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Best possible sufficient conditions for the Fourier transform to satisfy the Lipschitz or Zygmund condition

by

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Abstract. We consider complex-valued functions $f \in L^1(\mathbb{R})$, and prove sufficient conditions in terms of f to ensure that the Fourier transform $\hat{f}$ belongs to one of the Lipschitz classes $\mathrm{Lip}(\alpha)$ and $\mathrm{lip}(\alpha)$ for some $0 < \alpha \leq 1$, or to one of the Zygmund classes $\mathrm{Zyg}(\alpha)$ and $\mathrm{zyg}(\alpha)$ for some $0 < \alpha \leq 2$. These sufficient conditions are best possible in the sense that they are also necessary in the case of real-valued functions f for which either $xf(x) \geq 0$ or $f(x) \geq 0$ almost everywhere.

1. Introduction. We consider complex-valued functions $f : \mathbb{R} \to \mathbb{C}$ which are integrable in Lebesgue's sense over $\mathbb{R} := (-\infty, \infty)$, in symbols: $f \in L^1(\mathbb{R})$. As is well known, the Fourier transform of f defined by

$$\hat{f}(t) := \frac{1}{2\pi} \int_{\mathbb{R}} f(x) e^{-itx}\, dx, \quad t \in \mathbb{R}, \tag{1.1}$$

is a continuous function and $\hat{f}(t) \to 0$ as $|t| \to \infty$. For more information see, e.g., [2, Chapter I].

We recall that $\hat{f}$ is said to satisfy the *Lipschitz condition of order* $\alpha > 0$, in symbols: $\hat{f} \in \mathrm{Lip}(\alpha)$, if

$$|\hat{f}(t+h) - \hat{f}(t)| \leq Ch^\alpha \quad \text{for all } t \in \mathbb{R} \text{ and } h > 0, \tag{1.2}$$

where the constant C does not depend on t or h. Furthermore, $\hat{f}$ is said to belong to the *little Lipschitz class* $\mathrm{lip}(\alpha)$ for some $\alpha > 0$ if

$$\lim_{h\to 0} h^{-\alpha}[\hat{f}(t+h) - \hat{f}(t)] = 0 \quad \text{uniformly in } t \in \mathbb{R}.$$

Since $\hat{f}$ is bounded on $\mathbb{R}$ and vanishes at $\pm\infty$, it is enough to require the fulfillment of (1.2) for $0 < h \leq 1$.

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We recall that the Fourier transform $\hat{f}$ is said to satisfy the *Zygmund condition of order* $\alpha > 0$, in symbols: $\hat{f} \in \mathrm{Zyg}(\alpha)$, if

$$|\hat{f}(t+h) - 2\hat{f}(t) + \hat{f}(t-h)| \le Ch^\alpha \quad \text{for all } t \in \mathbb{R} \text{ and } h > 0, \tag{1.3}$$

where the constant C does not depend on t or h. Furthermore, $\hat{f}$ is said to belong to the *little Zygmund class* $\mathrm{zyg}(\alpha)$ for some $\alpha > 0$ if

$$\lim_{h\to 0} h^{-\alpha}[\hat{f}(t+h) - 2\hat{f}(t) + \hat{f}(t-h)] = 0 \quad \text{uniformly in } t \in \mathbb{R}.$$

Again, it is enough to require the fulfillment of (1.3) for $0 < h \le 1$.

It is well known (see, e.g., [1, Chapter 2] or [3, Chapter 2, §3]) that if $\hat{f} \in \mathrm{lip}(1)$, in particular if $\hat{f} \in \mathrm{Lip}(\alpha)$ for some $\alpha > 1$, then $\hat{f} \equiv 0$. Furthermore, if $\hat{f} \in \mathrm{zyg}(2)$, in particular if $\hat{f} \in \mathrm{Zyg}(\alpha)$ for some $\alpha > 2$, then $\hat{f} \equiv 0$.

2. Main results. Our main results are formulated in the following four theorems.

Theorem 1.

(i) *Suppose* $f : \mathbb{R} \to \mathbb{C}$ *is such that* $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. *If for some* $0 < \alpha \le 1$,

$$\int_{|x|<y} |xf(x)|\, dx = O(y^{1-\alpha}) \quad \text{for all } y > 0, \tag{2.1}$$

then $f \in L^1(\mathbb{R})$ *and* $\hat{f} \in \mathrm{Lip}(\alpha)$.

(ii) *Conversely, suppose* $f \in L^1(\mathbb{R})$ *and* $xf(x) \ge 0$ *for almost every* $x \in \mathbb{R}$. *If* $\hat{f} \in \mathrm{Lip}(\alpha)$ *for some* $0 < \alpha \le 1$, *then condition* (2.1) *holds.*

Theorem 2.

(i) *Suppose* $f : \mathbb{R} \to \mathbb{C}$ *is such that* $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. *If for some* $0 < \alpha \le 2$,

$$\int_{|x|<y} x^2|f(x)|\, dx = O(y^{2-\alpha}) \quad \text{for all } y > 0, \tag{2.2}$$

then $f \in L^1(\mathbb{R})$ *and* $\hat{f} \in \mathrm{Zyg}(\alpha)$.

(ii) *Conversely, suppose* $f \in L^1(\mathbb{R})$ *and* $f(x) \ge 0$ *for almost every* $x \in \mathbb{R}$. *If* $\hat{f} \in \mathrm{Zyg}(\alpha)$ *for some* $0 < \alpha \le 2$, *then condition* (2.2) *holds.*

Modifying the proofs of Theorems 1 and 2, in Section 4 we obtain the following two theorems.

Theorem 3. *In case* $0 < \alpha < 1$, *both statements in Theorem* 1 *remain valid if the right-hand side in* (2.1) *is replaced by* $o(y^{1-\alpha})$ *as* $y \to \infty$, *and* $f \in \mathrm{Lip}(\alpha)$ *is replaced by* $f \in \mathrm{lip}(\alpha)$.

Theorem 4. *In case $0 < \alpha < 2$, both statements in Theorem 2 remain valid if the right-hand side in (2.2) is replaced by $o(y^{2-\alpha})$ as $y \to \infty$, and $f \in \mathrm{Zyg}(\alpha)$ is replaced by $f \in \mathrm{zyg}(\alpha)$.*

3. Auxiliary results. In this section, we consider nonnegative-valued, measurable functions g defined on $\mathbb{R}_+ := [0, \infty)$. We will prove two lemmas, which are of interest in themselves.

Lemma 1.

(i) *If $\delta > \gamma \geq 0$ and*

$$\int_0^y u^\delta g(u)\, du = O(y^\gamma) \quad \text{for all } y > 0, \tag{3.1}$$

then $g \in L^1(y, \infty)$ and

$$\int_y^\infty g(u)\, du = O(y^{\gamma-\delta}) \quad \text{for all } y > 0. \tag{3.2}$$

(ii) *Conversely, if $\delta \geq \gamma > 0$ and condition* (3.2) *holds, then condition* (3.1) *also holds.*

We note that Lemma 1 fails in the endpoint cases not included above. For example, if $\delta = \gamma > 0$ in (i), then for $g(u) := u^{-1}$ condition (3.1) is satisfied, while (3.2) is not. If $\delta > \gamma = 0$ in (ii), then for $g(u) := u^{-1-\delta}$ condition (3.2) is satisfied, while (3.1) is not.

Proof of Lemma 1. (i) By (3.1), there exists a constant $C = C(g)$ such that for all $y > 0$,

$$y^\delta \int_y^{2y} g(u)\, du \leq \int_y^{2y} u^\delta g(u)\, du \leq C(2y)^\gamma,$$

whence it follows that

$$\int_y^{2y} g(u)\, du \leq 2^\gamma C y^{\gamma-\delta}, \tag{3.3}$$

and since $\gamma < \delta$, we conclude that

$$\begin{aligned} \int_y^\infty g(u)\, du &\leq 2^\gamma C y^{\gamma-\delta} \sum_{m=0}^\infty \int_{2^m y}^{2^{m+1} y} g(u)\, du \\ &\leq 2^\gamma C y^{\gamma-\delta} \sum_{m=0}^\infty 2^{m(\gamma-\delta)} = O(y^{\gamma-\delta}), \quad y > 0. \end{aligned} \tag{3.4}$$

This proves (3.2).

(ii) By (3.2), there exists another constant $C = C(g)$ such that for all $y > 0$,

$$\int_{y/2}^{y} u^\delta g(u)\,du \le y^\delta \int_{y/2}^{y} g(u)\,du \le 2^{\delta-\gamma} C y^\gamma, \tag{3.5}$$

and since $\gamma > 0$, we conclude that

$$\begin{aligned}\int_0^y u^\delta g(u)\,du &= \sum_{m=-\infty}^{0} \int_{2^{m-1}y}^{2^m y} u^\delta g(u)\,du \\ &\le 2^{\delta-\gamma} C y^\gamma \sum_{m=-\infty}^{0} 2^{m\gamma} = O(y^\gamma), \quad y > 0.\end{aligned} \tag{3.6}$$

This proves (3.1). ∎

Modifying the proof of Lemma 1, we obtain

LEMMA 2.

(i) *If* $\delta > \gamma > 0$ *and*

$$\int_0^y u^\delta g(u)\,du = o(y^\gamma) \quad \textit{as } y \to \infty, \tag{3.7}$$

then $g \in L^1(y, \infty)$ *for large enough* y *and*

$$\int_y^\infty g(u)\,du = o(y^{\gamma-\delta}) \quad \textit{as } y \to \infty. \tag{3.8}$$

(ii) *Conversely, if* $\delta > \gamma > 0$, $u^\delta g(u) \in L^1_{\text{loc}}(\mathbb{R}_+)$, *and condition* (3.8) *holds, then condition also holds.*

We note that the endpoint case $\delta > \gamma = 0$ in (i) makes no sense, unless $g(u) = 0$ almost everywhere, since the left-hand side in (3.7) is an increasing function of y. In the other endpoint case $\delta = \gamma \ge 0$, both (3.7) and (3.8) are trivially satisfied if $g \in L^1(\mathbb{R})$.

Proof of Lemma 2. (i) By (3.7), for every $\varepsilon > 0$ there exists $y_0 = y_0(\varepsilon)$ such that for all $y \ge y_0$, (3.3) is satisfied with ε in place of C. Analogously to (3.4), it follows that

$$\int_y^\infty g(u)\,du \le 2^\gamma \varepsilon y^{\gamma-\delta} \sum_{n=0}^{\infty} 2^{m(\gamma-\delta)}, \quad y \ge y_0.$$

Since $\delta > \gamma$ and $\varepsilon > 0$ is arbitrary, this proves (3.8).

(ii) By (3.8), for every $\varepsilon > 0$ there exists another $y_0 = y_0(\varepsilon)$ such that for all $y \geq y_0$, (3.5) is satisfied with ε in place of C, that is,

$$\int_{y/2}^{y} u^\delta g(u)\,du \leq 2^{\delta-\gamma}\varepsilon y^\gamma, \quad y \geq y_0. \tag{3.9}$$

Due to the assumption $u^\delta g(u) \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, there exists $y_1 = y_1(\varepsilon, y_0) > 2y_0$ such that

$$\int_0^{y_0} u^\delta g(u)\,du \leq \varepsilon y_1^\gamma. \tag{3.10}$$

Given any $y \geq y_1$, there exists an integer $m_0 = m_0(y_1) \leq -1$ for which

$$2^{-m_0-1}y < y_0 \leq 2^{m_0}y.$$

Now, by (3.9) and (3.10), we conclude (cf. (3.6)) that for all $y \geq y_1$ we have

$$\begin{aligned}\int_0^{y} u^\delta g(u)\,du &\leq \Big\{\int_0^{y_0} + \sum_{m=-m_0}^{0} \int_{2^{m-1}y}^{2^m y}\Big\} u^\delta g(u)\,du \\ &\leq \varepsilon y_1^\gamma + \sum_{m=-m_0}^{0} 2^{\delta-\gamma}\varepsilon(2^m y)^\gamma \leq \varepsilon y^\gamma\Big(1 + 2^{\delta-\gamma}\sum_{m=-m_0}^{0} 2^{m\gamma}\Big).\end{aligned}$$

Since $\gamma > 0$ and $\varepsilon > 0$ is arbitrary, this proves (3.7). ∎

4. Proofs of theorems

Proof of Theorem 1. (i) For any $t \in \mathbb{R}$ and $h > 0$, by (1.1) we have

$$\begin{aligned}2\pi|\hat{f}(t+h) - \hat{f}(t)| &= \Big|\int_{\mathbb{R}} f(x)e^{-itx}(e^{-ihx} - 1)\,dx\Big| \\ &\leq \Big\{\int_{|x|<1/h} + \int_{|x|>1/h}\Big\}|f(x)|\,|e^{-ihx} - 1| =: I_h + J_h,\end{aligned} \tag{4.1}$$

say. Since

$$|e^{-ihx} - 1| = \Big|2\sin\frac{hx}{2}\Big| \leq \min\{2, h|x|\},$$

by (2.1) we estimate as follows:

$$|I_h| \leq h\int_{|x|<1/h} |xf(x)|\,dx = hO\Big(\Big(\frac{1}{h}\Big)^{1-\alpha}\Big) = O(h^\alpha). \tag{4.2}$$

Applying Lemma 1(i) in the case of (2.1), we find that

$$|J_h| \leq 2\int_{|x|>1/h} |f(x)|\,dx = O\Big(\Big(\frac{1}{h}\Big)^{-\alpha}\Big) = O(h^\alpha). \tag{4.3}$$

Combining (4.1)–(4.3) gives $\hat{f} \in \mathrm{Lip}(\alpha)$.

(ii) Assume $\hat{f} \in \mathrm{Lip}(\alpha)$ for some $0 < \alpha \le 1$. By (1.1), we have

$$2\pi|\hat{f}(t) - \hat{f}(0)| = \Big|\int_{\mathbb{R}} f(x)(e^{-itx} - 1)\,dx\Big| \le Ct^\alpha, \quad t > 0,$$

where the constant C does not depend on t. Taking only the imaginary part of the integral between the absolute value bars, we even have

$$\Big|\int_{\mathbb{R}} f(x)\sin tx\,dx\Big| \le Ct^\alpha, \quad t > 0. \tag{4.4}$$

We may integrate the integral in (4.4) with respect to t over the interval $(0, h)$, where $h > 0$. By Fubini's theorem, we obtain

$$\Big|\int_{\mathbb{R}} f(x)\frac{1-\cos hx}{x}\,dx\Big| = \int_{\mathbb{R}} \frac{f(x)}{x}\,2\sin^2\frac{hx}{2}\,dx \le C\frac{h^{\alpha+1}}{\alpha+1}, \tag{4.5}$$

where the constant C does not depend on h, and we took into account that $xf(x) \ge 0$. Using the well-known inequality

$$\sin u \ge \frac{2}{\pi}u \quad \text{for } 0 \le u \le \pi/2, \tag{4.6}$$

it follows from (4.5) that

$$\frac{2h^2}{\pi^2}\int_{|x|<1/h} xf(x)\,dx \le C\frac{h^{\alpha+1}}{\alpha+1},$$

that is,

$$\int_{|x|<1/h} xf(x)\,dx \le \frac{C\pi^2}{2(\alpha+1)}h^{\alpha-1} = O\Big(\Big(\frac{1}{h}\Big)^{1-\alpha}\Big), \quad h > 0.$$

This proves (2.1) with $y := 1/h$, $h > 0$. ∎

Proof of Theorem 3. It runs along the same lines as the proof of Theorem 1, using Lemma 2 instead of Lemma 1. The details are left to the reader. ∎

Proof of Theorem 2. (i) For any $t \in \mathbb{R}$ and $h > 0$, by (1.1) we have

$$\begin{aligned}(4.7)\quad & 2\pi|\hat{f}(t+h) - 2\hat{f}(t) + \hat{f}(t-h)| \\ &= \Big|\int_{\mathbb{R}} f(x)e^{-itx}(e^{-ihx} - 2 + e^{ihx})\,dx\Big| \\ &\le \Big\{\int_{|x|<1/h} + \int_{|x|>1/h}\Big\}|f(x)|\,|e^{-ihx} - 2 + e^{ihx}|dx =: I_h + J_h,\end{aligned}$$

say. Since

$$|e^{-ihx} - 2 + e^{ihx}| = |2(\cos hx - 1)| = 4\sin^2\frac{hx}{2} \le \min\{4, h^2x^2\},$$

by (2.2) we estimate as follows:

$$(4.8)\qquad |I_h| \le h^2 \int_{|x|<1/h} x^2|f(x)|\,dx = h^2 O\Big(\Big(\frac{1}{h}\Big)^{2-\alpha}\Big) = O(h^\alpha).$$

Applying Lemma 1(i) in the case of (2.2), we find that

$$(4.9)\qquad |J_h| \le 4 \int_{|x|>1/h} |f(x)|\,dx = O\Big(\Big(\frac{1}{h}\Big)^{-\alpha}\Big) = O(h^\alpha).$$

Combining (4.7)–(4.9) gives $\hat{f} \in \mathrm{Zyg}(\alpha)$.

(ii) Assume $\hat{f} \in \mathrm{Zyg}(\alpha)$ for some $0 < \alpha \le 2$. By (1.1), we have

$$(4.10)\qquad 2\pi|\hat{f}(h) - 2\hat{f}(0) + \hat{f}(-h)| = \Big|\int_{\mathbb{R}} f(x)(2\cos hx - 2)\,dx\Big|$$
$$= 4\int_{\mathbb{R}} f(x)\sin^2\frac{hx}{2}\,dx \le Ch^\alpha, \quad h>0,$$

where the constant C does not depend on h, and we took into account that $f(x) \ge 0$. Making use of inequality (4.6), it follows from (4.10) that

$$\frac{4h^2}{\pi^2}\int_{|x|<1/h} x^2 f(x)dx \le Ch^\alpha,$$

that is,

$$\int_{|x|<1/h} x^2 f(x)\,dx \le \frac{C\pi^2}{4}h^{\alpha-2} = O\Big(\Big(\frac{1}{h}\Big)^{2-\alpha}\Big), \quad h>0.$$

This proves (2.2) with $y := 1/h$, $h > 0$. ∎

Proof of Theorem 4. It is a repetition of the proof of Theorem 2 with appropriate modifications, using Lemma 2 instead of Lemma 1. The details are left to the reader. ∎

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