STUDIA MATHEMATICA 199 (2) (2010)

The norms and singular numbers of polynomials of the classical Volterra operator in $L_2(0,1)$

by

YURI LYUBICH (Haifa) and DASHDONDOG TSEDENBAYAR (Ulaanbaatar)

Abstract. The spectral problem $(s^2 I - \phi(V)^*\phi(V))f = 0$ for an arbitrary complex polynomial ϕ of the classical Volterra operator V in $L_2(0,1)$ is considered. An equivalent boundary value problem for a differential equation of order $2n$, $n = \deg(\phi)$, is constructed. In the case $\phi(z) = 1 + az$ the singular numbers are explicitly described in terms of roots of a transcendental equation, their localization and asymptotic behavior is investigated, and an explicit formula for the $\|I + aV\|_2$ is given. For all $a \neq 0$ this norm turns out to be greater than 1.

1. Introduction. For any compact linear operator A in a Hilbert space the singular numbers $s_k(A)$ are the distances from A to the set of all operators of rank less than or equal to $k-1$, $k \geq 1$. Their squares are the eigenvalues of the compact selfadjoint nonnegative operator A^*A counted according to their multiplicities (see e.g. [1]). In particular, $s_1(A) = \|A\|$. The latter has been used by Halmos [2] to calculate the L_2-norm $\|\cdot\|_2$ of the classical Volterra operator

$$(Vf)(x) = \int_0^x f(t)\,dt, \qquad 0 \leq x \leq 1.$$

Actually, the Halmos calculation yields

$$s_k(V) = \frac{2}{(2k-1)\pi} \tag{1.1}$$

for all $k \geq 1$, in particular,

$$\|V\|_2 = 2/\pi. \tag{1.2}$$

The point is that the spectral problem

$$(V^*Vf)(x) = s^2 f(x), \qquad 0 \leq x \leq 1,$$

2010 *Mathematics Subject Classification*: 47A10, 47A35, 47G10.
Key words and phrases: Volterra operator, singular numbers, boundary value problem.

DOI: 10.4064/sm199-2-3

is equivalent to the boundary value problem

$$f''(x) + \lambda f(x) = 0, \quad f'(0) = 0, \quad f(1) = 0,$$

where $\lambda = 1/s^2$. This yields $f(x) = \cos\sqrt{\lambda}x$ (under the normalization $f(0) = 1$) and $\cos\sqrt{\lambda} = 1$, which immediately implies (1.1).

A similar equivalence for the powers V^n was established by Thorpe [4]. The corresponding boundary value problem is

$$(-1)^n f^{(2n)} = \lambda f, \quad f^{(l)}(0) = f^{(n+l)}(1) = 0, \quad 0 \le l \le n-1.$$

In the present paper we generalize these results to the arbitrary complex polynomials

$$\phi(V) = \sum_{i=0}^{n} a_i V^i, \quad a_n \neq 0, \quad n \ge 1. \tag{1.3}$$

Note that the operator $\phi(V)$ is not compact if $a_0 \neq 0$, but in any case $\phi(V)^*\phi(V) = |a_0|^2 I + K$ where I is the identity operator and K is a compact self-adjoint operator. This suggests defining the *singular numbers* of $\phi(V)$ as the nonnegative square roots of the eigenvalues of $\phi(V)^*\phi(V)$. In fact, they are positive, since the operator $\phi(V)$ is injective. Indeed, it is invertible if $a_0 \neq 0$, otherwise, it is of the form $V^l\psi(V)$, $l \ge 1$, with $\psi(V)$ invertible. Actually, the singleton $\{a_0\}$ is the spectrum of $\phi(V)$.

The singular numbers of $\phi(V)$ constitute a countable set S_ϕ converging to $|a_0|$. We have

$$\sup S_\phi = \|\phi(V)\|_2 \ge |a_0|. \tag{1.4}$$

In Section 2 we consider the case $n = 1$ and obtain an explicit formula for the singular numbers, in particular, for $\|I + aV\|_2)$ in terms of roots of a transcendental equation that comes from a boundary value problem. We describe the localization of these roots in much detail. The general case $n \ge 1$ is considered in Section 3 where we construct a boundary value problem equivalent to the spectral problem in question. In Section 4 the problem of equality in (1.4) is discussed.

2. The case $n = 1$. Let $\phi(V) = a_0 I + a_1 V$, $a_1 \neq 0$. Since the case $a_0 = 0$ trivally reduces to that of [2], one can assume $a_0 \neq 0$. Without loss of generality one can set $a_0 = 1$ and then denote a_1 by a, for short. Thus, we will consider $\phi(V) = I + aV$ with $a \neq 0$.

Our spectral problem

$$(I + \overline{a}V^*)(I + aV)f = s^2 f$$

can be rewritten as

$$(s^2 - 1)f - \overline{a}V^*f - aVf - |a|^2 V^*Vf = 0. \tag{2.1}$$

We proceed from this integral equation to a differential equation by applying the operator $D = d/dx$ twice. Note that $DV = I$, while $DV^* = -I$ since

$$(V^*f)(x) = \int_x^1 f(t)\,dt, \quad 0 \le x \le 1.$$

Thus, (2.1) yields

$$(s^2-1)f' - (a-\overline{a})f + |a|^2 Vf = 0 \tag{2.2}$$

and then

$$(s^2-1)f'' - (a-\overline{a})f' + |a|^2 f = 0. \tag{2.3}$$

Now we insert $x = 0$ and $x = 1$ into (2.1) and (2.2) taking into account that

$$(Vf)_{x=0} = (V^*f)_{x=1} = 0,$$

and

$$(Vf)_{x=1} = (V^*f)_{x=0} = J_1 \equiv \int_0^1 f\,dt,$$

hence,

$$(V^*Vf)_{x=1} = 0, \quad (V^*Vf)_{x=0} = J_2 \equiv \int_0^1 (Vf)\,dt.$$

In this way we obtain four linear homogeneous equations for the six values: $f(0)$, $f'(0)$, $f(1)$, $f'(1)$, J_1, J_2. Eliminating J_1 and J_2 we get two boundary conditions for the differential equation (2.3). Let us emphasize that in [2] and [4] no integrals remain after substitution of $x = 0$ and $x = 1$ into the corresponding integral equation and its derivatives. In our case this is true only for one of the four equalities, namely, we get

$$(s^2-1)f'(0) - (a-\overline{a})f(0) = 0 \tag{2.4}$$

when putting $x = 0$ in (2.2). However,

$$(s^2-1)f(1) - aJ_1 = 0, \quad (s^2-1)f'(1) - (a-\overline{a})f(1) + |a|^2 J_1 = 0$$

for $x = 1$ in (2.1) and (2.2), respectively. Eliminating J_1 we obtain

$$(s^2-1)f'(1) + (\overline{a}s^2 - a)f(1) = 0. \tag{2.5}$$

Lemma 2.1. *For every s the integral equation* (2.1) *is equivalent to the differential equation* (2.3) *with the boundary conditions* (2.4) *and* (2.5).

Proof. We already know that each solution f to the equation (2.1) satisfies (2.3)–(2.5). In the converse direction we start with f satisfying (2.3)–(2.5) and set

$$g = ((I + \overline{a}V^*)(I + aV) - s^2 I)f.$$

We have to show that $g = 0$. To this end we note that $g'' = 0$ by (2.3), hence $g(x)$ is a linear function of x. Furthermore,

$$g'(0) = (s^2 - 1)f'(0) - (a - \overline{a})f(0),$$

hence $g'(0) = 0$ by (2.4). Thus, $g(x)$ is a constant, so

$$\overline{a}g(1) = \overline{a}g(1) + g'(1) = (s^2 - 1)f'(1) - (\overline{a}s^2 - a)f(1)$$

by (2.5). Since $a \neq 0$, we get $g(1) = 0$. ∎

The singular numbers in question are just those s for which the boundary value problem (2.3)–(2.5) has a solution $f \neq 0$. It is easy to see that $s \neq 1$. Indeed, otherwise from (2.3) it follows that

$$(a - \overline{a})f' - |a|^2 f = 0. \tag{2.6}$$

If a is real then $f = 0$ at once. If a is not real then f satisfies the linear differential equation (2.6) and, in addition, $f(0) = 0$ by (2.4). Hence, $f = 0$ again.

Now we are in a position to prove the following theorem.

Theorem 2.2. *Let $a = \alpha + i\beta$. The singular numbers of the operator $I + aV$ are given by the formula*

$$s = \sqrt{\frac{\alpha^2 + \Delta}{\alpha^2 + \beta^2}} \tag{2.7}$$

where α and β are the real and the imaginary parts of a and Δ runs over all real roots of the equation

$$\sqrt{\Delta} \cot \frac{\sqrt{\Delta}(\alpha^2 + \beta^2)}{\Delta - \beta^2} = \alpha. \tag{2.8}$$

Though the value $\sqrt{\Delta}$ is determined only up to the factor ± 1, the right-hand side of (2.8) is uniquely determined since the function $\cot(\cdot)$ is odd. Also note that if Δ is a root of the equation (2.8) then

$$\Delta \neq \beta^2, \quad \frac{\sqrt{\Delta}(\alpha^2 + \beta^2)}{\Delta - \beta^2} \neq m\pi$$

for all integers m, in particular, $\Delta \neq 0$. However, $\Delta = 0$ becomes admssible by passing to the limit as $\Delta \to 0$. The limit equality is

$$\frac{\beta^2}{\alpha^2 + \beta^2} = -\alpha,$$

or equivalently,

$$\beta^2 = -\frac{\alpha^3}{\alpha + 1} \tag{2.9}$$

where automatically $-1 < \alpha < 0$ and $\beta \neq 0$, so a is not real. Under the relation (2.9) Theorem 2.2 extends by including $\Delta = 0$ into (2.7), so we have the *special singular number*

$$s^0 = \frac{|\alpha|}{\sqrt{\alpha^2+\beta^2}} = \sqrt{1+\alpha} \tag{2.10}$$

in this situation.

For $\Delta < 0$ the equation (2.8) can be rewritten as

$$\sqrt{|\Delta|}\coth\frac{\sqrt{|\Delta|}(\alpha^2+\beta^2)}{|\Delta|+\beta^2} = -\alpha. \tag{2.11}$$

Proof of Theorem 2.2. In our current notation the equation (2.3) is

$$(s^2-1)f'' - 2i\beta f' + (\alpha^2+\beta^2)f = 0, \tag{2.12}$$

and the boundary conditions (2.4) and (2.5) are

$$(s^2-1)f'(0) - 2i\beta f(0) = 0 \tag{2.13}$$

and

$$(s^2-1)f'(1) + ((s^2-1)\alpha - (s^2+1)i\beta)f(1) = 0. \tag{2.14}$$

The characteristic equation for the differential equation (2.12) is

$$(s^2-1)r^2 - 2i\beta r + (\alpha^2+\beta^2) = 0. \tag{2.15}$$

Its roots are

$$r_1 = i\frac{\beta+\sqrt{\Delta}}{s^2-1}, \quad r_2 = i\frac{\beta-\sqrt{\Delta}}{s^2-1}, \tag{2.16}$$

where

$$\Delta = \beta^2 + (s^2-1)(\alpha^2+\beta^2). \tag{2.17}$$

The latter is equivalent to (2.7). We have to prove that (2.8) (including the equality $\Delta = 0$ in the case (2.9)) is equivalent to the nontrivial solvability of the boundary value problem (2.12)–(2.14). Let us start with $\Delta \neq 0$, i.e. $r_1 \neq r_2$.

The general solution to (2.12) is

$$f(x) = c_1e^{r_1x} + c_2e^{r_2x},$$

where c_1 and c_2 are arbitrary constants. Hence,

$$f(0) = c_1 + c_2, \qquad f'(0) = c_1r_1 + c_2r_2$$

and

$$f(1) = c_1e^{r_1} + c_2e^{r_2}, \qquad f'(1) = c_1r_1e^{r_1} + c_2r_2e^{r_2}.$$

By substitution into (2.13) and (2.14) we obtain a system of linear homogeneous equations for c_1 and c_2. This system has a nontrivial solution if and only if its determinant is equal to zero. This equality reduces to

$$((s^2-1)r_1 - \alpha - i\beta)e^{r_1} = ((s^2-1)r_2 - \alpha - i\beta)e^{r_2}$$

by an elementary calculation taking (2.15) into account. Using (2.16) we obtain

$$(\alpha - i\sqrt{\Delta})e^{r_1} = (\alpha + i\sqrt{\Delta})e^{r_2}. \tag{2.18}$$

Note that $\alpha - i\sqrt{\Delta} \neq 0$, otherwise, $\alpha^2 + \Delta = 0$, which implies $s = 0$ by (2.17). Therefore, one can rewrite (2.18) as

$$e^{r_1 - r_2} = \frac{\alpha + i\sqrt{\Delta}}{\alpha - i\sqrt{\Delta}},$$

or equivalently,

$$\frac{e^{r_1 - r_2} + 1}{e^{r_1 - r_2} - 1} = \frac{\alpha}{i\sqrt{\Delta}}.$$

This is nothing but the equation (2.8) since

$$r_1 - r_2 = \frac{2i\sqrt{\Delta}}{s^2 - 1} = \frac{2i\sqrt{\Delta}(\alpha^2 + \beta^2)}{\Delta - \beta^2}$$

by (2.16) and (2.17).

In the case $\Delta = 0$ the only root of (2.15) is

$$r = \frac{i\beta}{s^2 - 1} = -\frac{i(\alpha^2 + \beta^2)}{\beta}$$

since

$$s^2 - 1 = -\frac{\beta^2}{\alpha^2 + \beta^2}$$

by (2.17). (Note that $\beta \neq 0$ since $s \neq 1$.) The general solution to (2.12) is now of the form

$$f(x) = (c_1 + c_2 x)e^{rx}.$$

Accordingly,

$$f(0) = c_1, \quad f'(0) = c_1 r_1 + c_2 r_2$$

and

$$f(1) = c_1 e^{r_1} + c_2 e^{r_2}, \quad f'(1) = c_1 r_1 e^{r_1} + c_2 r_2 e^{r_2}.$$

It is easy to check that the determinant of the corresponding linear system for c_1 and c_2 vanishes if and only if the relation (2.9) is valid. As we know, the latter is the limit form of (2.8) as $\Delta \to 0$. ∎

Now we investigate the equation (2.8) with unknown $\Delta \neq 0$ (written as (2.11) for $\Delta < 0$). We start with $\beta = 0$. For the new unknown

$$\xi = \alpha^2/\sqrt{|\Delta|} > 0 \tag{2.19}$$

we have the equation

$$\coth \xi = -\xi/\alpha \tag{2.20}$$

if $\Delta < 0$, but

$$\cot \xi = \xi/\alpha \tag{2.21}$$

if $\Delta > 0$. By (2.7) and (2.19) the corresponding singular number is

$$s = \sqrt{1 + \frac{\varepsilon\alpha^2}{\xi^2}} \tag{2.22}$$

where ξ is a root of (2.20) or (2.21) and $\varepsilon = -1$ or 1, respectively. By these equations one can rewrite (2.22) as

$$s = 1/\cosh \xi \quad \text{or} \quad s = 1/|\cos \xi|, \tag{2.23}$$

respectively.

The equation (2.20) *has no positive roots if $\alpha > 0$, but if $\alpha < 0$ then* (2.20) *has exactly one positive root.* Indeed, in the latter case the function $\coth \xi + \xi/\alpha$ on $(0, \infty)$ monotonically decreases from $+\infty$ to $-\infty$.

In contrast, the equation (2.21) has infinitely many positive roots for any $\alpha \neq 0$: *there is exactly one root ξ_k of* (2.21) *in each interval $((k-1)\pi, k\pi)$, $k \geq 1$.* Accordingly,

$$s_k = \sqrt{1 + \frac{\alpha^2}{\xi_k^2}} = \frac{1}{|\cos \xi_k|}, \quad k \geq 1. \tag{2.24}$$

Now let $\beta \neq 0$. Then we introduce the new unknown

$$\xi = \sqrt{|\Delta|}/|\beta| > 0, \tag{2.25}$$

instead of Δ, and the new real parameters

$$\gamma = \frac{\alpha^2 + \beta^2}{|\beta|}, \quad \delta = \frac{\alpha}{|\beta|}, \tag{2.26}$$

instead of α and β. In this setting we have

$$\xi \coth \frac{\gamma\xi}{\xi^2 + 1} = -\delta \tag{2.27}$$

if $\Delta < 0$, but

$$\xi \cot \frac{\gamma\xi}{\xi^2 - 1} = \delta \tag{2.28}$$

if $\Delta > 0$. The corresponding singular number is

$$s = \sqrt{\frac{\delta^2 + \varepsilon\xi^2}{\delta^2 + 1}} \tag{2.29}$$

where ξ is a root of (2.27) or (2.28) and ε is defined as in (2.22).

Since $\gamma > 0$, *the equation* (2.27) *has no positive roots if $\delta \geq 0$.* Let $\delta < 0$, i.e. $\alpha < 0$ by (2.26). Then all positive roots of (2.27) (if they exist) are less than $|\delta|$. With this restriction (2.27) is equivalent to $g(\xi) = 0$ where

$$g(\xi) = \frac{2\gamma\xi}{\xi^2 + 1} - \log \frac{|\delta| + \xi}{|\delta| - \xi}.$$

The derivative $g'(\xi)$ is

$$\gamma(\xi^2 - 1)(\xi^2 - \delta^2) - |\delta|(\xi^2 + 1)^2,$$

up to a positive factor. This biquadratic polynomial has at most two positive roots, so the same is true of $g'(\xi)$ and hence of $g(x)$ because of Rolle's theorem and $g(0+) = 0$. Thus, *the equation* (2.27) *has at most two positive roots if* $\delta < 0$.

It remains to investigate the positive roots of the equation (2.28). With $\delta = 0$ they are

$$\xi_k^0 = \frac{\gamma + \varepsilon_k \sqrt{\gamma^2 + (2k-1)^2\pi^2}}{(2k-1)\pi} \tag{2.30}$$

where k runs over all integers and $\varepsilon_k = \operatorname{sign}(2k-1)$. It is easy to see that $\xi_{k+1}^0 < \xi_k^0$ for $k \neq 0$ but $\xi_1^0 > \xi_0^0$. Moreover, $\xi_1^0 > 1$, while $\xi_0^0 < 1$. (It is useful to note that $\xi_{-(k-1)}^0 \xi_k^0 = 1$.) Obviously, the roots ξ_k^0 and ξ_{-k}^0 tend to 1 as $k \to +\infty$.

Now let $\delta \neq 0$. Then with $\xi \neq \xi_k^0$ $(k = 0, \pm 1, \pm 2, \ldots)$, (2.28) is equivalent to $h(\xi) = 0$ where

$$h(\xi) = h_0(\xi) - \arctan\left(\frac{\xi}{\delta}\right) - q(\xi)\pi, \quad h_0(\xi) = \frac{\gamma\xi}{\xi^2 - 1}$$

and an integer coefficient $q(\xi)$ is determined by the inequality

$$-\frac{\pi}{2} < h_0(\xi) - q(\xi)\pi < \frac{\pi}{2}.$$

The function $h_0(\xi)$ monotonically decreases from 0 to $-\infty$ on the interval $(0, 1)$ and from $+\infty$ to 0 on the interval $(1, \infty)$. Since

$$h_0(\xi_k^0) = \frac{(2k-1)\pi}{2},$$

we have $q(\xi) = k$ for $\xi_{k+1}^0 < \xi < \xi_k^0$, $k \neq 0$, and $q(\xi) = 0$ for $\xi < \xi_0^0$ or $\xi > \xi_1^0$. Thus,

$$h(\xi) = h_0(\xi) - \arctan\left(\frac{\xi}{\delta}\right) - k\pi$$

on the interval $J_k = (\xi_{k+1}^0, \xi_k^0)$, $k \neq 0$, and

$$h(\xi) = h_0(\xi) - \arctan\left(\frac{\xi}{\delta}\right)$$

on the intervals $J_0 = (0, \xi_0^0)$ and $J_\infty = (\xi_1^0, \infty)$. On each of these intervals the function $h(\xi)$ is continuous. Moreover, we have the one-sided limit values

$$h(\xi_k^0-) = -\left(\frac{\pi}{2} + \arctan\left(\frac{\xi_k^0}{\delta}\right)\right) < 0, \quad h(\xi_k^0+) = \frac{\pi}{2} - \arctan\left(\frac{\xi_k^0}{\delta}\right) > 0$$

for all k, and

$$h(0+) = 0, \quad h(+\infty) = -\frac{\pi}{2}\operatorname{sign}\delta. \tag{2.31}$$

Therefore, *the equation* (2.28) *has at least one root in each interval* J_k, $k \neq 0, \infty$. For $k = \infty$ this is true if $\delta > 0$. For $k = 0$ it is true if $\delta < 0$ and $|\delta|\gamma < 1$. Indeed, in this case $h'(0) > 0$, so $h(\xi) > 0$ for small $\xi > 0$.

In general, the derivative $h'(\xi)$ is

$$\gamma(\xi^2 + 1)(\xi^2 + \delta^2) + \delta(\xi^2 - 1)^2,$$

up to a negative factor. If $\delta > 0$ then $h'(\xi) < 0$ for all $\xi > 0$. In this case the equation (2.28) has exactly one root $\xi_k \in J_k, k \neq 0$, including J_∞, while J_0 does not contain roots at all.

If $\delta < 0$ then $h'(\xi)$ has at most two positive roots. These roots can lie either in a J_k, and then (2.28) has no more than three roots there, or they belong to some different J_k and J_l, and then the number of roots in each of them does not exceed 2. *Any other interval* J_m *contains exactly one root.* The number of roots in J_0 and in J_∞ does not exceed 2 because of (2.31).

Now we denote by ξ_k a root of $h(\xi)$ in the interval J_k. The sequences $(\xi_k : k \geq 1)$ and $(\xi_{-k} : k \geq 0)$ monotonically tend to 1 from above and from below, respectively. By (2.29) with $\varepsilon = 1$ the same is true for $(s_k : k \geq 1)$ and $(s_{-k} : k \geq 0)$. All these singular numbers are greater than $|\delta|/\sqrt{\delta^2 + 1}$. This lower bound is just the special singular number s^0 (see (2.10)) if $\Delta = 0$ is a root of the equation (2.8), i.e. if (2.9) is valid. There are at most two singular numbers below this bound since they appear only if $\Delta < 0$.

THEOREM 2.3. *Let* $a = \alpha + i\beta \neq 0$. *If* $\beta = 0$, *i.e.* $a = \alpha \in \mathbb{R} \setminus \{0\}$, *then*

$$\|I + aV\|_2 = \sqrt{1 + \frac{\alpha^2}{\xi_{\min}^2}} = \frac{1}{|\cos \xi_{\min}|} \tag{2.32}$$

where $\xi_{\min}$ *is the smallest positive root of the equation* (2.21).

If $\beta \neq 0$, *then*

$$\|I + aV\|_2 = \sqrt{\frac{\delta^2 + \xi_{\max}^2}{\delta^2 + 1}} \tag{2.33}$$

where $\xi_{\max}$ *is the greatest root of the equation* (2.28) *with* δ *and* γ *defined by* (2.26).

Proof. Combine formula (1.4) with (2.24) in the case $\beta = 0$ and with (2.29) in the case $\beta \neq 0$, $\varepsilon = 1$. ∎

COROLLARY 2.4. *If $\alpha = 0$, i.e. $a = i\beta$, where $\beta \in \mathbb{R} \setminus \{0\}$, then*

$$\|I + aV\|_2 = \frac{|\beta|}{\pi} + \sqrt{1 + \frac{\beta^2}{\pi^2}}. \tag{2.34}$$

Indeed, in this case we have (2.33) with $\delta = 0$ and $\xi_{\max} = \xi_1^0$ coming from (2.30) with $\gamma = |\beta|$.

As $\beta \to \infty$, formula (2.34) leads to Halmos's formula (1.2). Similarly, (1.1) follows from (2.29) and (2.30).

COROLLARY 2.5. *The inequality*

$$\|I + aV\|_2 > 1 \tag{2.35}$$

holds for all complex $a \neq 0$.

Proof. This follows from (2.32) if $\beta = 0$ and from (2.33) if $\beta \neq 0$ since $\xi_{\max} > 1$ in either case. ∎

Another way to get the inequality (2.35) (except for $a \leq 0$) is to recall that the operator $I + aV$ is not power bounded if a is not real nonpositive (see [5]). With $a < 0$ this operator becomes power bounded [5], but its norm remains greater than 1 by Corollary 2.5.

3. The general case. For an arbitrary polynomial ϕ of degree $n \geq 1$ we proceed from the integral equation

$$(s^2 I - \phi(V)^* \phi(V)) f = 0 \tag{3.1}$$

to

$$D^{2n}(s^2 I - \phi(V)^* \phi(V)) f = 0, \tag{3.2}$$

by differentiation of order $2n$. In more detail, if $\phi(V)$ is of the form (1.3) then the equation (3.1) can be rewritten as

$$\Big((s^2 - |a_0|^2) I - \sum \overline{a}_i a_k (V^*)^i V^k\Big) f = 0$$

with the summation over $0 \leq i, k \leq m$, $i + k \geq 1$. Accordingly, (3.2) takes the form

$$(s^2 - |a_0|^2) f^{(2n)} - \sum_{j=1}^{2n} c_j f^{(2n-j)} = 0 \tag{3.3}$$

where

$$c_j = \sum_{i+k=j} (-1)^i \overline{a}_i a_k. \tag{3.4}$$

Note that $c_{2n} = (-1)^n |a_n|^2 \neq 0$, so (3.3) cannot be an identity. This is a linear homogeneous differential equation with constant coefficients. The formal leading coefficient is $s^2 - |a_0|^2$, so the order of the equation (3.3) is $2n$ if $s \neq |a_0|$. The other cofficients do not depend on s.

Now we substitute $x = 0$ and $x = 1$ into the intermediate integro-differential equations

$$D^i(s^2 I - \phi(V)^*\phi(V))f = 0, \quad 0 \le i \le 2n-1, \tag{3.5}$$

including (3.1). This yields $4n$ linear homogeneous equations for $4n$ boundary values

$$f(0), f'(0), \dots, f^{(2n-1)}(0), f(1), f'(1), \dots, f^{(2n-1)}(1)$$

and the integrals

$$((V^*)^i V^k f)(0), \quad (V^k f)(1), \quad 1 \le i, k \le n.$$

The total number of these integrals is $n(n+2)$, but each one is a linear combination of $2n$ power moments

$$M_l = \int_0^1 t^l f(t)\, dt, \quad 0 \le l \le 2n-1.$$

Indeed, from the classical formula

$$(V^k f)(x) = \frac{1}{(k-1)!} \int_0^x (x-t)^{k-1} f(t)\, dt$$

and its version

$$((V^*)^i f)(x) = \frac{1}{(i-1)!} \int_x^1 (t-x)^{i-1} f(t)\, dt$$

it follows that

$$(V^k f)(1) = \frac{1}{(k-1)!} \int_0^1 (1-t)^{k-1} f(t)\, dt$$

and

$$((V^*)^i V^k f)(0) = \frac{1}{(i-1)!(k-1)!} \int_0^1 f(t)\, dt \int_t^1 (s-t)^{k-1} s^{i-1}\, ds.$$

All these integrals are linear combinations of M_l, $0 \le l \le k-1$, and M_{i+k-1}. The orders of these moments are less than $2n$.

Now let F be the $4n$-column consisting of the values $f^{(m)}(0)$ and $f^{(m)}(1)$, $0 \le m \le 2n-1$, and let M be the $2n$-column consisting of the values M_l, $0 \le l \le 2n-1$. Then we have an equation

$$AF + BM = 0 \tag{3.6}$$

where A and B are matrices of sizes $4n \times 4n$ and $4n \times 2n$, respectively.

To eliminate M from (3.6) we take any $4n$-row w such that $wB = 0$. Then we obtain a boundary condition $(wA)F = 0$. In fact, only linearly independent systems of boundary conditions are interesting. Accordingly, we choose a basis $\{w_j : 1 \le j \le r\}$ in the space of solutions of the equation $wB = 0$ and get

$$(w_j A)F = 0, \quad 1 \le j \le r. \tag{3.7}$$

Note that $r \ge 2n$ since the vector equation $wB = 0$ can be rewritten as a system of $2n$ linear homogeneous scalar equations with $4n$ unknowns.

THEOREM 3.1. *The integral equation* (3.1) *is equivalent to the differential equation* (3.2) *with the boundary conditions* (3.7), *where* $r = 2n$ *and* $\{w_j\}$ *is any linearly independent system of* $2n$ *solutions of the equation* $wB = 0$.

Proof. We only need to prove that every solution f of the boundary value problem under consideration satisfies (3.1). In other words, we have to prove that $g = 0$ for

$$g = (s^2 I - \phi(V)^* \phi(V))f.$$

By (3.2) we have $D^{2n} g = 0$, therefore, g is a polynomial of degree less than or equal to $2n - 1$. The column G consisting of the values $g^{(m)}(0)$ and $g^{(m)}(1)$, $0 \le m \le 2n - 1$, satisfies the boundary conditions

$$w_j G = 0, \quad 1 \le j \le 2n, \tag{3.8}$$

since $G = AF + BM$ and $w_j AF = 0$ by (3.7), while $w_j B = 0$ by the choice of w_j.

The upper and the lower halves of the column G are G_0 and G_1 whose entries are $g^{(m)}(0)$ and $g^{(m)}(1)$, $0 \le m \le 2n - 1$, respectively. The Taylor expansion at $x = 0$ shows that $G_1 = CG_0$ where C is a $2n \times 2n$-matrix. The matrix C is invertible by the Taylor expansion at $x = 1$. The equations (3.8) can be rewritten as

$$(u_j + v_j C)G_0 = 0, \quad 1 \le j \le 2n, \tag{3.9}$$

where u_j and v_j are the left and the right halves of the row w_j. From (3.9) it follows that $G_0 = 0$, and then $g = 0$ since the rows $u_j + v_j C$, $1 \le j \le 2n$, are linearly independent. The latter is true since the rows w_j, $1 \le j \le 2n$, are linearly independent and the matrix C is invertible. ∎

By Theorem 3.1 the singular numbers of the operator $\phi(V)$ are just those s for which the boundary linear functionals $F \mapsto (w_j A)F$, $1 \le j \le 2n$, on the space of solutions of the differential equation (3.2) are linearly dependent. Equivalently, the latter means that

$$\det(w_j AF_i) = 0 \tag{3.10}$$

where $\{f_i : 1 \le i \le 2n\}$ is an arbitrary basis of that space (a "fundamental system" of solutions of (3.2)). Thus, *the singular numbers are just the roots of the equation* (3.10). The unknown s is contained in the matrix A and in the columns F_i corresponding to the solutions f_i, $1 \le i \le 2n$. Actually, the matrix A is lower triangular and all its diagonal entries are equal to $s^2 - |a_0|^2$. The other entries of A do not depend on s.

Theorem 3.1 also implies that *for every singular number s the eigenspace of the operator $\phi(V)^*\phi(V)$ corresponding to the eigenvalue s^2 is of dimension $\le 2n$.*

4. Appendix. An unsolved problem. According to Corollary 2.5 the only operator of L_2-norm 1 in the family $\{I + aV : a \in \mathbb{C}\}$ is I.

PROBLEM. *Describe the class $\mathcal{I}$ of functions $\phi(z)$ such that $\phi(0) = 1$ and $\|\phi(V)\|_2 = 1$.*

In this context $\phi(z)$ can be any function analytic in a neighborhood of $z = 0$, $\phi(0) = 1$. Since the spectrum of $\phi(V)$ is $\{1\}$ we have $\|\phi(V)\|_2 \ge 1$ a priori.

The class $\mathcal{I}$ is not empty: the function $\mathbf{1}(z) \equiv 1$ belongs to $\mathcal{I}$. In total, $\mathcal{I}$ is a convex multiplicative semigroup. On the other hand, if $\phi \in \mathcal{I}$ and $\phi \ne \mathbf{1}$ then $\mathbf{1}/\phi \notin \mathcal{I}$, i.e. $\|\phi(V)^{-1}\|_2 > 1$. Indeed, otherwise, $\phi(V)$ is an isometry, hence $\phi(V) = I$ by the clasical Gelfand theorem.

For $\phi \in \mathcal{I}$ the operator $\phi(V)$ is power bounded. From Theorem 1.1 of [3] it follows that *if $\phi \in \mathcal{I}\setminus\{\mathbf{1}\}$ then $\phi'(0)$ is real negative.* This necessary condition is not sufficient even for linear functions, as we already know. Moreover, we do not know any nontrivial polynomial $\phi \in \mathcal{I}$. We conjecture that *there are no such polynomials.* However, there are some functional examples.

PROPOSITION 4.1. *If all roots of a polynomial ϕ are real negative and $\phi(0) = 1$ then $\mathbf{1}/\phi \in \mathcal{I}$.*

Proof. Halmos proved (see [2, Problem 150]) that $\|(I + V)^{-1}\|_2 = 1$. This means that the function $(1 + z)^{-1}$ belongs to $\mathcal{I}$. The same proof shows that the function $(1 + az)^{-1}$ belongs to $\mathcal{I}$ for every $a > 0$. The general case is settled by the decomposition of $\phi(z)$ into factors of the form $1 + az$ where $-a$ runs over the reciprocals of the roots of ϕ. ∎

REMARK 4.2. If a is not real nonnegative then $\phi(z) = (1 + az)^{-1}$ does not belong to $\mathcal{I}$ since $\phi'(0) = -a$. In other words, $\|(I + aV)^{-1}\|_2 > 1$ for all complex a, except for $a \ge 0$.

Halmos's proof cited above is based on the inequality $\mathrm{Re}\,(Vf, f) \ge 0$ that means that the operator $-V$ is dissipative. Hence, if $a \le 0$ then $\|\exp(aV)\|_2 \le 1$, so eventually $\|\exp(aV)\|_2 = 1$, thus $\exp(az) \in \mathcal{I}$. The sufficient condition $a \le 0$ is also necessary since $a = \phi'(0)$.

Acknowledgments. The authors are grateful to Prof. Jaroslav Zemánek for useful discussions.

The authors were supported by the European Community project "Operator theory methods for differential equations" (TODEQ).

References

[1] I. C. Gohberg and M. G. Kreĭn, *Introduction to Theory of Linear Nonselfadjoint Operators*, Transl. Math. Monogr. 18, Amer. Math. Soc., Providence, RI, 1969.

[2] P. R. Halmos, *A Hilbert Space Problem Book*, Van Nostrand, Princeton, 1967.

[3] Yu. Lyubich, *The power boundedness and resolvent conditions for functions of the classical Volterra operator*, Studia Math. 196 (2010), 41–63.

[4] B. Thorpe, *The norm of powers of the indefinite integral operator in* $(0,1)$, Bull. London Math. Soc. 30 (1998), 543–548.

[5] D. Tsedenbayar, *On the power boundedness of certain Volterra operator pencils*, Studia Math. 156 (2003), 59–66.

Yuri Lyubich
Technion
Haifa 32000, Israel
E-mail: lyubich@tx.technion.ac.il

Dashdondog Tsedenbayar
Department of Mathematics
Mongolian University of Science and Technology
P.O. Box 46/520
Ulaanbaatar, Mongolia
E-mail: tsdnbr@must.edu.mn

Received January 16, 2010 (6802)