STUDIA MATHEMATICA 161 (1) (2004)

Ideals in big Lipschitz algebras of analytic functions

by

THOMAS VILS PEDERSEN (Frederiksberg)

Abstract. For $0 < \gamma \le 1$, let Λ_γ^+ be the big Lipschitz algebra of functions analytic on the open unit disc $\mathbb{D}$ which satisfy a Lipschitz condition of order γ on $\overline{\mathbb{D}}$. For a closed set E on the unit circle $\mathbb{T}$ and an inner function Q, let $J_\gamma(E,Q)$ be the closed ideal in Λ_γ^+ consisting of those functions $f \in \Lambda_\gamma^+$ for which

(i) $f = 0$ on E,
(ii) $|f(z) - f(w)| = o(|z-w|^\gamma)$ as $d(z,E), d(w,E) \to 0$,
(iii) $f/Q \in \Lambda_\gamma^+$.

Also, for a closed ideal I in Λ_γ^+, let $E_I = \{z \in \mathbb{T} : f(z) = 0 \text{ for every } f \in I\}$ and let Q_I be the greatest common divisor of the inner parts of non-zero functions in I. Our main conjecture about the ideal structure in Λ_γ^+ is that $J_\gamma(E_I, Q_I) \subseteq I$ for every closed ideal I in Λ_γ^+. We confirm the conjecture for closed ideals I in Λ_γ^+ for which E_I is countable and obtain partial results in the case where $Q_I = 1$. Moreover, we show that every wk* closed ideal in Λ_γ^+ is of the form $\{f \in \Lambda_\gamma^+ : f = 0 \text{ on } E \text{ and } f/Q \in \Lambda_\gamma^+\}$ for some closed set $E \subseteq \mathbb{T}$ and some inner function Q.

1. Introduction. Throughout this paper, we let $0 < \gamma \le 1$ unless otherwise stated and denote all constants by C. Let Λ_γ be the big Lipschitz algebra of functions f on the unit circle $\mathbb{T}$ for which

$$|f(z) - f(w)| \le C|z-w|^\gamma$$

for $z, w \in \mathbb{T}$. Equipped with the norm

$$\|f\|_{\Lambda_\gamma} = \|f\|_\infty + \sup\left\{\frac{|f(z)-f(w)|}{|z-w|^\gamma} : z, w \in \mathbb{T},\ z \ne w\right\} \qquad (f \in \Lambda_\gamma),$$

it is well known to be a Banach algebra. We shall be concerned with the closed subalgebra

$$\Lambda_\gamma^+ = \{f \in \Lambda_\gamma : \widehat{f}(n) = 0 \text{ for } n < 0\}$$

of Λ_γ (where $\widehat{f}(n)$ is the nth Fourier coefficient of f). Since every function in Λ_γ^+ has an extension to a function analytic in the open unit disc $\mathbb{D}$, we

2000 *Mathematics Subject Classification*: 46J20, 46J15, 30H05.

This work was carried out at Université Bordeaux 1 while the author was holding a TMR Marie Curie postdoctoral grant from the European Commission.

deduce that

$$\Lambda_\gamma^+ = \Lambda_\gamma \cap \mathcal{A}(\overline{\mathbb{D}}),$$

where $\mathcal{A}(\overline{\mathbb{D}})$ is the usual disc algebra. Moreover, a function f analytic on $\mathbb{D}$ belongs to Λ_γ^+ if and only if

$$(1) \qquad |f'(z)| \le C(1-|z|)^{\gamma-1} \quad (z \in \mathbb{D}),$$

and

$$\|f\|_{\Lambda_\gamma^+} = \|f\|_\infty + \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|)^{1-\gamma} \quad (f \in \Lambda_\gamma^+)$$

defines an equivalent norm on Λ_γ^+ ([3, Theorem 5.1]). In particular, we have $f \in \Lambda_1^+$ if and only if $f' \in \mathcal{H}^\infty$ (the algebra of bounded analytic functions on $\mathbb{D}$). In passing, we mention that Dyakonov ([4]) has shown that

$$\|f\|_\infty + \sup\left\{\frac{\big||f(z)|-|f(w)|\big|}{|z-w|^\gamma} : z, w \in \overline{\mathbb{D}},\ z \ne w\right\} \quad (f \in \Lambda_\gamma^+)$$

defines an equivalent norm on Λ_γ^+. This is a remarkable result since this norm only depends on the moduli of the functions. However, for practical purposes the norm $\|\cdot\|_{\Lambda_\gamma^+}$ is easier to estimate.

In this paper, we describe certain closed ideals in Λ_γ^+ by means of zero sets and inner functions. For $f \in \Lambda_\gamma^+$, let

$$Z(f) = \{z \in \overline{\mathbb{D}} : f(z) = 0\}$$

be the zero set of f (counting multiplicities on $\mathbb{D}$). Also, for a closed ideal I in Λ_γ^+, let

$$Z_I = \bigcap_{f\in I} Z(f)$$

be the hull of I, let

$$E_I = Z_I \cap \mathbb{T}$$

and let Q_I be the greatest common divisor of the inner parts of non-zero functions in I ([6, p. 85]). We shall use the following result of Havin and Shamoyan several times. (See, for instance, [15].)

Theorem 1.1. *If $f \in \Lambda_\gamma^+$ and Q is an inner function for which $f/Q \in \mathcal{H}^\infty$, then $f/Q \in \Lambda_\gamma^+$ and*

$$\|f/Q\|_{\Lambda_\gamma^+} \le C\|f\|_{\Lambda_\gamma^+}.$$

In particular, if f belongs to a closed ideal I in Λ_γ^+, then $f/Q_I \in \Lambda_\gamma^+$.

Recall that a closed set $E \subseteq \mathbb{T}$ is called a *Carleson set* if

$$\int_{\mathbb{T}} \log d(e^{i\theta}, E)\, d\theta > -\infty.$$

Carleson ([2, Theorem 1]) proved that E is a Carleson set if and only if there exists a function $f \in \Lambda_\gamma^+$ with $E = Z(f)$. In this case

$$I_\gamma(E) = \{f \in \Lambda_\gamma^+ : f = 0 \text{ on } E\}$$

is a closed ideal in Λ_γ^+ with $E_{I_\gamma(E)} = E$ and $Q_{I_\gamma(E)} = 1$. Now, let $Q = BS$ be an inner function, where B is a Blaschke product and S a singular inner function. Let $Z(B)$ be the zeros of B (in $\mathbb{D}$) and let $\mathrm{supp}(S)$ be the support of the singular measure on $\mathbb{T}$ that defines S. It follows from [9, Theorems 2 and 4] that there exists a function $f \in I_\gamma(E)$ with inner factor Q if and only if

$$\begin{cases} \int_{\mathbb{T}} \log d(e^{i\theta}, E \cup Z(B))\, d\theta > -\infty, \\ \mathrm{supp}(S) \subseteq E, \\ \overline{Z(B)} \setminus Z(B) \subseteq E. \end{cases} \tag{2}$$

In this case $f/Q \in \Lambda_\gamma^+$ by the previous theorem and

$$I_\gamma(E, Q) = \{f \in I_\gamma(E) : f/Q \in \Lambda_\gamma^+\}$$

is a closed ideal in Λ_γ^+ with $E_{I_\gamma(E,Q)} = E$ and $Q_{I_\gamma(E,Q)} = Q$. Clearly, $I_\gamma(E, Q)$ is the largest closed ideal I in Λ_γ^+ with $E_I = E$ and $Q_I = Q$.

For $0 < \gamma < 1$, our results are motivated by the ideal structure in the little Lipschitz algebra λ_γ^+, which is the closed subalgebra of Λ_γ^+ of functions f satisfying

$$|f(z) - f(w)| = o(|z - w|^\gamma)$$

uniformly as $|z - w| \to 0$. Matheson ([11]) showed that

$$I = \{f \in \lambda_\gamma^+ : f = 0 \text{ on } E_I \text{ and } f/Q_I \in \mathcal{H}^\infty\} = I_\gamma(E_I, Q_I) \cap \lambda_\gamma^+$$

for every closed ideal I in λ_γ^+. In the non-separable algebra Λ_γ^+, it is not possible to obtain such a result. This is most easily seen for $\gamma = 1$. Let χ be a character on $\mathcal{H}^\infty$ belonging to the fiber at $z = 1$, that is, $\chi(\alpha) = 1$, where α denotes the function $z \mapsto z$ (see, for example, [6, Chapter 10]). Then

$$I_\chi = \{f \in I_1(\{1\}) : \chi(f') = 0\}$$

is a closed ideal in Λ_1^+ with $E_{I_\chi} = \{1\}$ and $Q_{I_\chi} = 1$. Moreover, $I_{\chi_1} \neq I_{\chi_2}$ if $\chi_1 \neq \chi_2$. Similarly, for $0 < \gamma < 1$, we shall see that there are uncountably many closed ideals I in Λ_γ^+ with $E_I = \{1\}$ and $Q_I = 1$. Nevertheless, we shall obtain certain results about the ideal structure in Λ_γ^+.

In the algebra Λ_γ on $\mathbb{T}$, Sherbert ([14, Theorem 5.1]) proved that, for a closed set $E \subseteq \mathbb{T}$, the closed ideal

$$\{f \in \Lambda_\gamma : f = 0 \text{ on } E \text{ and } |f(z) - f(w)| = o(|z - w|^\gamma) \text{ as } d(z, E), d(w, E) \to 0\}$$

is the smallest closed ideal in Λ_γ which has E as hull. We shall prove a similar result for Λ_γ^+. For a Carleson set $E \subseteq \mathbb{T}$, let

$$J_\gamma(E) = \{f \in I_\gamma(E) : |f(z) - f(w)| = o(|z-w|^\gamma) \text{ as } d(z,E), d(w,E) \to 0\}.$$

It is easily seen that $J_\gamma(E)$ is a closed ideal in Λ_γ^+. Also, for a closed set $E \subseteq \mathbb{T}$ and an inner function Q satisfying (2), let

$$J_\gamma(E,Q) = \{f \in J_\gamma(E) : f/Q \in \mathcal{H}^\infty\}.$$

It follows from Theorem 1.1 that $J_\gamma(E,Q)$ is a closed ideal in Λ_γ^+, and $E_{J_\gamma(E,Q)} = E$ and $Q_{J_\gamma(E,Q)} = Q$ by [9, Theorem 4]. The main result in this paper is that the following conjecture holds when E_I is countable.

Conjecture. *Let I be a closed ideal in Λ_γ^+. Then $J_\gamma(E_I, Q_I) \subseteq I$.*

The proof of Matheson's result (and of other similar results in separable algebras—see, for instance, [1], [10] and [16]) was to a high extent based on the so-called Carleman transform. (See the next section for the definition.) Apparently, Hedenmalm ([5]) was the first to apply the Carleman transform to a non-separable Banach algebra, when he obtained certain results about the ideal structure in the algebra $\mathcal{H}^\infty$.

The proof of our main result uses the Carleman transform and ideas by Bennett and Gilbert ([1]). The Carleman transform of a linear functional φ depends only on the restriction of φ to the separable subalgebra λ_γ^+ and we therefore find it interesting that it can be used to obtain results about Λ_γ^+. Moreover, we use a representation of the Carleman transform which is different from the one used in [1], and by following the lines of our proof, one can actually obtain a simpler proof of the main result in [1].

The organization of the paper is as follows. We first obtain some basic facts about the Carleman transform (Section 2) and the ideal $J_\gamma(E,Q)$ (Section 3). In Section 4 we prove our main result, and in Section 5 we partially confirm our conjecture for closed ideals I in Λ_γ^+ with $Q_I = 1$. Finally, in Section 6 we show that the wk* closed ideals in Λ_γ^+ are exactly the ideals $I_\gamma(E,Q)$, where the closed set $E \subseteq \mathbb{T}$ and the inner function Q satisfy (2).

2. The Carleman transform. For $\varphi \in (\Lambda_\gamma^+)^*$, we define the *Carleman transform* Φ of φ on $\mathbb{C} \setminus \overline{\mathbb{D}}$ by

$$\Phi(z) = \langle (z-\alpha)^{-1}, \varphi \rangle \quad (z \in \mathbb{C} \setminus \overline{\mathbb{D}}).$$

With $\widehat{\varphi}(n) = \langle \alpha^n, \varphi \rangle$ for $n \in \mathbb{N}_0$, we have

$$\Phi(z) = \sum_{n=0}^{\infty} \widehat{\varphi}(n) z^{-(n+1)} \quad (z \in \mathbb{C} \setminus \overline{\mathbb{D}}).$$

For $f \in \Lambda_\gamma^+$ and $0 < r < 1$, let $f_r(z) = f(rz)$ $(z \in \overline{\mathbb{D}})$. For notational convenience, let

$$\lambda_1^+ = \{f \in \Lambda_1^+ : f' \in \mathcal{A}(\overline{\mathbb{D}})\}.$$

For $f \in \lambda_\gamma^+$, it is well known (see, for example, [8, I.2.13]) that $f_r \to f$ in λ_γ^+ as $r \to 1_-$. Hence

$$\begin{aligned}\langle f, \varphi\rangle &= \lim_{r\to 1_-} \langle f_r, \varphi\rangle = \lim_{r\to 1_-} \sum_{n=0}^{\infty} \widehat{f}(n) r^n \widehat{\varphi}(n) \\ &= \lim_{s\to 1_+} \frac{1}{2\pi} \int_{\mathbb{T}} f(e^{i\theta}) e^{i\theta} \Phi(se^{i\theta})\, d\theta\end{aligned}$$

and this was used by Matheson in his proof. However, for $f \in \Lambda_\gamma^+ \setminus \lambda_\gamma^+$, we do not have $f_r \to f$ in Λ_γ^+ as $r \to 1_-$, so this method does not work in our case.

Let I be a closed ideal in Λ_γ^+, let

$$I^\perp = \{\varphi \in (\Lambda_\gamma^+)^* : \langle f, \varphi\rangle = 0 \text{ for every } f \in I\}$$

be the annihilator of I and let $\pi : \Lambda_\gamma^+ \to \Lambda_\gamma^+/I$ be the quotient map. Suppose that $\varphi \in I^\perp$ $(= (\Lambda_\gamma^+/I)^*)$. It is well known that the character space of the algebra Λ_γ^+/I equals Z_I, so the spectrum of $\pi(\alpha)$ equals Z_I and the function

$$\Phi(z) = \langle (z - \pi(\alpha))^{-1}, \varphi\rangle \qquad (z \in \mathbb{C} \setminus Z_I)$$

thus extends the domain of Φ to $\mathbb{C} \setminus Z_I$.

For $f \in \Lambda_\gamma^+$ and $z \in \mathbb{D}$, define $S_z f$ by

$$(S_z f)(w) = \begin{cases} \dfrac{f(z) - f(w)}{z - w} & \text{for } w \in \overline{\mathbb{D}} \setminus \{z\}, \\ f'(z) & \text{for } w = z. \end{cases}$$

Then $S_z f \in \Lambda_\gamma \cap \mathcal{A}(\overline{\mathbb{D}}) = \Lambda_\gamma^+$. It is easily seen that

$$\|(z - \alpha)^{-1}\|_{\Lambda_\gamma} \le C\big|1 - |z|\big|^{-(1+\gamma)} \qquad (z \in \mathbb{C} \setminus \mathbb{T}), \tag{3}$$

so we have

$$\|S_z f\|_{\Lambda_\gamma^+} \le C(1 - |z|)^{-(1+\gamma)} \qquad (z \in \mathbb{D}). \tag{4}$$

We shall often use the following representation of Φ.

LEMMA 2.1. *Let I be a closed ideal in Λ_γ^+ and let $\varphi \in I^\perp$. Then*

$$\Phi(z) = \frac{\langle S_z g, \varphi\rangle}{g(z)} \qquad (z \in \mathbb{D} \setminus Z(g))$$

for $g \in I$.

Proof. For $g \in I$ and $z \in \mathbb{D} \setminus Z(g)$, we have $(z - \alpha) S_z g = g(z) - g$ and thus $(z - \pi(\alpha))^{-1} = \pi(S_z g)/g(z)$, so the result follows. ∎

The normal approach to the Carleman transform (see, for example, [1], [10], [11] and [16]) is to define Φ on $\mathbb{D} \setminus Z_I$ by the expression $\Phi(z) = \langle S_z g, \varphi\rangle / g(z)$ and then show that Φ extends analytically to $\mathbb{C} \setminus Z_I$. With the present definition, we obtained this as an immediate consequence of the general fact from Banach algebra theory that the character space of the algebra Λ_γ^+/I equals Z_I.

The following result is similar to [1, Theorem 2.4].

LEMMA 2.2. *Let I be a closed ideal in Λ_γ^+ and let $\varphi \in I^\perp$. Suppose that $z_0 \in Z_I \cap \mathbb{D}$ is of multiplicity k. Then Φ has a pole of order at most k at z_0.*

Proof. There exist $g \in I$ and $h \in \Lambda_\gamma^+$ with $h(z_0) \neq 0$ such that $g = (\alpha - z_0)^k h$. By the previous lemma, we thus have

$$(z - z_0)^k \Phi(z) = (z - z_0)^k \frac{\langle S_z g, \varphi\rangle}{g(z)} = \frac{\langle S_z g, \varphi\rangle}{h(z)}$$

for z in a neighborhood of z_0, which proves the lemma. ∎

For $\varphi \in (\Lambda_\gamma^+)^*$ and $f \in \Lambda_\gamma^+$, we define $\varphi_f (= f\varphi) \in (\Lambda_\gamma^+)^*$ by

$$\langle g, \varphi_f\rangle = \langle fg, \varphi\rangle \quad (g \in \Lambda_\gamma^+).$$

If I is a closed ideal in Λ_γ^+ and $\varphi \in I^\perp$, then $\varphi_f \in I^\perp$ for $f \in \Lambda_\gamma^+$. We denote the Carleman transform of φ_f by Φ_f. Whereas Φ depends only on the restriction of φ to λ_γ^+, the function Φ_f depends only on the restriction of φ to the subalgebra $\lambda_\gamma^+ f$ of Λ_γ^+. Heuristically, this is the reason why the Carleman transform can be successfully applied to the non-separable algebra Λ_γ^+.

LEMMA 2.3. *Let $f \in \Lambda_\gamma^+$, let I be a closed ideal in Λ_γ^+ and let $\varphi \in I^\perp$. Then*

$$\Phi_f(z) = f(z)\Phi(z) - \langle S_z f, \varphi\rangle$$

for $z \in \mathbb{D} \setminus Z_I$.

Proof. Let $z \in \mathbb{D} \setminus Z_I$ and choose $g \in I$ such that $g(z) \neq 0$. Since $gS_z f \in I$, we have

$$\begin{aligned}\Phi_f(z) - f(z)\Phi(z) &= \frac{\langle S_z g, \varphi_f\rangle - f(z)\langle S_z g, \varphi\rangle}{g(z)} \\ &= \frac{\langle (f - f(z))S_z g, \varphi\rangle}{g(z)} = \frac{\langle (g - g(z))S_z f, \varphi\rangle}{g(z)} = -\langle S_z f, \varphi\rangle\end{aligned}$$

as required. ∎

3. The ideal $J_\gamma(E, Q)$. In this section, we prove some basic facts about $J_\gamma(E, Q)$. In order to use the characterization (1) of Λ_γ^+, we need to describe $J_\gamma(E)$ in terms of derivatives.

PROPOSITION 3.1. *For a closed set* $E \subseteq \mathbb{T}$ *and* $f \in \Lambda_\gamma^+$, *the following conditions are equivalent*:

(a) $f \in J_\gamma(E)$.
(b) $f \in I_\gamma(E)$ *and* $|f'(z)| = o((1-|z|)^{\gamma-1})$ *as* $d(z,E) \to 0$.

Proof. (a)⇒(b). Given $\varepsilon > 0$, we choose $\delta > 0$ such that $|f(z)-f(w)| < \varepsilon|z-w|^\gamma$ for $z, w \in \overline{\mathbb{D}}$ with $d(z,E), d(w,E) < \delta$. Let $z \in \mathbb{D}$ with $d(z,E) < \delta/2$ and let $r = 1-|z| < \delta/2$. Then $d(w,E) < \delta$ for $|w-z| = r$, so Cauchy's formula

$$f'(z) = \frac{1}{2\pi i} \oint_{|w-z|=r} \frac{f(w)-f(z)}{(w-z)^2}\,dw$$

shows that $|f'(z)| < \varepsilon r^{\gamma-1}$ as required.

(b)⇒(a). Let $\varepsilon > 0$ and choose $\delta_1 > 0$ such that $|f'(z)| < \varepsilon(1-|z|)^{\gamma-1}$ for $z \in \mathbb{D}$ with $d(z,E) < \delta_1$. Choose $\delta_2 > 0$ such that $|f(z)| < \varepsilon\delta_1^\gamma$ for $z \in \overline{\mathbb{D}}$ with $d(z,E) < \delta_2$ and let $\delta = \min\{\delta_1, \delta_2\}$. Let $z_1, z_2 \in \overline{\mathbb{D}}$ with $d(z_k, E) < \delta/3$ $(k=1,2)$. If $|z_2 - z_1| \ge \delta_1/3$, then

$$|f(z_2) - f(z_1)| < 2\varepsilon\delta_1^\gamma \le 2\cdot 3^\gamma \varepsilon |z_2-z_1|^\gamma,$$

so we may assume that $|z_2 - z_1| < \delta_1/3$. With $z_k = r_k e^{i\theta_k}$ $(k=1,2)$, we may also assume that $r_1 \ge r_2$ and that $0 \le \theta_2 - \theta_1 \le \pi$. First, suppose that $|z_2 - z_1| \le 1 - r_1$. Since $d(w,E) < \delta$ and $|w| \le r_1$ for every point w on the line segment from z_1 to z_2, we deduce that

$$|f(z_2) - f(z_1)| < |z_2 - z_1|\varepsilon(1-r_1)^{\gamma-1} \le \varepsilon|z_2-z_1|^\gamma.$$

Now, suppose that $|z_2 - z_1| \ge 1 - r_1$. Let $\varrho = 1 - |z_2 - z_1|$ and let Γ be the curve

$$\Gamma = \{re^{i\theta_1} : \varrho \le r \le r_1\} \cup \{\varrho e^{i\theta} : \theta_1 \le \theta \le \theta_2\}$$
$$\cup\, \{re^{i\theta_2} : r \text{ is between } \varrho \text{ and } r_2\}.$$

Then $d(w,E) < \delta$ for $w \in \Gamma$, so

$$|f(\varrho e^{i\theta_1}) - f(z_1)| < \varepsilon \int_\varrho^1 (1-r)^{\gamma-1}\,dr = (\varepsilon/\gamma)|z_2-z_1|^\gamma.$$

Similarly, if $\varrho \le r_2$, then

$$|f(z_2) - f(\varrho e^{i\theta_2})| < \varepsilon|z_2-z_1|^\gamma.$$

If $\varrho \ge r_2$, then

$$\begin{aligned}|f(z_2) - f(\varrho e^{i\theta_2})| &< \varepsilon \int_{r_2}^{\varrho} (1-r)^{\gamma-1}\,dr \le (\varepsilon/\gamma)((1-r_2)^\gamma - (1-r_1)^\gamma) \\ &\le \varepsilon(r_1-r_2)^\gamma \le \varepsilon|z_2-z_1|^\gamma.\end{aligned}$$

Moreover,

$$|f(\varrho e^{i\theta_2}) - f(\varrho e^{i\theta_1})| < \varepsilon(\theta_2 - \theta_1)(1-\varrho)^{\gamma-1} \le C\varepsilon|z_2 - z_1|^\gamma,$$

so we obtain

$$|f(z_2) - f(z_1)| < (C+2)\varepsilon|z_2 - z_1|^\gamma$$

as required. ∎

For $\gamma = 1$, the previous proposition takes the following form. Let

$$\mathcal{H}_E^\infty = \{f \in \mathcal{H}^\infty : f'(z) \to 0 \text{ as } d(z,E) \to 0\}$$

for a closed set $E \subseteq \mathbb{T}$.

Corollary 3.2. *For a closed set $E \subseteq \mathbb{T}$ and $f \in \Lambda_1^+$, we have $f \in J_1(E)$ if and only if $f \in I_1(E)$ and $f' \in \mathcal{H}_E^\infty$.*

We shall use the notation

$$J_{\gamma,0} = J_\gamma(\{1\}), \qquad I_{\gamma,0} = I_\gamma(\{1\}).$$

Also, for $s > 0$, let ψ_{-s} be the singular inner function defined by

$$\psi_{-s}(z) = \exp\left(-s\,\frac{1+z}{1-z}\right) \qquad (z \in \overline{\mathbb{D}} \setminus \{1\})$$

and write

$$J_{\gamma,s} = J_\gamma(\{1\}, \psi_{-s}), \qquad I_{\gamma,s} = I_\gamma(\{1\}, \psi_{-s}).$$

For $n \in \mathbb{N}$, let

$$K_n = \frac{1-\alpha}{1+1/n-\alpha}.$$

For many separable Banach algebras of analytic functions on $\mathbb{D}$, it is well known that the sequence (K_n) is an approximate identity for the maximal ideal of functions vanishing at $z = 1$. In our case, the local condition at $z = 1$ imposed on functions in $J_{\gamma,0}$ enables us to prove the following result.

Lemma 3.3. *For $f \in J_{\gamma,0}$, we have $K_n f \to f$ in Λ_γ^+ as $n \to \infty$. In particular, for $\gamma < 1$, the sequence (K_n) is an approximate identity for the ideal $J_{\gamma,0}$.*

Proof. Let $f \in J_{\gamma,0}$ and let $p_n = 1 - K_n = n^{-1}(1+1/n-\alpha)^{-1}$ $(n \in \mathbb{N})$. Since $p_n \to 0$ uniformly on compact subsets of $\overline{\mathbb{D}} \setminus \{1\}$ as $n \to \infty$, it follows that

$$\sup_{z\in\mathbb{D}} |p_n(z) f'(z)|(1-|z|)^{1-\gamma} \to 0$$

as $n \to \infty$. Also,

$$\begin{aligned}|p_n'(z)f(z)|(1-|z|)^{1-\gamma} &\le \varepsilon(|1-z|)\left|\frac{1-z}{n(1+1/n-z)^2}\right| \\ &= \varepsilon(|1-z|)\left|\frac{1-z}{1+1/n-z}\right|\left|\frac{1}{n(1+1/n-z)}\right| \\ &\le \varepsilon(|1-z|) \quad (z \in \overline{\mathbb{D}}),\end{aligned}$$

where $\varepsilon(t) \to 0$ as $t \to 0$. Since

$$\frac{1-z}{n(1+1/n-z)^2} \to 0$$

uniformly on compact subsets of $\overline{\mathbb{D}} \setminus \{1\}$ as $n \to \infty$, it thus follows that

$$\sup_{z\in\mathbb{D}} |p_n'(z)f(z)|(1-|z|)^{1-\gamma} \to 0$$

as $n \to \infty$. Hence $p_n f \to 0$ in Λ_γ^+ as $n \to \infty$. ∎

We finish this section with a description of the ideals $J_{\gamma,s}$ in terms of generators.

Lemma 3.4. *Let $\beta, s > 0$ and let $f = (1-\alpha)^\beta \psi_{-s}$. Then $f \in \Lambda_\gamma^+$ if and only if $\beta \ge 2\gamma$ and $f \in J_{\gamma,s}$ if and only if $\beta > 2\gamma$.*

Proof. We have

$$f' = -\beta(1-\alpha)^{\beta-1}\psi_{-s} - 2s(1-\alpha)^{\beta-2}\psi_{-s}.$$

For $z \in \mathbb{D}$, we write $1 - z = re^{i\theta}$. Then $1 - |z|^2 = r(2\cos\theta - r)$, so $2\cos\theta > r > 0$. Also,

$$\operatorname{Re}\left(\frac{1+z}{1-z}\right) = \frac{2\cos\theta}{r} - 1,$$

so

$$\begin{aligned}&|1-z|^{\beta-2}|\psi_{-s}(z)|(1-|z|^2)^{1-\gamma} \\ &\quad = r^{\beta-2}\exp\left(-s\left(\frac{2\cos\theta}{r}-1\right)\right)(r(2\cos\theta - r))^{1-\gamma} \\ &\quad = r^{\beta-2\gamma}\exp\left(-s\left(\frac{2\cos\theta}{r}-1\right)\right)\left(\frac{2\cos\theta}{r}-1\right)^{1-\gamma},\end{aligned}$$

and the result follows. ∎

Proposition 3.5. (a) *For $\beta > \gamma$, we have*

$$J_{\gamma,0} = \overline{\Lambda_\gamma^+(1-\alpha)^\beta}.$$

(b) *For $s > 0$ and $\beta > 2\gamma$, we have*

$$J_{\gamma,s} = \overline{\Lambda_\gamma^+(1-\alpha)^\beta\psi_{-s}}.$$

Proof. (a) Since $(1-\alpha)^\beta \in J_{\gamma,0}$, we have the inclusion $\overline{\Lambda_\gamma^+(1-\alpha)^\beta} \subseteq J_{\gamma,0}$. Moreover, it follows from Lemma 3.3 that $J_{\gamma,0} = \overline{J_{\gamma,0}(1-\alpha)}$ and thus $J_{\gamma,0} \subseteq \overline{\Lambda_\gamma^+(1-\alpha)^m}$ for $m \in \mathbb{N}$, which proves the reverse inclusion.

(b) It follows from the previous lemma that

$$\overline{\Lambda_\gamma^+(1-\alpha)^\beta\psi_{-s}} \subseteq J_{\gamma,s}.$$

Conversely, let $f \in J_{\gamma,s}$. By Lemma 3.3, we have $K_n f \to f$ in Λ_γ^+ as $n \to \infty$. Fix $n \in \mathbb{N}$ and let $g = K_n f$. Let $0 < a < 1$ and let

$$T^\varepsilon(z) = \exp(-\varepsilon(1-z)^{-a}) \quad (z \in \overline{\mathbb{D}}).$$

Then $|T| \le 1$ on $\overline{\mathbb{D}}$ and

$$T'(z) = -a(1-z)^{-(a+1)}T(z) \quad (z \in \overline{\mathbb{D}}),$$

so (T^ε) is a semigroup of outer functions in Λ_γ^+. (In Section 6, we shall make use of a more general version of this semigroup.) Moreover,

$$(T^\varepsilon g)' = T^\varepsilon g' + \varepsilon(T'/T)T^\varepsilon g \quad (\varepsilon > 0).$$

Since $T^\varepsilon \to 1$ uniformly on compact subsets of $\overline{\mathbb{D}} \setminus \{1\}$ and since $|g(z)| \le C|1-z|^{\gamma+1}$ for $z \in \overline{\mathbb{D}}$, we deduce that $T^\varepsilon g \to g$ in Λ_γ^+ as $\varepsilon \to 0$. Finally, using (a), we choose a sequence (g_m) in Λ_γ^+ such that

$$g_m(1-\alpha)^\beta \to g/\psi_{-s}$$

in Λ_γ^+ as $m \to \infty$. It is easily seen that $T^\varepsilon\psi_{-s} \in \Lambda_\gamma^+$, so

$$T^\varepsilon g_m(1-\alpha)^\beta\psi_{-s} \to T^\varepsilon g$$

in Λ_γ^+ as $m \to \infty$ for $\varepsilon > 0$, and it follows that $f \in \overline{\Lambda_\gamma^+(1-\alpha)^\beta\psi_{-s}}$. ∎

4. Ideals with countable hull. Our main aim in this paper is to prove the following result.

THEOREM 4.1. *We have*

$$J_\gamma(E_I, Q_I) \subseteq I$$

for every closed ideal I in Λ_γ^+ for which E_I is countable.

Before proceeding to the proof of the theorem, we present a few consequences. It follows from Theorem 1.1 that if $f \in I_\gamma(E,Q)$, then $f/Q \in I_\gamma(E)$. We do not know whether the corresponding result for $J_\gamma(E,Q)$ holds in general, but for E countable it follows easily from the theorem.

COROLLARY 4.2. *Suppose that a closed set $E \subseteq \mathbb{T}$ and an inner function Q satisfy* (2) *and that E is countable. If $f \in J_\gamma(E,Q)$, then $f/Q \in J_\gamma(E)$.*

Proof. Consider the closed ideal

$$I = \{f \in J_\gamma(E,Q) : f/Q \in J_\gamma(E)\}$$

in Λ_γ^+. We have $E_I = E$ and $Q_I = Q$, so the previous theorem entails that $J_\gamma(E,Q) \subseteq I$ and the conclusion follows. ∎

For $\gamma = 1$, Theorem 4.1 can be restated as follows with the use of Corollary 3.2.

COROLLARY 4.3. *Let I be a closed ideal I in Λ_1^+ and suppose that E_I is countable. Then*

$$\{f \in I_1(E_I, Q_I) : f' \in \mathcal{H}^\infty_{E_I}\} \subseteq I.$$

For the primary ideals, more can be said.

COROLLARY 4.4. *Let $s \geq 0$. The closed ideals I in Λ_γ^+ with $E_I = \{1\}$ and $Q_I = \psi_{-s}$ are exactly the closed subspaces I of Λ_γ^+ with*

$$J_{\gamma,s} \subseteq I \subseteq I_{\gamma,s}.$$

Proof. Let I be a closed subspace of Λ_γ^+ with $J_{\gamma,s} \subseteq I \subseteq I_{\gamma,s}$. For $f \in \Lambda_\gamma^+$ and $g \in I$, we have

$$(f - f(1))g \in I_{\gamma,0} \cdot I_{\gamma,s} \subseteq J_{\gamma,s} \subseteq I,$$

so $fg \in I$ and the result follows. ∎

In his paper [5] on the ideal structure in $\mathcal{H}^\infty$, Hedenmalm stated the following result, which is now easily deduced from our results.

COROLLARY 4.5. *Let I be a closed ideal in Λ_1^+ with $E_I = \{1\}$ and $Q_I = 1$. Then there is a closed subspace $\mathcal{Z}$ in $\mathcal{H}^\infty$ containing $\mathcal{H}^\infty_{\{1\}}$ such that*

$$I = \{f \in I_1(\{1\}) : f' \in \mathcal{Z}\}.$$

Conversely, every such set I is a closed ideal in Λ_1^+ with $E_I = \{1\}$ and $Q_I = 1$.

Proof. For $f \in I_1(\{1\})$, we have $\|f\|_\infty \leq 2\|f'\|_\infty$, so $f \mapsto \|f'\|_\infty$ defines a norm on $I_1(\{1\})$ which is equivalent to the Λ_1^+ norm. Hence $I \mapsto I' = \{f' : f \in I\}$ defines a bijective correspondence between the closed subspaces I in Λ_1^+ with $J_1(\{1\}) \subseteq I \subseteq I_1(\{1\})$ and the closed subspaces $\mathcal{Z}$ in $\mathcal{H}^\infty$ with $\mathcal{H}^\infty_{\{1\}} \subseteq \mathcal{Z}$, so the result follows from the previous corollary. ∎

Finally, we shall show that there are uncountably many closed ideals between $J_{\gamma,s}$ and $I_{\gamma,s}$.

LEMMA 4.6. *Let $f_s = (1-\alpha)^{2\gamma}\psi_{-s}$ $(s > 0)$. For $0 < t_0 < s_0$, we have*

$$\|f_s - f_t\|_{\Lambda_\gamma^+/J_{\gamma,0}} \geq C$$

for $t_0 \leq t < s \leq s_0$.

Proof. We have

$$\|f\|_{\Lambda_\gamma^+/J_{\gamma,0}} \geq \limsup_{z\to 1} |f'(z)|(1-|z|)^{1-\gamma}$$

for $f \in \Lambda_\gamma^+$. Also,

$$f_s' = -2\gamma(1-\alpha)^{2\gamma-1}\psi_{-s} - 2s(1-\alpha)^{2\gamma-2}\psi_{-s},$$

so

$$\|f_s - f_t\|_{\Lambda_\gamma^+/J_{\gamma,0}} \geq \limsup_{z\to 1} \bigl|2(s-t)(1-z)^{2\gamma-2}\psi_{-s}(z) + 2t(1-z)^{2\gamma-2}(\psi_{-s}(z)-\psi_{-t}(z))\bigr|(1-|z|)^{1-\gamma}.$$

As in the proof of Lemma 3.4, we write $1-z = re^{i\theta}$ for $z \in \mathbb{D}$. Then

$$\operatorname{Im}\left(\frac{1+z}{1-z}\right) = \frac{2\sin\theta}{r},$$

so there exists a sequence (z_n) tending to 1 such that

$$\operatorname{Im}\left(\frac{1+z_n}{1-z_n}\right) = \frac{(2n+1)\pi}{s-t}$$

and thus

$$|\psi_{-(s-t)}(z_n) - 1| \geq 1 - \operatorname{Re}\psi_{-(s-t)}(z_n) \geq 1.$$

It thus follows from the proof of Lemma 3.4 that

$$\limsup_{n\to\infty} \bigl|(1-z_n)^{2\gamma-2}(\psi_{-s}(z_n)-\psi_{-t}(z_n))\bigr|(1-|z_n|)^{1-\gamma} \geq C.$$

Hence there exists $\delta > 0$ such that

$$\|f_s - f_t\|_{\Lambda_\gamma^+/J_{\gamma,0}} \geq t_0 C$$

for $0 < s-t < \delta$ and the result follows. ∎

COROLLARY 4.7. *For $s \geq 0$, there are uncountably many closed ideals I in Λ_γ^+ with $J_{\gamma,s} \subseteq I \subseteq I_{\gamma,s}$.*

Proof. The inclusion map $\iota : I_{\gamma,s} \to I_{\gamma,0}$ induces a bounded linear map $\widetilde{\iota} : I_{\gamma,s}/J_{\gamma,s} \to I_{\gamma,0}/J_{\gamma,0}$. Since $f_t \in I_{\gamma,s}$ for $t \geq s$, we deduce from the previous lemma that $I_{\gamma,s}/J_{\gamma,s}$ is non-separable, so the result follows from Corollary 4.4. ∎

We now turn to the proof of Theorem 4.1. Recall the following definitions (with a few modifications) from [1]:

H_+: consists of the analytic functions f on $\mathbb{D}$ for which $|f(z)| \leq C(1-|z|)^{-N}$ for $z \in \mathbb{D}$ for some $N \in \mathbb{N}$,

H_-: consists of the analytic functions f on $\mathbb{C}\setminus\overline{\mathbb{D}}$ with $|f(z)| \leq C(|z|-1)^{-N}$ for $z \in \mathbb{C}\setminus\overline{\mathbb{D}}$ for some $N \in \mathbb{N}$ and $f(z) \to 0$ as $|z| \to \infty$,

$\mathcal{G}$: consists of the analytic functions f on $\mathbb{C}\setminus\mathbb{T}$ for which $f\in H_-$ on $\mathbb{C}\setminus\overline{\mathbb{D}}$ and $f=g/h$ with $g\in H_+$ and $h\in\mathcal{H}^\infty$ on $\mathbb{D}$.

The following result as well as its proof are similar to [1, Theorem 4.3].

PROPOSITION 4.8. *Let I be a closed ideal in Λ_γ^+ and let $\varphi\in I^\perp$. If $f\in J_\gamma(E_I,Q_I)$, then Φ_f does not have any isolated singularities.*

Proof. It follows from Lemma 2.2 that $Q_I\Phi$ and thus $f\Phi$ is analytic on $\mathbb{D}$. Hence Φ_f is analytic on $\mathbb{D}$ by Lemma 2.3, so the singularities of Φ_f belong to $Z_I\cap\mathbb{T}=E_I$. Moreover, by Lemmas 2.1 and 2.3, we have

$$\Phi_f(z)=\frac{(f(z)/Q(z))\langle S_zg,\varphi\rangle-(g(z)/Q(z))\langle S_zf,\varphi\rangle}{g(z)/Q(z)}\qquad(z\in\mathbb{D}\setminus Z(g))$$

for $g\in I$. From (3) and (4), we thus deduce that $\Phi_f\in\mathcal{G}$, so it follows from [1, Theorem 3.2(ii)] that any isolated singularity of Φ_f is a pole.

Suppose that Φ_f has a pole of order p at (say) $z=1$, so that the function Ψ defined by

$$\Psi=(1-\alpha)^p\Phi_f \tag{5}$$

is analytic in a neighborhood U of 1 and $a=\Psi(1)\neq 0$. Since $f\in J_{\gamma,0}$, we have $K_nf\to f$ in Λ_γ^+ as $n\to\infty$ by Lemma 3.3. Moreover, $K_n\in\lambda_\gamma^+$ and the polynomials are dense in λ_γ^+, so there exists a sequence (p_n) of polynomials with $p_n(1)=1$ and $p_nf\to 0$ in Λ_γ^+ as $n\to\infty$. Let $\varphi_n=\varphi_{p_nf}$ and let Φ_n be the Carleman transform of φ_n. Since $\varphi_n=(\varphi_f)_{p_n}$, it follows from Lemma 2.3 that

$$\Phi_n(z)=p_n(z)\Phi_f(z)-\langle S_zp_n,\varphi_f\rangle\qquad(z\in\mathbb{D}\setminus Z_I) \tag{6}$$

and $q_n(z)=\langle S_zp_n,\varphi_f\rangle$ is a polynomial in z. Combining (5) and (6), we obtain

$$(1-\alpha)^p\Phi_n=p_n\Psi-(1-\alpha)^pq_n$$

on U, so the function Ψ_n defined by $\Psi_n=(1-\alpha)^p\Phi_n$ is analytic in U and $\Psi_n(1)=a$.

Choose a circle Γ centered at 1 and contained in U and a function $g\in I$ such that $g(z)\neq 0$ for $z\in\Gamma\cap\overline{\mathbb{D}}$. We have

$$\|\varphi_n\|\le\|p_nf\|_{\Lambda_\gamma^+}\cdot\|\varphi\|\to 0 \tag{7}$$

as $n\to\infty$, so

$$|\Phi_n(z)|\le C(1-|z|)^{-(\gamma+1)}\qquad(z\in\Gamma\cap\mathbb{D})$$

by (4) and

$$|\Phi_n(z)|\le C(|z|-1)^{-(\gamma+1)}\qquad(z\in\Gamma\setminus\overline{\mathbb{D}})$$

by (3). It thus follows from the proof of [8, Lemma VI.8.3] that the sequence (Ψ_n) is uniformly bounded on some disc centered at 1. By (7), we have $\Phi_n\to 0$ pointwise on $\mathbb{C}\setminus Z_I$ as $n\to\infty$ and thus $\Psi_n\to 0$ pointwise on Γ

as $n \to \infty$. Hence $\Psi_n(1) \to 0$ as $n \to \infty$ by Cauchy's integral formula and Lebesgue's dominated convergence theorem, contradicting $\Psi_n(1) = a \neq 0$. ∎

Proof of Theorem 4.1. Let I be a closed ideal in Λ_γ^+, let $\varphi \in I^\perp$ and let $f \in J_\gamma(E_I, Q_I)$. We will use the same transfinite induction as in [1, p. 17] to prove that Φ_f is entire. Let $L_0 = E_I$ and inductively define L_σ for any ordinal σ in the following way: If $\sigma = \tau + 1$ is not a limit ordinal, we define L_σ to be the set of limit points of L_τ, and if σ is a limit ordinal, we let $L_\sigma = \bigcap_{\tau<\sigma} L_\tau$. If z_0 is a singularity of Φ_f, then $z_0 \in E_I = L_0$. Suppose that we have shown that $z_0 \in L_\tau$ for every ordinal $\tau < \sigma$. If $\sigma = \tau + 1$ is not a limit ordinal, then $L_\sigma \setminus L_\tau$ consists of isolated points, so it follows from the previous proposition that $z_0 \in L_\sigma$. The same conclusion clearly holds if σ is a limit ordinal, so we conclude that $z_0 \in L_\sigma$ for every ordinal σ. However, L_0 contains no perfect subsets, so $L_\sigma \subset L_\tau$ for every non-limit ordinal $\sigma = \tau + 1$, and it follows that there exists a first ordinal σ_0 such that L_{σ_0} is empty. This contradicts our earlier conclusion $z_0 \in L_{\sigma_0}$. Consequently, Φ_f does not have any singularities, so Φ_f is entire. Hence $\Phi_f = 0$ and since $\mathrm{span}\{(z-\alpha)^{-1} : z \in \mathbb{C} \setminus \overline{\mathbb{D}}\}$ is dense in λ_γ^+, this is equivalent to $\varphi_f = 0$ on λ_γ^+. Consequently,

$$\langle f, \varphi \rangle = \langle 1, \varphi_f \rangle = 0$$

and since $\varphi \in I^\perp$ was arbitary, we conclude that $f \in I$. ∎

For closed ideals with finite hull, we shall now give a proof of Theorem 4.1 which is more constructive and does not depend on Proposition 4.8. For simplicity, we consider only closed ideals I in Λ_γ^+ with $Z_I = \{1\}$. For $\gamma < 1$ and $Q_I = 1$, the main idea in the proof is to show that if $\varphi \in I^\perp$, then $\langle f, \varphi \rangle = af(1)$ for $f \in \lambda_\gamma^+$ for some $a \in \mathbb{C}$ (and similarly for $\gamma = 1$).

Proof of Theorem 4.1 when $Z_I = \{1\}$. First, suppose that $Q_I = 1$. For $\varphi \in I^\perp$, we have

$$|\Phi(z)| \le C(|z| - 1)^{-(1+\gamma)} \qquad (z \in \mathbb{C} \setminus \overline{\mathbb{D}}) \tag{8}$$

by (3). Moreover, for $g \in I$ and $z \in \mathbb{D}$, it follows from (4) and Lemma 2.1 that

$$|g(z)\Phi(z)| \le C(1 - |z|)^{-(1+\gamma)} \qquad (z \in \mathbb{D}).$$

Hence Φ has a pole at $z = 1$ by [1, Theorem 3.2(ii)]. We first consider the case where $0 < \gamma < 1$. Then $z = 1$ is a simple pole of Φ by (8) and since $\Phi(z) \to 0$ as $|z| \to \infty$, we deduce that

$$\Phi(z) = a(z-1)^{-1} \qquad (z \in \mathbb{C} \setminus \{1\})$$

for some $a \in \mathbb{C}$. Let $\delta_1 \in (\Lambda_\gamma^+)^*$ denote the point evaluation at $z = 1$. Then

$$\langle (z-\alpha)^{-1}, \delta_1 \rangle = (z-1)^{-1} \qquad (z \in \mathbb{C} \setminus \overline{\mathbb{D}}),$$

so $\varphi = a\delta_1$ on the closed span of $\{(z-\alpha)^{-1} : |z| > 1\}$, that is, on λ_γ^+. In particular, $\langle 1-\alpha, \varphi\rangle = 0$. The Hahn–Banach theorem thus implies that $1-\alpha \in I$, so $J_{\gamma,0} \subseteq I$ by Proposition 3.5. For $\gamma = 1$, the same method works with the following changes. From (8), we deduce that Φ has a pole of order 2 at $z = 1$, so

$$\Phi(z) = a(z-1)^{-1} + b(z-1)^{-2} \quad (z \in \mathbb{C} \setminus \{1\})$$

for some $a, b \in \mathbb{C}$. On λ_1^+, we define δ_1' by $\langle g, \delta_1'\rangle = g'(1)$ $(g \in \lambda_1^+)$. Then

$$\langle (z-\alpha)^{-1}, \delta_1'\rangle = (z-1)^{-2} \quad (z \in \mathbb{C} \setminus \overline{\mathbb{D}}),$$

so $\varphi = a\delta_1 + b\delta_1'$ on λ_1^+. In particular, $\langle (1-\alpha)^2, \varphi\rangle = 0$, so $J_{1,0} \subseteq I$ by Proposition 3.5.

Now, suppose that $Q_I = \psi_{-s}$ for some $s > 0$. We have $(1-\alpha)^2\psi_{-s} \in \Lambda_\gamma^+$ and the division ideal

$$\widetilde{I} = \{f \in I_\gamma(\{1\}) : (1-\alpha)^2\psi_{-s}f \in I\}$$

satisfies $E_{\widetilde{I}} = \{1\}$ and $Q_{\widetilde{I}} = 1$, so $J_{\gamma,0} \subseteq \widetilde{I}$ by the first part of the proof. Since $(1-\alpha)^2 \in J_{\gamma,0}$, we thus have $(1-\alpha)^4\psi_{-s} \in I$, so the conclusion follows from Proposition 3.5. ∎

5. Ideals with $Q_I = 1$. Our aim in this section is to prove the following result.

THEOREM 5.1. *Let $E \subseteq \mathbb{T}$ be a Carleson set and let $F \in J_\gamma(E)$ be an outer function with $Z(F) = E$. Then*

$$\overline{\Lambda_\gamma^+ F} = J_\gamma(E).$$

REMARKS. (1) We do not know whether a closed ideal I in Λ_γ^+ with $Q_I = 1$ necessarily contains an outer function F with $Z(F) = E_I$. However, if this is the case, then the theorem verifies our conjecture for this class of closed ideals. This is seen as follows: Let $H \in J_\gamma(E_I)$ be an outer function with $Z(H) = E_I$. Then $FH \in I \cap J_\gamma(E_I)$ and $Z(FH) = E_I$, so it follows from the theorem that

$$J_\gamma(E_I) = \overline{\Lambda_\gamma^+ FH} \subseteq \overline{\Lambda_\gamma^+ F} \subseteq I$$

as required.

(2) We do not know how to prove a version of the theorem for the ideals $J_\gamma(E, Q)$ with $Q \neq 1$.

For a closed set $E \subseteq \mathbb{T}$ and $p \in \mathbb{N}$, let

$$I_\gamma^p(E) = \{f \in \Lambda_\gamma^+ : |f(z)| \le Cd(z, E)^p \ (z \in \mathbb{T})\}.$$

For $f \in I_\gamma^p(E)$, we have $|f(z)| \le C|z-w|^p$ $(z \in \mathbb{T},\ w \in E)$ and since $(\alpha-w)^p$ is outer, this holds for $z \in \overline{\mathbb{D}}$, so it follows that $|f(z)| \le Cd(z, E)^p$ $(z \in \overline{\mathbb{D}})$.

Theorem 5.1 is an immediate consequence of the following two results.

PROPOSITION 5.2. *Let $E \subseteq \mathbb{T}$ be a Carleson set, let $F \in J_\gamma(E)$ be an outer function with $Z(F) = E$ and let $p \in \mathbb{N}$ with $p > 2\gamma$. Then*

$$J_\gamma(E) \cap I_\gamma^p(E) \subseteq \overline{\Lambda_\gamma^+ F}.$$

PROPOSITION 5.3. *Let $E \subseteq \mathbb{T}$ be a Carleson set and let $p \in \mathbb{N}$. Then $J_\gamma(E) \cap I_\gamma^p(E)$ is dense in $J_\gamma(E)$.*

For an outer function F and a measurable set $\Gamma \subseteq \mathbb{T}$, let

$$F_\Gamma(z) = \exp\left(\frac{1}{2\pi}\int_\Gamma \frac{e^{i\theta}+z}{e^{i\theta}-z}\log|F(e^{i\theta})|\,d\theta\right) \quad (z \in \mathbb{D}). \tag{9}$$

Observe that $|F_\Gamma| = |F|$ a.e. on Γ and $|F_\Gamma| = 1$ a.e. on $\mathbb{T} \setminus \Gamma$. Also, $F_\Gamma \to 1$ pointwise on $\mathbb{D}$ as $m(\Gamma) \to 0$. The following proof is inspired by [13].

Proof of Proposition 5.2. Let $f \in J_\gamma(E) \cap I_\gamma^p(E)$ and write $\mathbb{T} \setminus E = \bigcup_{n=1}^\infty V_n$, where (V_n) is a sequence of pairwise disjoint, open arcs on $\mathbb{T}$ with endpoints a_n and b_n. For $N \in \mathbb{N}$, let $\Gamma_N = \bigcup_{n=N+1}^\infty V_n$ and let $F_N = F_{\Gamma_N}$. We shall prove that

(i) $F_N f \to f$ in Λ_γ^+ as $N \to \infty$.

(ii) $F_N f \in \overline{\Lambda_\gamma^+ F}$ for $N \in \mathbb{N}$.

(i): We have

$$(F_N f - f)' = (F_N - 1)f' + F_N' f.$$

Also, $F_N \to 1$ uniformly on compact subsets of $\overline{\mathbb{D}} \setminus E$, so $F_N f \to f$ uniformly on $\overline{\mathbb{D}}$ and

$$\sup_{z \in \mathbb{D}} |(F_N(z) - 1)f'(z)|(1-|z|)^{1-\gamma} \to 0$$

as $N \to \infty$. We shall now prove that

$$|F_N'(z)f(z)| = o((1-|z|)^{\gamma-1}) \tag{10}$$

as $d(z,E) \to 0$ uniformly in N. For $N \in \mathbb{N}$, let $E_N = E \cap \overline{\Gamma}_N = \partial\Gamma_N$ and let

$$\begin{aligned}
G_{1N} &= \{z = re^{it} \in \mathbb{D} : d(e^{it}, E_N) \le (1-r)^{1/2}\},\\
G_{2N} &= \{z = re^{it} \in \mathbb{D} : d(e^{it}, E_N) > (1-r)^{1/2} \text{ and } e^{it} \notin \Gamma_N\},\\
G_{3N} &= \{z = re^{it} \in \mathbb{D} : d(e^{it}, E_N) > (1-r)^{1/2} \text{ and } e^{it} \in \Gamma_N\}.
\end{aligned}$$

For $z = re^{it} \in G_{1N}$, choose $e^{i\theta} \in E_N$ such that

$$\begin{aligned}
d(z, E_N)^2 = |z - e^{i\theta}|^2 &= (1-r)^2 + 4r\sin^2(\theta - t)/2\\
&= (1-r)^2 + r d(e^{it}, E_N)^2 \le 1 - r.
\end{aligned}$$

By Cauchy's inequalities, $|F_N'(z)| \le C(1-r)^{-1}$, so

$$|F_N'(z)f(z)| \le C d(z,E)^p (1-r)^{-1} \le C(1-r)^{p/2-1}.$$

For $z = re^{it} \in G_{2N}$, we have $d(e^{it}, \Gamma_N) = d(e^{it}, E_N)$ and thus

$$d(z,\Gamma_N)^2 = (1-r)^2 + rd(e^{it},\Gamma_N)^2 = d(z,E_N)^2.$$

Moreover,

$$F_N'(z) = \frac{1}{\pi}\int\limits_{\Gamma_N} \frac{e^{i\theta}}{(e^{i\theta}-z)^2}\log|F(e^{i\theta})|\,d\theta\cdot F_N(z),$$

so

$$|F_N'(z)| \le C\int\limits_{\mathbb{T}} \big|\log|F(e^{i\theta})|\big|\,d\theta\cdot d(z,E_N)^{-2}$$

and thus

$$|F_N'(z)f(z)| \le Cd(z,E_N)^{p-2}.$$

Also, $d(z,E_N)^2 = (1-r)^2 + rd(e^{it},E_N)^2 \ge 1-r$, so

$$|F_N'(z)f(z)| \le Cd(z,E)^{p-2\gamma}(1-r)^{\gamma-1}.$$

Now, let $z = re^{it} \in G_{3N}$. We have

$$F_N'(z) = \frac{F'(z)F_N(z)}{F(z)} - \frac{1}{\pi}\int\limits_{\mathbb{T}\setminus\Gamma_N} \frac{e^{i\theta}}{(e^{i\theta}-z)^2}\log|F(e^{i\theta})|\,d\theta\cdot F_N(z).$$

Since $d(z,\mathbb{T}\setminus\Gamma_N) \ge d(z,E_N)$, the second term can be estimated as for $z \in G_{2N}$. For the first term, we apply [13, Lemma 1] with $\Gamma = \mathbb{T}\setminus\Gamma_N$ and $\eta = 1/2$ and obtain $|F_N(z)/F(z)| \le C$. Since $F \in J_\gamma(E)$, we have verified (10).

For $\delta > 0$, let $E_\delta = \{z \in \mathbb{T} : d(z,E) < \delta\}$ and $U_\delta = \{z \in \mathbb{D} : d(z,E) < \delta\}$. Given $\varepsilon > 0$, it follows from (10) that there exists $\delta > 0$ such that

$$|F_N'(z)f(z)|(1-|z|)^{1-\gamma} \le \varepsilon$$

for $z \in U_\delta$ and $N \in \mathbb{N}$. Since $\overline{V}_n \cap E \ne \emptyset$ $(n \in \mathbb{N})$, there exists $N_0 \in \mathbb{N}$ such that $V_n \subseteq E_{\delta/2}$ for $n > N_0$ and thus $\Gamma_N \subseteq E_{\delta/2}$ for $N \ge N_0$. Hence $d(z,\Gamma_N) \ge \delta/2$ for $z \notin U_\delta$ and $N \ge N_0$, so

$$|F_N'(z)| \le Cd(z,\Gamma_N)^{-2}\int\limits_{\Gamma_N} \big|\log|F(e^{i\theta})|\big|\,d\theta \to 0$$

uniformly on $\mathbb{D}\setminus U_\delta$ as $N \to \infty$. We thus conclude that $F_N f \in \Lambda_\gamma^+$ and that $F_N f \to f$ in Λ_γ^+ as $N \to \infty$.

(ii): Fix $N \in \mathbb{N}$. Since $f \in J_\gamma(E)$, it follows from (10) that $F_N f \in J_\gamma(E)$. For $a \in \mathbb{T}$ and $\mu > 0$, let

$$K_{a\mu}(z) = \frac{a-z}{(1+\mu)a-z} \qquad (z \in \overline{\mathbb{D}}).$$

(This is a generalization of the sequence (K_n) introduced in Section 3.) With

$$\Phi_\mu = \Big(\prod_{n=1}^{N} K_{a_n\mu}K_{b_n\mu}\Big)^p,$$

it follows from Lemma 3.3 that

$$\Phi_\mu F_N f \to F_N f \tag{11}$$

in Λ_γ^+ as $\mu \to 0$. Now, fix $\mu > 0$. For $\varepsilon > 0$ and $n = 1, \ldots, N$, let $V_{n\varepsilon}$ be the subarc of V_n whose endpoints c_n and d_n are at a distance ε from a_n and b_n respectively. Let $D_\varepsilon = \bigcup_{n=1}^N V_{n\varepsilon}$ and let

$$\Phi_{\mu\varepsilon} = \Big(\prod_{n=1}^N K_{c_n\mu} K_{d_n\mu}\Big)^p.$$

We shall show that

(a) $\Phi_{\mu\varepsilon} F_{D_\varepsilon}^{-1} \in \Lambda_\gamma^+$ for $\varepsilon > 0$

(b) $\Phi_{\mu\varepsilon} F_{D_\varepsilon}^{-1} F f \to \Phi_\mu F_N f$ in Λ_γ^+ as $\varepsilon \to 0$.

It then follows from (11) that $F_N f \in \overline{\Lambda_\gamma^+ F}$. For simplicity, we only prove (a) and (b) for $N = 1$, but the proof is essentially the same in the general case.

(a): Let $\varepsilon > 0$. It follows from the proof of (10) that

$$|\Phi_{\mu\varepsilon}(z) F'_{D_\varepsilon}(z)| \le C(1-|z|)^{\gamma-1} \quad (z \in \mathbb{D}).$$

Also, the outer function F_{D_ε} is bounded away from zero on $\mathbb{T}$ and thus on $\overline{\mathbb{D}}$, so

$$|\Phi_{\mu\varepsilon}(z)(F_{D_\varepsilon}^{-1})'(z)| \le C(1-|z|)^{\gamma-1} \quad (z \in \mathbb{D}),$$

and (a) follows.

(b): For $\varepsilon > 0$, let $W_\varepsilon = V_1 \setminus V_{1\varepsilon}$ so that $\partial W_\varepsilon = \{a_1, c_1, d_1, b_1\}$. Then

$$F_{V_{1\varepsilon}}^{-1} F = F_{W_\varepsilon} F_1,$$

so

$$\begin{aligned} (\Phi_{\mu\varepsilon} F_{V_{1\varepsilon}}^{-1} F f - \Phi_\mu F_1 f)' &= (\Phi_{\mu\varepsilon} F_{W_\varepsilon} F_1 - \Phi_\mu F_1) f' + (\Phi_{\mu\varepsilon} F_{W_\varepsilon} - \Phi_\mu) F_1' f \\ &\quad + (\Phi'_{\mu\varepsilon} F_{W_\varepsilon} - \Phi'_\mu) F_1 f + \Phi_{\mu\varepsilon} F'_{W_\varepsilon} F_1 f. \end{aligned} \tag{12}$$

As $\varepsilon \to 0$, we have $\Phi_{\mu\varepsilon} \to \Phi_\mu$ and $\Phi'_{\mu\varepsilon} \to \Phi'_\mu$ uniformly on $\overline{\mathbb{D}}$ and $F_{W_\varepsilon} \to 1$ uniformly on compact subsets of $\overline{\mathbb{D}} \setminus \{a_1, b_1\}$, so

$$\begin{aligned} \sup_{z\in\mathbb{D}} \Big| [(\Phi_{\mu\varepsilon} F_{W_\varepsilon} F_1 - \Phi_\mu F_1) f' + (\Phi_{\mu\varepsilon} F_{W_\varepsilon} - \Phi_\mu) F_1' f \\ + (\Phi'_{\mu\varepsilon} F_{W_\varepsilon} - \Phi'_\mu) F_1 f](z) \Big| \cdot (1-|z|)^{1-\gamma} \to 0. \end{aligned}$$

In order to estimate the last term on the right-hand side of (12), we shall imitate the proof of (10). For $\varepsilon > 0$, let $f_\varepsilon = \Phi_{\mu\varepsilon} f \in I_\gamma^p(\partial W_\varepsilon)$ and let

$$\begin{aligned} G_{1\varepsilon} &= \{z = re^{it} \in \mathbb{D} : d(e^{it}, \partial W_\varepsilon) \le (1-r)^{1/2}\}, \\ G_{2\varepsilon} &= \{z = re^{it} \in \mathbb{D} : d(e^{it}, \partial W_\varepsilon) > (1-r)^{1/2} \text{ and } e^{it} \notin W_\varepsilon\}, \\ G_{3\varepsilon} &= \{z = re^{it} \in \mathbb{D} : d(e^{it}, \partial W_\varepsilon) > (1-r)^{1/2} \text{ and } e^{it} \in W_\varepsilon\}. \end{aligned}$$

For $z = re^{it} \in G_{1\varepsilon}$, we have $d(z, \partial W_\varepsilon)^2 \le 1 - r$, so

$$|F'_{W_\varepsilon}(z) f_\varepsilon(z)| \le C(1-r)^{p/2-1}$$

uniformly for $\varepsilon > 0$. Moreover, $F'_{W_\varepsilon} \to 0$ uniformly on compact subsets of $\mathbb{D}$, so

$$\sup_{z \in G_{1\varepsilon}} |F'_{W_\varepsilon}(z) f_\varepsilon(z)|(1-|z|)^{1-\gamma} \to 0$$

as $\varepsilon \to 0$. Also, for $z = re^{it} \in G_{2\varepsilon}$, we have

$$|F'_{W_\varepsilon}(z) f_\varepsilon(z)|(1-r)^{1-\gamma} \le C \int_{W_\varepsilon} \big|\log|F(e^{i\theta})|\big|\, d\theta \to 0$$

as $\varepsilon \to 0$. For $z = re^{it} \in G_{3\varepsilon}$, we have $d(z, \partial W_\varepsilon)^2 \le 2d(e^{it}, \partial W_\varepsilon)^2 \le 2\varepsilon^2$. Moreover,

$$F'_{W_\varepsilon}(z) = \frac{F'(z) F_{W_\varepsilon}(z)}{F(z)} - \frac{1}{\pi} \int_{\mathbb{T} \setminus W_\varepsilon} \frac{e^{i\theta}}{(e^{i\theta} - z)^2} \log|F(e^{i\theta})|\, d\theta \cdot F_{W_\varepsilon}(z),$$

and $|F_{W_\varepsilon}(z)/F(z)| \le C$ by [13, Lemma 1], so

$$\begin{aligned} &|F'_{W_\varepsilon}(z) f_\varepsilon(z)|(1-r)^{1-\gamma} \\ &\quad \le C d(z, \partial W_\varepsilon)^p ((1-r)^{\gamma-1} + d(z, \partial W_\varepsilon)^{-2})(1-r)^{1-\gamma} \\ &\quad \le C(d(z, \partial W_\varepsilon)^p + d(z, \partial W_\varepsilon)^{p-2+2(1-\gamma)}) \le C(\varepsilon^p + \varepsilon^{p-2\gamma}). \end{aligned}$$

All in all, we conclude that

$$\sup_{z \in \mathbb{D}} |F'_{W_\varepsilon}(z) f_\varepsilon(z)|(1-|z|)^{1-\gamma} \to 0$$

as $\varepsilon \to 0$, so (b) follows from (12). ■

We now turn to the proof of Proposition 5.3. In the proof of the corresponding result for λ_γ^+ ([12, Theorem A]), the first step is that if $f \in \lambda_\gamma^+$ with $f = FQ$, where F is an outer and Q an inner function, then

$$f_t = F^{1+t} Q \to f$$

in λ_γ^+ as $t \to 0$, and moreover $f_t \in I_\gamma^{(1+t)\gamma}(Z(F))$. In our case, for $f \in J_\gamma(E)$, we only have $f_t \to f$ in Λ_γ^+ as $t \to 0$ if $Z(F) = E$, and this complicates the proof of Proposition 5.3. We shall need the following factorization result, which we find interesting in itself.

Proposition 5.4. *Let $F \in \Lambda_\gamma^+$ be an outer function and suppose that $Z(F) = E_1 \cup E_2$, where $E_1, E_2 \subseteq \mathbb{T}$ are closed, disjoint sets. Then there exist outer functions $F_1, F_2 \in \Lambda_\gamma^+$ such that $F = F_1 F_2$ and $Z(F_k) = E_k$ $(k = 1, 2)$.*

Proof. Choose open sets $U_1, U_2, V_1, V_2 \subseteq \mathbb{T}$ such that $E_k \subseteq U_k, \overline{U}_k \subseteq V_k$ $(k = 1, 2)$ and such that V_1 and V_2 are disjoint, and choose $\chi_1, \chi_2 \in \Lambda_\gamma$

such that $\chi_1 + \chi_2 = 1$ on $\mathbb{T}$ and $\chi_k = 1$ on U_k $(k = 1, 2)$. For $k = 1, 2$, let $\varphi_k = \chi_k \log |F|$ and define an outer function F_k by

$$F_k(z) = \exp\left(\frac{1}{2\pi}\int_{\mathbb{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,\varphi_k(e^{i\theta})\,d\theta\right) \qquad (z \in \mathbb{D}).$$

Then $Z(F_k) = E_k$ and $F = F_1F_2$. Choose $\psi_k \in \Lambda_\gamma$ such that $\psi_k = \varphi_k$ on $\mathbb{T} \setminus U_k$ and let

$$G_k(z) = \exp\left(\frac{1}{2\pi}\int_{\mathbb{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,\psi_k(e^{i\theta})\,d\theta\right),$$

$$H_k(z) = \exp\left(\frac{1}{2\pi}\int_{\mathbb{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,(\varphi_k(e^{i\theta}) - \psi_k(e^{i\theta}))\,d\theta\right)$$

for $z \in \mathbb{D}$, so that $F_k = G_kH_k$. Since Λ_γ is closed under harmonic conjugation ([17, Theorem III.13.29]), it follows that $\log G_k \in \Lambda_\gamma^+$ and thus $G_k, G_k^{-1} \in \Lambda_\gamma^+$. For $e^{i\theta} \in U_1$, the function $z \mapsto (e^{i\theta}+z)/(e^{i\theta}-z)$ belongs to $\Lambda_\gamma(\mathbb{T} \setminus V_1)$, so we deduce that $H_1 \in \Lambda_\gamma(\mathbb{T} \setminus V_1)$ and thus $F_1 \in \Lambda_\gamma(\mathbb{T} \setminus V_1)$. Similarly $F_2 \in \Lambda_\gamma(\mathbb{T} \setminus V_2)$, so $F_1 = F/F_2 \in \Lambda_\gamma(\mathbb{T} \setminus V_2)$ since F_2 has no zeros on $\mathbb{T} \setminus V_2$. Hence $F_1 \in \Lambda_\gamma$ and thus $F_1 \in \Lambda_\gamma^+$. Similarly $F_2 \in \Lambda_\gamma^+$. ■

Proof of Proposition 5.3. Let $f \in J_\gamma(E)$ with $f = FQ$, where F is an outer and Q an inner function, and let $\varepsilon > 0$. Choose $0 < \delta \le \varepsilon$ such that

$$|f'(z)| < \varepsilon(1 - |z|)^{\gamma-1}$$

for $z \in U_\delta$, where U_δ and E_δ are as in the proof of Proposition 5.2. It is easily seen that there exist closed, disjoint sets $E_1, E_2 \subseteq \mathbb{T}$ with $E \subseteq E_1 \subseteq E_\delta$ and $Z(F) = E_1 \cup E_2$, so it follows from the previous proposition that $F = F_1F_2$, where $F_1, F_2 \in \Lambda_\gamma^+$ are outer functions with $Z(F_k) = E_k$ $(k = 1, 2)$. For $t > 0$, let

$$f_t = F_1^{1+t}F_2Q = F_1^t f,$$

so that

$$f_t' = tF_1^{t-1}F_1'f + F_1^tf' = F_1^t(tF_1'F_2Q + f'). \tag{13}$$

Since $F_1 = 0$ on $E_1 \supseteq E$, we deduce that $f_t \in J_\gamma(E) \cap I_\gamma^{(1+t)\gamma}(E)$. Moreover,

$$(f_t - f)' = tF_1^tF_1'F_2Q + (F_1^t - 1)f'.$$

Since $Z(F_1) \subseteq E_\delta$, we have $F_1^t \to 1$ uniformly on $\mathbb{D} \setminus U_\delta$ as $t \to 0$, so

$$\limsup_{t\to 0} \|f_t - f\|_{\Lambda_\gamma^+} \le C \sup_{z\in U_\delta} |f'(z)|(1 - |z|)^{1-\gamma} < C\varepsilon.$$

Write $\mathbb{T} \setminus E_1 = \bigcup_{n=1}^\infty W_n$, where (W_n) is a sequence of pairwise disjoint, open arcs on $\mathbb{T}$. For $N \in \mathbb{N}$, let $\Omega_N = \bigcup_{n=N+1}^\infty W_n$ and let

$$F_{1N} = (F_1)_{\Omega_N}$$

(see (9)). Fix $t > 0$ and let $q \in \mathbb{N}$. We have $F_{1N}^q \to 1$ uniformly on compact subsets of $\overline{\mathbb{D}} \setminus E_1$ and $f_t = 0$ on E_1, so $F_{1N}^q f_t \to f_t$ uniformly on $\overline{\mathbb{D}}$. To estimate $(F_{1N}^q)' f_t = q F_{1N}^{q-1} F_{1N}' f_t$ on $\mathbb{D} \setminus U_\delta$, we choose $N_0 \in \mathbb{N}$ such that $\Omega_N \subseteq E_{\delta/2}$ for $N \geq N_0$. We have

$$|F_{1N}'(z)| \leq C d(z, \Omega_N)^{-2} \int_{\Omega_N} \big|\log |F(e^{i\theta})|\big| \, d\theta \leq C\delta^{-2} \int_{\Omega_N} \big|\log |F(e^{i\theta})|\big| \, d\theta \to 0$$

uniformly for $z \in \mathbb{D} \setminus U_\delta$ as $N \to \infty$. To estimate $(F_{1N}^q)' f_t$ on U_δ, we repeat the proof of [12, Theorem B] (for q sufficiently large) with $d(z) = d(z, E_1)$ and use the fact that

$$|f(z)| \leq C|F_1(z)| \leq C d(z, E_1)^\gamma \leq C\varepsilon^\gamma,$$

and obtain

$$\limsup_{N\to\infty} \sup_{z \in U_\delta} |(F_{1N}^q)'(z) f_t(z)| (1 - |z|)^{1-\gamma} = \kappa(\varepsilon),$$

where $\kappa(\varepsilon) \to 0$ as $\varepsilon \to 0$. Moreover, by (13), we have

$$\sup_{z \in U_\delta} |f_t'(z)|(1 - |z|)^{1-\gamma} \leq C \sup_{z \in U_\delta} |F_1^t(z)| \leq C\delta^{t\gamma} \leq C\varepsilon^{t\gamma},$$

so

$$\limsup_{N\to\infty} \|f_t - F_{1N}^q f_t\|_{\Lambda_\gamma^+} = \widetilde{\kappa}(\varepsilon)$$

where $\widetilde{\kappa}(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Now, fix $N \in \mathbb{N}$. It follows from the above that $F_{1N}^q f_t \in J_\gamma(E)$. Moreover,

$$|F_{1N}^q(z)| \leq C d(z, \partial\Omega_N)^p \quad (z \in \overline{\mathbb{D}})$$

for $q \geq p/\gamma$. Since $\partial(\mathbb{T} \setminus E_1) = E_1$, we deduce that $E \setminus \partial\Omega_N$ is finite, say $E \setminus \partial\Omega_N = \{a_1, \ldots, a_M\}$. By Lemma 3.3, we then have

$$\Big(\prod_{m=1}^{M} K_{a_m\mu}\Big)^p F_{1N}^q f_t \to F_{1N}^q f_t$$

in Λ_γ^+ as $\mu \to 0$, and since

$$\Big(\prod_{m=1}^{M} K_{a_m\mu}\Big)^p F_{1N}^q \in J_\gamma(E) \cap I_\gamma^p(E),$$

this finishes the proof. ∎

6. Weak-star closed ideals. In this section, we characterize the wk* closed ideals in Λ_γ^+. We begin by describing the wk* topology on Λ_γ and Λ_γ^+. For $z \in \mathbb{T}$, let $\delta_z \in \Lambda_\gamma^*$ be the point evaluation functional at z, and let

$$Y_\gamma = \overline{\operatorname{span}\{\delta_z : z \in \mathbb{T}\}}$$

(norm closure in Λ_γ^*). Johnson ([7, Section 4]) proved that

$$Y_\gamma^* = \Lambda_\gamma.$$

Moreover, a bounded net in Λ_γ converges wk* to zero in Λ_γ if and only if it converges pointwise to zero on $\mathbb{T}$, and in this case it actually converges uniformly to zero on $\mathbb{T}$. When $0 < \gamma < 1$, we further have $Y_\gamma = \lambda_\gamma^*$ and thus $\Lambda_\gamma = \lambda_\gamma^{**}$ ([7, Theorem 4.7]).

LEMMA 6.1. *Multiplication is separately wk* continuous in Λ_γ.*

Proof. The space $Y_\gamma^{**} = \Lambda_\gamma^*$ is a Banach Λ_γ-module under the action

$$\langle f, g\varphi\rangle = \langle fg, \varphi\rangle \quad (f, g \in \Lambda_\gamma,\ \varphi \in Y_\gamma^{**}).$$

For $z \in \mathbb{T}$ and $g \in \Lambda_\gamma$, we have

$$\langle f, g\delta_z\rangle = f(z)g(z) \quad (f \in \Lambda_\gamma),$$

so $g\delta_z = g(z)\delta_z$. Hence Y_γ is a Λ_γ-submodule and the conclusion follows. ∎

Let (f_n) be a sequence in Λ_γ^+ which converges wk* to f in Λ_γ as $n \to \infty$. Then $\widehat{f_n}(m) \to \widehat{f}(m)$ as $n \to \infty$ for $m \in \mathbb{Z}$ by Lebesgue's dominated convergence theorem. Hence $f \in \Lambda_\gamma^+$, so Λ_γ^+ is wk* closed by the Krein–Šmulian theorem. Denoting the quotient space $Y_\gamma/{}^\perp(\Lambda_\gamma^+)$ by Y_γ^+, we thus have

$$\Lambda_\gamma^+ = (Y_\gamma^+)^*.$$

The next result often provides us with the easiest way to show wk* convergence in Λ_γ^+.

LEMMA 6.2. *Let (f_n) be a bounded sequence in Λ_γ^+ which converges pointwise to zero on $\mathbb{D}$ as $n \to \infty$. Then $f_n \to 0$ wk* in Λ_γ^+ as $n \to \infty$.*

Proof. Let $z \in \mathbb{T}$ and $\varepsilon > 0$. Choose $w \in \mathbb{D}$ with $|z - w| < \varepsilon$. Since $f_n(w) \to 0$ as $n \to \infty$ and since (f_n) is bounded in Λ_γ^+, it follows that $\limsup_{n\to\infty} |f_n(z)| \le C\varepsilon^\gamma$. Hence $f_n \to 0$ pointwise on $\mathbb{T}$ as $n \to \infty$ and the result follows. ∎

We now turn our attention to wk* closed ideals in Λ_γ^+.

PROPOSITION 6.3. *Suppose that a closed set $E \subseteq \mathbb{T}$ and an inner function Q satisfy* (2). *Then $I_\gamma(E, Q)$ is a wk* closed ideal in Λ_γ^+.*

Proof. Let (f_n) be a sequence in $I_\gamma(E, Q)$ and suppose that $f_n \to f$ wk* in Λ_γ^+ as $n \to \infty$ for some $f \in \Lambda_\gamma^+$. Then $f \in I_\gamma(E)$ and it follows from Theorem 1.1 that (f_n/Q) is a bounded sequence in Λ_γ^+. Moreover, $f_n/Q \to f/Q$ pointwise on $\mathbb{T}$ as $n \to \infty$, so we deduce that $f_n/Q \to f/Q$ wk* in Λ_γ^+ as $n \to \infty$. In particular, $f \in I_\gamma(E, Q)$. The Krein–Šmulian theorem thus implies that $I_\gamma(E, Q)$ is wk* closed. ∎

The aim of this section is to prove the following result, which states that the ideals $I_\gamma(E, Q)$ are the only wk* closed ideals in Λ_γ^+.

THEOREM 6.4. *Let I be a wk* closed ideal in Λ_γ^+. Then*

$$I = I_\gamma(E_I, Q_I).$$

The proof of the theorem takes up the rest of this paper. The idea in the proof is similar to that of [10] and [11]. Firstly, the Carleman transform is used to show that a wk* closed ideal I in Λ_γ^+ with $Q_I = 1$ necessarily contains a certain class of functions. Secondly, we show that every function in $I_\gamma(E, Q)$ can be approximated by sufficiently smooth functions. Finally, the result is deduced from these two facts.

For a (wk*) closed ideal I in Λ_γ^+, we let

$$^\perp I = \{\varphi \in Y_\gamma^+ : \langle \varphi, f\rangle = 0 \text{ for every } f \in I\} = I^\perp \cap Y_\gamma^+.$$

Also, for an inner function Q, a closed set $Z \subseteq \overline{\mathbb{D}}$ and $p > 0$, let

$$I_\gamma^p(Z, Q) = \{f \in \Lambda_\gamma^+ : f/Q \in \Lambda_\gamma^+ \text{ and } |f(z)| \le Cd(z, Z)^p \ (z \in \mathbb{T})\},$$

so that $I_\gamma^p(E) = I_\gamma^p(E, 1)$ for a closed set $E \subseteq \mathbb{T}$ (see the previous section). For $f \in \Lambda_\gamma^+$, we have $\|f_r\|_{\Lambda_\gamma^+} \le \|f\|_{\Lambda_\gamma^+}$ for $r < 1$ and thus $f_r \to f$ wk* in Λ_γ^+ as $r \to 1_-$, so we can use a method from [10] in the proof of the next result.

LEMMA 6.5. *Let I be a wk* closed ideal in Λ_γ^+ with $Q_I = 1$. Then*

$$I_\gamma^{2(1+\gamma)}(E_I, 1) \subseteq I.$$

Proof. Let $f \in I_\gamma^{2(1+\gamma)}(E_I, 1)$ and suppose that $\varphi \in {}^\perp I$. Then

$$\langle \varphi, f\rangle = \lim_{r\to 1_-} \langle \varphi, f_r\rangle = \lim_{s\to 1_+} \frac{1}{2\pi} \int_{\mathbb{T}} f(e^{i\theta}) e^{i\theta} \Phi(se^{i\theta})\, d\theta.$$

From the proof of [10, Lemma 3.3] (see also [11, Theorem 5]), we deduce that

$$|\Phi(z)| \le Cd(z, E_I)^{-2(1+\gamma)} \quad (z \in \mathbb{C} \setminus \overline{\mathbb{D}}),$$

so it follows from Lebesgue's dominated convergence theorem that

$$\langle \varphi, f\rangle = \frac{1}{2\pi} \int_{\mathbb{T}} f(e^{i\theta}) e^{i\theta} \Phi(e^{i\theta})\, d\theta.$$

By the Beurling–Rudin theorem, the space I is dense in the Hardy space $\mathcal{H}^2$, so there exists a sequence (f_n) in I converging to 1 in $\mathcal{H}^2$. Since $ff_n \in I$, we thus have

$$\langle \varphi, f\rangle = \lim_{n\to\infty} \frac{1}{2\pi} \int_{\mathbb{T}} f(e^{i\theta}) f_n(e^{i\theta}) e^{i\theta} \Phi(e^{i\theta})\, d\theta = \lim_{n\to\infty} \langle \varphi, ff_n\rangle = 0.$$

Hence $f \in I$ by the Hahn–Banach theorem. ∎

The main difficulty in the proof of Theorem 6.4 is contained in the following approximation result.

Proposition 6.6. *Let $p > 0$ and suppose that a closed set $E \subseteq \mathbb{T}$ and an inner function Q satisfy* (2). *Let $Z = E \cup Z(B)$. Then $I^p_\gamma(Z, Q)$ is wk* dense in $I_\gamma(E, Q)$.*

In order to prove the proposition, we shall need a series of lemmas. The following result should be compared with the comments before the proof of Proposition 5.4.

Lemma 6.7. *Let $f = FQ \in \Lambda^+_\gamma$, where F is an outer and Q an inner function. Then $f_t = F^{1+t}Q \in \Lambda^+_\gamma$ for $t > 0$ and $f_t \to f$ wk* in Λ^+_γ as $t \to 0$.*

Proof. We have $F \in \Lambda^+_\gamma$ by Theorem 1.1. Since $f' = F'Q + FQ'$, it thus follows that

$$\sup_{z\in\mathbb{D}} |F(z)Q'(z)|(1 - |z|)^{1-\gamma} < \infty.$$

Moreover, $f'_t = (1 + t)F^tF'Q + F^{1+t}Q'$, so we deduce that (f_t) is bounded in Λ^+_γ as $t \to 0$. Finally, $f_t \to f$ pointwise on $\mathbb{T}$ as $t \to 0$, so $f_t \to f$ wk* in Λ^+_γ as $t \to 0$. ■

For $a \in \mathbb{T}$ and $\mu > 0$, let $K_{a\mu}$ be as in the previous section. For $f \in \Lambda^+_\gamma$ with $f(a) = 0$, it follows from the proof of Lemma 3.3 that

$$\sup_{z\in\mathbb{D}} |K'_{a\mu}(z)f(z)|(1 - |z|)^{1-\gamma} \le C$$

for $\mu > 0$. Hence $(K_{a\mu}f)$ is bounded in Λ^+_γ, and since $K_{a\mu}f \to f$ pointwise on $\mathbb{T}$, we deduce that $K_{a\mu}f \to f$ wk* in Λ^+_γ as $\mu \to 0$. From this, it is easy to deduce the following result.

Lemma 6.8. *Let $p \ge 1$, let $f \in \Lambda^+_\gamma$ and let $\{a_1, \ldots, a_N\} \subseteq Z(f) \cap \mathbb{T}$. Then*

$$\Big(\prod_{n=1}^{N} K_{a_n\mu}\Big)^p f \to f$$

wk in Λ^+_γ as $\mu \to 0$.*

For an outer function F and a measurable set $\Gamma \subseteq \mathbb{T}$, recall the definition of F_Γ from (9). From the proof of [12, Theorem B], we obtain the following result.

Lemma 6.9. *Let F be an outer function, Q an inner function and suppose that $FQ \in \Lambda^+_\gamma$. Let $t > 0$ and let $f = F^{1+t}Q$. Then there exists q_0 such that, for $q \ge q_0$, we have*

$$F^q_\Gamma f \in \Lambda^+_\gamma \quad \textit{with} \quad \|F^q_\Gamma f\|_{\Lambda^+_\gamma} \le C$$

for every open set $\Gamma \subseteq \mathbb{T}$ with $\partial\Gamma \subseteq Z(f)$ (where $\partial\Gamma$ denotes the boundary of Γ in $\mathbb{T}$).

Proof of Proposition 6.6. By Lemma 6.7, it is sufficient to prove that, whenever a function $f \in I_\gamma(E, Q)$ is of the form $f = F^{1+t}Q$, where $t > 0$,

F is an outer function and Q an inner function such that $FQ \in \Lambda_\gamma^+$, then f can be approximated in the wk* topology on Λ_γ^+ by functions from $I_\gamma^p(Z,Q)$. Let $q = \max\{q_0, p/\gamma\}$. As in the proof of Proposition 5.2, let $\mathbb{T} \setminus E = \bigcup_{n=1}^\infty V_n$, where (V_n) is a sequence of pairwise disjoint, open arcs on $\mathbb{T}$ with endpoints a_n and b_n, and for $N \in \mathbb{N}$, let $\Gamma_N = \bigcup_{n=N+1}^\infty V_n$ and $F_N = F_{\Gamma_N}$. As $N \to \infty$, we have $m(\Gamma_N) \to 0$ and thus $F_N \to 1$ pointwise on $\mathbb{D}$, so it follows from Lemmas 6.2 and 6.9 that $F_N^q f \to f$ wk* in Λ_γ^+ for every $q \geq q_0$.

Let $N \in \mathbb{N}$ be fixed. We have $E \setminus \overline{\Gamma}_N \subseteq \{a_1, b_1, \ldots, a_N, b_N\}$ and

$$\Big(\prod_{n=1}^N K_{a_n\mu} K_{b_n\mu}\Big)^p F_N^q f \to F_N^q f$$

wk* in Λ_γ^+ as $\mu \to 0$ by Lemma 6.8,

Fix $\mu > 0$. For $\varepsilon > 0$ and $n = 1, \ldots, N$, let $V_{n\varepsilon}$ be the subarc of V_n whose endpoints c_n and d_n are at a distance ε from a_n and b_n respectively. Let

$$g_\varepsilon = \Big(\prod_{n=1}^N K_{a_n\mu} K_{c_n\mu} K_{d_n\mu} K_{b_n\mu}\Big)^{p/2} \Big(\prod_{n=1}^N F_{V_n \setminus V_{n\varepsilon}}\Big)^q F_N^q f.$$

It follows from the proof of [12, Theorem B] that (g_ε) is bounded in Λ_γ^+ as $\varepsilon \to 0$, so

$$g_\varepsilon \to \Big(\prod_{n=1}^N K_{a_n\mu} K_{b_n\mu}\Big)^p F_N^q f$$

wk* in Λ_γ^+ as $\varepsilon \to 0$ by Lemma 6.2.

Finally, fix $\varepsilon > 0$. For $z \in \overline{\Gamma}_N$, we have $|F_N(z)| = |f(z)|$, and for $z \in V_n \setminus V_{n\varepsilon}$ for some $n \in \{1, \ldots, N\}$, we have $|F_{V_n \setminus V_{n\varepsilon}}(z)| = |f(z)|$. In both cases, we thus have

$$|g_\varepsilon(z)| \leq C|f(z)|^q \leq Cd(z,Z)^p.$$

Clearly, this also holds for $z \in \bigcup_{n=1}^N \overline{V}_{n\varepsilon}$, so $g_\varepsilon \in I_\gamma^p(Z,Q)$, which finishes the proof. ∎

It follows from Lemma 6.5 and Proposition 6.6 that Theorem 6.4 holds for closed ideals I with $Q_I = 1$. We now finish the proof of the general case.

Proof of Theorem 6.4. Korenblum ([9], see also [10]) has shown that there exists an outer function T satisfying the following conditions:

(i) $T^\varepsilon Q_I \in \Lambda_\gamma^+$ for every $\varepsilon > 0$,
(ii) $Z(T) = E_I$,
(iii) $|T'(z)/T(z)| \leq Cd(z, Z_I)^{-2}$ $(z \in \mathbb{T})$.

Let $\varepsilon > 0$ and consider the division ideal

$$I_\varepsilon = \{f \in \Lambda_\gamma^+ : T^\varepsilon Q_I f \in I\}$$

in Λ_γ^+. Since multiplication is separately wk* continuous in Λ_γ^+ (Lemma 6.1), it follows that I_ε is wk* closed. Moreover, for $g \in I$, we have $g/Q_I \in I_\varepsilon$, so we deduce that $Q_{I_\varepsilon} = 1$ and $E_{I_\varepsilon} = E_I$. As mentioned before the proof, we thus have $I_\varepsilon = I_\gamma(E_I, 1)$.

Now, let $g \in I_\gamma^2(Z_I, Q_I)$. Then $g/Q_I \in I_\gamma(E_I, 1) = I_\varepsilon$, so $T^\varepsilon g \in I$. It follows from (iii) that

$$|(T^\varepsilon)'(z)g(z)| = |\varepsilon T^\varepsilon(z)(T'(z)/T(z))g(z)| \le C \quad (z \in \mathbb{T})$$

for $\varepsilon > 0$. Hence $T^\varepsilon g$ is bounded in Λ_γ^+ as $\varepsilon \to 0$ and since $T^\varepsilon g \to g$ pointwise on $\mathbb{T}$ as $\varepsilon \to 0$, we have $T^\varepsilon g \to g$ wk* in Λ_γ^+ as $\varepsilon \to 0$, so $g \in I$. Finally, $I_\gamma^2(Z_I, Q_I)$ is wk* dense in $I_\gamma(E_I, Q_I)$ by Proposition 6.6, so the result follows. ∎

References

[1] C. Bennett and J. E. Gilbert, *Homogeneous algebras on the circle. I. Ideals of analytic functions*, Ann. Inst. Fourier (Grenoble) 22 (1972), no. 3, 1–19.

[2] L. Carleson, *Sets of uniqueness for functions regular in the unit circle*, Acta Math. 87 (1952), 325–345.

[3] P. L. Duren, *Theory of H^p Spaces*, Academic Press, San Diego, 1970.

[4] K. M. Dyakonov, *Equivalent norms on Lipschitz-type spaces of holomorphic functions*, Acta Math. 178 (1997), 143–167.

[5] H. Hedenmalm, *Bounded analytic functions and closed ideals*, J. Anal. Math. 48 (1987), 142–166.

[6] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, NJ, 1962.

[7] J. A. Johnson, *Banach spaces of Lipschitz functions and vector-valued Lipschitz functions*, Trans. Amer. Math. Soc. 148 (1970), 147–169.

[8] Y. Katznelson, *An Introduction to Harmonic Analysis*, Wiley, New York, 1968.

[9] B. I. Korenblum, *Functions holomorphic in a disc and smooth in its closure*, Dokl. Akad. Nauk SSSR 200 (1971), 24–27 (in Russian); English transl.: Soviet Math. Dokl. 12 (1971), 1312–1315.

[10] —, *Closed ideals in the ring A^n*, Funktsional. Anal. i Prilozhen. 6 (1972), no. 3, 38–52 (in Russian); English transl.: Funct. Anal. Appl. 6 (1972), 203–214.

[11] A. L. Matheson, *Closed ideals in rings of analytic functions satisfying a Lipschitz condition*, in: Banach Spaces of Analytic Functions (Kent, OH, 1976), J. Baker *et al.* (eds.), Lecture Notes in Math. 604, Springer, Berlin, 1977, 67–72.

[12] —, *Approximation of analytic functions satisfying a Lipschitz condition*, Michigan Math. J. 25 (1978), 289–298.

[13] —, *Cyclic vectors for invariant subspaces in some classes of analytic functions*, Illinois J. Math. 36 (1992), 136–144.

[14] D. R. Sherbert, *The structure of ideals and point derivations in Banach algebras of Lipschitz functions*, Trans. Amer. Math. Soc. 111 (1964), 240–272.

[15] N. A. Shirokov, *Division and multiplication by inner functions in spaces of analytic functions smooth up to the boundary*, in: Complex Analysis and Spectral Theory (Leningrad, 1979/80), V. P. Khavin and N. K. Nikol'skiĭ (eds.), Lecture Notes in Math. 864, Springer, Berlin, 1981, 413–439.

[16] B. A. Taylor and D. L. Williams, *Ideals in rings of analytic functions with smooth boundary values*, Canad. J. Math. 22 (1970), 1266–1283.

[17] A. Zygmund, *Trigonometric Series*, vol. 1, Cambridge Univ. Press, 2nd ed., 1959.

Department of Mathematics and Physics
The Royal Veterinary and Agricultural University
Thorvaldsensvej 40
DK-1871 Frederiksberg C, Denmark
E-mail: vils@dina.kvl.dk

Received May 13, 2002
Revised version March 31, 2003 (4945)