $\varepsilon > 0$. By definition, there exists $\delta_1 > 0$ such that if $\{[s_i, t_i]\}_{i=1}^m$ is a finite set of nonoverlapping intervals with $s_i, t_i \in \sigma$ for all i and $\sum_{i=1}^m |t_i - s_i| < \delta_1$, then $\sum_{i=1}^m |f(t_i) - f(s_i)| < \varepsilon/2$. Choose n such that the measure of τ_n is less than δ_1, and write τ_n as the disjoint union of closed intervals $J_1, \ldots, J_l$.

Since $\iota(f)|\sigma_n \in \mathrm{AC}(\sigma_n)$, we can find $\delta_2 > 0$ such that if $\{[s_i, t_i]\}_{i=1}^m$ is a finite set of nonoverlapping intervals with $s_i, t_i \in \sigma_n$ for all i and $\sum_{i=1}^m |t_i - s_i| < \delta_2$, then $\sum_{i=1}^m |\iota(f)(t_i) - \iota(f)(s_i)| < \varepsilon/2$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Suppose that $\{[c_i, d_i]\}_{i=1}^m$ is a finite set of nonoverlapping subintervals of J with $\sum_{i=1}^m |d_i - c_i| < \delta$. Since σ_n has only finitely many components, the set $(\bigcup_{i=1}^m [c_i, d_i]) \cap \sigma_n$ can be written as a finite union of disjoint closed intervals $\bigcup_{i=1}^{m_1} [c_i^1, d_i^1]$. Similarly we write $(\bigcup_{i=1}^m [c_i, d_i]) \cap \tau_n = \bigcup_{i=1}^{m_2} [c_i^2, d_i^2]$. Now, by Propositions 2.2 and 2.3,

$$\sum_{i=1}^{m_1} \mathrm{var}(\iota(f), [c_i^1, d_i^1]) \leq \sum_{i=1}^{l} \mathrm{var}(\iota(f), J_i) = \sum_{i=1}^{l} \mathrm{var}(f, J_i) < \varepsilon/2.$$

On the other hand, $\sum_{i=1}^{m_2} \mathrm{var}(\iota(f), [c_i^2, d_i^2]) < \varepsilon/2$ and so

$$\sum_{i=1}^{m} \mathrm{var}(\iota(f), [c_i, d_i]) = \sum_{i=1}^{m_1} \mathrm{var}(\iota(f), [c_i^1, d_i^1]) + \sum_{i=1}^{m_2} \mathrm{var}(\iota(f), [c_i^2, d_i^2]) < \varepsilon.$$

Thus $\iota(f) \in \mathrm{AC}(J)$. ∎

COROLLARY 2.14. *The map* $\iota|\mathrm{AC}(\sigma)$ *is a linear isometry from* $\mathrm{AC}(\sigma)$ *into* $\mathrm{AC}(J)$.

COROLLARY 2.15. *If* $f \in \mathrm{AC}(\sigma)$ *then* $f \in \mathrm{BV}(\sigma)$.

Proof. If $f \in \mathrm{AC}(\sigma)$ then by Theorem 2.13, $\iota(f) \in \mathrm{AC}(J)$. Hence $\iota(f) \in \mathrm{BV}(J)$. By Proposition 2.2, $f = \iota(f)|\sigma \in \mathrm{BV}(\sigma)$. ∎

THEOREM 2.16. *Let* $\sigma \subset \mathbb{R}$ *be compact. Then* $\mathrm{AC}(\sigma)$ *is a Banach subalgebra of* $\mathrm{BV}(\sigma)$.

Proof. Let $f, g \in \mathrm{AC}(\sigma)$ and $k \in \mathbb{C}$. Then for $s, t \in \sigma$ the following hold:

$$|(f+g)(t) - (f+g)(s)| \leq |f(t) - f(s)| + |g(t) - g(s)|,$$
$$|(fg)(t) - (fg)(s)| \leq \|f\|_\infty |g(t) - g(s)| + \|g\|_\infty |f(t) - f(s)|,$$
$$|kf(t) - kf(s)| = |k|\,|f(t) - f(s)|.$$

From these and the ε, δ definition of $\mathrm{AC}(\sigma)$ we deduce $\mathrm{AC}(\sigma)$ is a subalgebra of $\mathrm{BV}(\sigma)$.

It remains to show completeness. Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathrm{AC}(\sigma)$. Then $\{\iota(f_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathrm{AC}(J)$ and so converges, say to $F \in \mathrm{AC}(J)$. By Lemma 2.10, $F|\sigma \in \mathrm{AC}(\sigma)$. Also, by Lemma 2.6, $\iota(F|\sigma) = F$. Then

$$\lim_{n\to\infty} \|f_n - F|\sigma\|_{\mathrm{BV}(\sigma)} \le \lim_{n\to\infty} \|\iota(f_n - F|\sigma)\|_{\mathrm{BV}(J)} = \lim_{n\to\infty} \|\iota(f_n) - F\|_{\mathrm{BV}(J)} = 0. \ \blacksquare$$

THEOREM 2.17. *The set $\mathcal{P}$ of polynomials is dense in* $\mathrm{AC}(\sigma)$.

Proof. Let $f \in \mathrm{AC}(\sigma)$. Fix $\varepsilon > 0$. By the density of $\mathcal{P}$ in $\mathrm{AC}(J)$ there exists $p \in \mathcal{P}$ such that $\|\iota(f) - p\|_{\mathrm{BV}(J)} < \varepsilon$. Then

$$\|f - p\|_{\mathrm{BV}(\sigma)} = \|(\iota(f) - p)|\sigma\|_{\mathrm{BV}(\sigma)} \le \|\iota(f) - p\|_{\mathrm{BV}(J)} < \varepsilon. \ \blacksquare$$

It is an easy consequence of the results in this section that if $T \in B(X)$ has an $\mathrm{AC}(\sigma(T))$ functional calculus then it also admits an $\mathrm{AC}(J)$ functional calculus and hence is well-bounded. The converse is also true. Details can be found in [2] or [3].

3. $\mathrm{BV}(\sigma)$ **for** $\sigma \subset \mathbb{C}$ **compact.** Suppose now that σ is a nonempty compact subset of $\mathbb{C}$. (Throughout we shall identify $\mathbb{C}$ and $\mathbb{R}^2$.) A first step in defining $\mathrm{BV}(\sigma)$ is to make a sensible definition for $\mathrm{var}(f, \sigma)$ for a function $f : \sigma \to \mathbb{C}$. The idea behind our construction is to consider the variation, denoted $\mathrm{cvar}(f, \gamma)$, along finite length curves γ in the plane. One is then left with the problem of how to separate the variation that is due to the function from the variation which is due to the geometry of the curve. This is done by assigning a weight factor $\varrho(\gamma) \in [0, 1]$ to each curve γ. The weight factor is large for straight lines and low for very sinuous ones. The two-dimensional variation is then defined as the supremum of $\varrho(\gamma)\,\mathrm{cvar}(f, \gamma)$ over all curves γ. In this way the affine invariance properties are more or less built into the definition. The first difficulty lies in showing that this definition has the appropriate multiplicativity properties to enable it to be used to define a Banach algebra norm. One also needs to show that all sufficiently well-behaved functions (such as polynomials and Lipschitz functions) will have bounded variation under this definition and that this definition reduces to that of the previous section if $\sigma \subset \mathbb{R}$.

3.1. *Weight factors.* By a curve in the plane we shall mean an element of the set $\Gamma = C([0, 1])$. Note that it will sometimes be important to distinguish between a curve (which includes its parameterization) and its image in $\mathbb{C}$.

If $\gamma_1, \gamma_2 \in \Gamma$ and $\gamma_1(1) = \gamma_2(0)$ let $\gamma_1 \circ \gamma_2 \in \Gamma$ be defined by

$$(\gamma_1 \circ \gamma_2)(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \le t \le 1/2, \\ \gamma_2(2t-1) & \text{if } 1/2 < t \le 1. \end{cases}$$

If $\gamma_1, \gamma_2 \in \Gamma$ and if there exists $h : [0,1] \to [0,1]$ where h is a continuous nondecreasing or nonincreasing surjective function such that $\gamma_1(t) = \gamma_2(h(t))$ for all $t \in [0,1]$ then we write $\gamma_1 \cong \gamma_2$.

Let $\gamma \in \Gamma$. Then $t \in [0,1]$ is said to be an *entry point* of γ on a line l if either

(i) $t = 0$ and $\gamma(0) \in l$, or
(ii) $\gamma(t) \in l$ and for all $\varepsilon > 0$ there exists $s \in (t - \varepsilon, t) \cap [0,1]$ such that $\gamma(s) \notin l$.

Similarly $t \in [0,1]$ is said to be an *exit point* of γ on a line l if either

(i) $t = 1$ and $\gamma(1) \in l$, or
(ii) $\gamma(t) \in l$ and for all $\varepsilon > 0$ there exists $s \in (t, t + \varepsilon) \cap [0,1]$ such that $\gamma(s) \notin l$.

There are similar definitions for entry and exit points of γ on a line segment or for $\gamma \in C([a,b])$ rather than $\gamma \in C([0,1])$. Figure 1 illustrates a curve $\gamma \in \Gamma$ with four entry points t_1, t_2, t_3 and t_4 on a line l.

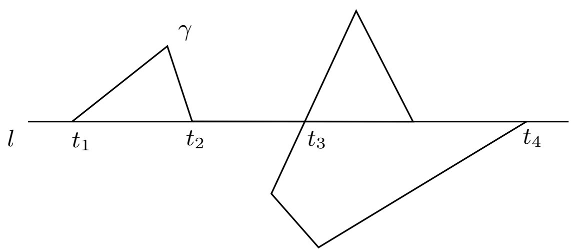

Fig. 1. Entry points $t_1, \ldots, t_4$ of γ along l

Suppose $\gamma \in \Gamma$ and $\{\gamma_i\}_{i=1}^n \subset \Gamma$. Set $\mathrm{vf}(\gamma, l)$ to be the number of entry points of γ on l and set $\mathrm{vf}(\bigcup_{i=1}^n \gamma_i, l) = \sum_{i=1}^n \mathrm{vf}(\gamma_i, l)$. Clearly if $\gamma_1 \cong \gamma_2$ then $\mathrm{vf}(\gamma_1, l) = \mathrm{vf}(\gamma_2, l)$. We set $\mathrm{vf}(\gamma)$ and $\mathrm{vf}(\bigcup_{i=1}^n \gamma_i)$ to be the supremum of $\mathrm{vf}(\gamma, l)$ and $\mathrm{vf}(\bigcup_{i=1}^n \gamma_i, l)$ over all lines l. We write $\mathrm{vf}_{\mathrm{H}}(\gamma)$ for the supremum of $\mathrm{vf}(\gamma, l)$ over all horizontal lines l, vf_{V} for the supremum of $\mathrm{vf}(\gamma, l)$ over all vertical lines, and so on. Clearly $\mathrm{vf} \ge \mathrm{vf}_{\mathrm{H}}$ and $\mathrm{vf} \ge \mathrm{vf}_{\mathrm{V}}$. We write ϱ for $1/\mathrm{vf}$. For example $\varrho_{\mathrm{V}}(\bigcup_{i=1}^n \gamma_i) = 1/\mathrm{vf}_{\mathrm{V}}(\bigcup_{i=1}^n \gamma_i)$. If, for example, $\mathrm{vf}(\gamma) = \infty$ then we take the convention that $\varrho(\gamma) = 0$. It is also clear that $\varrho \le \varrho_{\mathrm{H}}$, $\varrho \le \varrho_{\mathrm{V}}$ and that if $\gamma_1 \cong \gamma_2$ then $\varrho(\gamma_1) = \varrho(\gamma_2)$. We can extend the notion of ϱ, ϱ_{V} and so on to include curves in $C([a,b])$ in the obvious way.

In Figure 2 there are three curves $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$. From the diagram one can see that $\mathrm{vf}(\bigcup_{i=1}^3 \gamma_i, l) = 6$. No line has more entry points on each curve

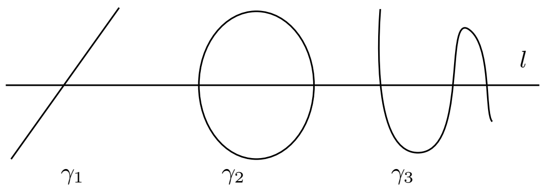

Fig. 2. $\varrho(\gamma_1) = 1$, $\varrho(\gamma_2) = 1/2$ and $\varrho(\gamma_3) = 1/3$.

than l does. Hence $\varrho(\gamma_1) = 1$, $\varrho(\gamma_2) = 1/2$ and $\varrho(\gamma_3) = 1/3$. It is easy to see that $\varrho_{\mathrm{H}}(\gamma_i) = \varrho(\gamma_i)$ for each i and that $\varrho_{\mathrm{V}}(\gamma_1) = \varrho_{\mathrm{V}}(\gamma_3) = 1$ and $\varrho_{\mathrm{V}}(\gamma_2) = 1/2$.

Let $\sigma \subset \mathbb{C}$ be compact and let l be a line parameterized by $\mathbb{R}$. Then $t \in \mathbb{R}$ is said to be an entry point of l on σ if $l(t) \in \sigma$ and for all $\varepsilon > 0$ there exists $s \in (t - \varepsilon, t)$ such that $l(s) \notin \sigma$. Again set $\mathrm{vf}(\sigma, l)$ to be the number of entry points of l on σ and $\mathrm{vf}(\sigma)$ to be the supremum of $\mathrm{vf}(\sigma, l)$ over all lines l. Clearly $\mathrm{vf}(\sigma, l)$ does not depend on the choice of parameterization of the line l.

Note that if $\gamma \in \mathit{\Gamma}$ then it does not follow that $\mathrm{vf}(\gamma) = \mathrm{vf}(\gamma([0,1]))$. For example if γ is given by

$$\gamma(t) = \begin{cases} 2t & \text{if } 0 \le t \le 1/2, \\ 2 - 2t & \text{if } 1/2 < t \le 1. \end{cases}$$

Then $\mathrm{vf}(\gamma([0,1])) = \mathrm{vf}([0,1]) = 1$ but $\mathrm{vf}(\gamma) = \mathrm{vf}(\gamma, \mathbb{R}) = 2$.

We now define a set of curves $\mathit{\Gamma}_L$ which we later show allows us to approximate any $\gamma \in \mathit{\Gamma}$ by a curve consisting of line segments. Let $j, n \in \mathbb{Z}^+$ and suppose that $j < n$. For $t \in \left[\frac{j-1}{n-1}, \frac{j}{n-1}\right]$ define

$$\alpha_{j,n}(t) = (n-1)t - (j-1).$$

Hence $\alpha_{j,n}$ maps $\left[\frac{j-1}{n-1}, \frac{j}{n-1}\right]$ homeomorphically onto $[0,1]$. Let $z_1, \dots, z_n \in \mathbb{C}$. Write $\mathit{\Pi}(z_1, \dots, z_n)$ for the function $[0,1] \to \mathbb{C}$ defined on each interval $\left[\frac{j-1}{n-1}, \frac{j}{n-1}\right]$ for $1 \le j \le n-1$ by

$$\mathit{\Pi}(z_1, \dots, z_n)(t) = (1 - \alpha_{j,n}(t)) z_j + \alpha_{j,n}(t) z_{j+1}.$$

Hence $\mathit{\Pi}(z_1, \dots, z_n) \in \mathit{\Gamma}$ and is a curve consisting of line segments whose endpoints are $z_1, \dots, z_n$ and which is parameterized by $[0,1]$.

Set

$$\mathit{\Gamma}_L = \{\gamma \in \mathit{\Gamma} : \gamma \cong \mathit{\Pi}(z_1, \dots, z_n) \text{ for some } z_i \in \mathbb{C} \text{ and } n \in \mathbb{N}\}.$$

Let $\gamma \in \mathit{\Gamma}$. Let $S = \{s_i\}_{i=1}^n \in \mathit{\Lambda}([0,1])$. Set

$$\gamma_S = \mathit{\Pi}(\gamma(s_1), \dots, \gamma(s_n)) \in \mathit{\Gamma}_L.$$

The curve γ_S is said to be the S *approximation of* γ.

LEMMA 3.1. *Let $\gamma \in \Gamma$ and suppose* $\mathrm{vf}(\gamma) < \infty$. *Then*

$$\lim_{S \in \Lambda([0,1])} \varrho(\gamma_S) = \varrho(\gamma).$$

Proof. Fix $S = \{s_i\}_{i=1}^n \in \Lambda([0,1])$. Let l be a line. If $t \in [0,1]$ is an entry point of γ_S on l then there exist $1 \le j \le n-1$ such that $t \in \left[\frac{j-1}{n-1}, \frac{j}{n-1}\right]$. Since l is a line and $\gamma_S\left(\left[\frac{j-1}{n-1}, \frac{j}{n-1}\right]\right)$ is a line segment, there is no other entry point t' of γ_S on l such that $t' \in \left[\frac{j-1}{n-1}, \frac{j}{n-1}\right]$. But since γ is continuous and $\gamma(s_j) = \gamma_S\left(\frac{j-1}{n-1}\right)$ and $\gamma(s_{j+1}) = \gamma_S\left(\frac{j}{n-1}\right)$, it follows there is at least one entry point s of γ on l such that $s_j \le s \le s_{j+1}$. Hence $\mathrm{vf}(\gamma_S, l) \le \mathrm{vf}(\gamma, l)$ and so $\varrho(\gamma_S) \ge \varrho(\gamma)$.

To conclude the proof we show that there exists $S \in \Lambda([0,1])$ such that $\varrho(\gamma_S) \le \varrho(\gamma)$ and $\varrho(\gamma_{S'}) \le \varrho(\gamma)$ for any refinement S' of S. Since $\mathrm{vf}(\gamma) < \infty$ there exists a line l such that $\mathrm{vf}(\gamma, l) = \mathrm{vf}(\gamma) := m$. Let $\{t_i\}_{i=1}^m \in \Lambda([0,1])$ be the entry points of γ on l, ordered so that $t_1 < t_2 < \cdots < t_m$. Assume for the moment that $t_1 \ne 0$ and $t_m \ne 1$. Let $\{s_i\}_{i=1}^{m+1} \in \Lambda([0,1])$ be such that $s_1 < t_1 < s_2 < t_2 < \cdots < s_m < t_m < s_{m+1}$. Then for all i, $\gamma(s_i) \notin l$. Set $S = \{s_1, t_1, s_2, t_2, \ldots, s_m, t_m, s_{m+1}\}$. Then for each $1 \le j \le m$, $(2j-1)/2m$ is an entry point of γ_S on l which corresponds to the entry point t_j of γ on l. Furthermore, every entrance point of γ_S on l is of the form $(2j-1)/(2m)$. Hence $\mathrm{vf}(\gamma_S) \ge \mathrm{vf}(\gamma_S, l) = \mathrm{vf}(\gamma, l) = \mathrm{vf}(\gamma)$ and so $\varrho(\gamma_S) \le \varrho(\gamma)$. A similar proof holds if $t_1 = 0$ or $t_m = 1$. Finally, note we can apply the above procedure by adding more points to S between each of the t_i and not change $\varrho(\gamma_S)$. ∎

3.2. *Variation in two variables.* Here we define the variation of a function defined on a nonempty compact set σ in the complex plane. We show in Proposition 3.6 that this definition reduces to the usual definition when $\sigma = J$ and the definition in Section 2 when $\sigma \subset \mathbb{R}$.

Let $\gamma \in \Gamma$ and let $\emptyset \ne \sigma \subset \mathbb{C}$ be compact. We say that $\{z_i\}_{i=1}^n$ is a *partition of* γ *over* σ if $z_i \in \sigma$ for all i and if there exists $\{s_i\}_{i=1}^n \in \Lambda([0,1])$ such that $z_i = \gamma(s_i)$ for all i. Let $\Lambda(\sigma, \gamma)$ be the set of partitions of γ over σ. Clearly $\Lambda(\sigma, \gamma)$ inherits the lattice structure of $\Lambda([0,1])$.

Let $f : \sigma \to \mathbb{C}$ and $\gamma \in \Gamma$. We define the *variation along the curve* γ by

$$\mathrm{cvar}(f, \gamma, \sigma) = \mathrm{cvar}(f, \gamma) = \sup_{\{z_i\}_{i=1}^n \in \Lambda(\sigma,\gamma)} \sum_{i=1}^{n-1} |f(z_{i+1}) - f(z_i)|.$$

Clearly $\mathrm{cvar}(f, \gamma) = \mathrm{var}(\iota(f \circ \gamma), [0,1])$ where ι is the map described by equation (1) in Section 2. Again, since $\Lambda(\sigma, \gamma)$ is a lattice one can use the triangle inequality and replace the supremum in the above expression by a limit. There is a version of Proposition 2.1 for cvar.

Proposition 3.2. *Let $\sigma_1 \subset \sigma \subset \mathbb{C}$ both be compact. Let $f, g : \sigma \to \mathbb{C}$ and $k \in \mathbb{C}$. Suppose $\gamma = \gamma_1 \circ \gamma_2 \in \Gamma$ with $\gamma_1(1) \in \sigma$. Then*

(i) $\operatorname{cvar}(f+g,\gamma) \le \operatorname{cvar}(f,\gamma) + \operatorname{cvar}(g,\gamma)$,
(ii) $\operatorname{cvar}(fg,\gamma) \le \|f\|_\infty \operatorname{cvar}(g,\gamma) + \|g\|_\infty \operatorname{cvar}(f,\gamma)$,
(iii) $\operatorname{cvar}(kf,\gamma) = |k| \operatorname{cvar}(f,\gamma)$,
(iv) $\operatorname{cvar}(f,\gamma) = \operatorname{cvar}(f,\gamma_1) + \operatorname{cvar}(f,\gamma_2)$,
(v) $\operatorname{cvar}(f,\gamma_1) \le \operatorname{cvar}(f,\gamma)$,
(vi) $\operatorname{cvar}(f,\gamma,\sigma_1) \le \operatorname{cvar}(f,\gamma,\sigma)$.

Proof. The proofs are the same as for Proposition 2.1. ∎

Note that the variation along a curve does not depend on the parameterization.

Lemma 3.3. *Let $f : \sigma \to \mathbb{C}$. Let $\gamma_1, \gamma_2 \in \Gamma$ and suppose that $\gamma_1 \cong \gamma_2$. Then* $\operatorname{cvar}(f,\gamma_1) = \operatorname{cvar}(f,\gamma_2)$.

Definition 3.4. Let $f : \sigma \to \mathbb{C}$. Then the *variation* of f on σ is defined to be

$$\operatorname{var}(f,\sigma) = \sup_{\gamma\in\Gamma} \varrho(\gamma) \operatorname{cvar}(f,\gamma). \tag{2}$$

Here we take the convention that if $\gamma \in \Gamma$ is such that $\varrho(\gamma) = 0$ and if $\operatorname{cvar}(f,\gamma) = \infty$ then $\varrho(\gamma) \operatorname{cvar}(f,\gamma) = 0$. As we shall show in Proposition 3.6 this notation is not ambiguous since it agrees with the notation given in Section 2 if $\sigma \subset \mathbb{R}$.

In practice, Γ is usually too large a set to work with. As the next lemma shows, one can replace Γ with Γ_L (or indeed any of a number of sets of simpler curves) and obtain the same definition of variation over σ.

Lemma 3.5. *Let $f : \sigma \to \mathbb{C}$. Then*

$$\sup_{\gamma\in\Gamma_L} \varrho(\gamma) \operatorname{cvar}(f,\gamma) = \sup_{\gamma\in\Gamma} \varrho(\gamma) \operatorname{cvar}(f,\gamma).$$

Proof. Clearly $\sup_{\gamma\in\Gamma_L} \varrho(\gamma) \operatorname{cvar}(f,\gamma) \le \sup_{\gamma\in\Gamma} \varrho(\gamma) \operatorname{cvar}(f,\gamma)$. Let $\gamma\in\Gamma$. We may assume that $\varrho(\gamma) > 0$. Let $S = \{s_i\}_{i=1}^n \in \Lambda(\sigma,\gamma)$. Then

$$\sum_{i=1}^{n-1} |f(s_{i+1}) - f(s_i)| \le \operatorname{cvar}(f,\gamma_S).$$

Hence by Lemma 3.1,

$$\begin{aligned}
\varrho(\gamma) \operatorname{cvar}(f,\gamma) &= \lim_{S=\{s_i\}_{i=1}^n \in \Lambda(\sigma,\gamma)} \varrho(\gamma) \sum_{i=1}^{n-1} |f(s_{i+1}) - f(s_i)| \\
&\le \lim_{S\in\Lambda(\sigma,\gamma)} \varrho(\gamma) \operatorname{cvar}(f,\gamma_S) = \lim_{S\in\Lambda(\sigma,\gamma)} \varrho(\gamma_S) \operatorname{cvar}(f,\gamma_S) \\
&\le \sup_{\gamma'\in\Gamma_L} \varrho(\gamma') \operatorname{cvar}(f,\gamma'). \ \blacksquare
\end{aligned}$$

We shall show now that the new definition of variation agrees with the previous one if $\sigma \subset \mathbb{R}$. It suffices to consider the case where $\sigma = [0,1]$. One can use the function ι when σ is an arbitrary compact subset of $\mathbb{R}$.

PROPOSITION 3.6. *Let* $f \in \mathrm{BV}([0,1])$. *Then*

$$\operatorname{cvar}(f, \Pi(0,1)) = \sup_{\gamma \in \Gamma} \varrho(\gamma) \operatorname{cvar}(f, \gamma).$$

Proof. Let $\gamma = \Pi(0,1) \in \Gamma_L$. Then $\varrho(\gamma) = 1$ and $\operatorname{cvar}(f,\gamma) = \operatorname{var}(f,[0,1])$ and so $\sup_{\gamma \in \Gamma_L} \varrho(\gamma) \operatorname{cvar}(f,\gamma) \geq \operatorname{cvar}(f, \Pi(0,1))$.

Let $\gamma \in \Gamma_L$. Suppose that $S = \{s_1, \ldots, s_n\}$ are the entry and exit points of γ on $[0,1]$, ordered so that $s_1 < \cdots < s_n$. Then $\operatorname{cvar}(f,\gamma) \leq \operatorname{cvar}(f,\gamma_S)$. Also $\varrho(\gamma) \leq \varrho_{\mathrm{V}}(\gamma) = \varrho_{\mathrm{V}}(\gamma_S)$. Let $\{x_i\}_{i=1}^m = \{\gamma(s_i)\}_{i=1}^n$ be ordered so that $x_1 < \cdots < x_m$. Then

$$\operatorname{cvar}(f,\gamma_S) = \alpha_1 \operatorname{cvar}(f, \Pi(x_1,x_2)) + \cdots + \alpha_{m-1} \operatorname{cvar}(f, \Pi(x_{m-1},x_m))$$

where $\alpha_i = \operatorname{vf}(\gamma_S, \Pi(x_i,x_{i+1}))$ for all i. But $\alpha_i \leq \operatorname{vf}(\gamma_S,[0,1]) \leq \operatorname{vf}_{\mathrm{V}}(\gamma_S)$. Therefore

$$\operatorname{cvar}(f,\gamma_S) \leq \operatorname{vf}_{\mathrm{V}}(\gamma_S)(\operatorname{cvar}(f,\Pi(x_1,x_2)) + \cdots + \operatorname{cvar}(f,\Pi(x_{m-1},x_m))).$$

Hence

$$\begin{aligned}
&\varrho(\gamma) \operatorname{cvar}(f,\gamma) \leq \varrho(\gamma) \operatorname{cvar}(f,\gamma_S) \\
&\quad \leq \varrho(\gamma_S) \operatorname{vf}_{\mathrm{V}}(\gamma_S)(\operatorname{cvar}(f,\Pi(x_1,x_2)) + \cdots + \operatorname{cvar}(f,\Pi(x_{m-1},x_m))) \\
&\quad \leq \varrho_{\mathrm{V}}(\gamma_S) \operatorname{vf}_{\mathrm{V}}(\gamma_S)(\operatorname{cvar}(f,\Pi(x_1,x_2)) + \cdots + \operatorname{cvar}(f,\Pi(x_{m-1},x_m))) \\
&\quad = \operatorname{cvar}(f,\Pi(x_1,x_m)) \leq \operatorname{cvar}(f,\Pi(0,1)).
\end{aligned}$$

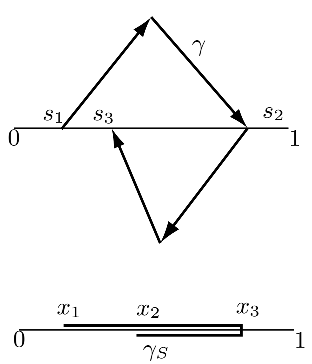

Fig. 3. Idea of the proof of Proposition 3.6

The proof now follows from Lemma 3.5. Figure 3 illustrates the idea of the proof for a curve $\gamma \in \Gamma_L$. The curve γ has entry points $\{s_1, s_2, s_3\}$ on $[0,1]$. Then $x_1 = s_1$, $x_2 = s_3$ and $x_3 = s_2$. Clearly $\operatorname{vf}_{\mathrm{V}}(\gamma) = 2$ and

$\operatorname{cvar}(f, \gamma_S) \le 2(\operatorname{cvar}(f, [t_1, t_2]) + \operatorname{cvar}(f, [t_2, t_3]))$. Hence $\varrho(\gamma) \operatorname{cvar}(f, \gamma) \le \operatorname{cvar}(f, \Pi(0, 1))$. ∎

The next proposition follows easily from Proposition 3.2.

PROPOSITION 3.7. *Let $\sigma_1 \subset \sigma \subset \mathbb{C}$ both be compact. Let $f, g : \sigma \to \mathbb{C}$ and $k \in \mathbb{C}$. Then*

(i) $\operatorname{var}(f + g, \sigma) \le \operatorname{var}(f, \sigma) + \operatorname{var}(g, \sigma)$,
(ii) $\operatorname{var}(fg, \sigma) \le \|f\|_\infty \operatorname{var}(g, \sigma) + \|g\|_\infty \operatorname{var}(f, \sigma)$,
(iii) $\operatorname{var}(kf, \sigma) = |k| \operatorname{var}(f, \sigma)$,
(iv) $\operatorname{var}(f, \sigma_1) \le \operatorname{var}(f, \sigma)$.

3.3. *The Banach algebra* $\mathrm{BV}(\sigma)$. For $f : \sigma \to \mathbb{C}$, set $\|f\|_{\mathrm{BV}(\sigma)} = \|f\|_\infty + \operatorname{var}(f, \sigma)$. The functions of bounded variation with domain σ are defined to be

$$\mathrm{BV}(\sigma) = \{f : \sigma \to \mathbb{C} : \|f\|_{\mathrm{BV}(\sigma)} < \infty\}.$$

THEOREM 3.8. $(\mathrm{BV}(\sigma), \|\cdot\|_{\mathrm{BV}(\sigma)})$ *is a Banach algebra.*

Proof. Checking that $\|\cdot\|_{\mathrm{BV}(\sigma)}$ has the properties of an algebra norm is straightforward. For example using Proposition 3.7 we have

$$\begin{aligned}
\|fg\|_{\mathrm{BV}(\sigma)} &= \|fg\|_\infty + \operatorname{var}(fg, \gamma) \\
&\le \|f\|_\infty \|g\|_\infty + \|f\|_\infty \operatorname{var}(g, \sigma) + \|g\|_\infty \operatorname{var}(f, \sigma) \\
&\le \|f\|_\infty \|g\|_\infty + \|f\|_\infty \operatorname{var}(g, \sigma) + \|g\|_\infty \operatorname{var}(f, \sigma) + \operatorname{var}(f, \sigma) \operatorname{var}(g, \sigma) \\
&= (\|f\|_\infty + \operatorname{var}(f, \sigma))(\|g\|_\infty + \operatorname{var}(g, \sigma)) = \|f\|_{\mathrm{BV}(\sigma)} \|g\|_{\mathrm{BV}(\sigma)}.
\end{aligned}$$

It remains to show that $\mathrm{BV}(\sigma)$ is complete. Let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in $\mathrm{BV}(\sigma)$. Fix $\varepsilon > 0$. By the definition of $\|\cdot\|_{\mathrm{BV}(\sigma)}$, $\{f_n\}_{n=1}^\infty$ converges uniformly to a function f. Choose N_1 so that $n \ge N_1$ implies $\|f - f_n\|_\infty < \varepsilon/2$. Being a Cauchy sequence in $\mathrm{BV}(\sigma)$ means there exists an N_2 so that $m, n > N_2$ implies that for all $\gamma \in \Gamma$ and all $\{z_i\}_{i=1}^n \in \Lambda(\sigma, \gamma)$ we have

$$\varrho(\gamma) \sum_{i=1}^{n-1} |(f_n - f_m)(z_{i+1}) - (f_n - f_m)(z_i)| < \frac{\varepsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$. Let $n > N$, $\gamma \in \Gamma$ and $\{z_i\}_{i=1}^n \in \Lambda(\sigma, \gamma)$. Then

$$\begin{aligned}
&\varrho(\gamma) \sum_{i=1}^{n-1} |(f_n - f)(z_{i+1}) - (f_n - f)(z_i)| \\
&\qquad = \lim_m \varrho(\gamma) \sum_{i=1}^{n-1} |(f_n - f_m)(z_{i+1}) - (f_n - f_m)(z_i)| < \frac{\varepsilon}{2}.
\end{aligned}$$

Hence

$$\begin{aligned}\operatorname{var}(f-f_n,\sigma) &= \sup_{\gamma\in\Gamma_L} \varrho(\gamma)\operatorname{cvar}(f-f_n,\gamma)\\ &= \sup_{\gamma\in\Gamma_L}\sup_{\{z_i\}_{i=1}^n\in\Lambda(\sigma,\gamma)} \varrho(\gamma)\sum_{i=1}^{n-1}|(f_n-f)(z_{i+1})-(f_n-f)(z_i)| \le \frac{\varepsilon}{2}.\end{aligned}$$

Finally, $\|f-f_n\|_{\mathrm{BV}(\sigma)} = \|f-f_n\|_\infty + \operatorname{var}(f-f_n,\sigma) \le \varepsilon/2+\varepsilon/2 = \varepsilon$. ∎

These algebras respect domain inclusion in the expected manner.

Lemma 3.9. *Suppose that $\sigma_1 \subset \sigma_2 \subset \mathbb{C}$ are both compact and $f \in \mathrm{BV}(\sigma_2)$. Then $\|f|\sigma_1\|_{\mathrm{BV}(\sigma_1)} \le \|f\|_{\mathrm{BV}(\sigma_2)}$ and so $f|\sigma_1 \in \mathrm{BV}(\sigma_1)$.*

Proof. By Proposition 3.7(iv),

$$\|f|\sigma_1\|_{\mathrm{BV}(\sigma_1)} = \|f|\sigma_1\|_\infty + \operatorname{var}(f|\sigma_1,\sigma_1) \le \|f\|_\infty + \operatorname{var}(f,\sigma_2) = \|f\|_{\mathrm{BV}(\sigma_2)}. \ ∎$$

3.4. *Affine invariance.* One of the objectives in this paper was to have an algebra which has the same sort of affine invariance properties as $C(\sigma)$. Let $f\in \mathrm{BV}(\sigma)$. Define $\theta_{\alpha,\beta}(f) : \alpha\sigma+\beta \to \mathbb{C}$ by

$$\theta_{\alpha,\beta}(f)(z) = f(\alpha^{-1}(z-\beta)).$$

Proposition 3.10. *For any $\alpha,\beta\in\mathbb{C}$, $\alpha\ne 0$, the map $\theta_{\alpha,\beta}$ is an isometric isomorphism from* $\mathrm{BV}(\sigma)$ *onto* $\mathrm{BV}(\alpha\sigma+\beta)$.

Proof. Clearly $\theta_{\alpha,\beta}$ is a linear homomorphism. Let $f\in\mathrm{BV}(\sigma)$ and $\gamma\in\Gamma$. Then $\alpha\gamma+\beta\in\Gamma$. Hence

$$\begin{aligned}\operatorname{cvar}(f,\gamma,\sigma) &= \sup_{\{z_i\}_{i=1}^n\in\Lambda(\sigma,\gamma)} \sum_{i=1}^{n-1}|f(z_{i+1})-f(z_i)|\\ &= \sup_{\{w_i\}_{i=1}^n\in\Lambda(\alpha\sigma+\beta,\alpha\gamma+\beta)} \sum_{i=1}^{n-1}|f(\alpha^{-1}(w_{i+1}-\beta))-f(\alpha^{-1}(w_i-\beta))|\\ &= \sup_{\{w_i\}_{i=1}^n\in\Lambda(\alpha\sigma+\beta,\alpha\gamma+\beta)} \sum_{i=1}^{n-1}|\theta_{\alpha,\beta}(f)(w_{i+1})-\theta_{\alpha,\beta}(f)(w_i)|\\ &= \operatorname{cvar}(\theta_{\alpha,\beta}(f),\alpha\gamma+\beta,\alpha\sigma+\beta).\end{aligned}$$

Since $\varrho(\gamma) = \varrho(\alpha\gamma+\beta)$ it follows that $\operatorname{var}(\theta_{\alpha,\beta}(f),\alpha\sigma+\beta) = \operatorname{var}(f,\sigma)$. It is clear that $\|\theta_{\alpha,\beta}(f)\|_\infty = \|f\|_\infty$. Hence $\|\theta_{\alpha,\beta}(f)\|_{\mathrm{BV}(\alpha\sigma+\beta)} = \|f\|_{\mathrm{BV}(\sigma)}$. Finally, note that $(\theta_{\alpha,\beta})^{-1} = \theta_{\alpha^{-1},-\alpha^{-1}\beta}$. ∎

3.5. *Compositions of functions.* It is possible to generalize Proposition 3.6 to the following proposition. This result allows us to conclude that many important AC operators (such as the trigonometrically well-bounded operators) are also $\mathrm{AC}(\sigma)$ operators for some σ.

PROPOSITION 3.11. *Let $\sigma \subset \mathbb{R}$ be compact. Let $g \in \mathrm{BV}(\sigma) \cap C(\sigma)$. Suppose that $\varrho(\iota(g)) > 0$. Then*

$$\|f \circ g\|_{\mathrm{BV}(\sigma)} \le \frac{1}{\varrho(\iota(g))} \|f\|_{\mathrm{BV}(g(\sigma))}$$

for all $f \in \mathrm{BV}(g(\sigma))$.

Proof. Since $g \in C(\sigma)$ it is clear that $\iota(g) \in C(J)$ where J is the smallest interval which contains σ and so $\varrho(\iota(g))$ makes sense. For $S = \{z_i\}_{i=1}^n \in \Lambda(\sigma)$, set $g_S = \Pi(g(z_1), \dots, g(z_n)) \in \Gamma$. For such S,

$$\begin{aligned}\sum_{i=1}^{n-1} |(f \circ g)(z_{i+1}) - (f \circ g)(z_i)| \le \operatorname{cvar}(f, g_S) &= \frac{\varrho(g_S) \operatorname{cvar}(f, g_S)}{\varrho(g_S)} \\ &\le \frac{\operatorname{var}(f, g(\sigma))}{\varrho(g_S)}.\end{aligned}$$

By Lemma 3.1 it follows that $\lim_{S\in\Lambda(\sigma)} \varrho(g_S) = \varrho(\iota(g))$, so taking the limit over $S \in \Lambda(\sigma)$ shows that

$$\operatorname{var}(f \circ g, \sigma) \le \frac{1}{\varrho(\iota(g))} \operatorname{var}(f, \sigma).$$

Since $0 < \varrho(\iota(g)) \le 1$ it follows that

$$\|f \circ g\|_\infty = \|f\|_\infty \le \frac{1}{\varrho(\iota(g))} \|f\|_\infty$$

and so the result follows. ∎

3.6. *Subsets of* $\mathrm{BV}(\sigma)$. The above definition of $\mathrm{BV}(\sigma)$ is obviously of limited use unless this set contains a sufficiently rich collection of functions. We shall now look at some classes of functions which lie in $\mathrm{BV}(\sigma)$. In particular we shall see that polynomials, $C^\infty(\sigma)$ functions and Lipschitz functions are of bounded variation, as are characteristic functions of polygonal regions.

Given $f \in \mathrm{BV}(\mathrm{Re}(\sigma))$ define $u(f) : \sigma \to \mathbb{C}$ by $u(f)(x + iy) = f(x)$. Similarly if $g \in \mathrm{BV}(\mathrm{Im}(\sigma))$ define $v(g) : \sigma \to \mathbb{C}$ by $v(g)(x + iy) = g(y)$.

LEMMA 3.12. *The map u is a unital norm-decreasing linear homomorphism from $\mathrm{BV}(\mathrm{Re}(\sigma))$ into $\mathrm{BV}(\sigma)$. Similarly v is a unital norm-decreasing linear homomorphism from $\mathrm{BV}(\mathrm{Im}(\sigma))$ into $\mathrm{BV}(\sigma)$.*

Proof. The only thing not clear is that u and v are norm-decreasing. Let $f \in \mathrm{BV}(\mathrm{Re}(\sigma))$ and let $\gamma \in \Gamma_L$. Recall that $\mathrm{Re}(\gamma)$ is defined by $\mathrm{Re}(\gamma)(t) = \mathrm{Re}(\gamma(t))$. Clearly $\mathrm{Re}(\gamma) \in \Gamma_L$. From

$$|u(f)(t) - u(f)(s)| = |f(\mathrm{Re}(t)) - f(\mathrm{Re}(s))|$$

it follows that $\operatorname{cvar}(u(f), \gamma, \sigma) = \operatorname{cvar}(f, \mathrm{Re}(\gamma), \mathrm{Re}(\sigma))$. Also, using a similar argument to that used in Proposition 3.6, we find that $\operatorname{cvar}(f, \mathrm{Re}(\gamma), \mathrm{Re}(\sigma))$

$\leq \mathrm{vf}_{\mathrm{V}}(\gamma)\operatorname{var}(f,\mathrm{Re}(\gamma))$. Then

$$\begin{aligned}
&\varrho(\gamma)\operatorname{cvar}(u(f),\gamma,\sigma)\\
&\qquad = \varrho(\gamma)\operatorname{cvar}(f,\mathrm{Re}(\gamma),\mathrm{Re}(\sigma)) \leq \varrho_{\mathrm{V}}(\gamma)\operatorname{cvar}(f,\mathrm{Re}(\gamma),\mathrm{Re}(\sigma))\\
&\qquad \leq \varrho_{\mathrm{V}}(\gamma)\,\mathrm{vf}_{\mathrm{V}}(\gamma)\operatorname{var}(f,\mathrm{Re}(\gamma)) = \operatorname{var}(f,\mathrm{Re}(\gamma)) \leq \operatorname{var}(f,\mathrm{Re}(\sigma)).
\end{aligned}$$

Taking the supremum over all $\gamma \in \Gamma_L$ and using Lemma 3.5 gives the result. The proof for v is very similar. ∎

To show that all the polynomials are in $\mathrm{BV}(\sigma)$, it suffices to show that the function $\lambda_\sigma : \sigma \to \mathbb{C}$, $\lambda_\sigma(z) = z$, lies in $\mathrm{BV}(\sigma)$. Where there is little chance of confusion we shall write λ rather than λ_σ. Let $\mathcal{P}_2$ denote the polynomials in z and $\overline{z}$.

Corollary 3.13. $\lambda, \overline{\lambda} \in \mathrm{BV}(\sigma)$.

Proof. $\lambda = u(\lambda_{\mathrm{Re}(\sigma)}) + iv(\lambda_{\mathrm{Im}(\sigma)})$. ∎

Corollary 3.14. $\mathcal{P}_2 \subset \mathrm{BV}(\sigma)$.

Given a compact set $\sigma \subset \mathbb{C}$ let

$$C_\sigma = \operatorname{var}(\lambda,\sigma).$$

Given $\gamma = \Pi(z_1,\dots,z_n) \in \Gamma_L$ we write $l(\gamma)$ for the length of γ. That is, $l(\gamma) = \sum_{i=1}^{n-1}|z_{i+1}-z_i|$. Then $l(\gamma) = \operatorname{cvar}(\lambda,\gamma)$ and so $\varrho(\gamma)l(\gamma) \leq C_\sigma$. Since σ is compact there exist $z, w \in \sigma$ such that $\operatorname{diam}(\sigma) = |z-w|$. In this case $\operatorname{diam}(\sigma) = |z-w| = \operatorname{cvar}(\lambda,\Pi(z,w)) \leq \operatorname{var}(\lambda,\sigma)$. In general this inequality is strict. For example let $\sigma = [0,1]\times[0,1]$. If $\gamma = \Pi(0,1,1+i,i,0) \in \Gamma_L$ then $\varrho(\gamma) = 1/2$ and $\operatorname{cvar}(\lambda,\gamma) = 4$. Hence $\operatorname{diam}(\sigma) = \sqrt{2} < 2 = \varrho(\gamma)\operatorname{cvar}(\lambda,\gamma) \leq \operatorname{var}(\lambda,\sigma)$.

Recall that we write $\mathrm{Lip}(\sigma)$ for the Lipschitz functions with domain σ and $L(f)$ for the Lipschitz constant of $f \in \mathrm{Lip}(\sigma)$.

Lemma 3.15. *Let $f \in \mathrm{Lip}(\sigma)$. Then* $\operatorname{var}(f,\sigma) \leq L(f)C_\sigma$.

Proof. Suppose that $\gamma \in \Gamma$. Then

$$\begin{aligned}
\operatorname{cvar}(f,\gamma) &= \sup_{\{s_i\}_{i=1}^n \in \Lambda(\sigma,\gamma)} \sum_{i=1}^{n-1} |f(s_{i+1}) - f(s_i)|\\
&\leq \sup_{\{s_i\}_{i=1}^n \in \Lambda(\sigma,\gamma)} L(f)\sum_{i=1}^{n-1}|s_{i+1}-s_i| = L(f)l(\gamma)
\end{aligned}$$

and so $\varrho(\gamma)\operatorname{cvar}(f,\gamma) \leq L(f)\varrho(\gamma)l(\gamma) \leq L(f)C_\sigma$. ∎

Corollary 3.16. *If $f \in \mathrm{Lip}(\sigma)$ then $f \in \mathrm{BV}(\sigma)$.*

Corollary 3.17. *Let $\{f_n\}_{n=1}^\infty \subset \mathrm{Lip}(\sigma)$ and $f \in \mathrm{Lip}(\sigma)$. Then*

$$\lim_{n\to\infty}\|f-f_n\|_{\mathrm{Lip}(\sigma)} = 0 \quad \textit{implies} \quad \lim_{n\to\infty}\|f-f_n\|_{\mathrm{BV}(\sigma)} = 0.$$

PROPOSITION 3.18. *If* $f \in \mathrm{BV}(\sigma)$ *then* $\mathrm{Re}(f), \mathrm{Im}(f) \in \mathrm{BV}(\sigma)$.

Proof. For $\mathrm{Re}(f)$ it suffices to prove that $\mathrm{var}(\mathrm{Re}(f), \sigma) \le \mathrm{var}(f, \sigma)$. This follows from the identity $|\mathrm{Re}(f)(z) - \mathrm{Re}(f)(w)| \le |f(z) - f(w)|$. That $\mathrm{Im}(f) \in \mathrm{BV}(\sigma)$ follows similarly. ∎

The following shows $\mathrm{BV}(\sigma)$ is inverse closed.

PROPOSITION 3.19. *Let* $f \in \mathrm{BV}(\sigma)$ *and suppose* $M = \inf_{z\in\sigma} |f(z)| > 0$. *Then* $1/f \in \mathrm{BV}(\sigma)$.

Proof. Since

$$\left|\frac{1}{f(z)} - \frac{1}{f(w)}\right| = \left|\frac{f(w) - f(z)}{f(w)f(z)}\right| \le M^2|f(z) - f(w)|,$$

we have $\mathrm{var}\,(1/f, \sigma) \le M^2 \,\mathrm{var}(f, \sigma)$. ∎

Characteristic functions of polygons are of bounded variation.

PROPOSITION 3.20. *Let* $A \subset \mathbb{C}$ *be a closed convex* n*-sided polygonal region. Then* $\chi_{A\cap\sigma} \in \mathrm{BV}(\sigma)$ *and* $\|\chi_{A\cap\sigma}\|_{\mathrm{BV}(\sigma)} \le n + 1$.

Proof. Clearly χ_A can be written as $\prod_{j=1}^n \chi_{A_j}$ where each A_i is a half-plane. It follows from Proposition 3.2(ii) that $\mathrm{var}(\chi_A, \sigma) \le \sum_{j=1}^n \mathrm{var}(\chi_{A_j}, \sigma)$. Lemma 3.12 and Propositions 3.10 and 3.2(iv) show that $\mathrm{var}(\chi_{A_j}, \sigma) \le 1$ for all j and so $\mathrm{var}(\chi_A, \sigma) \le n$. The result follows. ∎

There is, just as in the one-variable case, a severe restriction on the form of idempotent functions in $\mathrm{BV}(\sigma)$. It is not too hard to show that if the polygon A sits within the interior of σ then the above estimate is sharp. Indeed, sets formed by taking a finite number of set operations involving polygons are essentially the only sets whose characteristic functions are in $\mathrm{BV}(\sigma)$. Making this precise is slightly delicate, since what really matters is how the set A intersects with σ. These questions will be pursued in more detail in [3].

If $\sigma = J \times K$ is a rectangle (with sides parallel to the axes), it is natural to ask how this new definition compares to the more classical notion (due to Hardy and Krause) which was used by Berkson and Gillespie in their definition of AC-operators [7]. We shall denote by $\mathrm{BV}_{\mathrm{HK}}(J \times K)$ the Banach algebra of functions on $J \times K$ which are of bounded variation in the Hardy–Krause sense. We show in [4] that

(i) $\mathrm{BV}_{\mathrm{HK}}(J \times K) \subset \mathrm{BV}(J \times K)$.
(ii) The inclusion map $\mathrm{BV}_{\mathrm{HK}}(J \times K) \hookrightarrow \mathrm{BV}(J \times K)$ is continuous.
(iii) If J and K are nondegenerate, then $\mathrm{BV}_{\mathrm{HK}}(J \times K) \ne \mathrm{BV}(J \times K)$.

4. $\mathrm{AC}(\sigma)$ **for** $\sigma \subset \mathbb{C}$ **compact.** From an operator-theoretic point of view, one would like to be able to deduce structural information about an operator T from bounds on $\|p(T)\|$ for p in some small algebra of functions. If X is reflexive and $\sigma(T) \subset \mathbb{R}$, then a bound of the form $\|p(T)\| \le C\|p\|_\infty$ is sufficient to show that T can be written as an integral with respect to a countably additive spectral measure, whereas a weaker bound of the form $\|p(T)\| \le C\|p\|_{\mathrm{AC}}$ implies that T has an integral representation with respect to a spectral family of projections. If the spectrum is not real then it is unrealistic to expect to be able to prove much unless the algebra contains at least $\mathcal{P}_2$, the polynomials in two variables. This leads to our definition of the absolutely continuous functions defined on a nonempty compact subset σ of $\mathbb{C}$. These form a Banach subalgebra $\mathrm{AC}(\sigma)$ of $\mathrm{BV}(\sigma)$.

In this section we look at some classes of functions in $\mathrm{AC}(\sigma)$. We show, for example, that $C^\infty(\sigma) \subset \mathrm{AC}(\sigma)$. Rather surprisingly however, Example 4.13 shows that unlike the situation when $\sigma \subset \mathbb{R}$, Lipschitz functions are not necessarily absolutely continuous.

As before let σ be a nonempty compact set in the complex plane and let $J \times K$ be the smallest rectangle containing σ. Let $\mathrm{AC}(\sigma) = \overline{\mathcal{P}}_2$, where the closure is taken in $\mathrm{BV}(\sigma)$ norm. By Corollary 3.14 these polynomials are all functions of bounded variation and so this makes sense. The set $\mathrm{AC}(\sigma)$ is then a Banach subalgebra of $\mathrm{BV}(\sigma)$. If $\sigma = [a,b] \subset \mathbb{R}$ then $\mathrm{AC}(\sigma)$ coincides with the usual notion of absolute continuity. Furthermore, by Theorem 2.17, if $\sigma \subset \mathbb{R}$ then this definition coincides with that in Section 2. We also get the affine invariance properties that one would hope for. In the next theorem $\theta_{\alpha,\beta}$ is the map defined in Section 3.4.

THEOREM 4.1. *Let $\alpha, \beta \in \mathbb{C}$ where $\alpha \ne 0$. Then $\theta_{\alpha,\beta}|\mathrm{AC}(\sigma)$ is an isometric isomorphism from $\mathrm{AC}(\sigma)$ onto $\mathrm{AC}(\alpha\sigma + \beta)$.*

Proof. All we need show is that if $f \in \mathrm{AC}(\sigma)$ then $\theta_{\alpha,\beta}(f) \in \mathrm{AC}(\alpha\sigma+\beta)$. Let $\{p_n\}_{n=1}^\infty$ be a sequence of polynomials approximating f in $\mathrm{BV}(\sigma)$ norm. Then since $\theta_{\alpha,\beta}$ is isometric, $\{\theta_{\alpha,\beta} \circ p_n\}_{n=1}^\infty$ is a sequence of polynomials that approximates $\theta_{\alpha,\beta}(f)$. ■

As one might hope, absolutely continuous functions are continuous.

LEMMA 4.2. *Let $\sigma \subset \mathbb{C}$ be compact. Then $\mathrm{AC}(\sigma) \subset C(\sigma)$.*

Proof. Let $f \in \mathrm{AC}(\sigma)$ and let $\{p_n\}_{n=1}^\infty \subset \mathcal{P}_2$ with $\lim_{n\to\infty} \|f - p_n\|_{\mathrm{BV}(\sigma)} = 0$. Then by definition of the norm on $\mathrm{BV}(\sigma)$, $\lim_{n\to\infty} \|f - p_n\|_\infty = 0$ and so $f \in C(\sigma)$ ■

If σ is simple enough (for example $\sigma = \{0, 1, 1/2, 1/3, \ldots\}$) then $\mathrm{AC}(\sigma) = C(\sigma) \cap \mathrm{BV}(\sigma)$, but clearly this is not the case in general.

Cross sections of absolutely continuous functions are absolutely continuous functions of one variable.

LEMMA 4.3. *Let $f \in \mathrm{AC}(\sigma)$ and let $\gamma \in \Gamma_L$ where $\gamma(0), \gamma(1) \in \sigma$. Then $\iota(f \circ \gamma) \in \mathrm{AC}([0,1])$.*

Proof. There exists $\{p_n\}_{n=1}^{\infty} \subset \mathcal{P}_2$ with $\lim_{n\to\infty} \|f - p_n\|_{\mathrm{BV}(\sigma)} = 0$. Then

$$\lim_{n\to\infty} \mathrm{var}(\iota((f-p_n)\circ\gamma), [0,1]) = \lim_{n\to\infty} \mathrm{cvar}(f-p_n, \gamma) = 0.$$

But $\iota(p_n \circ \gamma)$ is continuous piecewise $C^{\infty}([0,1])$ and hence in $\mathrm{AC}([0,1])$. Therefore $\iota(f \circ \gamma) \in \mathrm{AC}([0,1])$. ∎

The previous lemma does not characterize functions in $\mathrm{AC}(\sigma)$, as we shall see in Example 4.13.

Absolutely continuous functions of one variable extend naturally to absolutely continuous functions on σ. Recall that if $f \in \mathrm{BV}(\mathrm{Re}(\sigma))$ and $g \in \mathrm{BV}(\mathrm{Im}(\sigma))$ then $u(f) : \sigma \to \mathbb{C}$ is defined by $u(f)(x+iy) = f(x)$ and $v(g) : \sigma \to \mathbb{C}$ is defined by $v(g)(x+iy) = g(y)$.

PROPOSITION 4.4. *The map $u|\mathrm{AC}(\mathrm{Re}(\sigma))$ is a norm-decreasing linear homomorphism from $\mathrm{AC}(\mathrm{Re}(\sigma))$ into $\mathrm{AC}(\sigma)$. The map $v|\mathrm{AC}(\mathrm{Im}(\sigma))$ is a norm-decreasing linear homomorphism from $\mathrm{AC}(\mathrm{Im}(\sigma))$ into $\mathrm{AC}(\sigma)$.*

Proof. Consider the map u. By Lemma 3.12, u is a norm-decreasing linear homomorphism and so it remains to show that u maps $\mathrm{AC}(\mathrm{Re}(\sigma))$ into $\mathrm{AC}(\sigma)$. Let $f \in \mathrm{AC}(\mathrm{Re}(\sigma))$. Then there exists $\{p_n\}_{n=1}^{\infty} \in \mathcal{P}_2$ such that $\lim_{n\to\infty} \|f - p_n\|_{\mathrm{BV}(\mathrm{Re}(\sigma))} = 0$. Then $u(p_n) \in \mathcal{P}_2$ for all n, and

$$\begin{aligned}\lim_{n\to\infty} \|u(f) - u(p_n)\|_{\mathrm{BV}(\sigma)} &\le \lim_{n\to\infty} \|u(f-p_n)\|_{\mathrm{BV}(\sigma)} \\ &\le \lim_{n\to\infty} \|f-p_n\|_{\mathrm{BV}(\mathrm{Re}(\sigma))} = 0.\end{aligned}$$

Hence $u(f) \in \mathrm{AC}(\sigma)$. A similar proof holds for v. ∎

LEMMA 4.5. *Let $\sigma_1 \subset \sigma_2 \subset \mathbb{C}$ both be compact. If $f \in \mathrm{AC}(\sigma_2)$ then $f|\sigma_1 \in \mathrm{AC}(\sigma_1)$.*

Proof. Let $\{p_n\}_{n=1}^{\infty} \subset \mathcal{P}_2$ and suppose that $\lim_{n\to\infty} \|f - p_n\|_{\mathrm{BV}(\sigma_2)} = 0$. Then by Lemma 3.9, $\lim_{n\to\infty} \|f - p_n\|_{\mathrm{BV}(\sigma_1)} \le \lim_{n\to\infty} \|f - p_n\|_{\mathrm{BV}(\sigma_2)} = 0$. Hence $f|\sigma_1 \in \mathrm{AC}(\sigma_1)$. ∎

Since the absolutely continuous functions have been defined as the closure of the polynomials, one usually has to employ approximation arguments to prove things about them. Often it turns out to be more convenient to use some other dense set instead of the polynomials. Let $C^{\infty}(\sigma)$ be the set of all $f : \sigma \to \mathbb{C}$ which have a C^{∞} extension to an open neighbourhood of σ.

LEMMA 4.6. *Let $\sigma = J \times K$ be a rectangle. If $f \in C^2(J \times K)$ has continuous second order derivatives then $f \in \mathrm{AC}(J \times K)$.*

Proof. By the two-dimensional mean value theorem there exists $\{p_n\}_{n=1}^{\infty} \subset \mathcal{P}_2$ such that $\lim_{n\to\infty} \|f - p_n\|_{\mathrm{Lip}(J\times K)} = 0$. The result now follows from Corollary 3.17. ∎

PROPOSITION 4.7. *$C^\infty(\sigma)$ is a dense subset of* $\mathrm{AC}(\sigma)$.

Proof. Let $f \in C^\infty(\sigma)$. By definition there exists $F \in C^\infty(U)$, an extension of f defined on an open neighbourhood U of σ. We can then choose V open with minimally smooth boundary (see [16, Sect. 6.3.3]) and $\sigma \subset V \subset U$. Then $F|V$ can be extended to a function, also denoted F, in $C^\infty(J \times K)$. Hence by Lemma 4.6, $F \in \mathrm{AC}(J \times K)$, and so by Lemma 4.5, $f = F|\sigma \in \mathrm{AC}(\sigma)$. The density follows from the fact that polynomials are in $C^\infty(\sigma)$. ∎

Proposition 4.7 enables simple proofs that absolutely continuous functions are stable under simple operations.

COROLLARY 4.8. *If $f \in \mathrm{AC}(\sigma)$ then* $\mathrm{Re}(f), \mathrm{Im}(f) \in \mathrm{AC}(\sigma)$.

Proof. Let $\{p_n\}_{n=1}^\infty$ be a sequence of polynomials with the property that $\lim_{n\to\infty} \|f - p_n\|_{\mathrm{BV}(\sigma)} = 0$. Then $\{\mathrm{Re}(p_n)\}_{n=1}^\infty \subset C^\infty(\sigma)$. By Proposition 3.18, $\lim_{n\to\infty} \|\mathrm{Re}(f) - \mathrm{Re}(p_n)\|_{\mathrm{BV}(\sigma)} \le \lim_{n\to\infty} \|f - p_n\|_{\mathrm{BV}(\sigma)} = 0$. Hence $\mathrm{Re}(f) \in \mathrm{AC}(\sigma)$. Similarly $\mathrm{Im}(f) \in \mathrm{AC}(\sigma)$. ∎

COROLLARY 4.9. *If $f \in \mathrm{AC}(\sigma)$ and $f(z) \neq 0$ on σ then $1/f \in \mathrm{AC}(\sigma)$.*

Proof. Let $\{p_n\}_{n=1}^\infty$ be a sequence of polynomials approximating f in $\mathrm{BV}(\sigma)$ norm. Let $M = \inf_{z\in\sigma} |f(z)|$ and $M_n = \inf_{z\in\sigma} |f(z)p_n(z)|$. Since σ is closed and f is continuous it follows that $M > 0$. Clearly $\lim_{n\to\infty} M_n = M^2$. For large enough n we have $1/p_n \in C^\infty(\sigma)$. Then

$$\lim_{n\to\infty} \left\| \frac{1}{fp_n} \right\|_\infty = \lim_{n\to\infty} M_n^{-1} = M^{-2}.$$

Also, by Proposition 3.19,

$$\lim_{n\to\infty} \mathrm{var}\left(\frac{1}{fp_n}, \sigma\right) \le \lim_{n\to\infty} M_n^2 \,\mathrm{var}(p_n f, \sigma) = M^4 \,\mathrm{var}(f^2, \sigma) < \infty.$$

Then

$$\lim_{n\to\infty} \mathrm{var}\left(\frac{1}{f} - \frac{1}{p_n}, \sigma\right) = \lim_{n\to\infty} \mathrm{var}\left(\frac{p_n - f}{p_n f}, \sigma\right)$$
$$\le \lim_{n\to\infty} \mathrm{var}(p_n - f, \sigma) \left\| \frac{1}{p_n f} \right\|_\infty + \lim_{n\to\infty} \mathrm{var}\left(\frac{1}{p_n f}, \sigma\right) \|p_n - f\|_\infty = 0. \ ∎$$

In some cases it is more convenient to work with an appropriate analogue of continuous piecewise linear functions. We shall now define such an analogue and prove that this class of functions is always dense in $\mathrm{AC}(\sigma)$. We say that a finite partition $\{A_i\}_{i=1}^n$ is a *triangulation* of a rectangle $J \times K$ if

(i) for each i, A_i is a non-degenerate (topologically) closed triangle,
(ii) for all i, j with $i \neq j$, $\mathrm{int}(A_i) \cap \mathrm{int}(A_j) = \emptyset$,
(iii) $\bigcup_{i=1}^n A_i = J \times K$.

A function $F : J \times K \to \mathbb{C}$ is said to be *continuous and piecewise triangularly planar* if F is continuous and there is a triangulation $\{A_i\}_{i=1}^n$ such that for each i, $F|A_i$ is planar. The set of continuous piecewise triangularly planar functions with domain $J \times K$ is denoted $\mathrm{CTPP}(J \times K)$. It is easy to see that if $F, G \in \mathrm{CTPP}(J \times K)$ then there exists a triangulation $\{A_i\}_{i=1}^n$ such that $F + G|A_i$ is planar for all i. Hence $\mathrm{CTPP}(J \times K)$ is a vector space.

LEMMA 4.10. *The set* $\mathrm{CTPP}(J \times K)$ *is dense in* $\mathrm{AC}(J \times K)$.

Proof. This follows from the two-dimensional mean value theorem. In particular we can always approximate any polynomial by a continuous piecewise planar function in Lipschitz norm and hence in $\mathrm{BV}(J \times K)$ norm. ∎

We say that $A \subset \sigma$ is a *triangle relative to* σ if there exists $A' \subset J \times K$ such that A' is a topologically closed triangle and if $A = A' \cap \sigma$. We say $\{A_i\}_{i=1}^n$ is a *triangulation of* σ if $A_i \subset \sigma$ for all i and there exists a triangulation $\{A_i'\}_{i=1}^m$ of $J \times K$ such that $A_i = A_i' \cap \sigma$ for all $1 \le i \le n$. We say a function f is *continuous and piecewise triangularly planar relative to* σ if f is continuous and there is some triangulation $\{A_i\}_{i=1}^n$ of σ such that $f|A_i$ is planar for all i. The set of continuous and piecewise triangularly planar functions relative to σ is denoted $\mathrm{CTPP}(\sigma)$. This agrees with the previous definition of $\sigma = J \times K$. Clearly $f \in \mathrm{CTPP}(\sigma)$ if and only if there exists $F \in \mathrm{CTPP}(J \times K)$ such that $F|\sigma = f$.

LEMMA 4.11. *The set* $\mathrm{CTPP}(\sigma)$ *is dense in* $\mathrm{AC}(\sigma)$.

Proof. Suppose that $f \in \mathrm{AC}(\sigma)$ and $\varepsilon > 0$. Then there exists a polynomial p such that $\|p - f\|_{\mathrm{BV}(\sigma)} < \varepsilon/2$. Now, by Lemma 4.10 there exists $G \in \mathrm{CTPP}(J \times K)$ such that $\|G - p\|_{\mathrm{BV}(J \times K)} < \varepsilon/2$. Thus, if $g = G|\sigma$, then $g \in \mathrm{CTPP}(\sigma)$ and

$$\|f - g\|_{\mathrm{BV}(\sigma)} \le \|f - p\|_{\mathrm{BV}(\sigma)} + \|p - g\|_{\mathrm{BV}(\sigma)} < \varepsilon/2 + \|G - p\|_{\mathrm{BV}(J \times K)} < \varepsilon. \ \blacksquare$$

If $\sigma \subset \mathbb{R}$ then all Lipschitz functions are absolutely continuous. However for $\sigma \subset \mathbb{C}$ it is not necessarily true that all Lipschitz functions are in $\mathrm{AC}(\sigma)$. We show this in Example 4.13. First a technical lemma.

LEMMA 4.12. *Let* $f \in \mathrm{AC}([0,1] \times [0,1])$. *For* $\eta \in [0,1]$ *set*

$$\gamma_\eta(t) = \Pi(0 + i\eta, 1 + i\eta) \in \Gamma_L.$$

Then $\lim_{\eta \to 0^+} \mathrm{cvar}(f, \gamma_\eta) = \mathrm{cvar}(f, \gamma_0)$.

Proof. Suppose first that p is a polynomial in two variables. Then by Lemma 4.3,

$$\mathrm{cvar}(p, \gamma_\eta) = \int_0^1 \left| \frac{\partial p}{\partial x}(x, \eta) \right| dx.$$

Fix $\varepsilon > 0$. If η is small enough then for all $x \in [0,1]$,

$$\left| \frac{\partial p}{\partial x}(x, \eta) - \frac{\partial p}{\partial x}(x, 0) - \eta \, \frac{\partial^2 p}{\partial x \partial y} \, (x, 0) \right| < \varepsilon.$$

Hence

$$
\begin{aligned}
\operatorname{cvar}(p,\gamma_\eta) &= \int_0^1 \left|\frac{\partial p}{\partial x}(x,\eta)\right| dx \\
&\le \int_0^1 \left|\frac{\partial p}{\partial x}(x,0)\right| dx + \int_0^1 \eta\left|\frac{\partial^2 p}{\partial x \partial y}(x,0)\right| dx + \int_0^1 \varepsilon\, dx \\
&= \operatorname{cvar}(p,\gamma_0) + \eta\int_0^1 \left|\frac{\partial^2 p}{\partial x \partial y}(x,0)\right| dx + \varepsilon,
\end{aligned}
$$

so $\lim_{\eta\to 0^+}\operatorname{cvar}(p,\gamma_\eta) \le \operatorname{cvar}(p,\gamma_0)+\varepsilon$. Using a similar argument one can show that $\operatorname{cvar}(p,\gamma_0) \le \lim_{\eta\to 0^+}\operatorname{cvar}(p,\gamma_\eta)+\varepsilon$. Therefore $\lim_{\eta\to 0^+}\operatorname{cvar}(p,\gamma_\eta) = \operatorname{cvar}(p,\gamma_0)$.

For arbitrary $f \in \mathrm{AC}([0,1]\times[0,1])$ fix $\varepsilon > 0$ and choose $p \in \mathcal{P}_2$ such that $\|f-p\|_{\mathrm{BV}([0,1]\times[0,1])} < \varepsilon/2$. Then

$$
\lim_{\eta\to 0^+} |\operatorname{cvar}(f,\gamma_\eta) - \operatorname{cvar}(f,\gamma_0)| \le \lim_{\eta\to 0^+} |\operatorname{cvar}(p,\gamma_\eta) - \operatorname{cvar}(p,\gamma_0)| + \varepsilon = \varepsilon. \blacksquare
$$

We are now able to construct an example which shows that Lipschitz functions are not necessarily absolutely continuous. This example also shows that even though all cross sections of a function are absolutely continuous it does not necessarily follow that the function is absolutely continuous.

EXAMPLE 4.13. For each $n\in\mathbb{N}$ let $l_n = [0,1]\times\{2^{-n}\}$. Let $h_n : [0,1]\to\mathbb{R}$ be the sawtooth function with n teeth, each of height 1 and such that $h_n(0) = 0$. For each n define

$$
f\left(x, \frac{1}{2^n}\right) = \frac{h_n(x)}{2^{n+1}}.
$$

Define $f([0,1]\times\{0\}) = 0$. Then $f \in \mathrm{Lip}(\overline{\bigcup_n l_n})$ with $L(f) = 1$. Now f can be extended to a function $F : [0,1]\times[0,1]\to\mathbb{R}$ such that $l(F) = 1$. If $\gamma_{1/n}$ is defined as in Lemma 4.12 then $\operatorname{cvar}(F,\gamma_{1/n}) = L(F) = 1$ for all $n\in\mathbb{N}$. But $\operatorname{cvar}(F,\gamma_0) = 0$. Hence by Lemma 4.12, $F \notin \mathrm{AC}([0,1]\times[0,1])$. Also note that for all $\gamma\in\Gamma_L$ where $\gamma(0),\gamma(1)\in J\times K$ we have $\iota(F\circ\gamma)\in\mathrm{Lip}([0,1])\subset\mathrm{AC}([0,1])$.

5. Operator theory. We shall say that an operator $T \in B(X)$ is an $\mathrm{AC}(\sigma)$ *operator* if it admits an $\mathrm{AC}(\sigma)$ functional calculus, that is, if there exists a continuous Banach algebra homomorphism $\Psi : \mathrm{AC}(\sigma)\to B(X)$ such that $\Psi(1) = I$ and $\Psi(\lambda) = T$. It is easy to see that if T is a normal operator on a Hilbert space, or more generally, a scalar-type spectral operator, then T is an $\mathrm{AC}(\sigma(T))$ operator.

As we noted in the introduction, the theory of $\mathrm{AC}(\sigma)$ operators will be pursued more fully in [3] and [4]. There are however a few results which are worth recording here. The first is to confirm that this theory does indeed generalize the well-bounded theory. The "if" part of the next theorem follows from Lemmas 3.9 and 4.5. For the converse direction, it is obvious that every well-bounded operator is an $\mathrm{AC}(\sigma)$ operator. That one can choose $\sigma = \sigma(T)$ is shown in [2] or [3].

THEOREM 5.1. *An operator $T \in B(X)$ is well-bounded if and only if it is an $\mathrm{AC}(\sigma(T))$ operator and $\sigma(T) \subset \mathbb{R}$.*

Part of the motivation for our new definitions was to ensure that the class of $\mathrm{AC}(\sigma)$ operators is closed under affine transformations. The following is an immediate consequence of Theorem 4.1.

THEOREM 5.2. *If $T \in B(X)$ is an $\mathrm{AC}(\sigma)$ operator then for all $\alpha, \beta \in \mathbb{C}$, $\alpha T + \beta I$ is an $\mathrm{AC}(\alpha\sigma + \beta)$ operator.*

Berkson and Gillespie [7] defined an operator to be an AC operator if it admits a functional calculus for the algebra of functions which are absolutely continuous in the Hardy–Krause sense. We show in [4] that given any rectangle $J \times K$ we have $\mathrm{AC}_{\mathrm{HK}}(J \times K) \subset \mathrm{AC}(J \times K)$ and that the inclusion map is continuous. An immediate consequence is the following theorem.

THEOREM 5.3. *If $T \in B(X)$ is an $\mathrm{AC}(\sigma)$ operator then T is an* AC *operator* (*in the sense of Berkson and Gillespie*), *and hence there exist commuting well-bounded operators $A, B \in B(X)$ such that $T = A + iB$.*

The converse of this theorem is false. The example from [6] of an AC operator T such that $(1 + i)T$ is not an AC operator is also an example of an AC operator which is not an $\mathrm{AC}(\sigma)$ operator (for any σ).

One of the most important subclasses of AC operators has been the family of trigonometrically well-bounded operators. The following result is a consequence of Proposition 3.11 and the definition of being trigonometrically well-bounded [8].

THEOREM 5.4. *Every trigonometrically well-bounded operator is an $\mathrm{AC}(\mathbb{T})$ operator.*

It is true, but slightly delicate to prove, that the norm on $\mathrm{BV}(\mathbb{T})$ is equivalent to the natural one introduced in [8]. Consequently, on reflexive Banach spaces, $\mathrm{AC}(\mathbb{T})$ operators are precisely trigonometrically well-bounded operators. Details will appear in [3] and [4].

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