Essential norms of weighted composition operators between Hardy spaces $H^p$ and $H^q$ for $1 \leq p, q \leq \infty$

by

R. DEMAZEUX (Lens)

Abstract. We complete the different cases remaining in the estimation of the essential norm of a weighted composition operator acting between the Hardy spaces $H^p$ and $H^q$ for $1 \leq p, q \leq \infty$. In particular we give some estimates for the cases $1 = p \leq q \leq \infty$ and $1 \leq q < p \leq \infty$.

1. Introduction. Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ denote the open unit disk in the complex plane. Given two analytic functions $u$ and $\varphi$ defined on $D$ such that $\varphi(D) \subset D$, one can define the weighted composition operator $uC_\varphi$ that maps any analytic function $f$ defined on $D$ to the function $uC_\varphi(f) = u(f \circ \varphi)$. In [12], de Leeuw showed that the isometries in the Hardy space $H^1$ are weighted composition operators, while Forelli [8] obtained this result for the Hardy space $H^p$ when $1 < p < \infty$, $p \neq 2$. Another example is the study of composition operators on the half-plane. A composition operator in a Hardy space of the half-plane is bounded if and only if a certain weighted composition operator is bounded on the Hardy space of the unit disk (see [14] and [15]).

When $u \equiv 1$, we just have the composition operator $C_\varphi$. The continuity of these operators on the Hardy space $H^p$ is ensured by Littlewood’s subordination principle, which says that $C_\varphi(f)$ belongs to $H^p$ whenever $f \in H^p$ (see [4, Corollary 2.24]). As a consequence, the condition $u \in H^\infty$ suffices for the boundedness of $uC_\varphi$ on $H^p$. Considering the image of the constant functions, a necessary condition is that $u$ belongs to $H^p$. Nevertheless a weighted composition operator need not be continuous on $H^p$, and it is easy to find examples where $uC_\varphi(H^p) \nsubseteq H^p$ (see Lemma 2.1 of [3] for instance).

In this note we deal with weighted composition operators between $H^p$ and $H^q$ for $1 \leq p, q \leq \infty$. Boundedness and compactness are characterized

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in [3] for $1 \leq p \leq q < \infty$ by means of Carleson measures, while the essential norms of weighted composition operators are estimated in [5] for $1 < p \leq q < \infty$ by means of an integral operator. For the case $1 \leq q < p < \infty$, boundedness and compactness of $uC_\varphi$ are studied in [5], and Gorkin and MacCluer in [9] give an estimate of the essential norm of a composition operator acting between $H^p$ and $H^q$.

The aim of this paper is to complete the different cases remaining in the estimation of the essential norm of a weighted composition operator. In Sections 2 and 3, we give an estimate of the essential norm of $uC_\varphi$ acting between $H^p$ and $H^q$ when $p = 1$ and $1 \leq q < \infty$ and when $1 \leq p < \infty$ and $q = \infty$. Sections 4 and 5 are devoted to the case where $\infty \geq p > q \geq 1$.

Let $\overline{D}$ be the closure of the unit disk $D$ and $T = \partial D$ its boundary. We denote by $dm = dt/2\pi$ the normalised Haar measure on $T$. If $A$ is a Borel subset of $T$, the notation $m(A)$ as well as $|A|$ will designate the Haar measure of $A$. For $1 \leq p < \infty$, the Hardy space $H^p(D)$ is the space of analytic functions $f : D \to \mathbb{C}$ satisfying

$$\|f\|_p = \sup_{0 < r < 1} \left( \int_T |f(r\zeta)|^p \, dm(\zeta) \right)^{1/p} < \infty.$$ 

Endowed with this norm, $H^p(D)$ is a Banach space. The space $H^\infty(D)$ consists of all bounded analytic functions on $D$, and its norm is the supremum norm on $D$.

We recall that any function $f \in H^p(D)$ can be extended onto $T$ to a function $f^*$ by the formula $f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$. The limit exists almost everywhere by Fatou’s theorem, and $f^* \in L^p(T)$. Moreover, $f \mapsto f^*$ is an into isometry from $H^p(D)$ to $L^p(T)$ whose image, denoted by $H^p(T)$, is the closure (weak-star closure for $p = \infty$) of the set of polynomials in $L^p(T)$. So we can identify $H^p(D)$ and $H^p(T)$, and we will use the notation $H^p$ for both of these spaces. More on Hardy spaces can be found in [11] for instance.

The essential norm of an operator $T : X \to Y$, denoted $\|T\|_e$, is given by

$$\|T\|_e = \inf\{\|T - K\| \mid K \text{ is a compact operator from } X \text{ to } Y\}.$$ 

Observe that $\|T\|_e \leq \|T\|$, and $\|T\|_e$ is the norm of $T$ seen as an element of the quotient space $B(X,Y)/K(X,Y)$ where $B(X,Y)$ is the space of all bounded operators from $X$ to $Y$ and $K(X,Y)$ is the subspace consisting of all compact operators.

Notation: we will write $a \approx b$ whenever there exist two positive universal constants $c$ and $C$ such that $cb \leq a \leq Cb$. Throughout, $u$ will be a non-zero analytic function on $D$ and $\varphi$ will be a non-constant analytic function defined on $D$ and satisfying $\varphi(D) \subset D$. 
2. \( uC_\varphi \in B(H^1, H^q) \) for \( 1 \leq q < \infty \). Let us start with a characterization of the boundedness of \( uC_\varphi \) acting between \( H^p \) and \( H^q \):

**Theorem 2.1** (see [5, Theorem 4]). Let \( u \) be an analytic function on \( \mathbb{D} \) and \( \varphi \) an analytic self-map of \( \mathbb{D} \). Let \( 0 < p \leq q < \infty \). Then the weighted composition operator \( uC_\varphi \) is bounded from \( H^p \) to \( H^q \) if and only if

\[
\sup_{a \in \mathbb{D}} \left\{ \int_T |u(\zeta)|^q \left( \frac{1 - |a|^2}{|1 - \bar{a}\varphi(\zeta)|^2} \right)^{q/p} dm(\zeta) \right\}^{1/q} < \infty.
\]

As a consequence, \( uC_\varphi \) is a bounded operator as soon as \( uC_\varphi \) is uniformly bounded on the set \( \{ k_a^{1/p} \mid a \in \mathbb{D} \} \) where \( k_a \) is the normalized kernel defined by \( k_a(z) = (1 - |a|^2)/(1 - \bar{a}z)^2 \), \( a \in \mathbb{D} \). Note that \( k_a^{1/p} \in H^p \) and \( \| k_a^{1/p} \|_p = 1 \).

These kernels play a crucial role in the estimation of the essential norm of a weighted composition operator:

**Theorem 2.2** (see [5, Theorem 5]). Let \( u \) be an analytic function on \( \mathbb{D} \) and \( \varphi \) an analytic self-map of \( \mathbb{D} \). Assume that the weighted composition operator \( uC_\varphi \) is bounded from \( H^p \) to \( H^q \) with \( 1 < p \leq q < \infty \). Then

\[
\| uC_\varphi \|_e \approx \lim_{|a| \to 1^-} \left( \int_T |u(\zeta)|^q \left( \frac{1 - |a|^2}{|1 - \bar{a}\varphi(\zeta)|^2} \right)^{q/p} dm(\zeta) \right)^{1/q}.
\]

The aim of this section is to give the corresponding estimate for the case \( p = 1 \). We shall prove that the previous theorem is still valid for \( p = 1 \):

**Theorem 2.3.** Let \( u \) be an analytic function on \( \mathbb{D} \) and \( \varphi \) an analytic self-map of \( \mathbb{D} \). Suppose that the weighted composition operator \( uC_\varphi \) is bounded from \( H^1 \) to \( H^q \) for a certain \( 1 \leq q < \infty \). Then

\[
\| uC_\varphi \|_e \approx \lim_{|a| \to 1^-} \left( \int_T |u(\zeta)|^q \left( \frac{1 - |a|^2}{|1 - \bar{a}\varphi(\zeta)|^2} \right)^{q} dm(\zeta) \right)^{1/q}.
\]

Let us start with the upper estimate:

**Proposition 2.4.** Let \( uC_\varphi \in B(H^1, H^q) \) with \( 1 \leq q < \infty \). Then there exists a positive constant \( \gamma \) such that

\[
\| uC_\varphi \|_e \leq \gamma \lim_{|a| \to 1^-} \left( \int_T |u(\zeta)|^q \left( \frac{1 - |a|^2}{|1 - \bar{a}\varphi(\zeta)|^2} \right)^{q} dm(\zeta) \right)^{1/q}.
\]

The main tool of the proof is the use of Carleson measures. Assume that \( \mu \) is a finite positive Borel measure on \( \mathbb{D} \) and let \( 1 \leq p, q < \infty \). We say that \( \mu \) is a \( (p, q) \)-Carleson measure if the embedding \( J_\mu : f \in H^p \mapsto f \in L^q(\mu) \) is well defined. In this case, the closed graph theorem ensures that \( J_\mu \) is continuous. In other words, \( \mu \) is a \( (p, q) \)-Carleson measure if there exists a
constant $\gamma_1 > 0$ such that for every $f \in H^p$, 
\begin{equation}
\int_{\mathbb{D}} |f(z)|^q \, d\mu(z) \leq \gamma_1 \|f\|_p^q.
\end{equation}

Let $I$ be an arc in $\mathbb{T}$. By $S(I)$ we denote the Carleson window given by 
$$S(I) = \{z \in \mathbb{D} \mid 1 - |I| \leq |z| < 1, \ z/|z| \in I\}.$$ 

Let us denote by $\mu_{\mathbb{D}}$ and $\mu_{\mathbb{T}}$ the restrictions of $\mu$ to $\mathbb{D}$ and $\mathbb{T}$ respectively. The following result is a version of a theorem of Duren (see [7, p. 163]) for measures on $\mathbb{D}$:

**Theorem 2.5 (see [1, Theorem 2.5]).** Let $1 \leq p < q < \infty$. A finite positive Borel measure $\mu$ on $\mathbb{D}$ is a $(p, q)$-Carleson measure if and only if $\mu_{\mathbb{T}} = 0$ and there exists a constant $\gamma_2 > 0$ such that 
\begin{equation}
\mu_{\mathbb{D}}(S(I)) \leq \gamma_2 |I|^{q/p} \text{ for any arc } I \subset \mathbb{T}.
\end{equation}

Notice that the best constants $\gamma_1$ and $\gamma_2$ in (2.1) and (2.2) are comparable, meaning that there is a positive constant $\beta$ independent of the measure $\mu$ such that $(1/\beta)\gamma_2 \leq \gamma_1 \leq \beta \gamma_2$.

The notion of Carleson measure was introduced by Carleson in [2] as a part of his work on the corona problem. He gave a characterization of measures $\mu$ on $\mathbb{D}$ such that $H^p$ embeds continuously in $L^p(\mu)$.

Examples of such Carleson measures are provided by composition operators. Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map and let $1 \leq p, q < \infty$. The boundedness of the composition operator $C_\varphi : f \mapsto f \circ \varphi$ between $H^p$ and $H^q$ can be rephrased in terms of $(p, q)$-Carleson measures. Indeed, denote by $m_\varphi$ the pullback measure of $m$ by $\varphi$, which is the image of the Haar measure $m$ of $\mathbb{T}$ under the map $\varphi^*$, defined by 
$$m_\varphi(A) = m(\varphi^{-1}(A))$$

for every Borel subset $A$ of $\overline{\mathbb{D}}$. Then 
$$\|C_\varphi(f)\|_q^q = \int_{\mathbb{T}} |f \circ \varphi|^q \, dm = \int_{\mathbb{D}} |f|^q \, dm_\varphi = \|J_{m_\varphi}(f)\|_q^q$$

for all $f \in H^p$. Thus $C_\varphi$ maps $H^p$ boundedly into $H^q$ if and only if $m_\varphi$ is a $(p, q)$-Carleson measure.

We will denote by $r\mathbb{D}$ the open disk of radius $r$, in other words $r\mathbb{D} = \{z \in \mathbb{D} \mid |z| < r\}$ for $0 < r < 1$. We will need the following lemma concerning $(p, q)$-Carleson measures:

**Lemma 2.6.** Take $0 < r < 1$ and let $\mu$ be a finite positive Borel measure on $\overline{\mathbb{D}}$. Let 
$$N_r^* := \sup_{|a| \geq r} \int_{\mathbb{D}} |k_a(w)|^{q/p} \, d\mu(w).$$
If $\mu$ is a $(p,q)$-Carleson measure for $1 \leq p \leq q < \infty$ then so is $\mu_r := \mu|_{\mathbb{D}\setminus \mathbb{D}^r}$. Moreover one can find an absolute constant $M > 0$ satisfying $\|\mu_r\| \leq MN_r^*$ where $\|\mu_r\| := \sup_{I \subset \mathbb{T}} \mu_r(S(I))/|I|^{q/p}$.

We omit the proof of Lemma 2.6, which is a slight modification of the proof of Lemmas 1 and 2 in [5] using Theorem 2.5.

In the proof of the upper estimate of Theorem 2.2 in [5], the authors use a decomposition of the identity on $H^p$ of the form $I = K_N + R_N$ where $K_N$ is the partial sum operator defined by $K_N(\sum_{n=0}^{\infty} a_n z^n) = \sum_{n=0}^{N} a_n z^n$, and they use the fact that $(K_N)$ is a sequence of compact operators that is uniformly bounded in $B(H^p)$ and that $R_N$ converges pointwise to zero on $H^p$. Nevertheless the sequence $(K_N)$ is not uniformly bounded in $B(H^1)$. In fact, $(K_N)$ is uniformly bounded in $B(H^p)$ if and only if the Riesz projection $P : L^p \to H^p$ is bounded [16, Theorem 2], which occurs if and only if $1 < p < \infty$. Therefore we need to use a different decomposition for the case $p = 1$. Since $K_N$ is the convolution operator with the Dirichlet kernel on $H^p$, we shall consider the Fejér kernel $F_N$ of order $N$. Let us define $K_N : H^1 \to H^1$ to be the convolution operator associated to $F_N$ that maps $f \in H^1$ to $K_N f = F_N * f \in H^1$ and $R_N = I - K_N$. Then $\|K_N\| \leq 1$, $K_N$ is compact, and for every $f \in H^1$, $\|f - K_N f\|_1 \to 0$ by Fejér’s theorem. If $f(z) = \sum_{n \geq 0} \hat{f}(n) z^n \in H^1$, then

$$K_N f(z) = \sum_{n=0}^{N-1} \left(1 - \frac{n}{N}\right) \hat{f}(n) z^n.$$

**Lemma 2.7.** Let $1 \leq q < \infty$ and suppose that $uC_\varphi \in B(H^1, H^q)$. Then

$$\|uC_\varphi\|_e \leq \liminf_N \|uC_\varphi R_N\|_e.$$

**Proof.** We have

$$\|uC_\varphi\|_e = \|uC_\varphi K_N + uC_\varphi R_N\|_e = \|uC_\varphi R_N\|_e$$

since $K_N$ is compact

$$\leq \|uC_\varphi R_N\|$$

and the result follows by taking the lower limit.

We will need the following lemma to estimate the remainder $R_N$:

**Lemma 2.8.** Let $\varepsilon > 0$ and $0 < r < 1$. Then there exists $N_0 = N_0(r) \in \mathbb{N}$ such that for all $N \geq N_0$,

$$|R_N f(w)|^q < \varepsilon \|f\|^q_1$$

for every $|w| < r$ and every $f$ in $H^1$. 

Proof. Let \( K_w(z) = 1/(1 - \bar{w}z) \), \( w \in \mathbb{D} \), \( z \in \mathbb{D} \). Then \( K_w \) is a bounded analytic function on \( \mathbb{D} \). It is easy to see that for every \( f \in H^1 \),
\[
\langle R_N f, K_w \rangle = \langle f, R_N K_w \rangle
\]
where \( |w| < r \), \( N \geq 1 \) and
\[
\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} d\theta
\]
for \( f \in H^1 \) and \( g \in H^\infty \). Then \( |R_N f(w)| = |\langle R_N f, K_w \rangle| = |\langle f, R_N K_w \rangle| \leq \|f\|_1 \|R_N K_w\|_\infty \). Take \( |w| < r \) and choose \( N_0 \in \mathbb{N} \) so that for every \( N \geq N_0 \) one has \( r^N \leq \varepsilon^{1/2}(1 - r)/2 \) and \((1/N) \sum_{n=1}^{N-1} nr^n \leq (1/2)\varepsilon^{1/2} \). Since
\[
R_N K_w(z) = R_N \left( \sum_{n=0}^{\infty} \bar{w}^n z^n \right) = \sum_{n=0}^{N-1} \frac{n}{N} \bar{w}^n z^n + \sum_{n=N}^{\infty} \bar{w}^n z^n,
\]
one has
\[
\|R_N K_w\|_\infty < \frac{1}{N} \sum_{n=0}^{N-1} nr^n + \sum_{n=N}^{\infty} r^n \leq \varepsilon^{1/2}.
\]
Thus \( |R_N f(w)|^q \leq \varepsilon \|f\|^q_1 \) for every \( f \) in \( H^1 \).

Proof of Proposition 2.4. Denote by \( \mu \) the measure which is absolutely continuous with respect to \( m \) and whose density is \( |u|^q \), and let \( \mu_\varphi = \mu \circ \varphi^{-1} \)
be the pullback of \( \mu \) by \( \varphi \). Fix 0 < \( r < 1 \). For every \( f \in H^1 \),
\[
\| (u C_\varphi R_N) f \|_q^q = \int_\mathbb{T} |u(\zeta)|^q |((R_N f) \circ \varphi)(\zeta)|^q dm(\zeta)
\]
\[
= \int_\mathbb{T} |((R_N f) \circ \varphi)(\zeta)|^q d\mu(\zeta) = \int_{\mathbb{D}} |R_N f(w)|^q d\mu_\varphi(w)
\]
\[
= \int_{\mathbb{D} \setminus r \mathbb{D}} |R_N f(w)|^q d\mu_\varphi(w) + \int_{r \mathbb{D}} |R_N f(w)|^q d\mu_\varphi(w)
\]
\[
= I_1(N, r, f) + I_2(N, r, f).
\]

Let us first show that
\[
(2.4) \quad \lim_{N} \sup_{\|f\|_1 = 1} I_2(N, r, f) = 0.
\]
For \( \varepsilon > 0 \), Lemma 2.8 gives us an integer \( N_0(r) \) such that for every \( N \geq N_0(r) \),
\[
I_2(N, r, f) = \int_{r \mathbb{D}} |R_N f(w)|^q d\mu_\varphi(w) \leq \varepsilon \|f\|_1^q \mu_\varphi(r \mathbb{D})
\]
\[
\leq \varepsilon \|f\|_1^q \mu_\varphi(\mathbb{D}) \leq \varepsilon \|f\|_1^q \|u\|_q^q.
\]
So, \( r \) being fixed, we obtain (2.4).
Now we need an estimate of \( I_1(N, r, f) \). The continuity of \( uC_\varphi : H^1 \to H^q \) ensures that \( \mu_\varphi \) is a \((1, q)\)-Carleson measure, and therefore \( \mu_{\varphi, r} := \mu_{\varphi|_{\mathbb{D} \cap r \mathbb{D}}} \) is also a \((1, q)\)-Carleson measure by using Lemma 2.6 for \( p = 1 \). It follows that
\[
\int_{\mathbb{D} \cap r \mathbb{D}} |R_N f(w)|^q \, d\mu_{\varphi, r}(w) \leq \gamma_1 \|R_N f\|_1^q \leq \beta \|\mu_{\varphi, r}\| \|R_N f\|_1^q \leq 2^q \beta MN_r^* \|f\|_1^q
\]
using Lemma 2.6 and the fact that \( \|R_N\| \leq 1 + \|K_N\| \leq 2 \) for every \( N \in \mathbb{N} \). We take the supremum over \( B_{H^1} \) and the lower limit as \( N \) tends to infinity in (2.3) to obtain
\[
\liminf_{N \to \infty} \|uC_\varphi R_N\|^q \leq 2^q \beta MN_r^*.
\]
Now as \( r \) goes to 1 we have
\[
\lim_{r \to 1} N_r^* = \limsup_{|a| \to 1^-} \int_{\mathbb{T}} |k_a(w)|^q \, d\mu_\varphi(w) = \limsup_{|a| \to 1^-} \int_{\mathbb{T}} |u(\zeta)|^q \left( \frac{1 - |a|^2}{|1 - \bar{a}\varphi(\zeta)|^2} \right)^q \, dm(\zeta)
\]
and we obtain the desired estimate using Lemma 2.7.

Now let us turn to the lower estimate in Theorem 2.2. Let \( 1 \leq q < \infty \). Consider the Fejér kernel \( F_N \) of order \( N \), define \( K_N : H^q \to H^q \) to be the convolution operator associated to \( F_N \) and set \( R_N = I - K_N \). Then \( (K_N)_N \) is a sequence of uniformly bounded compact operators in \( B(H^q) \), and \( \|R_N f\|_q \to 0 \) for all \( f \in H^q \).

**Lemma 2.9.** There exists \( 0 < \gamma \leq 2 \) such that whenever \( uC_\varphi \) is a bounded operator from \( H^1 \) to \( H^q \) with \( 1 \leq q < \infty \), one has
\[
\frac{1}{\gamma} \limsup_{N} \|R_N uC_\varphi\| \leq \|uC_\varphi\|_e.
\]

**Proof.** Let \( K \in B(H^1, H^q) \) be a compact operator. Since \( (K_N) \) is uniformly bounded, one can find \( \gamma > 0 \) satisfying \( \|R_N\| \leq 1 + \|K_N\| \leq \gamma \) for all \( N > 0 \), and we have
\[
\|uC_\varphi + K\| \geq \frac{1}{\gamma} \|R_N (uC_\varphi + K)\| \geq \frac{1}{\gamma} \|R_N uC_\varphi\| - \frac{1}{\gamma} \|R_N K\|.
\]
Now use the fact that \( (R_N) \) goes pointwise to zero in \( H^q \), and consequently \( (R_N) \) converges strongly to zero over the compact set \( K(B_{H^1}) \) as \( N \) goes to infinity. It follows that \( \|R_N K\| \to 0 \), and
\[
\|uC_\varphi + K\| \geq \frac{1}{\gamma} \limsup_{N} \|R_N uC_\varphi\|
\]
for every compact operator \( K : H^1 \to H^q \). □
Proposition 2.10. Let \( u \) be an analytic function on \( \mathbb{D} \) and \( \varphi \) an analytic self-map of \( \mathbb{D} \). Assume that \( uC_{\varphi} \in B(H^1, H^q) \) with \( 1 \leq q < \infty \). Then

\[
\|uC_{\varphi}\|_e \geq \frac{1}{\gamma} \limsup_{|a| \to 1^-} \left( \int_T |u(\zeta)|^q \left( \frac{1 - |a|^2}{|1 - \overline{a}\varphi(\zeta)|^2} \right)^q dm(\zeta) \right)^{1/q}.
\]

Proof. Since \( k_a \) is a unit vector in \( H^1 \),

\[
\|R_NuC_{\varphi}\| = \|uC_{\varphi} - K_NuC_{\varphi}\| \geq \|uC_{\varphi}k_a\|_q - \|K_NuC_{\varphi}k_a\|_q.
\]

First case: \( q > 1 \). Since \((k_a)\) converges to zero for the topology of uniform convergence on compact sets in \( \mathbb{D} \) as \( |a| \to 1 \), so does \( uC_{\varphi}(k_a) \). The topology of uniform convergence on compact sets in \( \mathbb{D} \) and the weak topology agree on \( H^q \), so \( uC_{\varphi}(k_a) \) goes to zero for the weak topology in \( H^q \) as \( |a| \to 1 \). Since \( K_N \) is a compact operator, it is completely continuous and carries weak-null sequences to norm-null sequences. So \( \|K_N(uC_{\varphi}(k_a))\|_q \to 0 \) when \( |a| \to 1 \), and

\[
\|R_NuC_{\varphi}\| \geq \limsup_{|a| \to 1^-} \|uC_{\varphi}(k_a)\|_q.
\]

Taking the upper limit as \( N \to \infty \), we obtain the result using Lemma 2.9.

For the second case we will need the following computational lemma:

Lemma 2.11. Let \( \varphi \) be an analytic self-map of \( \mathbb{D} \). Take \( a \in \mathbb{D} \) and an integer \( N \geq 1 \). Denote by \( \alpha_p(a) \) the \( p \)th Fourier coefficient of \( C_{\varphi}(k_a/(1 - |a|^2)) \), so that for every \( z \in \mathbb{D} \) we have

\[
k_a(\varphi(z)) = (1 - |a|^2) \sum_{p=0}^{\infty} \alpha_p(a) z^p.
\]

Then there exists a positive constant \( M = M(N) > 0 \) depending on \( N \) such that \( |\alpha_p(a)| \leq M \) for all \( p \leq N \) and \( a \in \mathbb{D} \).

Proof. Write \( \varphi(z) = a_0 + \psi(z) \) with \( a_0 = \varphi(0) \in \mathbb{D} \) and \( \psi(0) = 0 \). If we develop \( k_a(z) \) as a Taylor series and replace \( z \) by \( \varphi(z) \) we obtain

\[
k_a(\varphi(z)) = (1 - |a|^2) \sum_{n=0}^{\infty} (n + 1)(\overline{a})^n \varphi(z)^n.
\]

Then

\[
\alpha_p(a) = \left\langle \sum_{n=0}^{\infty} (n + 1)(\overline{a})^n \varphi(z)^n, z^p \right\rangle
\]

\[
= \sum_{n=0}^{\infty} (n + 1)(\overline{a})^n \sum_{j=0}^{n} \binom{n}{j} a_0^{n-j} \langle \psi(z)^j, z^p \rangle.
\]
where $\langle f, g \rangle = \int_{\mathbb{T}} f \bar{g} \, dm$. Note that $\langle \psi(z)^j, z^p \rangle = 0$ if $j > p$ since $\psi(0) = 0$, and consequently

$$\alpha_p(a) = \sum_{n=0}^{\infty} (n+1)(\bar{a})^n \sum_{j=0}^{\min(n,p)} \binom{n}{j} a_0^{n-j} \langle \psi(z)^j, z^p \rangle$$

$$= \sum_{j=0}^{p} \sum_{n=j}^{\infty} (n+1)(\bar{a})^n \binom{n}{j} a_0^{n-j} \langle \psi(z)^j, z^p \rangle$$

$$= \sum_{j=0}^{p} \langle \psi(z)^j, z^p \rangle \sum_{n=j}^{\infty} (n+1)(\bar{a})^n \binom{n}{j} a_0^{n-j}.$$

In the case where $a_0 \neq 0$ we obtain

$$\alpha_p(a) = \sum_{j=0}^{p} \langle \psi(z)^j, z^p \rangle a_0^{-j} \sum_{n=j}^{\infty} (n+1) \binom{n}{j} (\bar{a}a_0)^n$$

$$= \sum_{j=0}^{p} \langle \psi(z)^j, z^p \rangle a_0^{-j} \frac{(j+1)(\bar{a}a_0)^j}{(1-\bar{a}a_0)^{j+2}}$$

$$= \sum_{j=0}^{p} \langle \psi(z)^j, z^p \rangle \frac{(j+1)(\bar{a})^j}{(1-\bar{a}a_0)^{j+2}}.$$

using the following equalities for $x = \bar{a}a_0 \in \mathbb{D}$:

$$\sum_{n=j}^{\infty} (n+1) \binom{n}{j} x^n = \left( \sum_{n=j}^{\infty} \binom{n}{j} x^{n+1} \right)' = \left( \frac{x^{j+1}}{(1-x)^{j+1}} \right)' = \frac{(j+1)x^j}{(1-x)^{j+2}}.$$

Note that the last expression obtained for $\alpha_p(a)$ is also valid for $a_0 = 0$. Thus, for $0 \leq p \leq N$ we have the following estimates:

$$|\alpha_p(a)| \leq \sum_{j=0}^{p} |\langle \psi(z)^j, z^p \rangle| \frac{j+1}{(1-|a_0|)^{j+2}} \leq \sum_{j=0}^{p} ||\psi^j||_\infty \frac{N+1}{(1-|a_0|)^{N+2}}$$

$$\leq \frac{(N+1)^2}{(1-|a_0|)^{N+2}} \max_{0 \leq j \leq N} ||\psi^j||_\infty \leq M,$$

where $M$ is a constant independent from $a$. ■

Second case: $q = 1$. In this case, it is no longer for the weak topology but for the weak-star topology of $H^1$ that $uC_{\psi}(k_a)$ tends to zero when $|a| \to 1$. Nevertheless, it is still true that $\|K_N uC_{\varphi}(k_a)\|_1 \to 0$ as $|a| \to 1$. Indeed, if $f(z) = \sum_{n \geq 0} \hat{f}(n)z^n \in H^1$, then

$$K_N f(z) = \sum_{n=0}^{N-1} \left( 1 - \frac{n}{N} \right) \hat{f}(n)z^n.$$
We have the following development:

\[ k_a(\varphi(z)) = (1 - |a|^2) \sum_{n=0}^{\infty} \alpha_n(a) z^n. \]

Denote by \( u_n \) the \( n \)th Fourier coefficient of \( u \), so that

\[ uC_\varphi(k_a)(z) = (1 - |a|^2) \sum_{n=0}^{\infty} \left( \sum_{p=0}^{n} \alpha_p(a) u_{n-p} \right) z^n, \quad \forall z \in \mathbb{D}. \]

It follows that

\[ \|K_NuC_\varphi(k_a)\|_1 \leq (1 - |a|^2) \sum_{n=0}^{N-1} \left( 1 - \frac{n}{N} \right) \left| \sum_{p=0}^{n} \alpha_p(a) u_{n-p} \right| \|z^n\|_1. \]

Now using the estimates of Lemma 2.11, one can find a constant \( M > 0 \) independent of \( a \) such that \( |\alpha_p(a)| \leq M \) for every \( a \in \mathbb{D} \) and \( 0 \leq p \leq N - 1 \). Use the fact that \( \|z^n\|_1 = 1 \) and \( |u_p| \leq \|u\|_1 \) to deduce that there is a constant \( M' > 0 \) independent of \( a \) such that

\[ \|K_NuC_\varphi(k_a)\|_1 \leq M'(1 - |a|^2) \|u\|_1 \]

for all \( a \in \mathbb{D} \). Thus \( K_NuC_\varphi(k_a) \) converges to zero in \( H^1 \) when \( |a| \to 1 \), and take the upper limit of 2.5 when \( a \to 1^- \) to obtain

\[ \|R_NuC_\varphi\| \geq \limsup_{|a|\to1} \|uC_\varphi(k_a)\|_1, \quad \forall N \geq 0. \]

We conclude with Lemma 2.9 and observe that \( \gamma = \sup \|R_N\| \leq 2 \) since \( \|R_N\| \leq 1 + \|K_N\| \leq 2. \) □

3. \( uC_\varphi \in B(H^p, H^\infty) \) for \( 1 \leq p < \infty \). Let \( u \) be a bounded analytic function. Characterizations of boundedness and compactness of \( uC_\varphi \) as a linear map between \( H^p \) and \( H^\infty \) have been studied in [3] for \( p \geq 1 \). Indeed,

\[ uC_\varphi \in B(H^p, H^\infty) \quad \text{if and only if} \quad \sup_{z \in \mathbb{D}} \frac{|u(z)|^p}{1 - |\varphi(z)|^2} < \infty \]

and

\[ uC_\varphi \quad \text{is compact if and only if} \quad \|\varphi\|_\infty < 1 \text{ or } \lim_{|\varphi(z)|\to1} \frac{|u(z)|^p}{1 - |\varphi(z)|^2} = 0. \]

In the case where \( \|\varphi\|_\infty = 1 \) we let

\[ M_\varphi(u) = \limsup_{|\varphi(z)|\to1} \frac{|u(z)|}{(1 - |\varphi(z)|^2)^{1/p}}. \]

In view of Theorem 1.7 in [13], it seems reasonable to expect that the essential norm of \( uC_\varphi \) is equivalent to the quantity \( M_\varphi(u) \). We first have a majorization:
PROPOSITION 3.1. Let \( u \) be an analytic function on \( \mathbb{D} \) and \( \varphi \) an analytic self-map of \( \mathbb{D} \). Suppose that \( uC_\varphi \) is a bounded operator from \( H^p \) to \( H^\infty \), where \( 1 \leq p < \infty \), and that \( \| \varphi \|_\infty = 1 \). Then

\[
\|uC_\varphi\|_e \leq 2M_\varphi(u).
\]

Proof. Let \( \varepsilon > 0 \), and pick \( r < 1 \) satisfying

\[
\sup_{|\varphi(z)| \geq r} \frac{|u(z)|}{(1 - |\varphi(z)|^2)^{1/p}} \leq M_\varphi(u) + \varepsilon.
\]

We approximate \( uC_\varphi \) by \( uC_\varphi K_N \) where \( K_N : H^p \to H^p \) is the convolution operator with the Fejér kernel of order \( N \), where \( N \) is chosen so that \( |R_N f(w)| < \varepsilon \|f\|_1 \) for all \( f \in H^1 \) and \( |w| < r \) (Lemma 2.8 for \( q = 1 \)). We want to show that \( \|uC_\varphi - uC_\varphi K_N\| = \|uC_\varphi R_N\| \leq \max(2M_\varphi(u) + 2\varepsilon, \varepsilon\|u\|_\infty) \), which will prove our assertion. If \( f \) is a unit vector in \( H^p \), then the norm of \( uC_\varphi R_N(f) \) is equal to

\[
\max \left( \sup_{|\varphi(z)| \geq r} |u(z)(R_N f) \circ \varphi(z)|, \sup_{|\varphi(z)| < r} |u(z)(R_N f) \circ \varphi(z)| \right).
\]

To estimate the first term, for \( \omega \in \mathbb{D} \), we denote by \( \delta_\omega \) the linear functional on \( H^p \) defined by \( \delta_\omega(f) = f(\omega) \). Then \( \delta_\omega \in (H^p)^* \) and we have \( \|\delta_w\|_{(H^p)^*} = 1/(1 - |w|^2)^{1/p} \) for every \( w \in \mathbb{D} \). Therefore

\[
\sup_{|\varphi(z)| \geq r} |u(z)(R_N f) \circ \varphi(z)| \leq \sup_{|\varphi(z)| \geq r} |u(z)||\delta_\varphi(z)||_{(H^p)^*} \|R_N f\|_p
\]

\[
\leq 2 \sup_{|\varphi(z)| \geq r} \frac{|u(z)|}{(1 - |\varphi(z)|^2)^{1/p}} \leq 2(M_\varphi(u) + \varepsilon),
\]

using the fact that \( \|R_N f\|_p \leq 2 \).

For the second term, since \( |\varphi(z)| < r \) we have

\[
|u(z)R_N f(\varphi(z))| \leq \|u\|_\infty |R_N f(\varphi(z))| \leq \varepsilon\|u\|_\infty \|f\|_1 \leq \varepsilon\|u\|_\infty,
\]

which ends the proof. \( \blacksquare \)

On the other hand, we have the lower estimate:

PROPOSITION 3.2. Let \( u \) be an analytic function on \( \mathbb{D} \) and \( \varphi \) an analytic self-map of \( \mathbb{D} \) satisfying \( \|\varphi\|_\infty = 1 \). Suppose that \( uC_\varphi \) is a bounded operator from \( H^p \) to \( H^\infty \), where \( 1 \leq p < \infty \). Then

\[
\frac{1}{2}M_\varphi(u) \leq \|uC_\varphi\|_e.
\]

Proof. Assume that \( uC_\varphi \) is not compact, implying \( M_\varphi(u) > 0 \). Let \( (z_n) \) be a sequence in \( \mathbb{D} \) satisfying

\[
\lim_n |\varphi(z_n)| = 1 \quad \text{and} \quad \lim_n \frac{|u(z_n)|}{(1 - |\varphi(z_n)|^2)^{1/p}} = M_\varphi(u).
\]
Consider the sequence \((f_n)\) defined by

\[
f_n(z) = k_{\varphi(z_n)}(z)^{1/p} = \frac{(1 - |\varphi(z_n)|^2)^{1/p}}{(1 - \varphi(z_n)z)^{2/p}}.
\]

Each \(f_n\) is a unit vector of \(H^p\). Let \(K : H^p \to H^\infty\) be a compact operator.

**First case:** \(p > 1\). Since the sequence \((f_n)\) converges to zero for the weak topology of \(H^p\) and \(K\) is completely continuous, the sequence \((Kf_n)\) converges to zero for the norm topology in \(H^\infty\). Use that

\[
\|uC\varphi + K\| \geq \limsup_n \|uC\varphi(f_n)\|_\infty - \|Kf_n\|_\infty
\]

and take the upper limit when \(n \to \infty\) to obtain

\[
\|uC\varphi + K\| \geq \frac{|u(z_n)|}{(1 - |\varphi(z_n)|^2)^{1/p}} \geq M\varphi(u).
\]

**Second case:** \(p = 1\). Let \(\varepsilon > 0\). Since the sequence \((f_n)\) is no longer weakly convergent to zero in \(H^1\), we cannot assert that \((Kf_n)_n\) goes to zero in \(H^\infty\). Nevertheless, passing to subsequences, one can assume that \((Kf_{n_k})_k\) converges in \(H^\infty\), and hence is a Cauchy sequence. So we can find an integer \(N > 0\) such that for all \(k \geq N\) and \(m > N\) we have \(\|Kf_{n_k} - Kf_{n_m}\| < \varepsilon\). We deduce that

\[
\|uC\varphi + K\| \geq \frac{1}{2} \|uC\varphi(f_{n_k} - f_{n_m})\|_\infty - \frac{\varepsilon}{2} \geq \frac{1}{2} \|u(z_{n_k})\| |f_{n_k}(\varphi(z_{n_k})) - f_{n_m}(\varphi(z_{n_m}))| - \frac{\varepsilon}{2} \geq \frac{|u(z_{n_k})|}{2(1 - |\varphi(z_{n_k})|^2)} - \frac{|u(z_{n_k})|(1 - |\varphi(z_{n_m})|^2)}{2|1 - \varphi(z_{n_m})\varphi(z_{n_k})|^2} - \frac{\varepsilon}{2}.
\]

Now take the upper limit as \(m \to \infty\) (\(k\) being fixed) and recall that \(\lim_m |\varphi(z_{n_m})| = 1\) and \(|\varphi(z_{n_k})| < 1\) to obtain

\[
\|uC\varphi + K\| \geq \frac{|u(z_{n_k})|}{2(1 - |\varphi(z_{n_k})|^2)} - \frac{\varepsilon}{2}
\]

for every \(k \geq N\). It remains to let \(k \to \infty\) to have

\[
\|uC\varphi + K\| \geq \frac{1}{2} M\varphi(u) - \frac{\varepsilon}{2}.
\]

Combining Propositions 3.1 and 3.2 we obtain the following estimate:

**Theorem 3.3.** Let \(u\) be an analytic function on \(D\) and \(\varphi\) an analytic self-map of \(D\) with \(\|\varphi\|_\infty = 1\). Suppose that \(uC\varphi\) is a bounded operator
from $H^p$ to $H^\infty$, where $1 \leq p < \infty$. Then $\|uC_\varphi\|_e \approx M_\varphi(u)$. More precisely,

$$\frac{1}{2}M_\varphi(u) \leq \|uC_\varphi\|_e \leq 2M_\varphi(u).$$

Note that if $p > 1$ one can replace the constant 1/2 by 1.

4. $uC_\varphi \in B(H^\infty, H^q)$ for $\infty > q \geq 1$. In this setting, boundedness of the weighted composition operator $uC_\varphi$ is equivalent to saying that $u$ belongs to $H^q$, and $uC_\varphi$ is compact if and only if $u = 0$ or $|E_\varphi| = 0$ where $E_\varphi = \{\zeta \in \mathbb{T} \mid \varphi^*(\zeta) \in \mathbb{T}\}$ is the extremal set of $\varphi$ (see [9]). We give here some estimates of the essential norm of $uC_\varphi$ that appear in [9] for the special case of composition operators:

**Theorem 4.1.** Let $u \in H^q$ with $\infty > q \geq 1$ and $\varphi$ be an analytic self-map of $\mathbb{D}$. Then $\|uC_\varphi\|_e \approx (\int_{E_\varphi} |u(\zeta)|^q \, dm(\zeta))^{1/q}$. More precisely,

$$\frac{1}{2} \left( \int_{E_\varphi} |u(\zeta)|^q \, dm(\zeta) \right)^{1/q} \leq \|uC_\varphi\|_e \leq 2 \left( \int_{E_\varphi} |u(\zeta)|^q \, dm(\zeta) \right)^{1/q}.$$

We start with the upper estimate:

**Proposition 4.2.** Let $u \in H^q$ with $\infty > q \geq 1$ and $\varphi$ be an analytic self-map of $\mathbb{D}$. Then

$$\|uC_\varphi\|_e \leq 2 \left( \int_{E_\varphi} |u(\zeta)|^q \, dm(\zeta) \right)^{1/q}.$$

**Proof.** Take $0 < r < 1$. Since $\|r_\varphi\|_\infty \leq r < 1$, the set $E_{r_\varphi}$ is empty and therefore the operator $uC_{r_\varphi}$ is compact. Thus $\|uC_\varphi\|_e \leq \|uC_\varphi - uC_{r_\varphi}\|$. But

$$\|uC_{r_\varphi}\|^{q} = \sup_{\|f\|_\infty \leq 1} \int_{|E_\varphi|} |u(\zeta)|^q |f(\varphi(\zeta)) - f(r(\varphi(\zeta)))|^q \, dm(\zeta).$$

If $|E_\varphi| = 1$ then the integral in (4.1) coincides with

$$\int_{E_\varphi} |u(\zeta)|^q |f(\varphi(\zeta)) - f(r(\varphi(\zeta)))|^q \, dm(\zeta),$$

which is less than $2^q \int_{E_\varphi} |u(\zeta)|^q \, dm(\zeta)$. If $|E_\varphi| < 1$ we let $F_\varepsilon = \{\zeta \in \mathbb{T} \mid |\varphi^*(\zeta)| < 1 - \varepsilon\}$ for $\varepsilon > 0$, which is a nonempty set for $\varepsilon$ sufficiently small. (Let us mention here that an element $\zeta \in \mathbb{T}$ need not satisfy either $\zeta \in E_\varphi$ or $\zeta \in \bigcup_{\varepsilon > 0} F_\varepsilon$. It can happen that the radial limit $\varphi^*(\zeta)$ does not exist, but this occurs only for $\zeta$ belonging to a set of measure zero). We will use the pseudohyperbolic distance $\rho$ defined for $z$ and $w$ in the unit disk by $\rho(z, w) = |z - w|/|1 - \bar{w}z|$. The Pick–Schwarz theorem ensures that $\rho(f(z), f(w)) \leq \rho(z, w)$ for every $f \in B_{H^\infty}$. As a consequence, $|f(z) - f(w)| \leq 2\rho(z, w)$ for all $w$ and $z$ in $\mathbb{D}$. 

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If \( \zeta \in F_\varepsilon \) then
\[
\rho(\varphi(\zeta), r\varphi(\zeta)) = \frac{(1 - r)|\varphi(\zeta)|}{1 - r|\varphi(\zeta)|^2} \leq \frac{1 - r}{1 - r(1 - \varepsilon)^2}.
\]
One can choose \( 0 < r < 1 \) satisfying \( \sup_{F_\varepsilon} \rho(\varphi(\zeta), r\varphi(\zeta)) < \varepsilon/2 \), and so
\[
|f(\varphi(\zeta)) - f(r\varphi(\zeta))| \leq 2 \sup_{F_\varepsilon} \rho(\varphi(\zeta), r\varphi(\zeta)) \leq \varepsilon
\]
for all \( \zeta \in F_\varepsilon \) and every \( f \) in the closed unit ball of \( H^\infty \). It follows from these estimates and (4.1) that
\[
\|uC_\varphi - uC_{r\varphi}\|^q \leq \sup_{\|f\|_\infty \leq 1} \left( \int_{F_\varepsilon} |u(\zeta)|^q \varepsilon^q \, dm(\zeta) + \int_{T \setminus F_\varepsilon} 2^q |u(\zeta)|^q \, dm(\zeta) \right)
\]
\[
\leq \varepsilon^q \|u\|_q^q + 2^q \int_{T \setminus F_\varepsilon} |u(\zeta)|^q \, dm(\zeta).
\]
Let \( \varepsilon \) tend to zero to deduce the upper estimate. \( \blacksquare \)

Let us turn to the lower estimate:

**Proposition 4.3.** Suppose that \( \varphi \) is an analytic self-map of \( \mathbb{D} \) and let \( u \in H^q \) with \( \infty > q \geq 1 \). Then
\[
\|uC_\varphi\|_\varepsilon \geq \frac{1}{2} \left( \int_{E_\varphi} |u(\zeta)|^q \, dm(\zeta) \right)^{1/q}.
\]

**Proof.** Take a compact operator \( K \in B(H^\infty, H^q) \). Since the sequence \((z^n)_{n \in \mathbb{N}}\) is bounded in \( H^\infty \), there exists an increasing sequence \((n_k)_{k \geq 0}\) of integers such that \( (K(z^{n_k}))_{k \geq 0} \) converges in \( H^q \). For any \( \varepsilon > 0 \) one can find \( N \in \mathbb{N} \) such that for all \( k, m \geq N \) we have \( \|Kz^{n_k} - Kz^{n_m}\|_q < \varepsilon \). If \( 0 < r < 1 \), we let \( g_r(z) = g(rz) \) for a function \( g \) defined on \( \mathbb{D} \). Take \( k \geq N \). Then there exists \( 0 < r < 1 \) such that
\[
\|(u\varphi^{n_k})_r\|_q \geq \|u\varphi^{n_k}\|_q - \varepsilon.
\]
For all \( m \geq N \) we have
\[
\|uC_\varphi + K\| \geq \|(uC_\varphi + K)(z^{n_k} - z^{n_m})/2\|_q \geq \frac{1}{2} \|u(\varphi^{n_k} - \varphi^{n_m})\|_q - \varepsilon/2
\]
\[
\geq \frac{1}{2} \|(u\varphi^{n_k})_r - (u\varphi^{n_m})_r\|_q - \varepsilon/2
\]
\[
\geq \frac{1}{2} (\|(u\varphi^{n_k})_r\|_q - \|(u\varphi^{n_m})_r\|_q) - \varepsilon/2
\]
\[
\geq \frac{1}{2} (\|u\varphi^{n_k}\|_q - \|u\varphi^{n_m}\|_q) - \varepsilon.
\]
Let \( m \to \infty \), keeping in mind that \( 0 < r < 1 \) and \( \|\varphi_r\|_\infty < 1 \):
\[
\|(u\varphi^{n_m})_r\|_q \leq \|u\|_q \|\varphi_r\|^{n_m}_\infty \leq \|u\|_q \|\varphi_r\|^{n_m}_\infty \to 0.
\]
Thus \(\|uC_\varphi + K\| \geq (1/2)\|w\varphi^n_k\|_q - \varepsilon\) for all \(k \geq N\). We conclude by noticing that
\[
\|w\varphi^n_k\|_q = \left( \int_\mathbb{T} |u(\zeta)\varphi(\zeta)^n_k|^q \, dm(\zeta) \right)^{1/q} \to \left( \int_{E_\varphi} |u(\zeta)|^q \, dm(\zeta) \right)^{1/q}.
\]

5. \(uC_\varphi \in B(H^p, H^q)\) for \(\infty > p > q \geq 1\). In \([9]\), the authors give an estimate of the essential norm of a composition operator between \(H^p\) and \(H^q\) for \(1 < q < p < \infty\). The proof makes use of the Riesz projection from \(L^q\) onto \(H^q\), which is a bounded operator for \(1 < q < \infty\). Since it is not bounded from \(L^1\) to \(H^1\) (\(H^1\) is not even complemented in \(L^1\)) there is no way to use a similar argument. So we need a different approach to get some estimates for \(q = 1\). A solution is to make use of Carleson measures. First, we give a characterization of the boundedness of \(uC_\varphi\) in terms of a Carleson measure. In the case where \(p > q\), Carleson measures on \(\overline{\mathbb{D}}\) are characterized in \([1]\). Denote by \(\Gamma(\zeta)\) the Stolz domain generated by \(\zeta \in \mathbb{T}\), i.e. the interior of the convex hull of the set \(\{\zeta\} \cup (\alpha \mathbb{D})\), where \(0 < \alpha < 1\) is arbitrary but fixed.

**Theorem 5.1** (see \([1\text{, Theorem 2.2}]\)). Let \(\mu\) be a measure on \(\overline{\mathbb{D}}\), \(1 \leq q < p < \infty\) and \(s = p/(p-q)\). Then \(\mu\) is a \((p,q)\)-Carleson measure on \(\overline{\mathbb{D}}\) if and only if \(\zeta \mapsto \int_{\Gamma(\zeta)} \frac{d\mu(z)}{1-|z|^2}\) belongs to \(L^s(\mathbb{T})\) and \(\mu_T = F \, d\mathbb{m}\) for some \(F \in L^s(\mathbb{T})\).

This leads to a characterization of the continuity of a weighted composition operator between \(H^p\) and \(H^q\):

**Corollary 5.2.** Let \(u\) be an analytic function on \(\mathbb{D}\) and \(\varphi\) an analytic self-map of \(\mathbb{D}\). For \(1 \leq q < p < \infty\), the weighted composition operator \(uC_\varphi : H^p \to H^q\) is bounded if and only if \(G : \zeta \in \mathbb{T} \mapsto G(\zeta) = \int_{\Gamma(\zeta)} \frac{d\mu_\varphi(z)}{1-|z|^2}\) belongs to \(L^s(\mathbb{T})\) for \(s = p/(p-q)\) and \(\mu_\varphi = F \, d\mathbb{m}\) for some \(F \in L^s(\mathbb{T})\), where \(d\mu = |u|^q \, d\mathbb{m}\) and \(\mu_\varphi = \mu \circ \varphi^{-1}\) is the pullback of \(\mu\) by \(\varphi\).

**Proof.** \(uC_\varphi\) is a bounded operator if and only if there exists \(\gamma > 0\) such that for any \(f \in H^p\), \(\int_{\mathbb{T}} |u(\zeta)|^q |f \circ \varphi(\zeta)|^q \, dm(\zeta) \leq \gamma \|f\|_p^q\), which is equivalent (via a change of variables) to \(\int_{\mathbb{D}} |f(z)|^q \, d\mu_\varphi(z) \leq \gamma \|f\|_p^q\) for every \(f \in H^p\). This exactly means that \(\mu_\varphi\) is a \((p,q)\)-Carleson measure. This is equivalent by **Theorem 5.1** to the condition stated.

If \(f \in H^p\), the Hardy–Littlewood maximal nontangential function \(Mf\) is defined by \(Mf(\zeta) = \sup_{z \in \Gamma(\zeta)} |f(z)|\) for \(\zeta \in \mathbb{T}\). For \(1 < p < \infty\), \(M\) is a bounded operator from \(H^p\) to \(L^p\) and we will denote its norm by \(\|M\|_p\). The following lemma is the analogue of **Lemma 2.6** for the case \(p > q\).

**Lemma 5.3.** Let \(\mu\) be a positive Borel measure on \(\overline{\mathbb{D}}\). Assume that \(\mu\) is a \((p,q)\)-Carleson measure for \(1 \leq q < p < \infty\). Let \(0 < r < 1\) and \(\mu_r :=\)
Then $\mu_r$ is a $(p,q)$-Carleson measure, and there exists a positive constant $\gamma$ such that for every $f \in H^p$,

$$
\int_\mathbb{D} |f(z)|^q \, d\mu_r(z) \leq \|F\|_s + \gamma \|M\|_p^2 \|\tilde{G}_r\|_s \|f\|_p^q
$$

where $d\mu_T = F \, dm$ and $\tilde{G}_r(\zeta) = \int_{\Gamma(\zeta)} \frac{d\mu_r(z)}{1-|z|^2}$. In addition, $\|\tilde{G}_r\|_s \to 0$ as $r \to 1$.

We use the notation $\tilde{G}_r$ to avoid any confusion with the notation introduced before for $\varphi$ and its radial function $\varphi_r$.

**Proof.** Being a $(p,q)$-Carleson measure only depends on the ratio $p/q$ (see [1, Lemma 2.1]), so we have to show that $\mu_r$ is a $(p/q,1)$-Carleson measure. From the definition it is clear that $\tilde{G}_r \leq G \in L^s(\mathbb{T})$. Moreover $d\mu_T = d\mu_T = F \, dm \in L^s(\mathbb{T})$. Corollary 5.2 ensures that $\mu_r$ is a $(p,q)$-Carleson measure.

Let $f$ be in $H^p$. Then

$$
\int_\mathbb{T} |f(\zeta)|^q \, d\mu_r(\zeta) = \int_\mathbb{T} |f(\zeta)|^q \, d\mu(\zeta) = \int_\mathbb{T} |f(\zeta)|^q |F(\zeta)| \, dm(\zeta)
$$

$$
\leq \left( \int_\mathbb{T} |f(\zeta)|^p \, dm(\zeta) \right)^{q/p} \|F\|_s \leq \|f\|_p^q \|F\|_s
$$

using Hölder’s inequality with conjugate exponents $p/q$ and $s$.

For $z \neq 0$, $z \in \mathbb{D}$, let $\tilde{I}(z) = \{\zeta \in \mathbb{T} \mid z \in \Gamma(\zeta)\}$. In other words we have $\zeta \in \tilde{I}(z) \Leftrightarrow z \in \Gamma(\zeta)$. Then

$$
m(\tilde{I}(z)) \approx 1 - |z|
$$

and

$$
\int_\mathbb{D} |f(z)|^q \, d\mu_r(z) \approx \int_\mathbb{D} |f(z)|^q \left( \int_{\tilde{I}(z)} dm(\zeta) \right) \frac{d\mu_r(z)}{1-|z|^2}
$$

$$
= \int_\mathbb{T} \int_{\Gamma(\zeta)} |f(z)|^q \frac{d\mu_r(z)}{1-|z|^2} \, dm(\zeta)
$$

$$
\leq \int_\mathbb{T} Mf(\zeta)^q \int_{\Gamma(\zeta)} \frac{d\mu_r(z)}{1-|z|^2} \, dm(\zeta)
$$

where $Mf(\zeta) = \sup_{z \in \Gamma(\zeta)} |f(z)|$ is the Hardy–Littlewood maximal nontangential function. We apply Hölder’s inequality to obtain

$$
\int_\mathbb{D} |f(z)|^q \, d\mu_r(z) \leq \gamma\|Mf\|_p^q \|G_r\|_s \leq \gamma\|M\|_p^q \|\tilde{G}_r\|_s \|f\|_p^q,
$$

where $\gamma$ is a positive constant that comes from (5.2). Combining (5.1) and (5.3) shows that
\[ \int |f(z)|^q \, d\mu_r(z) \leq (\|F\|_s + \gamma \|M\|_p^q \|\tilde{G}_r\|_s) \|f\|_p^q. \]

It remains to show that \( \|\tilde{G}_r\|_s \to 0 \) when \( r \to 1 \). We will make use of Lebesgue’s dominated convergence theorem. Clearly \( 0 \leq \tilde{G}_r \leq G \in L^s(\mathbb{T}) \), so we need to show that \( \tilde{G}_r(\zeta) \to 0 \) as \( r \to 1 \) for \( m \)-almost every \( \zeta \in \mathbb{T} \). Let \( A = \{ \zeta \in \mathbb{T} \mid G(\zeta) < \infty \} \). Then \( m(A) = 1 \) since \( G \in L^s(\mathbb{T}) \).

Write \( \tilde{G}_r(\zeta) = \int_{r(\zeta)} \tilde{f}_r(z) \, d\mu(z) \) with \( \tilde{f}_r(z) = \mathbb{1}_{\mathbb{D}\setminus r\mathbb{D}}(z)(1-|z|^2)^{-1} \), \( z \in \Gamma(\zeta) \).

For every \( \zeta \in A \) one has
\[
|\tilde{f}_r(z)| \leq \frac{1}{1-|z|^2} \in L^1(\Gamma(\zeta), \mu) \quad \text{since} \quad \zeta \in A,
\]
\[ \tilde{f}_r(z) \to 0 \quad \text{for all} \quad z \in \Gamma(\zeta) \subset \mathbb{D}. \]

Lebesgue’s dominated convergence theorem in \( L^1(\Gamma(\zeta), \mu) \) ensures that \( \|\tilde{G}_r(\zeta) = \|\tilde{f}_r\|_{L^1(\Gamma(\zeta), \mu)} \) tends to zero as \( r \to 1 \) for \( m \)-almost every \( \zeta \in \mathbb{T} \), which ends the proof.

**Theorem 5.4.** Let \( u \) be an analytic function on \( \mathbb{D} \) and \( \varphi \) an analytic self-map of \( \mathbb{D} \). Assume that \( UC\varphi \) is a bounded operator from \( H^p \) to \( H^q \), with \( \infty > p > q \geq 1 \). Then
\[
\|UC\varphi\|_e \leq 2\|C\varphi\|_{p/q}^{1/q} \left( \int_{E\varphi} |u(\zeta)|^{pq/(p-q)} \, dm(\zeta) \right)^{p-q/pq},
\]
where \( \|C\varphi\|_{p/q} \) denotes the norm of \( C\varphi \) acting on \( H^{p/q} \).

**Proof.** We follow the same lines as in the proof of the upper estimate in Proposition 2.4, we have the decomposition \( I = K_N + R_N \) in \( B(H^p) \), where \( K_N \) is the convolution operator with the Fejér kernel, and
\[ \|UC\varphi\|_e \leq \lim_{N} \inf \|UC\varphi R_N\|. \]

We also have, for every \( 0 < r < 1 \),
\[
\|(UC\varphi R_N)f\|_q^q = \int_{\mathbb{D}\setminus r\mathbb{D}} |R_N f(w)|^q \, d\mu_\varphi(w) + \int_{r\mathbb{D}} |R_N f(w)|^q \, d\mu_\varphi(w)
\]
\[ = I_1(N, r, f) + I_2(N, r, f). \]

As in the \( p \leq q \) case, we show that
\[ \lim_{N} \sup_{\|f\|_p \leq 1} I_2(N, r, f) = 0. \]

The measure \( \mu_\varphi \) being a \( (p,q) \)-Carleson measure, we use Lemma 5.3 to obtain
\[ I_1(N, r, f) \leq (\|F\|_s + \gamma \|M\|_p^q \|\tilde{G}_r\|_s) \|R_N f\|_p^q \]
for every \( f \in H^p \). As a consequence,
\[ \|uC_\varphi\|_e \leq \liminf_N \left( \sup_{\|f\|_p \leq 1} I_1(N,r,f) \right)^{1/q} \leq 2(\|F\|_s + \gamma \|M\|_p^q \|G_r\|_s)^{1/q} \]

using the fact that \( \sup_N \|R_N\| \leq 2 \). Now we let \( r \to 1 \), keeping in mind that \( \|G_r\|_s \to 0 \). We obtain

\[ \|uC_\varphi\|_e \leq 2\|F\|_s^{1/q}. \]

It remains to see that we can choose \( F \) in such a way that

\[ \|F\|_s \leq \|C_\varphi\|_{p/q} \left( \int_{E_\varphi} |u(\zeta)|^{pa} d\mu(\zeta) \right)^{1/s}. \]

Indeed, if \( f \in C(\mathbb{T}) \cap H^{p/q} \), we apply Hölder’s inequality with conjugate exponents \( p/q \) and \( s \) to obtain

\[ \left| \int_{\mathbb{T}} f d\mu_{\varphi,T} \right| = \left| \int_{E_\varphi} |u|^q f \circ \varphi dm \right| \leq \int_{E_\varphi} |u|^q |f \circ \varphi| dm \leq \|C_\varphi(f)\|_{p/q} \left( \int_{E_\varphi} |u|^{qs} dm \right)^{1/s}, \]

meaning that \( \mu_{\varphi,T} \in (H^{p/q})^* \), which is isometrically isomorphic to \( L^s(\mathbb{T})/H^s_0 \), where \( H^s_0 \) is the subspace of \( H^s \) consisting of functions vanishing at zero. If we denote by \( N(\mu_{\varphi,T}) \) the norm of \( \mu_{\varphi,T} \) viewed as an element of \( (H^{p/q})^* \), then one can choose \( F \in L^s(\mathbb{T}) \) satisfying

\[ \|F\|_s = N(\mu_{\varphi,T}) \leq \|C_\varphi\|_{p/q} \left( \int_{E_\varphi} |u|^{pa} d\mu(\zeta) \right)^{1/s}, \]

and \( \mu_{\varphi,T} = F dm \) (see for instance [11, p. 194]). Finally we have

\[ \|uC_\varphi\|_e \leq 2\|C_\varphi\|_{p/q} \left( \int_{E_\varphi} |u(\zeta)|^{pa} d\mu(\zeta) \right)^{p-q}. \]

Although we have not been able to give a corresponding lower bound of this form for the essential norm of \( uC_\varphi \), we have the following result:

**Proposition 5.5.** Let \( 1 \leq q < p < \infty \), and assume that \( uC_\varphi \in B(H^p,H^q) \). Then

\[ \|uC_\varphi\|_e \geq \left( \int_{E_\varphi} |u(\zeta)|^q d\mu(\zeta) \right)^{1/q}. \]

**Proof.** Take a compact operator \( K \) from \( H^p \) to \( H^q \). Since it is completely continuous, and the sequence \( (z^n) \) converges weakly to zero in \( H^p \), \( (K(z^n))_n \) converges to zero in \( H^q \). Hence

\[ \|uC_\varphi + K\| \geq \|(uC_\varphi + K)z^n\|_q \geq \|uC_\varphi(z^n)\|_q - \|K(z^n)\|_q. \]
for every $n \geq 0$. Taking the limit as $n \to \infty$, we have
\[
\|uC_\varphi\|_e \geq \left( \int_{E_\varphi} |u(\zeta)|^q \, dm(\zeta) \right)^{1/q}.
\]

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**References**


R. Demazeux
Univ Lille Nord de France, France
UArtois, Laboratoire de Mathématiques de Lens EA 2462
Fédération CNRS Nord-Pas-de-Calais FR 2956
F-62 300 Lens, France
E-mail: romain.demazeux@euler.univ-artois.fr

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