

Hilbert C^* -modules and amenable actions

by

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Abstract. We study actions of discrete groups on Hilbert C^* -modules induced from topological actions on compact Hausdorff spaces. We show non-amenability of actions of non-amenable and non-a-T-menable groups, provided there exists a quasi-invariant probability measure which is sufficiently close to being invariant.

The notion of topological amenability of group actions has found many applications in recent years, particularly in index theory. Yu proved [23] that the coarse Baum–Connes conjecture and the Novikov conjecture hold for groups which satisfy property A, a weak version of amenability. Property A turned out to be equivalent to existence of a topologically amenable action on some compact space [13] and to exactness of the reduced group C^* -algebra $C_r^*(G)$ [11, 19]. Because of the interest of finding counterexamples to the above conjectures, it is natural to study conditions which would imply non-amenability of topological actions.

Given a topological action of a non-amenable group on a compact space, the existence of a finite invariant probability measure implies that the action is not topologically amenable (see Definition 1 and the remarks following it). However, apart from this fact there are practically no results which would give sufficient conditions for non-amenability of an action unless one assumes the existence of an invariant probability measure for the action. In this paper we study the situation in which we are given a topological action on a compact space X and a probability measure ν such that the action of G preserves the measure class. This means the translate $g^*\nu$ and ν are absolutely continuous with respect to each other and, in particular, the Radon–Nikodym derivatives $dg^*\nu/d\nu$ are defined almost everywhere for every $g \in G$. The general idea is that if there is a probability measure for the action which is sufficiently close to being invariant, then we can still prove non-amenability of actions using this probability measure.

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Our first result is that if the Radon–Nikodym derivatives of the translated measures satisfy some global integrability conditions, then a topologically amenable action gives rise to a proper, affine isometric action on a Hilbert space. The latter property, known as a-T-menability or the Haagerup property, was defined by Gromov [10]. As a consequence we get our first result, namely that for groups which do not admit such actions, e.g. groups with property (T), our condition on the Radon–Nikodym derivatives implies that the action of G cannot be topologically amenable.

Our second result is that if a non-amenable group G acts via measure class preserving homeomorphisms and the probability measure satisfies a certain metric condition then the action is not amenable. The condition is expressed in terms of an inequality between the bottom of the positive spectrum of the discrete Laplacian on G and the average Hellinger distance between ν and its translates by generators. The Hellinger distance is a bounded metric on the space of probability measures which quantifies how far the probability measure is from being invariant. In the latter case the distance between ν and its translates is always zero. In the last section we discuss some examples and applications.

The main tool that we use is the fact that the action of a group G on a compact space X gives a linear representation of G into the group of non-adjointable, norm-bounded, linear isometries of the Hilbert C^* -module $\ell_2(G) \otimes C(X)$. If additionally we equip X with a quasi-invariant probability measure ν , then we can use the larger module $\ell_2(G) \otimes L_\infty(X, \nu)$. The fact that G preserves the class of the probability measure ν allows us to overcome the non-adjointability of the above isometric representation and to apply Hilbert space techniques to analyze the action and related representations.

1. Hilbert C^* -modules and unitary representations. Let G be a group generated by a finite, symmetric set S (i.e., $S = S^{-1}$) and let $|\cdot| : G \rightarrow \mathbb{R}$ denote the associated word length function. The word length metric on G is the left-invariant metric $d(g, h) = |g^{-1}h|$. Let X be a compact, Hausdorff space equipped with an action of G by homeomorphisms, $g \mapsto \Phi_g$. We denote the induced action of G on $f \in C(X)$ by automorphisms by

$$g * f(x) = f(\Phi_{g^{-1}}(x)),$$

where $f \in C(X)$. The group G also has a natural action on itself by left translations which induces the left regular representation denoted by

$$g \cdot \xi_h = \xi_{g^{-1}h}$$

for $\xi \in \ell_2(G)$.

1.1. The G -regular representation on Hilbert C^* -modules. For a group G we will consider the following linear representations on a Hilbert

C^* -module. Let the action of G on a compact topological space X be given. Consider the linear space

$$\mathbb{F} = \{\xi : G \rightarrow C(X) : \xi_g = 0 \text{ for all but finitely many } g\}.$$

Equip \mathbb{F} with the inner product $\langle \cdot, \cdot \rangle_{C(X)} : \mathbb{F} \rightarrow C(X)$ given by taking the regular scalar product defined for $x \in X$,

$$\langle \xi, \eta \rangle_{C(X)}(x) = \sum_{g \in G} \xi_g(x) \eta_g(x).$$

Finally, complete the resulting space in the norm

$$\|v\|_{\ell_2(G) \otimes C(X)} = \|\langle v, v \rangle_{C(X)}\|^{1/2}.$$

The resulting space is a Hilbert C^* -module $\ell_2(G) \otimes C(X)$. An analogous construction can also be done after replacing $C(X)$ with $L_\infty(X, \nu)$ for a probability measure ν . A standard reference on this material is Lance's book [15].

Given a Hilbert C^* -module \mathcal{E} , denote by $\text{Iso}(\mathcal{E})$ the group of linear isomorphisms which preserve the norm but which are not necessarily adjointable. Define a representation $L : G \rightarrow \text{Iso}(\ell_2(G) \otimes C(X))$ by setting

$$(L_g \xi)_h(x) = \xi_{g^{-1}h}(\Phi_{g^{-1}}(x))$$

for all $\xi \in \mathbb{F}$, $g, h \in G$ and $x \in X$, and extend to linear operators on $\ell_2(G) \otimes C(X)$. We abbreviate $L_g \xi = g * g \cdot \xi$. The action L_g is the diagonal action on $\ell_2(G) \otimes C(X)$. (Note that the order of applying g does not matter, since the actions \cdot and $*$ commute.) This representation satisfies

$$(1.1) \quad \langle L_g \xi, L_g \eta \rangle_{C(X)} = g * \langle g \cdot \xi, g \cdot \eta \rangle_{C(X)}$$

for all $\xi, \eta \in \ell_2(G) \otimes C(X)$ and $g \in G$. Note that the operators L_g are linear and bounded in norm but are not adjointable operators on the Hilbert C^* -module. However, we can use them to construct a unitary representation of G .

1.2. Unitary representations induced by G -unitary representations. Consider the Hilbert C^* -module $\ell_2(G) \otimes L_\infty(X, \nu)$. The module $\ell_2(G) \otimes C(X)$ is a submodule of $\ell_2(G) \otimes L_\infty(X, \nu)$ and the above representation extends. We introduce the following scalar product on $\ell_2(G) \otimes L_\infty(X, \nu)$:

$$(1.2) \quad \langle \xi, \eta \rangle = \int_X \langle \xi, \eta \rangle_{L_\infty(X, \nu)}(x) \, d\nu,$$

where $\langle \xi, \eta \rangle_{L_\infty(X, \nu)} \in L_\infty(X, \nu)$ is defined analogously to $\langle \xi, \eta \rangle_{C(X)} \in C(X)$. This turns the space $\ell_2(G) \otimes L_\infty(X, \nu)$ into a pre-Hilbert space and we obtain a Hilbert space \mathcal{H} by completion. Denote

$$\rho_g = \frac{dg^* \nu}{d\nu},$$

so that

$$\int_X g * f \rho_g \, d\nu = \int_X f \, d\nu$$

for every measurable function $f : X \rightarrow \mathbb{R}$. A standing assumption in this paper is that the Radon–Nikodym derivatives ρ_s , $s \in S$, exist and are elements of $L_\infty(X, \nu)$. Define

$$\pi_g = \rho_g^{1/2} L_g.$$

Since the Radon–Nikodym derivatives are elements of $L_\infty(X, \nu)$, each π_g is a bounded linear operator on the Hilbert module $\ell_2(G) \otimes L_\infty(X, \nu)$. Moreover, the Radon–Nikodym derivatives satisfy the cocycle condition

$$(1.3) \quad \rho_{gh} = \rho_g g * \rho_h,$$

which guarantees that $\pi_{gh} = \pi_g \pi_h$. Furthermore, one can easily check that

$$\langle \pi_g \xi, \pi_g \eta \rangle = \langle \xi, \eta \rangle.$$

Thus the extension of each π_g to \mathcal{H} (also denoted π_g) is a unitary operator and we obtain a unitary representation π of G on \mathcal{H} .

Unitary representations as above are often used in the context of measurable cocycles (see for instance [26]).

1.3. Topologically amenable actions. Topological amenability of homeomorphic actions on compact spaces was defined in [1] and was modeled on Zimmer’s definition of measurable amenable actions [24].

DEFINITION 1. Let X be a compact topological space on which G acts by homeomorphisms. The action is topologically amenable if for every $\varepsilon > 0$ there exists $\xi \in \mathbb{F}$ such that

- (a) $\xi_g \geq 0$ for every $g \in G$,
- (b) $\langle \xi, \xi \rangle_{C(X)} = 1_X$,
- (c) $\sup_{x \in X} \left(1 - \frac{1}{\#S} \sum_{s \in S} \langle \xi, L_s \xi \rangle_{C(X)}(x) \right) \leq \varepsilon$.

Amenability of an action of G does not depend on the choice of the (finite) set of generators since we can express the new generators as finite products of the old generators. If X is a single point, then the definition reduces to that of amenability of G . Another way to phrase amenability of an action is to say that the groupoid of the action of G on X is amenable (see [1]). This condition can also be rephrased in terms of isoperimetric inequalities with coefficients in a G - C^* -algebra [18].

It was proved in [14] that for measurably amenable ergodic actions the representation π is weakly contained in the regular representation.

It is well-known that if there exists a G -invariant mean, that is, a continuous linear positive and G -invariant functional on $C(X)$ (which, by the

Riesz representation theorem, corresponds to an invariant probability measure on X), then the action of G on X is amenable if and only if G is amenable. Given such a mean, an appropriate averaging procedure applied to the functions $\xi : G \rightarrow C(X)$ as in Definition 1 gives corresponding functions $\tilde{\xi} : G \rightarrow \mathbb{R}$ which satisfy $\|\tilde{\xi} - s \cdot \tilde{\xi}\| \rightarrow 0$.

2. Amenable actions and a-T-menability. In this section we will give conditions on the Radon–Nikodym derivatives which will imply the non-amenability of actions by groups not having the Haagerup property.

2.1. Affine isometric actions. An affine isometric action of a group G on a Banach space E is given by

$$A_g v = \pi_g v + b_g,$$

where $\pi : G \rightarrow \text{Iso}(E)$ is a representation in the linear isometry group of E and $b : G \rightarrow E$ satisfies the cocycle condition

$$b_{gh} = \pi_g b_h + b_g$$

for $g, h \in G$. The action is called *metrically proper* if for every $v \in E$ we have $\lim_{|g| \rightarrow \infty} \|A_g v\| = \infty$, which is equivalent to

$$\lim_{|g| \rightarrow \infty} \|b_g\| = \infty.$$

DEFINITION 2 ([10]). A group which admits a metrically proper affine isometric action on a Hilbert space is said to be *a-T-menable* or to have the *Haagerup property*.

See [5] for a detailed account of a-T-menability. Let

$$\underline{\rho}(x) = \inf_{g \in G} \rho_g(x), \quad \bar{\rho}(x) = \sup_{g \in G} \rho_g(x).$$

Both $\underline{\rho}$ and $\bar{\rho}$ are ν -measurable since G is countable and we have $\underline{\rho}(x) \leq 1 \leq \bar{\rho}(x)$ for all $x \in X$. Thus $\underline{\rho}$ is automatically an element of $L_\infty(X, \nu)$.

THEOREM 3. *Let G be a finitely generated group. Assume that G acts by homeomorphisms on a compact Hausdorff space X and that there is a probability measure ν on X such that at least one of the following conditions holds:*

- (1) $\bar{\rho} \in L_1(X, \nu)$,
- (2) $\int_X \underline{\rho}(x) d\nu > 0$.

If the action is topologically amenable then the group admits a proper affine isometric action on a Hilbert space.

If the probability measure ν is invariant, then $\rho_g = 1_X$ for every $g \in G$, and the above conditions are trivially satisfied. In particular, we get

COROLLARY 4. *With the conditions of the above theorem, if G is not a - T -menable then the action is not amenable.*

Examples of groups which are not a - T -menable include groups which have property (T) or relative property (T). See the monographs [5] and [3].

Boundedness conditions related to (1), but stronger, were studied by Greenleaf [9] and Feldman and Moore [6]. They showed that if the Radon–Nikodym derivatives are globally bounded, which in our case translates to $\bar{\rho} \in L_\infty(X, \nu)$, then there exists an equivalent invariant probability measure. Later, Zimmer [25] showed that if $\sup_{g \in G} \rho_g(x) < \infty$ for almost every $x \in X$ and, in addition, the action is ergodic, then again there is an equivalent invariant probability measure. For general actions it is not known whether the conditions in Theorem 3 imply the existence of an equivalent invariant probability measure. Condition (2), however, has not been studied in this context. Note that by the cocycle property we have

$$\underline{\rho}(x) > 0 \Leftrightarrow \sup_{g \in G} g^{-1} * \rho_g(x) < \infty.$$

The next statement shows that the existence of an amenable action on a space gives an affine isometric action on a Hilbert module, which is in addition assumed to be metrically proper in a certain stronger sense.

PROPOSITION 5. *Let G be a finitely generated group acting amenably on a compact space X . Then G admits an affine isometric action on a Hilbert module \mathcal{E} over $C(X)$ with a cocycle $b : G \rightarrow \mathcal{E}$ such that the following functional inequality holds:*

$$\phi(|g|)1_X \leq \langle b_g, b_g \rangle_{C(X)} \leq K|g|^2 1_X$$

for some $K > 0$ and some non-decreasing $\phi : [0, \infty) \rightarrow [0, \infty)$ satisfying $\lim_{t \rightarrow \infty} \phi(t) = \infty$.

Proof. We use the cocycle construction as in [2]. Let ξ_n be as in Definition 1 with $\varepsilon = 1/n$. Let $R_n \in \mathbb{R}$ be such that $\langle \xi_n, L_g \xi_n \rangle_{C(X)} = 0$ whenever $|g| \geq R_n$. Take the Hilbert module

$$\mathcal{E} = \bigoplus_{n \in \mathbb{N}} \ell_2(G) \otimes C(X)$$

and define the representation $T : G \rightarrow \text{Iso}(\mathcal{E})$ by

$$T_g = \bigoplus_n L_g.$$

Then define a function $b : G \rightarrow \mathcal{E}$ by

$$b_g = \bigoplus_{n \in \mathbb{N}} L_g \xi_n - \xi_n.$$

One can easily check that b_g is a cocycle for T ; we will now estimate its norm. For any generator $s \in S$,

$$\langle b_s, b_s \rangle_{C(X)} = \sum_{n \in \mathbb{N}} \langle L_s \xi_n - \xi_n, L_s \xi_n - \xi_n \rangle_{C(X)} \leq \sum_{n \in \mathbb{N}} \frac{1}{n^2} 1_X.$$

Thus letting $K = \sum_{n \in \mathbb{N}} n^{-2}$ we obtain the upper bound for $g \in S$. The bound for general $g \in G$ follows by writing g as a word in generators and applying the upper bound for each of these.

Since each ξ_n is finitely supported we have

$$\langle L_g \xi_n - \xi_n, L_g \xi_n - \xi_n \rangle_{C(X)} = 2 \cdot 1_X,$$

whenever $|g| \geq R_n$. For $g \in G$ let $\phi(|g|)$ be the largest n for which $|g| \geq R_n$. Then we have

$$\langle b_g, b_g \rangle_{C(X)} = \sum_{n \in \mathbb{N}} \langle L_g \xi_n - \xi_n, L_g \xi_n - \xi_n \rangle_{C(X)} \geq 2 \sum_{n=1}^{\phi(|g|)} 1_X = 2\phi(|g|)1_X.$$

It is not hard to see that $\lim_{t \rightarrow \infty} \phi(t) = \infty$ and thus we obtain the lower bound 2ϕ . ■

A crucial property of $\bar{\rho}$ and $\underline{\rho}$ is the following invariance.

LEMMA 6. *The following identities hold for any $g \in G$:*

- (1) $\rho_g^{1/2}(g * \bar{\rho}^{1/2}) = \bar{\rho}^{1/2}$,
- (2) $\rho_g^{1/2}(g * \underline{\rho}^{1/2}) = \underline{\rho}^{1/2}$.

Proof. We will prove (1); (2) is completely analogous. For any fixed $g \in G$ we have

$$\rho_g(x)^{1/2} g * \bar{\rho}(x)^{1/2} = \sup_{h \in G} \rho_g(x)^{1/2} g * \rho_h(x)^{1/2} = \sup_{h \in G} \rho_{gh}(x)^{1/2} = \bar{\rho}(x)^{1/2}$$

for all $x \in X$. ■

Proof of Theorem 3. First we prove the assertion under the assumption (2). Let $b : G \rightarrow \bigoplus \ell_2(G) \otimes C(X)$ be a cocycle for the action as in Proposition 5. It can also be viewed as a cocycle $b : G \rightarrow \bigoplus \ell_2(G) \otimes L_\infty(X, \nu)$ for the same representation, viewed now as a representation on $\bigoplus \ell_2(G) \otimes L_\infty(X, \nu)$. Define a function $\underline{b} : G \rightarrow \bigoplus \ell_2(X) \otimes L_\infty(X, \nu)$ by

$$\underline{b}_g = \underline{\rho}^{1/2} b_g.$$

Then \underline{b} is a cocycle for the unitary representation $U_g = \bigoplus \pi_g$ on the Hilbert space \mathcal{H} defined via the scalar product $\langle \bigoplus \xi_n, \bigoplus \eta_n \rangle = \sum \langle \xi_n, \eta_n \rangle$, where the summands are scalar products defined by equation (1.2). Indeed, we have

$$U_g = \bigoplus \rho_g^{1/2} L_g = \rho_g^{1/2} T_g$$

and, by Lemma 6,

$$\begin{aligned} U_g \underline{b}_h + \underline{b}_g &= \rho_g^{1/2} T_g \underline{\rho}^{1/2} b_h + \underline{\rho}^{1/2} b_g = \rho_g^{1/2} (g * \underline{\rho}^{1/2}) T_g b_h + \underline{\rho}^{1/2} b_g \\ &= \underline{\rho}^{1/2} (T_g b_h + b_g) = \underline{b}_{gh}. \end{aligned}$$

We now have

$$\|\underline{b}_g\|^2 = \int_X \langle \underline{b}_g, \underline{b}_g \rangle(x) \, d\nu = \int_X \underline{\rho}(x) \langle b_g, b_g \rangle(x) \, d\nu.$$

Applying the functional inequalities from Proposition 5 we obtain

$$\phi(|g|) \left(\int_X \underline{\rho}(x) \, d\nu \right) \leq \|\underline{b}_g\|^2 \leq K|g|^2 \left(\int_X \underline{\rho}(x) \, d\nu \right).$$

Thus if $\int_X \underline{\rho}(x) \, d\nu = \underline{C} > 0$ then the affine isometric action

$$\underline{A}_g v = U_g v + \underline{b}_g$$

on \mathcal{H} is well-defined and metrically proper since $\underline{C}\phi(t) \rightarrow \infty$ as $t \rightarrow \infty$.

To prove the assertion assuming (1), we take $\bar{b} = \bar{\rho}^{1/2} b$. Similarly, \bar{b} is a cocycle for π . In this case we need an additional argument. This is because given a vector $v \in \ell_2(G) \otimes C(X)$, the vector $\bar{\rho}^{1/2} b_g$ will not be an element of the Hilbert module $\ell_2(G) \otimes L_\infty(X, \nu)$ unless $\bar{\rho}$ is bounded (which is exactly what we are trying to avoid). However, if $\bar{\rho} \in L_1(X, \nu)$ then $\bar{\rho}^{1/2} b$ is an element of a Hilbert space $\bigoplus \ell_2(G) \otimes L_2(X, \nu)$. Repeating the above argument for the cocycle \bar{b} we obtain

$$\phi(|g|) \left(\int_X \bar{\rho}(x) \, d\nu \right) \leq \|\bar{b}_g\|^2 \leq K|g| \left(\int_X \bar{\rho}(x) \, d\nu \right).$$

If $\int_X \bar{\rho}(x) \, d\nu < \infty$ then the isometric affine action

$$\bar{A}_g v = U_g v + \bar{b}_g$$

is well-defined and metrically proper. ■

3. Actions of non-amenable groups. In this section we will give a different condition for the non-amenability of an action of a non-amenable group. As mentioned earlier, if there is an invariant probability measure for an action of such a group, it follows easily that the action is not topologically amenable. However, we are interested in the situation in which the probability measure is only quasi-invariant. Similar ideas were used in [17], but with different motivations.

3.1. The Hellinger distance for probability measures. Given probability measures μ_1 and μ_2 , both absolutely continuous with respect to the probability measure ν on X , we consider the formula

$$H(\mu_1, \mu_2) = \left(\frac{1}{2} \int \left(\sqrt{\frac{d\mu_1}{d\nu}} - \sqrt{\frac{d\mu_2}{d\nu}} \right)^2 \, d\nu \right)^{1/2}.$$

The quantity H does not depend on the choice of the dominating measure ν and is known as the *Hellinger distance* between probability distributions [21, 22]. We can also write

$$H(\mu_1, \mu_2) = (1 - A(\mu_1, \mu_2))^{1/2},$$

where the quantity

$$A(\mu_1, \mu_2) = \int_X \sqrt{\frac{d\mu_1}{d\nu} \frac{d\mu_2}{d\nu}} d\nu$$

is referred to as the *Hellinger affinity*. The Fubini theorem applied to A gives one of the fundamental properties of the Hellinger metric, namely its behavior with respect to product measures. The Hellinger metric satisfies the following inequalities [21, p. 61] with respect to the L_1 -metric:

$$H(\mu_1, \mu_2)^2 \leq \|\mu_1 - \mu_2\|_{L_1} \leq H(\mu_1, \mu_2).$$

The Hellinger distance is used in asymptotic statistics and in quantum mechanics (see [21, 22]). Note that $H(\mu_1, \mu_2) = 0$ if and only if $\mu_1 = \mu_2$, and $H(\mu_1, \mu_2) = 1$ if and only if μ_1 and μ_2 are singular.

3.2. Spectrum of the Laplacian. Let G be an infinite group generated by a finite set S . The bottom of the spectrum of the discrete Laplace operator on the Cayley graph $X = (V, E) = \mathcal{G}(G, S)$ is defined via the variational expression

$$\lambda_1 = \inf_{f \in \ell_2(G)} \frac{\langle df, df \rangle}{\langle f, f \rangle} = \inf_{f \in \ell_2(G)} \frac{\sum_{s \in S, g \in G} |f_g - f_{s^{-1}g}|^2}{\sum_{g \in G} |f_g|^2},$$

where $d : \ell_2(V) \rightarrow \ell_2(E)$ is defined by $df(x, y) = f(y) - f(x)$ for an edge $(x, y) \in E$. The group G is amenable if $\lambda_1 = 0$ for any Cayley graph of G , and if $\lambda_1 > 0$, then it gives a sort of measure of how non-amenable G is. The constant λ_1 is closely related to the *Cheeger constant* of G , defined by

$$h = \inf \left\{ \frac{\#\partial F}{\#F} : F \subset G \text{ is finite} \right\}.$$

In particular, $h > 0$ if and only if $\lambda_1 > 0$. See e.g. [4, 16] for background.

THEOREM 7. *Let G be a non-amenable group generated by a finite set S and X be a compact Hausdorff space equipped with a probability measure ν . Let G act on X by homeomorphisms which preserve the measure class of ν . If*

$$(3.1) \quad \frac{1}{\#S} \sum_{s \in S} H(\nu, s^* \nu)^2 < \frac{\lambda_1}{2},$$

then the action of G on X is not amenable.

Proof. Let

$$\beta = \frac{1}{\#S} \sum_{s \in S} \int_X \rho_s(x)^{1/2} d\nu.$$

Then $\frac{1}{\#S} \sum_{s \in S} H(\nu, s^*\nu)^2 = 1 - \beta$. Assume the action of G on X is amenable and consider ξ for the given $\varepsilon > 0$ as in Definition 1. Then ξ satisfies

$$\langle \xi, \xi \rangle = \int_X \langle \xi, \xi \rangle_{C(X)}(x) d\nu = \int_X 1 d\nu = 1.$$

Moreover, since the Radon–Nikodym derivatives are positive, we have

$$\begin{aligned} \frac{1}{\#S} \sum_{s \in S} \langle \xi, \pi_s \xi \rangle &= \frac{1}{\#S} \sum_{s \in S} \int_X \rho_s(x)^{1/2} \langle \xi, L_s \xi \rangle_{C(X)}(x) d\nu \\ &\geq \frac{1}{\#S} \sum_{s \in S} \int_X \rho_s(x)^{1/2} (1 - \varepsilon) d\nu \geq \beta(1 - \varepsilon). \end{aligned}$$

Since each such ξ is finitely supported, for each ξ there is an $R > 0$ such that $\langle L_g \xi, \xi \rangle_{C(X)} = 0$ for all $g \in G$ satisfying $|g| \geq R$. Consequently, for $|g| \geq R$ we have

$$\langle \pi_g \xi, \xi \rangle = \int_X \rho_g(x)^{1/2} \langle \xi_x, L_g \xi_x \rangle_{C(X)}(x) d\nu = 0.$$

If we set

$$\psi_g = \langle \pi_g \xi, \xi \rangle,$$

then ψ is a finitely supported positive definite function on G . Since ψ is finitely supported, it defines a bounded convolution operator T on $\ell_2(G)$ and thus an element of $C_r^*(G)$, the reduced C^* -algebra of G . By the positive definiteness of ψ , the operator T is positive and there exists a square root $Q \in C_r^*(G)$ such that

$$Q^*Q = T.$$

We now define $\eta \in \ell_2(G)$ by setting

$$\eta_g = (Q1_e)g,$$

where 1_e is the point mass at e . Thus η is the column labeled by e in the matrix representation of Q . By the definition of η we have

$$\langle \eta, g \cdot \eta \rangle = \psi_g.$$

Here $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\ell_2(G)$. We conclude that $\|\eta\| = 1$ and

$$\frac{1}{\#S} \sum_{s \in S} \langle \eta, s \cdot \eta \rangle = \frac{1}{\#S} \sum_{s \in S} \psi(s) = \frac{1}{\#S} \sum_{s \in S} \langle \pi_s \xi, \xi \rangle \geq \beta(1 - \varepsilon).$$

Together with the definition of the isoperimetric constant λ_1 , we obtain the inequality

$$\lambda_1 \leq \langle d\eta, d\eta \rangle = \frac{2}{\#S} \sum_{s \in S} (1 - \langle \eta, s \cdot \eta \rangle) \leq 2(1 - \beta(1 - \varepsilon))$$

for every $\varepsilon > 0$, and consequently $\lambda_1 \leq 2(1 - \beta)$, which is a contradiction. ■

4. Concluding remarks

4.1. Actions which are close to isometric actions. We can study the question of amenability of actions in a setting where we require an action to be close to an isometric action. See for example [7] for such a study in the context of rigidity. Given an action of a group G on a compact manifold, if there exists a subset U of positive volume on which the action of G distorts the volume by a uniformly small amount, then Theorem 3 applies. To keep calculations simple we consider the case of the circle.

Denote by S^1 the circle and by $\text{Diff}_+^1(S^1)$ the group of orientation preserving C^1 -diffeomorphisms of S^1 . Consider the “distance at x ” given by restricting the r -uniform distance to $x \in S^1$:

$$d_x(\varphi, \phi) = d_{S^1}(\varphi(x), \phi(x)) + |D\varphi(x) - D\phi(x)|,$$

where $\varphi, \phi \in \text{Diff}_+^1(S^1)$ and D denotes the derivative. Assume that G acts on S^1 by diffeomorphisms, with the action given by a homomorphism $\varphi : G \rightarrow \text{Diff}_+^1(S^1)$. Assume also that G is not a-T-menale. If there exists $U \subseteq S^1$ and an isometric action $\phi : G \rightarrow \text{Diff}_+^1(S^1)$ such that $\sup_{x \in U} d_x(\varphi_g, \phi_g) \leq C < 1$ for all $g \in G$, then the action φ_g is not topologically amenable.

Indeed, in that case, $|1 - D\varphi_g(x)| = |D\phi_g(x) - D\varphi_g(x)| \leq C < 1$, since ϕ is an isometry. This implies $\inf_{g \in G} D\varphi_g(x) > 0$. Since $D\varphi_g = \rho_g$ for every $x \in U$, the claim follows from Theorem 3. The above discussion generalizes easily to piecewise smooth homeomorphisms.

4.2. Non-amenable actions of the free group on S^1 . A similar fact holds when we consider a non-amenable group. In that case we can restrict our attention to the generators but we have to compare the distances on the whole circle. Let G be a non-amenable, finitely generated group acting on S^1 by C^1 -diffeomorphisms. If the generators of G are sufficiently close to isometries in the sense that

$$1 - \frac{1}{\#S} \sum_{s \in S} \int_X \sqrt{|D\varphi_s|} \, d\nu < \frac{\lambda_1}{2},$$

then the action is not topologically amenable.

An explicit example can be constructed as follows. Introduce the C^1 -topology on $\text{Diff}_+^1(S^1)$ by the metric

$$d(\varphi, \phi) = \sup_{x \in S^1} d_{S^1}(\varphi(x), \phi(x)) + \sup_{x \in S^1} |D\varphi(x) - D\phi(x)|.$$

The C^1 -topology turns $\text{Diff}_+^1(S^1)$ into a Baire space [20]. Adapting the transversality argument from [8, Proposition 4.5] we see that any generic (in the sense of Baire's category theorem) pair of diffeomorphisms in $\text{Diff}_+^1(S^1)$ generates a free group. Consider any pair of isometries $(i_1, i_2) \in \text{Diff}_+^1(S^1) \times \text{Diff}_+^1(S^1)$. Arbitrarily close to the pair (i_1, i_2) there exists a pair (q_1, q_2) which generates a free group. Clearly, $d(q_k, i_k) \leq \varepsilon$ implies

$$\sup_{x \in S^1} |Dq_k(x) - 1| \leq \varepsilon.$$

However, in this case the Dq_k , $k = 1, 2$, are the Radon–Nikodym derivatives and thus we conclude that they satisfy the conditions of Theorem 7 if ε is sufficiently small.

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