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Heat kernel estimates for a class of higher order operators on Lie groups

by

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Abstract. Let G be a Lie group of polynomial volume growth. Consider a differential operator H of order $2m$ on G which is a sum of even powers of a generating list $A_1, \ldots, A_{d'}$ of right invariant vector fields. When G is solvable, we obtain an algebraic condition on the list $A_1, \ldots, A_{d'}$ which is sufficient to ensure that the semigroup kernel of H satisfies global Gaussian estimates for all times. For G not necessarily solvable, we state an analytic condition on the list which is necessary and sufficient for global Gaussian estimates. Our results extend previously known results for nilpotent groups.

1. Introduction. Let G be a non-compact, connected, unimodular Lie group with Lie algebra $\mathfrak{g}$, and assume that G has polynomial volume growth. Consider a differential operator

$$H = H^{(m)} = (-1)^m \sum_{i=1}^{d'} A_i^{2m},$$

where m is a positive integer and $A_1, \ldots, A_{d'}$ are right invariant vector fields on G corresponding to a list of generators $a_1, \ldots, a_{d'} \in \mathfrak{g}$ of the Lie algebra. It is known that H generates a semigroup $S_t = e^{-tH}$, $t > 0$, in the spaces $L^p = L^p(G; dg)$, $1 \le p \le \infty$, where dg denotes a fixed Haar measure on G. Moreover, S_t acts via a convolution $S_t f = K_t * f$, $f \in L^p$, $t > 0$, where $K_t \colon G \to \mathbb{R}$ is a smooth function satisfying the following "local" Gaussian bounds: for each $t_0 \in (0, \infty)$, there exist $c, b > 0$ such that

$$|K_t(g)| \le c t^{-D'/(2m)} e^{-b(\varrho_A(g)^{2m}/t)^{1/(2m-1)}} \tag{1}$$

for all $t \in (0, t_0]$ and $g \in G$ (see [16] and [11]). Here $\varrho_A \colon G \to [0, \infty)$ is the standard Carathéodory modulus associated with $A_1, \ldots, A_{d'}$, and $D' \in \mathbb{N}$ is a local dimension associated with ϱ_A (for background, see [19, 18]).

Since G has polynomial growth, there is a $D \in \mathbb{N}$ such that an estimate $c^{-1} r^D \le dg(B(r)) \le c r^D$ holds for all $r \ge 1$, where $B(r) := \{g \in G : \varrho_A(g) < r\}$ is the ball of radius r associated with ϱ_A. Let us say that H

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satisfies *global Gaussian bounds* if there exist $c, b > 0$ such that

$$|K_t(g)| \le c t^{-D/(2m)} e^{-b(\varrho_A(g)^{2m}/t)^{1/(2m-1)}} \tag{2}$$

for all $t \ge 1$ and $g \in G$. For $m = 1$, it is well known that $H = H^{(1)}$ satisfies global Gaussian bounds (see, for example, [19]). If G is nilpotent, or more generally, G is a local direct product of a nilpotent group and a compact group, and m is arbitrary, then H satisfies global Gaussian bounds (see [13, 8], and earlier works [10, 3, 15] dealing with homogeneous groups).

However, when G is the universal cover of the solvable group of Euclidean motions of the plane, it was discovered in [12] that there are choices of $a_1, \ldots, a_{d'}$ for which H does not satisfy global Gaussian bounds when $m \ge 2$. For such choices one has $c^{-1} t^{-\mu} \le \|K_t\|_\infty \le c t^{-\mu}$ for all $t \ge 1$, with some $\mu > D/(2m)$. Note that the author [6] recently established the upper bound $\|K_t\|_\infty \le c t^{-D/(2m)}$ for $t \ge 1$, for any G and any list $a_1, \ldots, a_{d'}$.

The aim of this paper is to give a general condition on $a_1, \ldots, a_{d'}$ sufficient for global Gaussian bounds. To state it, suppose that G is solvable, and let $\mathfrak{n} \subseteq \mathfrak{g}$ denote the nilradical (the largest nilpotent ideal) of $\mathfrak{g}$. To the solvable algebra $\mathfrak{g} = (\mathfrak{g}, [\cdot,\cdot])$ we can associate its nilshadow $\mathfrak{g}_N$, which is a nilpotent Lie algebra with $\mathfrak{g} = \mathfrak{g}_N$ as vector spaces and Lie bracket $[\cdot,\cdot]_N$ (see, for example, [1, 2, 9]). Let $\mathfrak{g}_{N;1} \supseteq \mathfrak{g}_{N;2} \supseteq \cdots$ be the lower central series of $\mathfrak{g}_N$, defined by $\mathfrak{g}_{N;1} = \mathfrak{g}_N$ and $\mathfrak{g}_{N;j+1} = [\mathfrak{g}_N, \mathfrak{g}_{N;j}]_N \subseteq \mathfrak{g}_{N;j}$, $j \in \mathbb{N}$. We say that the generating list $a_1, \ldots, a_{d'}$ is *nice of order* k, where $k \in \mathbb{N}$, if there exists a Cartan subalgebra $\mathfrak{w}$ of $\mathfrak{g}$ such that each $a_i = v_i + y_i$, where $v_i \in \mathfrak{w}$, $y_i \in \mathfrak{n}$, and

$$(\operatorname{ad} v_i)^n y_i \in \mathfrak{g}_{N;n+1} \tag{3}$$

for all $n \in \{1, \ldots, k-1\}$ and $i \in \{1, \ldots, d'\}$. Here $(\operatorname{ad} x)y = [x, y]$, $x, y \in \mathfrak{g}$, is the adjoint representation of $\mathfrak{g}$, and we recall that a *Cartan subalgebra* is a nilpotent subalgebra which equals its own normalizer.

THEOREM 1.1. *Assume that G is solvable. If $a_1, \ldots, a_{d'}$ is nice of order m, then $H = H^{(m)}$ satisfies global Gaussian bounds.*

We conjecture that, for G solvable and simply connected, niceness of order m is necessary as well as sufficient for global Gaussian bounds for $H^{(m)}$.

Theorem 1.1 actually has an extension for any group G of polynomial growth, which is more complicated to state: see Remark 2.6 below.

Theorem 1.1 will be derived from the following theorem of independent interest (and which is not restricted to the solvable case). Denote by e the identity of G.

THEOREM 1.2. *Let $m \in \mathbb{N}$. Suppose there exist a family $(\eta_R)_{R \ge 1}$ of C^∞-smooth functions on G and a constant $c > 0$ such that $0 \le \eta_R \le 1$,*

$\eta_R(e) = 1$, *the support of* η_R *is contained in* $B(cR)$, *and*

$$\|A_i^k \eta_R\|_\infty \le cR^{-k}$$

for all $i \in \{1, \dots, d'\}$, $k \in \{1, \dots, m\}$ *and* $R \ge 1$. *Then* $H = H^{(m)}$ *satisfies global Gaussian bounds.*

The above theorems are proved in Section 2. The effort of the proofs is reduced by utilizing work of the author [7] which shows that, to obtain Gaussian bounds for group-invariant semigroups, it suffices to verify certain L^2 "off-diagonal" estimates.

Let us notice some special cases and examples of Theorem 1.1. Every generating list is nice of order 1. If G is nilpotent, then $\mathfrak{g} = \mathfrak{g}_N = \mathfrak{n}$ as Lie algebras so that any generating list is nice of all orders. When G is solvable, and $\mathfrak{w}$ is a Cartan subalgebra of $\mathfrak{g}$, any generating list satisfying $\{a_1, \dots, a_{d'}\} \subseteq \mathfrak{w} \cup \mathfrak{n}$ is nice of all orders. Since $\mathfrak{g} = \mathfrak{w} + \mathfrak{n}$ (see for example [9, pp. 64–65]) where the sum need not be direct, we see that nice generating lists certainly exist.

If G is solvable and if $\mathfrak{g}_N$ is an abelian Lie algebra, then niceness of order k, $k \ge 2$, means that for some Cartan subalgebra $\mathfrak{w}$, $a_i = v_i + y_i$ with $v_i \in \mathfrak{w}$, $y_i \in \mathfrak{n}$, and $[v_i, y_i] = 0$ for all i. Thus, for such G, the condition is independent of k when $k \ge 2$. On the other hand, by considering solvable groups G for which $\mathfrak{g}_N$ is non-abelian with large nilpotent rank, for any k one can construct examples of generating lists which are nice of order k but not of order $k+1$. We leave the details to the reader.

For the example of the group G considered in [12], $\mathfrak{g} = \operatorname{span}\{b_1, b_2, b_3\}$ is a three-dimensional Lie algebra with non-zero commutation relations $[b_1, b_2] = b_3$, $[b_1, b_3] = -b_2$, and $\mathfrak{n} = \operatorname{span}\{b_2, b_3\}$. Let a_1, a_2, a_3 be a vector space basis of $\mathfrak{g}$. Since the Cartan subalgebras are just the one-dimensional subspaces of $\mathfrak{g}$ complementary to $\mathfrak{n}$, and $\mathfrak{g}_N$ is abelian, one sees that a_1, a_2, a_3 is nice of order k ($k \ge 2$) if and only if exactly two of the a_i are in $\mathfrak{n}$. The latter condition was noticed in [12].

Let us note a converse to Theorem 1.2.

THEOREM 1.3. *Suppose that* $H = H^{(m)}$ *satisfies global Gaussian bounds. Then there exist smooth functions* $(\eta_R)_{R \ge 1}$ *on* G *with properties as in the statement of Theorem* 1.2.

To prove Theorem 1.3, choose a non-decreasing function $F \in C^\infty(\mathbb{R})$ with $F(x) = 0$ for all $x \le 2^{-1}$ and $F(x) = 1$ for $x \ge 1$, and set

$$\eta_R(g) = F(K_{R^{2m}}(g)/K_{R^{2m}}(e)), \quad g \in G.$$

To verify the desired properties of η_R is an easy adaption of arguments of [5, Section 2], and we omit further details.

Combining Theorems 1.2 and 1.3 gives the following interesting result.

COROLLARY 1.4. *Let $m_1, m_2 \in \mathbb{N}$ with $m_1 > m_2$. If $H^{(m_1)}$ satisfies global Gaussian bounds on G, then $H^{(m_2)}$ satisfies global Gaussian bounds on G.*

Finally, let us speculate about heat kernel estimates for an arbitrary choice of $a_1, \ldots, a_{d'}$. We conjecture that there exists a suitable function $\varrho_{A,m}: G \to [0,\infty)$, depending on $A_1, \ldots, A_{d'}$ and on m, such that $\varrho_{A,m} \le c\varrho_A$ and

$$|K_t(g)| \le ct^{-\widetilde{D}/(2m)} e^{-b(\varrho_{A,m}(g)^{2m}/t)^{1/(2m-1)}}, \qquad \|K_t\|_\infty \ge c^{-1}t^{-\widetilde{D}/(2m)},$$

for all $t \ge 1$ and $g \in G$. Here, $\widetilde{D} \ge D$ denotes a global dimension associated with $\varrho_{A,m}$, that is, $dg(\{g \in G: \varrho_{A,m}(g) < r\}) \simeq r^{\widetilde{D}}$ for $r \ge 1$. The main difficulty in proving this conjecture is apparently to define a suitable $\varrho_{A,m}$; the results already stated suggest that $\varrho_{A,m}$ will be sensitive to perturbations of $A_1, \ldots, A_{d'}$ or m.

2. Proofs

Proof of Theorem 1.2. In general, c, c', b, ω and so on denote positive constants whose value may change from line to line. We first construct functions φ_R with similar properties to the given functions η_R, and with the symmetry property $\varphi_R(g^{-1}) = \varphi_R(g)$.

LEMMA 2.1. *There exist $c > 0$ and functions $\varphi_R \in C_c^\infty(G)$, $R \ge 1$, such that $0 \le \varphi_R \le 1$, $\varphi_R(g^{-1}) = \varphi_R(g)$ for all $g \in G$, φ_R is supported in $B(cR)$, $\varphi_R(g) = 1$ whenever $g \in B(c^{-1}R)$, and*

$$\|A_i^k \varphi_R\|_\infty \le cR^{-k}$$

for all $R \ge 1$, $i \in \{1, \ldots, d'\}$ and $k \in \{1, \ldots, m\}$.

Proof. For $R \ge 1$ let $t_R = \int_G \eta_R^2$ and $\widetilde{\eta}_R(g) = \eta_R(g^{-1})$, and define

$$\widehat{\eta}_R := t_R^{-1}(\eta_R * \widetilde{\eta}_R),$$

where $*$ denotes convolution of functions on G. So $A_i^k \widehat{\eta}_R = t_R^{-1}(A_i^k \eta_R) * \widetilde{\eta}_R$, and there is an estimate $c^{-1}R^D \le t_R \le cR^D$ for all $R \ge 1$. It is then straightforward to check that the $\widehat{\eta}_R$ have the same properties as η_R, and moreover $\widehat{\eta}_R(g^{-1}) = \widehat{\eta}_R(g)$, $g \in G$. Since $\|A_i \widehat{\eta}_R\|_\infty \le cR^{-1}$ and $\widehat{\eta}_R(e) = 1$, one easily shows that there is $c' > 0$ with

$$\widehat{\eta}_R(g) \ge 2^{-1}$$

for all $g \in B((c')^{-1}R)$ and $R \ge 1$. Choose a C^∞-smooth function $F: [0,\infty) \to [0,1]$ such that $F(0) = 0$ and $F(x) = 1$ for all $x \ge 2^{-1}$. The functions $\varphi_R := F \circ \widehat{\eta}_R$ have the required properties. ∎

Borrowing an idea of [14], we define a function $\varrho: G \to [1,\infty)$ by

$$\varrho(g) = 1 + \sum_{j=1}^{\infty} (1 - \varphi_j(g)).$$

From the properties of φ_R, one sees that the sum is locally finite, so ϱ is a smooth function, and that

$$c^{-1}(1+\varrho_A) \le \varrho \le c(1+\varrho_A). \tag{4}$$

Thus ϱ is a smooth approximation of ϱ_A. Moreover, $\varrho(g) = \varrho(g^{-1})$ for all $g \in G$. For $g \in G$ with $\varrho_A(g) > 1$, the properties of φ_R yield

$$|(A_i^k \varrho)(g)| \le \sum_{c^{-1}\varrho_A(g) \le j \le c\varrho_A(g)} \|A_i^k \varphi_j\|_\infty \le c' \varrho_A(g)^{1-k}$$

for all $k \in \{1, \ldots, m\}$. Hence

$$|(A_i^k \varrho)(g)| \le c'' \varrho(g)^{1-k} \tag{5}$$

for all $g \in G$ and $k \in \{1, \ldots, m\}$. Taking $k = 1$ in (5), we deduce that

$$|\varrho(h) - \varrho(gh)| \le c\varrho_A(g) \sum_{i=1}^{d'} \|A_i \varrho\|_\infty \le c' \varrho(g)$$

for all $g, h \in G$, and consequently $\varrho(gh) \le c(\varrho(g) + \varrho(h))$ for all $g, h \in G$.

In what follows, write $M = 2m$, denote by U_λ the multiplication operator $f \mapsto e^{\lambda\varrho} f$ for $\lambda \in \mathbb{R}$, and let $L = L_G$ be the left regular representation of G with $(L(g)f)(h) = f(g^{-1}h)$, $g, h \in G$, for functions $f\colon G \to \mathbb{C}$. We will establish estimates

$$\begin{aligned} &\|U_\lambda S_t U_{-\lambda}\|_{2\to2} \le c e^{\omega\lambda^M t}, \\ &\|(I - L(g))U_\lambda S_t U_{-\lambda}\|_{2\to2} \le c\varrho(g) t^{-1/M} e^{\omega\lambda^M t} \end{aligned} \tag{6}$$

for all $t \ge 1$, $\lambda \in \mathbb{R}$ and $g \in G$ such that $\varrho(g) \le t^{1/M}$ (where $\|\cdot\|_{2\to2}$ denotes the operator norm for bounded operators in L^2). Then it follows from Theorem 2.3 and Remark 2.4 of [7], together with the local estimate (1), that

$$|K_n(g)| \le c n^{-D/M} \exp(-b(\varrho(g)^M/n)^{1/(M-1)})$$

for all $n \in \mathbb{N}$ and $g \in G$. In this inequality, we may replace ϱ with ϱ_A, because of (4). Then using the semigroup property $K_t = K_{t-n} * K_n$ and (1), we easily deduce global Gaussian bounds for H.

Therefore, to complete the proof of Theorem 1.2 it remains to verify (6).

The proof of (6) is an adaption of the usual Davies perturbation technique. Introduce the set $\mathcal{E}_m$ of smooth functions $\psi\colon G \to \mathbb{R}$ such that $\|A_i^k\|_\infty < \infty$ for all $i \in \{1, \ldots, d'\}$ and $k \in \{1, \ldots, m\}$; we do not assume that ψ is bounded. Because of the relation

$$e^\psi A_i(e^{-\psi} f) = A_i f - (A_i\psi) f, \tag{7}$$

we may consider $H_\psi := e^\psi H e^{-\psi}$ as a differential operator which is a perturbation of H by terms of order less than $2m$ with bounded coefficients.

Define a functional σ_m on $\mathcal{E}_m$ by

$$\sigma_m(\psi) = \sum_{i=1}^{d'} \sum_{k=1}^{m} (\|A_i^k \psi\|_\infty)^{1/k};$$

we usually abbreviate $\sigma_m = \sigma$. In the next two lemmas, we will see that this functional seems well adapted to the study of off-diagonal estimates for H. Notice that the lemmas apply to an arbitrary generating list $a_1, \ldots, a_{d'}$, that is, they make no use of the hypothesis of Theorem 1.2.

LEMMA 2.2. *There exists $c > 0$ such that*

$$|(Hf, f) - (H_\psi f, f)| \le 2^{-1}(Hf, f) + c\sigma(\psi)^M \|f\|_2^2$$

for all $f \in C_c^\infty(G)$ and $\psi \in \mathcal{E}_m$. Consequently, there is $c_1 > 0$ with

$$\mathrm{Re}\,(H_\psi f, f) \ge -c_1 \sigma(\psi)^M \|f\|_2^2$$

for all $f \in C_c^\infty$ and $\psi \in \mathcal{E}_m$.

Proof. The second estimate of the lemma follows easily from the first. Let us sketch the proof of the first estimate, which is a variation of standard arguments (see [4] and [5, pp. 58–59]). Write

$$(Hf, f) - (H_\psi f, f) = \sum_{i=1}^{d'} ((A_i^m f, A_i^m f) - (e^\psi A_i^m e^{-\psi} f, e^{-\psi} A_i^m e^\psi f)),$$

and apply (7) to expand the left side as a sum of terms each of the form

$$T(f) = c((A_i^{n_1}\psi) \cdots (A_i^{n_p}\psi) A_i^{k_1} f, A_i^{k_2} f),$$

where $p \in \mathbb{N}$, $n_1, \ldots, n_p \in \{1, \ldots, m\}$, $k_1, k_2 \in \{0, 1, \ldots, m\}$, and $n_1 + \cdots + n_p + k_1 + k_2 = 2m = M$. Now set $r = n_1 + \cdots + n_p \in \{1, \ldots, M\}$, and apply a standard interpolation inequality

$$\|A_i^k f\|_2 \le c\|A_i^m f\|_2^{k/m} \|f\|_2^{1-(k/m)} \le c(Hf, f)^{k/M} \|f\|_2^{1-(k/m)}, \tag{8}$$

which is valid for $k \in \{0, 1, \ldots, m\}$. We obtain

$$\begin{aligned} |T(f)| &\le c\sigma(\psi)^r \|A_i^{k_1} f\|_2 \|A_i^{k_2} f\|_2 \le c'(Hf, f)^{1-(r/M)} (\sigma(\psi)^M \|f\|_2^2)^{r/M} \\ &\le \varepsilon(Hf, f) + c\varepsilon^{-(M-r)/r} \sigma(\psi)^M \|f\|_2^2 \end{aligned}$$

for all $\varepsilon > 0$, and the lemma follows. ∎

The next lemma is deduced from Lemma 2.2 by a standard line of reasoning, and we omit the details (see [5, pp. 55–56] or [4]).

LEMMA 2.3. *The operator H_ψ generates a semigroup $S_t^\psi = e^\psi S_t e^{-\psi}$ in L^2, satisfying the estimates*

$$\|S_t^\psi\|_{2\to 2} + t\|H_\psi S_t^\psi\|_{2\to 2} \le ce^{\omega\sigma(\psi)^M t}, \qquad \|A_i^k S_t^\psi\|_{2\to 2} \le ct^{-k/M} e^{\omega\sigma(\psi)^M t}$$

for all $t > 0$, $\psi \in \mathcal{E}_m$, $i \in \{1, \ldots, d'\}$ and $k \in \{1, \ldots, m\}$.

To prove (6) we consider separately the two cases $|\lambda|\geq 2^{-1}$ and $|\lambda|<2^{-1}$. In the first case, observe from (5) that $\lambda\varrho\in\mathcal{E}_m$ with an estimate

$$\sigma(\lambda\varrho)\leq c\sum_{i=1}^{d'}\sum_{k=1}^{m}|\lambda|^{1/k}\leq c'|\lambda|.$$

Therefore, from Lemma 2.3 and the standard inequality (see [18, p. 268])

$$\|(I-L(g))f\|_2\leq \varrho_A(g)\Big(\sum_{i=1}^{d'}\|A_if\|_2^2\Big)^{1/2}, \tag{9}$$

we obtain (6) for $|\lambda|\geq 2^{-1}$.

To deal with the second case, define $\varrho_j=1+\sum_{s=j}^{\infty}(1-\varphi_s)$ for each $j\in\mathbb{N}$. Then $\varrho_1=\varrho$ and

$$\|\varrho-\varrho_j\|_\infty\leq j-1,\qquad \|A_i^k\varrho_j\|_\infty\leq cj^{1-k}$$

for all $j\in\mathbb{N}$ and $k\in\{1,\ldots,m\}$, where the second inequality is proved in a similar way to (5). Given $0<|\lambda|<2^{-1}$, let j be the greatest integer less than or equal to $|\lambda|^{-1}$. Then $2^{-1}\leq|\lambda|j\leq 1$, $\sigma(\lambda\varrho_j)\leq c|\lambda|$, and from Lemma 2.3 we get

$$\begin{aligned}\|e^{\lambda\varrho}S_te^{-\lambda\varrho}\|_{2\to2}&\leq\|e^{\lambda(\varrho-\varrho_j)}\|_{2\to2}\|e^{\lambda\varrho_j}S_te^{-\lambda\varrho_j}\|_{2\to2}\|e^{-\lambda(\varrho-\varrho_j)}\|_{2\to2}\\&\leq c'e^{\omega\lambda^Mt},\end{aligned}$$

where c',ω are constants independent of λ. This yields the first estimate of (6) for $|\lambda|<2^{-1}$ (the case $\lambda=0$ is trivial). The second estimate of (6) may be proved in a similar way by estimating the operators

$$A_ie^{\lambda\varrho}S_te^{-\lambda\varrho}=(A_i(e^{\lambda(\varrho-\varrho_j)}))\,S_t^{\lambda\varrho_j}e^{-\lambda(\varrho-\varrho_j)}+e^{\lambda(\varrho-\varrho_j)}A_iS_t^{\lambda\varrho_j}e^{-\lambda(\varrho-\varrho_j)}$$

for $0<|\lambda|<2^{-1}$, and applying (9). The proof of Theorem 1.2 is complete. ∎

Proof of Theorem 1.1. Let G be solvable and fix a Cartan subalgebra $\mathfrak{w}$ of $\mathfrak{g}$ such that $a_i=v_i+y_i$, $v_i\in\mathfrak{w}$, $y_i\in\mathfrak{n}$, and such that (3) holds for all $n\in\{1,\ldots,m-1\}$. From $\mathfrak{w}$ one may construct the nilshadow Lie group $G_N=G_N(\mathfrak{w})$, with Lie algebra $\mathfrak{g}_N$, such that $G=G_N$ are identified as manifolds and $\mathfrak{g}=\mathfrak{g}_N$ are identified as vector spaces. For details see [9, 1, 2]: in these references G_N is constructed starting from a subspace $\mathfrak{v}\subseteq\mathfrak{w}$ satisfying $\mathfrak{g}=\mathfrak{v}\oplus\mathfrak{n}$, but the construction is actually independent of the choice of $\mathfrak{v}$ within $\mathfrak{w}$ (see [9, pp. 78–80]). For $x\in\mathfrak{g}$, we will write

$$X=dL_G(x),\qquad \widetilde{X}=dL_{G_N}(x)$$

respectively for the G-right invariant vector field and the G_N-right invariant vector field corresponding to x. In particular, $A_i=V_i+Y_i$, where $V_i=dL_G(v_i)$, $Y_i=dL_G(y_i)$. The following lemma is contained in the analysis of [2, 9].

LEMMA 2.4. *One has*

$$(Xf)(g) = (dL_G(x)f)(g) = ((dL_{G_N}(\overline{T}(g^{-1})x))f)(g) \tag{10}$$

for all $x \in \mathfrak{g}$, $f \in C_c^\infty(G)$ *and* $g \in G$. *Here* $\overline{T}$ *is a certain representation of* G *in the vector space* $\mathfrak{g}$, *such that* $\overline{T}(g)x = x$ *for all* $x \in \mathfrak{w}$ *and* $g \in G$, *and* $\overline{T}(\exp y) = I$ *for all* $y \in \mathfrak{n}$. *Moreover*, *the* $\overline{T}(g)$ *are orthogonal transformations with respect to a suitably chosen inner product on* $\mathfrak{g}$, *and* $\overline{T}(g)(\mathfrak{n}) \subseteq \mathfrak{n}$, $\overline{T}(g)(\mathfrak{g}_{N;j}) \subseteq \mathfrak{g}_{N;j}$ *for all* j.

LEMMA 2.5. *There exists a family* $(\eta_R)_{R\geq 1}$ *of smooth functions on* G_N *such that* $0 \leq \eta_R \leq 1$, $\eta_R(e) = 1$, *and* η_R *is supported in* $B(cR)$. *Moreover*, *for any* $k \in \mathbb{N}$ *and* $z_1, \ldots, z_k \in \mathfrak{g}$ *with* $z_j \in \mathfrak{g}_{N;m_j}$ *for some* $m_1, \ldots, m_k \in \mathbb{N}$, *there is* $c' > 0$ *such that*

$$\|\widetilde{Z}_1 \cdots \widetilde{Z}_k \eta_R\|_\infty \leq c' R^{-(m_1+\cdots+m_k)}$$

for all $R \geq 1$, *where* $\widetilde{Z}_j = dL_{G_N}(z_j)$.

Proof. The distances on G and G_N are equivalent at infinity, that is, if $\widetilde{\varrho}$ denotes a Carathéodory modulus on G_N then one has an estimate $c^{-1}(1+\widetilde{\varrho}) \leq 1 + \varrho_A \leq c(1+\widetilde{\varrho})$ (see [9] or [2]). Then the lemma follows from results of [14] and the nilpotency of G_N.

Alternatively, one can prove the lemma more directly by fixing a suitable $\eta_1 \in C_c^\infty(G_N)$ and setting $\eta_R(g) = \eta_1(\tau_{R^{-1}}(g))$ for $R > 1$, where τ_δ, $\delta > 0$, are dilations "at infinity" on the nilpotent group G_N (see, for example, [1, Section 5] or [17] for these dilations). We skip the details. ∎

Set $w_{i,n} = (\operatorname{ad} v_i)^n y_i$ and $W_{i,n} = dL_G(w_{i,n})$ for $n \in \mathbb{N}_0 = \{0, 1, \ldots\}$. Expanding $A_i^m = (V_i + Y_i)^m$ and reordering terms yields an expression

$$A_i^m = V_i^m + Y_i^m + \sum c_{l_0,\ldots,l_s} W_{i,l_1} \ldots W_{i,l_s} V_i^{l_0}, \tag{11}$$

where the sum is over all $s \geq 1$ and $l_0, l_1, \ldots, l_s \in \{0, 1, \ldots, m-1\}$ with $l_0 + l_1 + \cdots + l_s + s = m$ and where $c_{l_0,\ldots,l_s}$ are constants.

Since $v_i \in \mathfrak{w}$, it follows from Lemma 2.4 that $V_i = dL_{G_N}(v_i)$, that is, V_i is a G_N-invariant vector field. Also, given $n \in \{0, 1, \ldots, m-1\}$ and $i \in \{1, \ldots, d'\}$, since $w_{i,n} \in \mathfrak{n} \cap \mathfrak{g}_{N;n+1}$ we can use Lemma 2.4 to express

$$W_{i,n} = \sum_j \xi_j \widetilde{Z}_j,$$

for some fields $\widetilde{Z}_j = dL_{G_N}(z_j)$ with $z_j \in \mathfrak{n} \cap \mathfrak{g}_{N;n+1}$. The ξ_j: $G \to \mathbb{R}$ are smooth bounded functions which are constant in the $\mathfrak{n}$-directions, that is, $\widetilde{Y}\xi_j = 0$ for any $y \in \mathfrak{n}$ (for further details, see for example [9, p. 183]).

By combining these observations with (11), we may express A_i^m as a sum of terms each of the form $\xi \widetilde{Z}_1 \ldots \widetilde{Z}_k$, where ξ are smooth bounded functions and $z_j \in \mathfrak{g}_{N;m_j}$ with $m_1 + \cdots + m_k \geq m$. Therefore, by Lemma 2.5 one has $\|A_i^m \eta_R\|_\infty \leq cR^{-m}$ for $R \geq 1$. Since $\|\eta_R\|_\infty = 1$, by interpolation we get

$\|A_i^k \eta_R\|_\infty \leq c' R^{-k}$ for all $k \in \{1, \ldots, m\}$. Theorem 1.1 now follows from Theorem 1.2. ∎

REMARK 2.6. Let us note a generalization of Theorem 1.1 for any Lie group G of polynomial growth. In this general case, $\mathfrak{g} \supseteq \mathfrak{q} \supseteq \mathfrak{n}$ where $\mathfrak{q}$ is the solvable radical and $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$. A subalgebra $\mathfrak{w} \subseteq \mathfrak{g}$ will be called a *generalized Cartan subalgebra* of $\mathfrak{g}$ if $\mathfrak{w} = \mathfrak{m} \oplus \mathfrak{w}_0$ where $\mathfrak{m}$ is a Levi subalgebra of $\mathfrak{g}$ (that is, $\mathfrak{m}$ is a semisimple subalgebra with $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{q}$) and $\mathfrak{w}_0 \subseteq \mathfrak{q}$ is a Cartan subalgebra of the algebra $\mathfrak{q}_0(\mathfrak{m}) := \{x \in \mathfrak{q} : [\mathfrak{m}, x] = \{0\}\}$. A generalized Cartan subalgebra satisfies $\mathfrak{g} = \mathfrak{w} + \mathfrak{n}$ (see, for instance, [9, pp. 64–65]). Let $\mathfrak{q}_{N;1} \supseteq \mathfrak{q}_{N;2} \supseteq \cdots$ be the lower central series of the nilshadow $\mathfrak{q}_N$ associated with $\mathfrak{q}$. Then we have:

THEOREM 2.7. *Let $a_1, \ldots, a_{d'}$ be a generating list such that there exists a generalized Cartan subalgebra $\mathfrak{w}$ with $a_i = v_i + y_i$, $v_i \in \mathfrak{w}$, $y_i \in \mathfrak{n}$, and $(\operatorname{ad} v_i)^n y_i \in \mathfrak{q}_{N;n+1}$ for all $n \in \{1, \ldots, m-1\}$ and all i. Then $H = H^{(m)}$ satisfies global Gaussian bounds.*

The proof is a straightforward extension of the above proof of Theorem 1.1, since one knows suitable generalizations of Lemmas 2.4 and 2.5 (see again [9, 2] for the structure theory of G). We omit the details.

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