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A note on a construction of J. F. Feinstein

by

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Abstract. In [6] J. F. Feinstein constructed a compact plane set X such that $R(X)$, the uniform closure of the algebra of rational functions with poles off X, has no non-zero, bounded point derivations but is not weakly amenable. In the same paper he gave an example of a separable uniform algebra A such that every point in the character space of A is a peak point but A is not weakly amenable. We show that it is possible to modify the construction in order to produce examples which are also regular.

1. Introduction. A uniform algebra A on a compact (Hausdorff) space X is said to be *regular on* X if for any point x in X and any compact subset K of $X \setminus \{x\}$ there is a function f in A such that $f(x) = 1$ and f is zero on K. We call A *regular* if it is regular on its character space, and *trivial* if it is $C(X)$, the uniform algebra of all continuous functions on X. The first example of a non-trivial regular uniform algebra was given by McKissick [8] (see also [7]; [11, Chapter 37]): the example was $R(X)$ for a compact plane set X.

The notion of weak amenability was introduced in [1]. A commutative Banach algebra A is said to be *weakly amenable* if there are no non-zero, continuous derivations from A into any commutative Banach A-bimodule. It is proved in [1] that this is equivalent to there being no non-zero continuous derivations into the dual module A'.

As point derivations may be regarded as derivations into 1-dimensional, commutative Banach modules it is a necessary condition for weak amenability that there be no non-zero, bounded point derivations. However, this condition is not sufficient, even for uniform algebras: in [6] J. F. Feinstein constructed a compact plane set X such that the uniform algebra $R(X)$ has no non-zero, bounded point derivations but $R(X)$ is not weakly amenable. It was not clear whether such an example could also be regular. In this note we show that it can.

Notation. Throughout this paper Q will refer to the compact plane set $\{x + iy : x, y \in [-1, 1]\}$.

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For a plane set X and a function $f \in C(X)$, $|f|_X$ will be the uniform norm of f on X, $\sup\{|f(z)| : z \in X\}$. For a compact plane set X we denote by $R_0(x)$ the set of restrictions to X of rational functions with poles off X. Hence the uniform algebra $R(X)$ is the uniform closure of $R_0(X)$ in $C(X)$.

If D is a disc in the plane then $r(D)$ shall refer to its radius.

Let μ be a complex measure on a compact plane set X such that the bilinear functional on $R_0(X) \times R_0(X)$ defined by

$$(f, g) \mapsto \int_X f'(x) g(x)\, d\mu(x)$$

is bounded. Then, as in [6], we may extend by continuity to $R(X) \times R(X)$ and obtain a continuous derivation D from $R(X)$ to $R(X)'$ such that for any f and g in $R_0(X)$ we have

$$D(f)(g) = \int_X f'(x) g(x)\, d\mu(x).$$

In the next section, we strengthen Körner's [7] version of McKissick's Lemma to allow greater control over the centres and radii of the discs removed. Using this we modify Feinstein's [6] construction so that $R(X)$ is regular. In fact we prove the following theorem.

THEOREM 1.1. *For each $C > 0$ there is a compact plane set X obtained by deleting from Q a countable union of Jordan domains such that ∂Q is a subset of X, $R(X)$ is regular and has no non-zero, bounded point derivations and, for all f, g in $R_0(X)$,*

$$\Big| \int_{\partial Q} f'(z) g(z)\, dz \Big| \le C |f|_X |g|_X.$$

If we let X be a compact plane set constructed as in Theorem 1.1 we have, by the discussion above, a non-zero continuous derivation D from $R(X)$ to $R(X)'$ such that

$$D(f)(g) = \int_{\partial Q} f'(z) g(z)\, dz$$

for $f, g \in R_0(X)$. So $R(X)$ is not weakly amenable.

2. The construction. Our main new tool will be the following theorem, which is a variation on McKissick's result in [8] (see also [7]).

THEOREM 2.1. *For any $C_0 > 0$ there is a compact plane set X_1 obtained by deleting from Q a countable union of open discs $(D_n)_{n=1}^{\infty}$ whose closures are in* $\operatorname{int}(Q)$ *such that $R(X_1)$ is regular and, letting s_n be the distance* *from D_n to ∂Q,*

$$\sum_{n=1}^{\infty} \frac{r(D_n)}{s_n^2} < C_0.$$

In order to prove Theorem 2.1 we require a series of lemmas which are variations on the results of [7].

The following is [7, Lemma 2.1].

LEMMA 2.2. *If $N \geq 2$ is an integer and $h_N(z) = 1/(1 - z^N)$ then the following hold*:

(i) $|h_N(z)| \leq 2|z|^{-N}$ *for* $|z|^N \geq 2$;
(ii) $|1 - h_N(z)| \leq 2|z|^N$ *for* $|z|^N \leq 2^{-1}$;
(iii) $h_N(z) \neq 0$ *for all* z.

Further if $(8 \log N)^{-1} > \delta > 0$ *then*:

(iv) $|h_N(z)| \leq 2\delta^{-1}$ *provided only that* $|z-w| \geq \delta N^{-1}$ *whenever* $w^N = 1$.

The following is a variant of [7, Lemma 2.2].

LEMMA 2.3. *If in Lemma 2.2 we set $N = n2^{2n}$ with n sufficiently large then*:

(i) $|h_N(z)| \leq (n+1)^{-4}$ *for* $|z| \geq 1 + 2^{-(2n+1)}$;
(ii) $|1 - h_N(z)| \leq (n+1)^{-4}$ *for* $|z| \leq 1 - 2^{-(2n+1)}$;
(iii) $h_N \neq 0$ *for all* z;
(iv) $|h_N(z)| \leq n^{-4}2^{2n+1}$ *provided only that* $|z-w| \geq n^{-5}2^{-4n}$ *whenever* $w^N = 1$.

Proof. Parts (i), (ii) and (iii) are the corresponding parts of [7, 2.2]. Part (iv) follows on putting $\delta = n^{-3}2^{-2n}$. ∎

From this we obtain the following variation on [7, Lemma 2.3].

LEMMA 2.4. *Provided only that n is sufficiently large we can find a finite collection $A(n)$ of disjoint open discs and a rational function g_n such that, letting $s_0(\Delta) = \operatorname{dist}(\Delta, \mathbb{R} \cup i\mathbb{R})$ for a disc Δ, the following hold*:

(i) $\sum_{\Delta \in A(n)} r(\Delta)/s_0(\Delta)^2 < n^{-2}$ *and so* $\sum_{\Delta \in A(n)} r(\Delta) \leq n^{-2}$;
(ii) *the poles of g_n lie in* $\bigcup_{\Delta \in A(n)} \Delta$;
(iii) $|g_n(z)| \leq (n+1)^{-4}$ *for* $|z| \geq 1 - 2^{-(2n+1)}$;
(iv) $|1 - h_N(z)| \leq (n+1)^{-4}$ *for* $|z| \leq 1 - 2^{-(2n-1)}$;
(v) $|g_n(z)| \leq n^4 2^{2n+1}$ *for* $z \notin \bigcup_{\Delta \in A(n)} \Delta$;
(vi) $g_n(z) \neq 0$ *for all* z;
(vii) $\bigcup_{\Delta \in A(n)} \Delta \subset \{z : 2^{-(2n-1)} \leq |z| \leq 1 - 2^{-(2n+1)}\}$.

Proof. Let $N = n2^{2n}$, $\omega = \exp(2\pi/N)$, $g_n = h_N(\omega^{-1/2}(1 - 2^{-2n})^{-1}z)$. If we take $A(n)$ to be the collection of discs with radii $n^{-5}2^{-4n}$ and centres

$(1-2^{-2n})\omega^{r+1/2}$ $(0 \le r \le N-1)$, results (ii)–(vii) are either trivial or follow from Lemma 2.3 on scaling by a factor of $\omega^{1/2}(1-2^{-2n})$.

To show part (i), consider first those discs with centres $(1-2^{-2n})\omega^{r+1/2}$ $(0 \le r \le (N-1)/8)$. For such a disc Δ we have

$$\begin{aligned} s_0(\Delta) &= (1-2^{-2n})\sin\left(\frac{(r+1/2)\pi}{n2^{2n}}\right) - n^{-5}2^{-4n} \\ &\ge \frac{(r+1/2)\pi}{n2^{2n+2}} - n^{-5}2^{-4n} \ge \left(\frac{1}{2}\right)\frac{2r+1}{n2^{2n}}. \end{aligned}$$

So

$$\frac{r(\Delta)}{s_0(\Delta)^2} \le n^{-5}2^{-4n}\frac{4n^2 2^{4n}}{(2r+1)^2} = \frac{4n^{-3}}{(2r+1)^2}$$

and

$$\sum_{k=1}^{N/8-1}\frac{r(\Delta_k)}{s_0(\Delta_k)} \le 4n^{-3}\sum_{k=0}^{N/8-1}(2k+1)^{-2} \le 4n^{-3}\sum_{k=0}^{\infty}k^{-2} \le \frac{n^{-2}}{8}$$

provided only that $n > K = 32\sum_{r=0}^{\infty} r^{-2}$. So, by symmetry,

$$\sum_{\Delta\in A(n)}\frac{r(\Delta)}{s_0(\Delta)^2} \le n^{-2}$$

provided only that n is sufficiently large. ∎

Multiplying the g_n together as in [7], we obtain the following.

LEMMA 2.5. *Given any $\varepsilon > 0$ there exists an $m = m(\varepsilon)$ such that if* (*adopting the notation of Lemma* 2.4) *we let $f_n = (m!)^{-4}\prod_{r=m}^{n} g_r$ and $\{\Delta_k\}$ be a sequence enumerating the discs of $\bigcup_{r=m}^{\infty} A(r)$ then the following hold*:

(a) $\sum_{k=1}^{\infty} r(\Delta_k)/s_0(\Delta_k)^2 < \varepsilon$ *and so then* $\sum_{k=1}^{\infty} r(\Delta_k) < \varepsilon$;
(b) *the poles of the f_n lie in* $\bigcup_{k=1}^{\infty}\Delta_k$;
(c) *the sequence $\{f_n\}$ tends uniformly to zero on* $\{z\in\mathbb{C} : |z| \ge 1\}$.

Proof. Observe that

$$\sum_{k=1}^{\infty} r(\Delta_k) \le \sum_{k=1}^{\infty}\frac{r(\Delta_k)}{s_0(\Delta_k)^2} = \sum_{l=m}^{\infty}\sum_{\Delta\in A(l)}\frac{r(\Delta)}{s_0(\Delta)^2} \le \sum_{l=m}^{\infty} l^{-2} < \varepsilon$$

provided only that $m(\varepsilon) > 2\varepsilon^{-1}+1$. Thus conclusions (a) and (b) are easy to verify. To prove (c), set $K = \prod_{r=1}^{\infty}(1+(r+1)^{-4})$ and observe that, provided $m(\varepsilon)$ is large enough that $(m+1)!^4 > 2(m+1)^6 2^{2(m+1)} + 2(m+1)^2$, if we let $z \notin \bigcup_{k=1}^{\infty}\Delta_k$, a simple induction gives

$$|f_n(z)| \le \begin{cases} (n+1)!^{-4} & \text{for } 1-2^{-(2n+1)} < |z|, \\ n^{-2} \le \prod_{r=m}^{n}(1+(r+1)^{-4}) \le K & \\ & \text{for } 1-2^{-(2n-1)} < |z| < 1-2^{-(2n+1)}, \\ \prod_{r=m}^{n}(1+(r+1)^{-4}) \le K & \text{for } |z| < 1-2^{-(2n-1)}. \end{cases}$$

Using the trivial equality

$$|f_{n+1}(z) - f_n(z)| = |f_n(z)|\,|1-g_{n+1}(z)|,$$

we see that for $z \notin \bigcup_{k=1}^{\infty} \Delta_k$,

$$|f_{n+1}(z) - f_n(z)| \le \begin{cases} K(n+1)^{-4} & \text{for } |z| \le 1-2^{-(2n+1)}, \\ (n+1)!^{-4}(1+(n+1)^4 2^{2n+1}) \le (n+1)^{-2} & \\ & \text{for } 1-2^{-(2n+1)} < |z|. \end{cases}$$

Thus $|f_{n+1}(z)-f_n(z)| \le K(n+1)^{-2}$ for all $z \notin \bigcup_{k=1}^{\infty} \Delta_k$ and, by for example the Weierstrass M test, f_n converges uniformly to f say. To see that $f(z) \ne 0$ for $|z|<1$, $z \notin \bigcup_{k=1}^{\infty} \Delta_k$, note that if $|z| \le 1-2^{2n-1}$ then $f_n(z) \ne 0$, and

$$\sum_{r=n+1}^{\infty} |1-g_r(z)| \le \sum_{r-n+1}^{\infty} (r+1)^{-4} < \infty.$$

So by a basic result on infinite products (see, for example, [10, 15.5]),

$$f(z) = f_n(z) \prod_{r=n+1}^{\infty} g_r(z) \ne 0. \ \blacksquare$$

Hence by dilation and translation we obtain the following.

LEMMA 2.6. *Given any closed disc D, with centre a and radius r, and any $\varepsilon > 0$, we can find a sequence $\{\Delta_k\}$ of open discs and a sequence $\{f_n\}$ of rational functions such that, letting $s_1(\Delta) := \operatorname{dist}(\Delta_k, a+\mathbb{R}\cup i\mathbb{R})$ for a disc Δ, the following hold:*

(a) $\sum_{k=1}^{\infty} r(\Delta_k)/s_1(\Delta_k)^2 < \varepsilon$ *and so* $\sum_{k=1}^{\infty} R(\Delta_k) < \varepsilon$;
(b) *the poles of the f_n lie in* $\bigcup_{k=1}^{\infty} \Delta_k$;
(c) *the sequence $\{f_n\}$ tends uniformly to zero on* $(\mathbb{C}\setminus D)\setminus \bigcup_{k=1}^{\infty} \Delta_k$.

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let $\{D_l\}_{l=1}^{\infty}$ be an enumeration of all closed discs of centre z and radius r with $z \in \mathbb{Q}+i\mathbb{Q}$ and $r \in \mathbb{Q}^+$ such that, letting $K = \{-1-i, -1+i, 1-i, 1+i\}$, one of the following holds:

(1) $z \in \operatorname{int}(Q)$ and $r < \operatorname{dist}(z, \partial Y)$;

(2) $z \in \partial Q \setminus K$ and $r < \operatorname{dist}(z, K)$;

(3) $z \in K$ and $r < 1$.

We apply Lemma 2.6 with $D = D_l$ and $\varepsilon = \varepsilon_l$ where

$$\varepsilon_l < \begin{cases} 2^{-l-1}C_0 \operatorname{dist}(D_l, \partial Q)^2 & \text{if } D_l \text{ is of type (1)},\\ 2^{-l-1}C_0 \operatorname{dist}(D_l, K)^2 & \text{if } D_l \text{ is of type (2)},\\ 2^{-l-1}C_0 & \text{if } D_l \text{ is of type (3)},\end{cases}$$

to obtain $(\Delta_{l,n})$ and $(f_{l,k})$. Let $\{U_N\}_{N=1}^{\infty}$ be a sequence enumerating the $\Delta_{n,a}$ and

$$X_0 = Q \setminus \bigcup_{N=1}^{\infty} U_N.$$

We have

$$\sum_{N=1}^{\infty} \frac{r(U_N)}{s(U_N)^2} = \sum_{l=1}^{\infty}\sum_{n=1}^{\infty} \frac{r(\Delta_{l,n})}{s(\Delta_{l,n})^2} < \sum_{l=1}^{\infty} C_0 2^{-l} = C_0.$$

Given any point z in X_0 and any compact set $B \subset X_0$ there exists D_l with $z \in D_l$ and $B \cap D_l = \emptyset$. Hence $f_l := \lim_{k\to\infty} f_{l,k} \in R(X_0)$ has $f_l(z) \neq 0$ and $f_l(B) \subset \{0\}$, so $R(X_0)$ is regular. ■

In order to prove Theorem 1.1 we need some further lemmas. The first one is trivial.

LEMMA 2.7. *Let X, Y be compact plane sets with $X \subset Y$. If $R(Y)$ is regular, then the same is true for X.*

The next two results are essentially the same as those used in [6].

LEMMA 2.8. *Let (χ_n) be a sequence of Jordan domains whose closures are contained in Q. Set $X_2 = Q \setminus \bigcup_{n=1}^{\infty} \chi_n$. Let s_n be the distance from χ_n to ∂Q and let c_n be the length of the boundary of χ_n. Let f and g be in $R_0(X_2)$. Then*

$$\Big| \int_{\partial Q} f'(z) g(z)\, dz \Big| \le 2|f|_X |g|_X \sum_{n=1}^{\infty} \frac{c_n}{s_n^2}.$$

Proof. The argument of [6, 2.1] applies. ■

LEMMA 2.9. *Let X be a compact subset of Q. Suppose that there is a sequence of real numbers $L_n \in (0,1)$ such that $L_n \to 1$ and, for each n, $R(X \cap L_n Q)$ has no non-zero bounded point derivations. Then $R(X)$ has no non-zero bounded point derivations.*

Proof. The argument of [6, 2.2] applies. ■

The following is [6, 2.3].

LEMMA 2.10. *Let X, Y be compact plane sets with $X \subset Y$. If $R(Y)$ has no non-zero, bounded point derivations, then the same is true for X.*

The following result was proved by Wermer [12].

PROPOSITION 2.11. *Let D be a closed disc in $\mathbb{C}$ and let $\varepsilon > 0$. Then there is a sequence of open discs $U_k \subset D$ such that $R(D \setminus \bigcup_{k=1}^{\infty} U_k)$ has no non-zero bounded point derivations but the sum of the radii of the discs U_k is less than ε.*

COROLLARY 2.12. *Let Y be a square set of the form $rQ + z$ and $\delta > 0$. Then there is a sequence of Jordan domains $\chi_l \subset Y$ such that $R(Y \setminus \bigcup_{l=1}^{\infty} \chi_l)$ has no non-zero bounded point derivations but the sum of the lengths of the boundaries of the χ_l is less than δ.*

Proof. Apply the previous proposition to any closed disc containing Y with $\varepsilon < \delta/(2\pi)$. Then, by Lemma 2.10, letting (χ_l) be a sequence enumerating all non-empty sets of the form $U_k \cap \operatorname{int}(Y)$ will suffice. ∎

Proof of Theorem 1.1. Let $C > 0$. Set $L_n = n/(n+1)$. Applying Corollary 2.12 to $L_n Q$ we may choose Jordan domains $\chi_{n,k} \subset L_n Q$ such that $R(L_n Q \setminus \bigcup_{k=1}^{\infty} \chi_{n,k})$ has no non-zero bounded point derivations and the sum of the lengths of the boundaries of the $(\chi_{n,k})_{k=0}^{\infty}$ is less than $2^{-(n+1)} C(1 - L_n)^2$. Set

$$X_2 = Q \setminus \bigcup_{n,k} \chi_{n,k}$$

and let

$$X_1 = Q \setminus \bigcup_{n} D_n$$

be the result of applying Theorem 2.1 with $C_0 = C/(4\pi)$. Finally set $X = X_1 \cap X_2$. Then Lemma 2.7 gives that $R(X)$ is regular and Lemmas 2.9 and 2.10 give that $R(X)$ has no non-zero bounded point derivations. Enumerating the sets $\chi_{n,k}$ and D_n as $C_1, C_2, \ldots$ we may apply Lemma 2.8 to obtain the required estimate on the integral. ∎

3. Regularity and peak points. A point x in the character space X of a uniform algebra A is said to be a *peak point for A* if there is $f \in A$ such that $f(x) = 1$ and $|f(y)| < 1$ for all $y \in X \setminus \{x\}$; and x is a *point of continuity for A* if, for every compact set $K \subset X \setminus \{x\}$, there is a function f in A such $f(x) = 1$ and $f(K) \subset \{0\}$. Notice that a uniform algebra is regular if and only if every point in its character space is a point of continuity. The following result, regarding systems of Cole root extensions (see [2]), is [5, 2.8].

PROPOSITION 3.1. *Let A and B be uniform algebras such that B is the result of applying a system of Cole root extensions to A. If A is regular so is B.*

In [6] an example of a separable uniform algebra, A, such that every point of the character space is a peak point for A and A is not weakly amenable is obtained by first modifying the original example so that every point, except possibly those on the outer boundary circle, is a point of continuity, and then applying an appropriate system of Cole root extensions. Following essentially the same argument and noting the above proposition we obtain the following.

THEOREM 3.2. *There exists a regular uniform algebra A whose character space is metrizable such that every point of the character space of A is a peak point but A is not weakly amenable.*

We note that both this algebra and the algebra constructed in Theorem 1.1 have dense invertible group (by results in [4]).

We finish by noting that we do not know whether or not either of the uniform algebras we have constructed are strongly regular—i.e. if, for each point x in the character space, the algebra of functions constant on a neighbourhood of x is dense in the original algebra.

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