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The group of automorphisms of L_∞ is algebraically reflexive

by

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Abstract. We study the reflexivity of the automorphism (and the isometry) group of the Banach algebras $L_\infty(\mu)$ for various measures μ. We prove that if μ is a non-atomic σ-finite measure, then the automorphism group (or the isometry group) of $L_\infty(\mu)$ is [algebraically] reflexive if and only if $L_\infty(\mu)$ is *-isomorphic to $L_\infty[0,1]$. For purely atomic measures, we show that the group of automorphisms (or isometries) of $\ell_\infty(\Gamma)$ is reflexive if and only if Γ has non-measurable cardinal. So, for most "practical" purposes, the automorphism group of $\ell_\infty(\Gamma)$ is reflexive.

Introduction. Let X be a Banach space and $L(X)$ the Banach algebra of all (linear, continuous) operators on X. Suppose S is any subset of $L(X)$. An operator T is said to be *locally in* S if for each $x \in X$ there is $L \in S$ (probably depending on x) such that $T(x) = L(x)$. If each operator which is locally in S belongs to S, we say that S is (*algebraically*) *reflexive*. This notion of reflexivity has been fruitfully used in the analysis of operator algebras. Although most of the early works on reflexivity were concerned with derivations [10, 14, 15], in recent years the study of local automorphisms (of Banach algebras) and local surjective isometries (of Banach spaces) spurred a considerable interest in operator theory [3–7, 18, 19, 21–23, 26].

In spite of these efforts, the problem of reflexivity of the automorphism group (and the isometry group) of the Banach algebra L_∞ remained open.

As usual, given a measure space (Ω, Σ, μ), we write $L_\infty(\mu)$ for the Banach algebra of all essentially bounded measurable functions $f : \Omega \to \mathbb{K}$ equipped with the essential supremum norm, "pointwise" operations, and the traditional convention about identifying functions equal almost everywhere. When μ is Lebesgue measure on the Borel subsets of $[0, 1]$ we simply write L_∞.

The main result of the paper is that both the automorphism group and the isometry group of L_∞ are (algebraically) reflexive.

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This is somewhat surprising since every surjective isometry of L_∞ is the adjoint of an isometry of L_1 and the isometry group of L_1 fails to be reflexive (in a very strong way; see [6]). Moreover, as a commutative C^*-algebra, L_∞ can be regarded as the space of all continuous functions on its spectrum and no point in the spectrum of L_∞ is G_δ. The existence of enough G_δ-points often plays an important rôle in obtaining positive results on reflexivity of the automorphism (and isometry) groups for spaces of continuous functions; see [3, 6, 21, 22]. We emphasize that, by former results of Batty and Molnár, the group of automorphisms of L_∞ is not *topologically* reflexive [3, Theorem 5].

Actually, we shall prove that if μ is a non-atomic σ-finite measure, then the automorphism group (or the isometry group) of $L_\infty(\mu)$ is reflexive if and only if μ is a separable measure—this is equivalent to the separability of the Banach space $L_1(\mu)$, which is the natural predual of $L_\infty(\mu)$. And all this happens if and only if $L_\infty(\mu)$ is *-isomorphic to L_∞.

As for purely atomic measures, we show that the group of automorphisms (or isometries) of $\ell_\infty(\Gamma)$ is reflexive if and only if Γ has non-measurable cardinal. So, for most "practical" purposes, the automorphism group of $\ell_\infty(\Gamma)$ is reflexive.

1. Homomorphisms of $L_\infty(\mu)$-algebras. Let (Ω, Σ, μ) be a measure space. Two measurable sets A and B will be called *equivalent* (*modulo* μ) if $\mu(A \triangle B) = 0$. Identifying equivalent sets, we obtain a Boolean algebra denoted by Σ/μ in what follows. Of course, the Boolean structure of Σ/μ comes from that of Σ by the rules

$$[A] \cup [B] = [A \cup B], \quad [A] \cap [B] = [A \cap B], \quad [A]^{\mathrm{c}} = [A^{\mathrm{c}}],$$

where $[A]$ denotes the class of A in Σ/μ and A^{c} is the complement of A in Ω. Countable operations in Σ/μ can be defined in the obvious way. It is clear that Σ/μ is isomorphic to the Boolean algebra of idempotents of $L_\infty(\mu)$ via characteristic functions. Note that if A and B are equivalent sets, then 1_A and 1_B are the same element in $L_\infty(\mu)$, and so, the notation $1_{[A]}$ makes perfect (and obvious) sense.

Consider now measure spaces $(\Omega_i, \Sigma_i, \mu_i)$ for $i = 1, 2$ and let $T : L_\infty(\mu_1) \to L_\infty(\mu_2)$ be a homomorphism (that is, a *linear and unital* ring homomorphism). Since ring homomorphisms preserve idempotents there is a unique mapping $\Phi : \Sigma_1/\mu_1 \to \Sigma_2/\mu_2$ such that

$$T(1_A) = 1_{\Phi[A]} \quad (A \in \Sigma_1).$$

It is clear that Φ is a Boolean homomorphism (it preserves finite unions and intersections, as well as complements). Moreover, if $\Phi : \Sigma_1/\mu_1 \to \Sigma_2/\mu_2$ is a Boolean homomorphism, one can define a homomorphism $T : L_\infty(\mu_1) \to$

$L_\infty(\mu_2)$ taking

$$T\Big(\sum_{i=1}^{n}\lambda_i 1_{A_i}\Big)=\sum_{i=1}^{n}\lambda_i 1_{\Phi[A_i]}$$

for simple functions and extending it by continuity. Note that $\|T\|\le 1$.

All this shows that there exists a precise correspondence between homomorphisms of $L_\infty(\mu)$-spaces and Boolean homomorphisms of the underlying algebras.

Let (A_n) be a sequence in Σ_1 and $\Phi:\Sigma_1/\mu_1\to\Sigma_2/\mu_2$ a Boolean homomorphism. It is clear that $\bigcup_{n=1}^\infty \Phi[A_n]\subset\Phi[\bigcup_{n=1}^\infty A_n]$. But, in general, that containment is proper: if $\chi:\ell_\infty\to\mathbb{K}$ is a non-trivial character—here, ℓ_∞ corresponds to counting measure on $\mathbb{N}$ and $\mathbb{K}$ is the algebra of functions on a single point of mass one, so that $\Sigma_2=\{\emptyset,\Omega_2\}$—then the associated Boolean homomorphism $\Phi:2^{\mathbb{N}}\to\{\emptyset,\Omega_2\}$ sends all finite subsets of $\mathbb{N}$ into $\emptyset$, while $\Phi(\mathbb{N})=\Omega_2$.

The following innocent observation is the key of the paper.

Lemma 1. *Let $T:L_\infty(\mu_1)\to L_\infty(\mu_2)$ be a linear map. Suppose T is a local isomorphism in the sense that for every $f\in L_\infty(\mu_1)$ there is an isomorphism $U:L_\infty(\mu_1)\to L_\infty(\mu_2)$ such that $Tf=Uf$. Then*

(a) *T is an injective homomorphism.*

(b) *The Boolean homomorphism Φ associated to T preserves countable operations.*

Proof. In the complex case, part (a) follows from [3, Proposition 2] ($L_\infty(\mu)$ is a von Neumann algebra provided μ is not too pathological) or [6, Remark 5 after Theorem 5] ($L_\infty(\mu)$ is always semisimple) but we give a simpler proof for this particular case which does not depend on the ground field.

Let T be a local isomorphism. It is clear that T is a norm preserving map sending 1_{Ω_1} into 1_{Ω_2}. Moreover, for each $A\in\Sigma_1$ there is $A'\in\Sigma_2$ such that $T(1_A)=1_{A'}$. Obviously, A' is unique, modulo μ_2-null sets. Let A and B be disjoint subsets of Σ_1. Then, from

$$1_{(A\oplus B)'}=T(1_{A\oplus B})=T(1_A)+T(1_B)=1_{A'}+1_{B'},$$

it follows that $(A\oplus B)'=A'\oplus B'$, up to a null set.

Suppose f and g are simple functions in $L_\infty(\mu_1)$. Then there is a partition $A_1,\dots,A_n$ of Ω_1 such that

$$f=\sum_{i=1}^{n}\alpha_i 1_{A_i}\quad\text{and}\quad g=\sum_{i=1}^{n}\beta_i 1_{A_i}.$$

Since $fg=\sum_{i=1}^n\alpha_i\beta_i 1_{A_i}$ and taking into account that passing from $[A]$

to $[A']$ preserves disjointness one has

$$T(fg) = \sum_{i=1}^{n} \alpha_i \beta_i T(1_{A_i}) = \sum_{i=1}^{n} \alpha_i \beta_i 1_{A'_i}$$
$$= \Big(\sum_{i=1}^{n} \alpha_i 1_{A'_i}\Big) \cdot \Big(\sum_{i=1}^{n} \beta_i 1_{A'_i}\Big) = (Tf) \cdot (Tg),$$

which proves (a).

We now prove (b). A moment of reflection shows that it suffices to see that

$$[\Omega_2] = \bigcup_{n=1}^{\infty} \Phi[A_n]$$

whenever (A_n) is a countable partition of Ω_1. Suppose $U : L_\infty(\mu_1) \to L_\infty(\mu_2)$ is a linear ring isomorphism. It is clear that if f is a function in $L_\infty(\mu_1)$ vanishing on no set of positive measure, then the same occurs to Uf. Hence T must send non-vanishing functions to non-vanishing functions.

Let (A_n) be a partition of Ω_1. Take a sequence λ_n converging to zero, with $\lambda_n > 0$ for all n. Clearly, the series $\sum_{n=1}^{\infty} \lambda_n 1_{A_n}$ is summable in $L_\infty(\mu_1)$. Hence,

$$T\Big(\sum_{n=1}^{\infty} \lambda_n 1_{A_n}\Big) = \sum_{n=1}^{\infty} \lambda_n 1_{\Phi[A_n]}$$

is non-zero μ_2-almost everywhere, which implies that $\Phi[A_n]$ form a partition of Ω_2. This proves part (b). ∎

We close the section with the following observation. If μ is any measure, every (linear, continuous) functional on $L_\infty(\mu)$ can be regarded as a finitely additive (finite) measure $\nu : \Sigma/\mu \to \mathbb{K}$. See [9, pp. 354–357]. Clearly, if $T : L_\infty(\mu_1) \to L_\infty(\mu_2)$ is a continuous homomorphism whose associated homomorphism $\Phi : \Sigma_1/\mu_1 \to \Sigma_2/\mu_2$ preserves countable operations, then the adjoint map T^* preserves countable additivity of measures. If μ is σ-finite, then $L_\infty(\mu)$ equals $L_1(\mu)^*$ in the obvious way and, moreover, every countably additive (finite) measure ν on Σ/μ belongs to $L_1(\mu)$ (the Radon–Nikodým theorem) in the sense that there is $g \in L_1(\mu)$ such that

$$\nu([A]) = \int_A g \, d\mu$$

for all $A \in \Sigma$. We have the following.

Lemma 2. *Let μ_1 and μ_2 be σ-finite measures. For a continuous homomorphism $T : L_\infty(\mu_1) \to L_\infty(\mu_2)$ the following statements are equivalent*:

(a) *The Boolean homomorphism associated to T preserves countable operations.*

(b) *The adjoint map* $T^* : L_\infty(\mu_2)^* \to L_\infty(\mu_1)^*$ *preserves countable additivity.*

(c) T *is weak** *continuous.*

Proof. The implications (a)⇒(b)⇒(c) have already been proved. We prove (c)⇒(a). Let (A_n) be a disjoint sequence in Σ_1. Put $A = \bigoplus_n A_n$. Then

$$1_A = \text{weak}^*\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} 1_{A_i}.$$

Since T is weak* continuous we have

$$1_{\Phi[A]} = T(1_A) = \text{weak}^*\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} T(1_{A_i}) = \text{weak}^*\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} 1_{\Phi[A_i]},$$

and so

$$\Phi[A] = \bigoplus_{n=1}^{\infty} \Phi[A_n],$$

which completes the proof. ∎

2. Local automorphisms of L_∞**.** In this section we prove that the automorphism group of $L_\infty(\mu)$ is reflexive if the measure algebra Σ/μ is not too "big". To be more precise, let us consider the following distance in Σ/μ:

$$d([A],[B]) = \mu(A \triangle B).$$

The measure μ is said to be *separable* if $(\Sigma/\mu, d)$ is a separable metric space (it has a countable dense subset). It is well known (and obvious) that μ is separable if and only if $L_1(\mu)$ is a separable Banach space. In that case $L_1(\mu)$ is (isometrically) lattice isomorphic either to one of the spaces $L_1 \oplus_1 \ell_1(\Gamma)$ or $\ell_1(\Gamma)$, where Γ is at most countable (and possibly empty) and therefore $L_\infty(\mu)$ is (isometrically) *-isomorphic either to $L_\infty \times \ell_\infty(\Gamma)$ or to $\ell_\infty(\Gamma)$, with Γ countable.

A complete classification of the algebras $L_\infty(\mu)$ (for arbitrary measures) seems to be out of reach. Nevertheless, by a famous result of Maharam, if μ is decomposable [9, Definition 19.25], then $L_\infty(\mu)$ can be represented as

$$L_\infty(\mu) = \ell_\infty(\Gamma) \times \Big(\prod_{i\in I} L_\infty(\lambda^{\mathfrak{m}_i})\Big)_\infty \tag{1}$$

where Γ and I are (possibly empty) sets, $\mathfrak{m}_i$ are infinite cardinals and λ denotes Lebesgue measure on the Borel sets of the unit interval. The subscript indicates that the product on the right-hand side of (1) carries the supremum norm. Observe that, for instance, $L_\infty(\lambda^{\aleph_0}) = L_\infty$. Of course, σ-finite measures are decomposable.

The following result is a rewording of Sikorski's generalization [27] of von Neumann's [24]. Although von Neumann's result requires "hard" measure theory, a very simple "functional-analytic" proof for L_∞ is now at hand. We present it, not only for the sake of completeness, but also to display some arguments which we shall use later.

THEOREM 1 (von Neumann–Sikorski). *Let $T : L_\infty \to L_\infty(\mu)$ be a weak* continuous homomorphism, where μ is σ-finite. Then there is a measurable function $\varphi : \Omega \to [0,1]$ such that $T(f) = f \circ \varphi$ for all $f \in L_\infty$. Moreover φ is unique in $L_\infty(\mu)$.*

Proof. We prove the theorem for real scalars. The complex case follows easily, by taking into account that T sends real functions to real functions. Let ι be the identity on $[0,1]$ and put $\varphi = T(\iota)$. We show that

$$T(f) = f \circ \varphi \quad (f \in L_\infty). \tag{2}$$

This is obvious if f is a polynomial function. Since polynomials are weak* dense in L_∞ and both T and the composition operator $f \mapsto f \circ \varphi$ are weak* continuous it is clear that (2) holds for all $f \in L_\infty$. ∎

PROPOSITION 1. *The group of automorphisms of L_∞ is algebraically reflexive.*

Proof. Let T be a local automorphism of L_∞ and let U be an automorphism such that $T(\iota) = U(\iota)$, where ι is the identity on $[0,1]$. We claim that $T = U$. According to Lemmas 1 and 2, both T and U are weak* continuous. The proof of Theorem 1 shows that

$$T(f) = f \circ \varphi = U(f) \quad (f \in L_\infty),$$

where $\varphi = T(\iota) = U(\iota)$, and completes the proof. ∎

COROLLARY 1. *The isometry group of L_∞ is algebraically reflexive.*

Proof. Suppose T is a local surjective isometry of L_∞ and let $u = T(1)$. It is clear that u is unimodular: each surjective isometry of L_∞ is an automorphism multiplied by some unimodular Borel function. Reasoning as in [6, Theorem 5] and taking into account that the Gleason–Kahane–Żelazko theorem applies to L_∞ even in the real case, one sees that the map given by $L(f) = u^{-1}T(f)$ is a (unital) endomorphism of L_∞. Moreover, L is weak* continuous—it leaves invariant the set of non-vanishing functions: see the proof of Lemma 1. It is also clear that L is locally a surjective isometry. Let I be a surjective isometry of L_∞ such that $L(\iota) = I(\iota)$. A moment's reflection shows that I is in fact an automorphism and also that $L = I$. Hence T is onto. ∎

3. Bigger measures. In this section we will prove that the automorphism group (hence the isometry group) of $L_\infty(\mu)$ is (algebraically) non-reflexive if μ is an atomless non-separable σ-finite measure. In view of the Maharam decomposition (1), it suffices to consider the case $\mu = \lambda^{\mathfrak{m}}$, where $\mathfrak{m}$ is an uncountable cardinal. It will be convenient to regard $\mathfrak{m}$ also as an index set.

So, let $[0,1]^{\mathfrak{m}}$ be the product of $\mathfrak{m}$ copies of the unit interval. This is a compact space whose algebra of Borel subsets will be denoted by $\mathfrak{B}_{\mathfrak{m}}$. Finally, $\lambda^{\mathfrak{m}}$ will stand for the (product) Lebesgue measure on $\mathfrak{B}_{\mathfrak{m}}$.

Our immediate aim is to show that functions in $L_\infty(\lambda^{\mathfrak{m}})$ depend only on countably many coordinates of $[0,1]^{\mathfrak{m}}$. This is obvious for characteristic functions of open subsets of $[0,1]^{\mathfrak{m}}$. Actually such functions depend only on finitely many coordinates. Now, suppose $B \in \mathfrak{B}_{\mathfrak{m}}$. Since $\lambda^{\mathfrak{m}}$ is regular, one has

$$\lambda^{\mathfrak{m}}(B) = \inf \lambda^{\mathfrak{m}}(A),$$

where A runs over all open sets containing B. It follows that there is a decreasing sequence (A_n) of open sets containing B such that $\lambda^{\mathfrak{m}}(A_n)$ converges to $\lambda^{\mathfrak{m}}(B)$. Hence, $[B] = [\bigcap_n A_n]$, and since 1_B is (almost everywhere) the pointwise limit of the sequence (1_{A_n}) it is clear that 1_B has a representative depending only on countably many coordinates; the same is true for simple members of $L_\infty(\lambda^{\mathfrak{m}})$. Finally, each $f \in L_\infty(\lambda^{\mathfrak{m}})$ can be written as a pointwise limit of simple $\mathfrak{B}_{\mathfrak{m}}$-measurable functions and, therefore, it depends on countably many coordinates only.

PROPOSITION 2. *Let $\mathfrak{m}$ be an uncountable cardinal. Then both the automorphism group and the isometry group of $L_\infty(\lambda^{\mathfrak{m}})$ fail to be algebraically reflexive.*

Proof. Let $\sigma : \mathfrak{m} \to \mathfrak{m}$ be any injective mapping whose image is a proper subset of $\mathfrak{m}$. Define $T : L_\infty(\lambda^{\mathfrak{m}}) \to L_\infty(\lambda^{\mathfrak{m}})$ by

$$(Tf)((t_i)_{i\in\mathfrak{m}}) = f((t_{\sigma(i)})_{i\in\mathfrak{m}}).$$

This is clearly an injective unital endomorphism. For $j \in \mathfrak{m}$, let ι_j denote the projection of $[0,1]^{\mathfrak{m}}$ onto the jth factor, that is,

$$\iota_j((t_i)_{i\in\mathfrak{m}}) = t_j.$$

It is clear that ι_j lies in the range of T if and only if j lies in that of σ, and so, T cannot be onto. However, T is a local automorphism. To see this, fix $f \in L_\infty(\lambda^{\mathfrak{m}})$ and let $\mathfrak{m}(f)$ be a countable subset of $\mathfrak{m}$ containing all coordinates on which f depends. Let τ be a bijection of $\mathfrak{m}$ such that $\tau(i) = \sigma(i)$ provided $\sigma(i) \in \mathfrak{m}(f)$. It is clear that the map given by

$$(Ug)((t_i)_{i\in\mathfrak{m}}) = g((t_{\tau(i)})_{i\in\mathfrak{m}})$$

is an automorphism of $L_\infty(\lambda^{\mathfrak{m}})$. Moreover one has

$$(Uf)((t_i)_{i\in\mathfrak{m}}) = f((t_{\tau(i)})_{i\in\mathfrak{m}}) = f((t_{\sigma(i)})_{i\in\mathfrak{m}}) = (Tf)((t_i)_{i\in\mathfrak{m}}).$$

Hence T is a local automorphism (and also a local surjective isometry). This completes the proof. ∎

We can summarize the results of Sections 2 and 3 as follows:

THEOREM 2. *Let μ be a σ-finite measure. The following statements are equivalent*:

(a) *μ is separable.*

(b) *$L_\infty(\mu)$ is *-isomorphic to either $\ell_\infty(\Gamma)$ or $\ell_\infty(\Gamma)\times L_\infty$, where Γ is a countable set.*

(c) *The isometry group of $L_\infty(\mu)$ is reflexive.*

(d) *The automorphism group of $L_\infty(\mu)$ is reflexive.*

Proof. The implication (a)⇒(b) follows from the Maharam decomposition (1). We show that the isometry group of $\ell_\infty(\Gamma)\times L_\infty$ is reflexive. It is clear that every surjective isometry of $\ell_\infty(\Gamma)\times L_\infty$ leaves invariant both $\ell_\infty(\Gamma)$ and L_∞. Hence every local surjective isometry can be decomposed as

$$T(g,f) = (R(g), S(f)) \quad (g\in\ell_\infty(\Gamma),\ f\in L_\infty),$$

where R and S are local surjective isometries of $\ell_\infty(\Gamma)$ and L_∞, respectively. Since the isometry groups of $\ell_\infty(\Gamma)$ and L_∞ are reflexive ([3] and Corollary 1, respectively) the implication obtains.

That (c) implies (d) is trivial. We close the circle by showing that (d) implies (a). Suppose the isometry group of $L_\infty(\mu)$ is reflexive. Then the Maharam decomposition of $L_\infty(\mu)$ contains no factor $L_\infty(\lambda^{\mathfrak{m}})$ with $\mathfrak{m}$ uncountable. Otherwise the local automorphism constructed in Proposition 2 would extend to a non-surjective local automorphism of $L_\infty(\mu)$. But μ is σ-finite, and so, it is necessarily separable. ∎

3.1. *Applications to rearrangement invariant spaces.* The reflexivity of the isometry group for function lattices other than L_p was left open in [6]. We now answer the question in the affirmative, not only for Orlicz spaces (as asked in [6]), but for almost all rearrangement invariant Banach function spaces on $[0,1]$. We refer the reader to [16, Section 2] for the notion of a rearrangement invariant space.

COROLLARY 2. *Let X be a* (*real or complex*) *rearrangement invariant Banach function space on* $[0,1]$ *which is not linearly isomorphic to L_p for $1\le p\le\infty$. Then the isometry group of X is algebraically reflexive.*

Proof. Putting together the results in [31] and [12], it is clear that the hypothesis on X implies that every surjective isometry T of X has the form

$$(Tf)(t) = \sigma(t)f(\varphi(t)),$$

where σ is some unimodular Borel function and φ is a measure preserving automorphism of $[0,1]$.

This obviously implies that if I is a local surjective isometry of X, then I maps L_∞ into itself, and $I : L_\infty \to L_\infty$ is a local surjective isometry for the supremum norm. We know from Corollary 1 that $I(L_\infty) = L_\infty$. Since L_∞ is dense in X we conclude that the isometry $I : X \to X$ must be onto. ∎

In the following result, L_0 stands for the space of all (classes of) measurable functions on $[0,1]$ equipped with the F-norm

$$\|f\|_0 = \int_0^1 \frac{|f(t)|}{1+|f(t)|}\,dt.$$

Using [13, Theorem 4.1] instead of [31, 12] and taking into account that L_∞ is still dense in L_0 one gets the following (see [6, Remark 4]):

COROLLARY 3. *The isometry group of* L_0 *is algebraically reflexive.* ∎

Of course, these results can be generalized for many function spaces in which a quasi-norm, metric or something like it is defined by an integral

$$I[f] = \int_0^1 \varphi(|f(t)|)\,dt,$$

where φ is a suitable function which is not a power. We refrain from entering into further details here. We remark, however, that in view of the construction given in the proof of Proposition 2 one has the following "continuous" analogue of [6, Example 1]:

COROLLARY 4. *Let* X *be a* (*real or complex*) *metric linear space of* (*classes of*) *measurable functions on* $[0,1]^{\mathfrak{m}}$, *where* $\mathfrak{m}$ *is uncountable. If every measure preserving automorphism of* $[0,1]^{\mathfrak{m}}$ *induces an isometry of* X, *then the isometry group of* X *is algebraically non-reflexive.* ∎

4. Purely atomic measures. So far, we have completely settled the problem of reflexivity of the automorphism group of $L_\infty(\mu)$ for σ-finite measures. The simplest non-σ-finite measures are purely atomic measures on uncountable sets, and so, we deal in this section with the automorphism group of the algebra $\ell_\infty(\Gamma)$. It will turn out that the reflexivity of these groups is closely related to "arithmetical" properties of the cardinal of Γ.

Recall that each (linear, continuous) functional on $\ell_\infty(\Gamma)$ can be represented as a finitely additive, finite measure on 2^Γ. To avoid any possible confusion, if μ is such a measure, we write $\langle \mu, f\rangle$ for the value of (the functional represented by) μ at $f \in \ell_\infty(\Gamma)$, while the measure of $A \subset \Gamma$ will be denoted by $\mu(A)$. Needless to say, we have $\langle \mu, 1_A\rangle = \mu(A)$ for all $A \subset \Gamma$.

It is well known (and obvious) that non-zero characters on $\ell_\infty(\Gamma)$ correspond to finitely additive measures that take values in $\{0,1\}$, with $\mu(\Gamma)=1$. These are called *zero-one measures*. Note that if μ is a zero-one measure and f belongs to $\ell_\infty(\Gamma)$, then $\langle \mu, f\rangle = a$ if and only if for each $\varepsilon>0$ one has

$$\mu(\{\gamma\in\Gamma : |f(\gamma)-a|\le\varepsilon\})=1.$$

A measure vanishing on all singletons is said to be *free*. Clearly, a zero-one measure is either free or fixed, that is, has the form

$$\delta_\gamma(A)=\begin{cases}1 & \text{if } \gamma\in A,\\ 0 & \text{if } \gamma\notin A,\end{cases}$$

for some $\gamma\in\Gamma$. These measures are evaluations at points in Γ.

Following [8, Section 12], let us say that Γ *has measurable cardinal* if there exists a zero-one *countably additive* measure on 2^Γ which is free. Otherwise we call the cardinal of Γ *non-measurable*.

No measurable cardinal is known. For instance,

$$\aleph_0,\aleph_1,\ldots,\aleph_\omega,\ldots,\aleph_{\omega_1},\ldots,\aleph_{\omega_\omega},\ldots \quad\text{and}\quad \mathfrak{c},2^{\mathfrak{c}},2^{2^{\mathfrak{c}}},\ldots$$

are all non-measurable ($\mathfrak{c}$ is the continuum). And so are cardinals that can be obtained from given non-measurable cardinals by the standard processes of cardinal arithmetic. As explained in [8], the existence of measurable cardinals cannot be proved in the standard settings of set theory. It is conceivable that no such cardinal exists at all, but as far as we know, this is an unsolved problem.

THEOREM 3. *The group of automorphisms of $\ell_\infty(\Gamma)$ is algebraically reflexive if and only if Γ has non-measurable cardinal.*

Proof. First, we show that if Γ has measurable cardinal, then there is a non-surjective local automorphism of $\ell_\infty(\Gamma)$. The construction is a refinement of the argument given in [3] to show that the group of automorphisms of ℓ_∞ is not topologically reflexive.

Let μ be a free, countably additive, zero-one measure on Γ. Note that the intersection of countably many subsets of measure 1 (with respect to μ) is still of measure 1. Thus, if $\langle\mu,f\rangle=a$, then the set

$$f^{-1}(a)=\{\gamma\in\Gamma: f(\gamma)=a\}=\bigcap_{n=1}^{\infty}\{\gamma\in\Gamma: |f(\gamma)-a|\le 1/n\}$$

has measure 1, and therefore, it is infinite, since μ is free.

Fix $\alpha\in\Gamma$, and take a bijection $\sigma:\Gamma\to\Gamma\backslash\{\alpha\}$. Define $T:\ell_\infty(\Gamma)\to\ell_\infty(\Gamma)$ by

$$Tf(\gamma)=\begin{cases} f(\sigma^{-1}(\gamma)) & \text{if } \gamma\ne\alpha,\\ \langle\mu,f\rangle & \text{if } \gamma=\alpha.\end{cases}$$

We claim that T is a local automorphism. Fix $f \in \ell_\infty(\mathit{\Gamma})$ and let

$$\mathit{\Gamma}_0 = \{\gamma \in \mathit{\Gamma} : f(\gamma) = \langle \mu, f\rangle\}.$$

It is clear that there is a bijection τ of $\mathit{\Gamma}$ such that $\tau(\gamma) = \sigma(\gamma)$ for all $\gamma \notin \mathit{\Gamma}_0$ and $\tau(\mathit{\Gamma}_0) = \sigma(\mathit{\Gamma}_0) \cup \{\alpha\}$. Define $U : \ell_\infty(\mathit{\Gamma}) \to \ell_\infty(\mathit{\Gamma})$ by

$$Ug(\gamma) = g(\tau^{-1}(\gamma)).$$

Obviously, U is an automorphism. Let us prove that $Uf = Tf$, that is,

$$Uf(\gamma) = Tf(\gamma) \qquad (\gamma \in \mathit{\Gamma}). \tag{3}$$

This is obvious if $\gamma = \alpha$. In that case, one has

$$Uf(\alpha) = f(\tau^{-1}(\alpha)) = \langle \mu, f\rangle = Tf(\alpha)$$

because $\tau^{-1}(\alpha) \in \mathit{\Gamma}_0$. If $\gamma \in \sigma(\mathit{\Gamma}_0)$, then both $\sigma^{-1}(\gamma)$ and $\tau^{-1}(\gamma)$ are in $\mathit{\Gamma}_0$ and so

$$Uf(\gamma) = f(\tau^{-1}(\gamma)) = \langle \mu, f\rangle = f(\sigma^{-1}(\gamma)) = Tf(\alpha).$$

Finally, for $\gamma \in \mathit{\Gamma}\backslash\sigma(\mathit{\Gamma}_0 \cup \{\alpha\})$ we have $\sigma^{-1}(\gamma) = \tau^{-1}(\gamma)$ and (3) is obvious. Hence T is a local automorphism.

However, the range of T does not contain 1_α. For if we assume that $Tf = 1_\alpha$, then since T must agree with some automorphism at f we would have $f = 1_\beta$, for some $\beta \in \mathit{\Gamma}$. But $T(1_\beta)(\alpha) = \langle \mu, 1_\beta\rangle = \mu(\{\beta\}) = 0$ for all $\beta \in \mathit{\Gamma}$, a contradiction. This proves the "only if" part of the theorem.

As for the converse, let T be a local automorphism of $\ell_\infty(\mathit{\Gamma})$. By the comments made after Lemma 1, the adjoint map T^* preserves countable additivity. On the other hand, T is a unital endomorphism, and so T^* sends zero-one measures (they are characters) into zero-one measures.

Now, the non-measurability of $\mathit{\Gamma}$ goes at work: a zero-one measure on $\mathit{\Gamma}$ is countably additive if and only if it is fixed. Thus, we can define a mapping $\sigma : \mathit{\Gamma} \to \mathit{\Gamma}$ taking $T^*(\delta_\gamma) = \delta_{\sigma(\gamma)}$. One has

$$Tf(\gamma) = \langle Tf, \delta_\gamma\rangle = \langle f, T^*\delta_\gamma\rangle = \langle f, \delta_{\sigma(\gamma)}\rangle$$

for all $f \in \ell_\infty(\mathit{\Gamma})$ and all $\gamma \in \mathit{\Gamma}$. Hence,

$$T(f) = f \circ \sigma.$$

The injectivity of T implies that σ is onto. It remains to see that σ is injective. Take $\gamma \in \mathit{\Gamma}$. Then

$$T(1_\gamma) = 1_\gamma \circ \sigma = 1_{\sigma^{-1}(\gamma)}$$

and since T is a local automorphism we see that $\sigma^{-1}(\gamma)$ is a singleton. This completes the proof. ∎

Arguing as in the proof of Corollary 1 we obtain the following amendment of [6, Example 3(a)].

Corollary 5. *The isometry group of $\ell_\infty(\Gamma)$ is algebraically reflexive if and only if Γ has non-measurable cardinal.* ∎

5. Concluding remarks. In view of Theorems 2 and 3 one might conjecture that the "mixed" algebra $\ell_\infty(\Gamma, L_\infty)$ has reflexive automorphism group if and only if Γ is non-measurable. Unfortunately, I have been unable to prove any part of that conjecture. Endomorphisms of $\ell_\infty(\Gamma)$ do not extend to (endomorphisms of) $\ell_\infty(\Gamma, L_\infty)$ because the latter algebra is much bigger than $C(\beta\Gamma, L_\infty) = \ell_\infty(\Gamma) \otimes L_\infty$. So, the "obvious" pattern to prove necessity cannot be followed. (Actually, the automorphism group of $\ell_\infty(\mathbb{N}, L(\ell_2))$ is topologically reflexive [20], while that of $\ell_\infty(\mathbb{N})$ is not [3].) As for sufficiency, the main difficulty is that (unlike $\ell_\infty(\Gamma)$) no character on the mixed algebra is weak* continuous—that is, in $\ell_1(\Gamma, L_1)$. Some information about countably additive members of $\ell_\infty(\Gamma, L_\infty)^*$ would be welcome. In essence, this is a problem on (finite, non-negative) countably additive measures on 2^Γ. Call the cardinal of Γ *decent* if every (finite, non-negative) countably additive measure on 2^Γ vanishing on all singletons is null—this is an *a priori* stronger form of non-measurability. *Indecent* cardinals are termed *real-valued measurable* by some authors, but the terminology is not completely standard.

It can be proved that if Γ has decent cardinal, then the isometry group and the automorphism group of $\ell_\infty(\Gamma, L_\infty)$ are reflexive. The key point is that the hypothesis on Γ implies that each countably additive member of $\ell_\infty(\Gamma, L_\infty)^*$ belongs to $\ell_1(\Gamma, L_1)$. We leave the details to the reader because not much is known about decent cardinals. As far as we know, questions around the existence of indecent cardinals stem from the problem of whether Lebesgue measure can be extended to a countably additive measure defined on every subset of the unit interval—"le problème de la mesure". Venerable oldies are Ulam's [30] and Banach–Kuratowski's [2], where it is proved that, under the Continuum Hypothesis (CH), the continuum is decent (and, therefore, that such an extension cannot exist).

The existence of indecent cardinals is unprovable in ZFC (the usual setting of set theory with the axiom of choice). Also, from a constructive viewpoint, all cardinals are decent: Scott proved in [25] that indecent cardinals do not exist assuming the usual axioms (Σ) of set theory and Gödel's constructibility axiom $V = L$. In any case, the basic question is whether or not the continuum is decent: Talamo's work [29] shows that if indecent cardinals do exist, then the continuum is one. So, under CH, all cardinals are decent. For related results in the intermediate world of Martin's Axiom, see [1, 17, 28].

Added in proof. Theorem 3 was obtained earlier (for complex spaces) by K. Jarosz and T. S. S. R. K. Rao, *Local isometries of function spaces*, Math. Z. 243 (2003), 449–469.

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