STUDIA MATHEMATICA 206 (3) (2011)

Growth of semigroups in discrete and continuous time

by

ALEXANDER GOMILKO (Toruń), HANS ZWART (Enschede) and
NIELS BESSELING (Enschede)

Abstract. We show that the growth rates of solutions of the abstract differential equations $\dot{x}(t) = Ax(t)$, $\dot{x}(t) = A^{-1}x(t)$, and the difference equation $x_d(n+1) = (A+I)(A-I)^{-1}x_d(n)$ are closely related. Assuming that A generates an exponentially stable semigroup, we show that on a general Banach space the lowest growth rate of the semigroup $(e^{A^{-1}t})_{t\geq 0}$ is $O(\sqrt[4]{t})$, and for $((A+I)(A-I)^{-1})^n$ it is $O(\sqrt[4]{n})$. The similarity in growth holds for all Banach spaces. In particular, for Hilbert spaces the best estimates are $O(\log(t))$ and $O(\log(n))$, respectively. Furthermore, we give conditions on A such that the growth rate of $((A+I)(A-I)^{-1})^n$ is $O(1)$, i.e., the operator is power bounded.

1. Introduction. Let X be a Banach space and let A be a closed, densely defined operator on X. For this A we consider the abstract differential equation

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0. \tag{1.1}$$

Assuming that A generates a strongly continuous semigroup on X, for any $x_0 \in X$, this equation possesses a unique (mild) solution (see e.g. [8]). An important property of the solutions, and thus of the corresponding semigroup, is the boundedness of the trajectories. In this paper we study the relation between the boundedness of the solutions of (1.1) and those of the differential equation

$$\dot{x}(t) = A^{-1}x(t), \qquad x(0) = x_0, \tag{1.2}$$

and of the difference equation

$$x(n+1) = V(A)x(n), \qquad x(0) = x_0. \tag{1.3}$$

Here V denotes the *Cayley transform* of A, i.e.,

$$V = V(A) := (A+I)(A-I)^{-1}, \tag{1.4}$$

where I is the identity operator.

2010 *Mathematics Subject Classification*: Primary 47D06; Secondary 34G10.
Key words and phrases: Banach space, C_0-semigroup, generator, Cayley transform.

DOI: 10.4064/sm206-3-3

If X is finite-dimensional, then (under the condition that A is invertible) the solutions of (1.1)–(1.3) share the same stability properties. In particular, if one of these equations has only bounded solutions, then so do the others. If A is an injective linear operator on a Banach space X with dense range generating a uniformly bounded analytic C_0-semigroup, then it is well known that A^{-1} also generates such a semigroup [14]. On the other hand, it was shown in [13] that there exists a Banach space X and an injective linear operator on X with dense range generating a uniformly bounded C_0-semigroup whose inverse does not generate a C_0-semigroup. In Hilbert spaces, if (1.1) and (1.2) have only bounded solutions, so does (1.3) (see [2, 9, 11]). We extend this result by showing that if (1.1) and (1.2) have all solutions bounded, then so does the difference equation $x(n+1) = V(\delta A)x(n)$ for all $\delta > 0$, and the bound is independent of $\delta > 0$. Furthermore, the converse holds. If (1.1) and the difference equations $x(n+1) = V(\delta A)x(n)$ have only bounded solutions, and $\sup_{n,\delta>0} \|V^n(\delta A)\| < \infty$, then the solutions of (1.2) are bounded. For Banach spaces such positive results are not known. However, we show that if there exists a generator of an exponentially stable semigroup for which the solution of (1.2) grows as $g(t)$, then on the same Banach space there exists a generator of an exponentially stable semigroup for which the solutions of (1.3) grow as $g(n)$. The converse of this result also holds.

We end this section with some notation. $\mathcal{E} = \mathcal{E}(X)$ denotes the set of densely defined, closed linear operators on X and $\mathcal{L} = \mathcal{L}(X)$ denotes the algebra of bounded linear operators on X. By $\mathcal{G} = \mathcal{G}(X)$ we denote the set of generators of uniformly bounded C_0-semigroups and by $\mathcal{G}_{\exp} = \mathcal{G}_{\exp}(X)$ the set of generators of exponentially stable C_0-semigroups acting on X. If $A \in \mathcal{G}$, then $(e^{At})_{t\geq 0}$ is the strongly continuous semigroup generated by A.

2. Growth of the Cayley transform. In this section we investigate the growth of the power of the Cayley transform, as defined in (1.4). We study this growth for infinitesimal generators of uniformly bounded semigroups and of exponentially stable semigroups, i.e., for $A \in \mathcal{G}(X)$ and $A \in \mathcal{G}_{\exp}(X)$.

LEMMA 2.1. *For the Cayley transform* (1.4) *we have the following estimates on the Banach space X. If $A \in \mathcal{G}(X)$, then*

$$\sup_{n\in\mathbb{N}} \frac{\|V^n(A)\|}{n^{1/2}} < \infty. \tag{2.1}$$

If $A \in \mathcal{G}_{\exp}(X)$, then

$$\sup_{n\in\mathbb{N}} \frac{\|V^n(A)\|}{n^{1/4}} < \infty. \tag{2.2}$$

Furthermore, the estimates are sharp: there exists a Banach space X and *an operator $A \in \mathcal{G}(X)$ such that*

$$\liminf_{n\in\mathbb{N}} \frac{\|V^n(A)\|}{n^{1/2}} > 0, \tag{2.3}$$

and there exists a Banach space X and an $A_{\exp} \in \mathcal{G}_{\exp}(X)$ such that

$$\liminf_{n\in\mathbb{N}} \frac{\|V^n(A_{\exp})\|}{n^{1/4}} > 0. \tag{2.4}$$

Proof. For $A \in \mathcal{G}$ the powers of $V = V(A)$ are given by the expression (see e.g. [9])

$$V^n = I - \int_0^\infty e^{-t/2} L_{n-1}^{(1)}(t) e^{At/2}\, dt, \qquad n = 1, 2, \ldots, \tag{2.5}$$

where $L_n^{(1)}$ are the first generalized Laguerre polynomials, i.e.,

$$L_n^{(1)}(t) = \sum_{k=0}^{n} \frac{(n+1)!}{(k+1)!} \cdot \frac{(-t)^k}{k!(n-k)!}.$$

From (2.5), we obtain

$$\|V^n\| \le 1 + M \int_0^\infty e^{-t/2} |L_{n-1}^{(1)}(t)|\, dt, \qquad n \in \mathbb{N}, \tag{2.6}$$

where $M = \sup_{t>0} \|e^{tA}\|$. On the other hand, the well-known estimates [1] for the Laguerre polynomials give

$$\int_0^\infty e^{-t/2} |L_{n-1}^{(1)}(t)|\, dt \le cn^{1/2}, \qquad n \in \mathbb{N}.$$

Combining this with (2.6), we obtain the inequality (2.1).

Next, let $A \in \mathcal{G}_{\exp}(X)$. Then there exist $M, \omega > 0$ such that

$$\|e^{At}\| \le M e^{-\omega t}, \qquad t \ge 0. \tag{2.7}$$

From (2.5) and (2.7) we find that

$$\|V^n\| \le 1 + M \int_0^\infty e^{-(1+\omega)t/2} |L_{n-1}^{(1)}(t)|\, dt, \qquad n \in \mathbb{N}.$$

Then using the estimate (see [16, Ch. 6, §3])

$$|L_n^{(1)}(t)| \le c e^{t/2} t^{-3/4} (n^{1/4} + t^{5/4}), \qquad n \in \mathbb{N},\ t > 0,$$

we have

$$\|V^n\| \le 1 + Mcn^{1/4} \int_0^\infty e^{-\omega t/2} (t^{-3/4} + t^{1/2})\, dt \le 1 + c_\omega M n^{1/4},$$

where the constant $c_\omega > 0$ depends only on $\omega > 0$. This proves (2.2).

It remains to show that the estimates are sharp. We follow a reasoning similar to the one in [5]. For (2.1), we choose the Banach space $X = L_1(\mathbb{R})$ and the differentiation operator $\mathcal{D}$, i.e., $(\mathcal{D}f)(s) = f'(s)$, with domain $D(\mathcal{D}) = W_1^1(\mathbb{R})$, the Sobolev space. It is well-known that $\mathcal{D}$ is the infinitesimal generator of the unitary shift, i.e.,

$$(e^{t\mathcal{D}}f)(s) = f(s+t), \qquad s,t \in \mathbb{R}.$$

Then, by (2.5),

$$(V^n(\mathcal{D})f)(s) = f(s) - \int_0^\infty e^{-t/2} L_{n-1}^{(1)}(t) f(s+t/2)\,dt, \qquad s \in \mathbb{R},\ n \in \mathbb{N}.$$

From this, using the appropriate formula for the norm of an integral operator on L_1 (see [12, Th. XI.1.4]), we obtain

$$1 + \|V^n(\mathcal{D})\|_{L_1(\mathbb{R})} \geq \int_0^\infty e^{-t/2} |L_{n-1}^{(1)}(t)|\,dt, \qquad n \in \mathbb{N}. \tag{2.8}$$

On the other hand, according to [1], there exists a constant $c > 0$ such that

$$\int_0^\infty e^{-t/2} |L_n(t)|\,dt \geq c n^{1/2}, \qquad n \in \mathbb{N}, \tag{2.9}$$

where $L_n(t)$ are the usual Laguerre polynomials. Next, using the relations

$$\frac{dL_{n+1}}{dt}(t) = -L_n^{(1)}(t), \qquad L_n(0) = 1,$$

we obtain

$$\begin{aligned} \int_0^\infty e^{-t/2} |L_n(t)|\,dt &\leq \int_0^\infty e^{-t/2} \Big(\int_0^t |L_{n-1}^{(1)}(s)|\,ds + 1 \Big)\,dt \\ &\leq 2 \int_0^\infty e^{-s/2} |L_{n-1}^{(1)}(s)|\,ds + 2, \qquad n \in \mathbb{N}. \end{aligned} \tag{2.10}$$

From (2.8)–(2.11) we have the estimate

$$\|V^n(\mathcal{D})\|_{L_1(\mathbb{R})} \geq c_1 n^{1/2}, \qquad n \in \mathbb{N},$$

for some constant $c_1 > 0$. Hence the estimate (2.1) is sharp.

Now we show that (2.2) is sharp. We choose A to be minus the differentiation operator on $X = L_1(0,1)$, i.e.,

$$(Af)(y) = -f'(y), \qquad D(A) = \{f \in W_1^1(0,1) \mid f(0) = 0\}.$$

Furthermore, we define $A_0 = 2A - I$. Then A_0 is the generator of a nilpotent C_0-semigroup, and thus in particular $A_0 \in \mathcal{G}_{\exp}(X)$. The Cayley transform of A_0 equals

$$V(A_0) = (A_0 + I)(A_0 - I)^{-1} = A(A - I)^{-1} = (I + J)^{-1}, \tag{2.11}$$

where J the classical Volterra operator,

$$(Jf)(y) = \int_0^y f(s)\,ds.$$

From [15] we have

$$\|(I+J)^{-n}\|_{L_1(0,1)} \asymp n^{1/4}, \qquad n \to \infty. \tag{2.12}$$

Combining (2.11) with (2.12) shows the sharpness of the estimate (2.2). ∎

If $A \in \mathcal{E}(X)$ is such that $A^{-1} \in \mathcal{E}(X)$ and $(A-I)^{-1} \in \mathcal{L}(X)$, then it is easy to see that $V(A) = -V(A^{-1})$. We end this section with an extension of this result.

LEMMA 2.2. *Let $A \in \mathcal{E}$ with spectrum $\sigma(A)$ contained in the half-plane $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \le 0\}$. Let $V := V(A)$ be the Cayley transform of A. Suppose that for some $s \in \mathbb{R}$ the inverse of $A - isI$ exists as a densely defined operator, i.e., $(A - isI)^{-1} \in \mathcal{E}$. Define*

$$A_s := (-isA + I)(A - isI)^{-1}, \qquad D(A_s) = \operatorname{ran}(A - isI). \tag{2.13}$$

Then

$$A_s = -isI + (s^2+1)(A - isI)^{-1} \in \mathcal{E}, \tag{2.14}$$

and the Cayley transform of A_s satisfies

$$V_s := V(A_s) = \alpha(s)V, \qquad \alpha(s) = (is-1)(is+1)^{-1}, \tag{2.15}$$

where $|\alpha(s)| = 1$, and hence the growth rates of V^n and V_s^n are the same.

Proof. From the assumptions on A and the definition of A_s, it follows that $A_s \in \mathcal{E}$ and $\sigma(A_s)$ lies in the half-plane $\operatorname{Re}\lambda \le 0$.

Equality (2.14) is easy to show, and so we concentrate on the other one. Using the equality

$$A_s - I = -(is+1)(A-I)(A-isI)^{-1},$$

we find that

$$\begin{aligned}(A_s - I)^{-1} &= -(is+1)^{-1}(A-isI)(A-I)^{-1}\\ &= -(is+1)^{-1}I + \alpha(s)(A-I)^{-1}.\end{aligned}$$

Thus

$$V(A_s) = I + 2(A_s - I)^{-1} = \alpha(s)I + 2\alpha(s)(A-I)^{-1} = \alpha(s)V. \ \blacksquare$$

REMARK 2.3. From (2.14) we conclude that $(A - isI)^{-1}$ is the generator of a C_0-semigroup if and only if A_s is. Moreover,

$$\|e^{A_s t}\| = \|e^{(s^2+1)(A-isI)^{-1}t}\|, \qquad t \ge 0. \tag{2.16}$$

The following is easily proved by using the definition of A_s (see (2.13)).

REMARK 2.4. If for some $s \in \mathbb{R}$, $s \neq 0$, the operators $(A - isI)^{-1}$ and $(A + is^{-1}I)^{-1}$ are in $\mathcal{E}$, then $A_s^{-1} = A_{-s^{-1}}$.

3. Relation between the Cayley transform and $e^{A^{-1}t}$, general case. As stated in the introduction, we relate the stability properties of the differential equations (1.1) and (1.2), and the difference equation (1.3). In this section we show that if there exists an A such that all solutions of (1.1) are exponentially stable, but for some $x_0 \in X$ the solution of (1.2) is unbounded, then it is possible to construct an infinitesimal generator A_0 such that with this new operator all solutions of (1.1) are exponentially stable, but for some $x_0 \in X$ the solution of (1.3) is unbounded. The proof is based on the following observation.

Let $A \in \mathcal{G}(X)$ and let V be its Cayley transform. Then using the equality $V = I + 2(A - I)^{-1}$ we have

$$e^{Vz} = e^{z} e^{2z(A-I)^{-1}}, \qquad z \in \mathbb{C}. \tag{3.1}$$

Hence if V is power bounded, then

$$\|e^{z(A-I)^{-1}}\| \leq Me^{(|z|-\operatorname{Re} z)/2} = Me^{R\sin^2(\theta/2)}, \qquad z = Re^{i\theta} \in \mathbb{C}.$$

Thus in particular, $\|e^{t(A-I)^{-1}}\| \leq M$, $t \geq 0$. So if V is power bounded, then the operator $(A - I)^{-1}$ lies in $\mathcal{G}$.

The following theorem shows that growth bounds on $V^n(A)$ for $A \in \mathcal{G}_{\exp}(X)$ yield similar growth bounds for $(e^{A^{-1}t})_{t\geq 0}$.

THEOREM 3.1. *Let X be a Banach space and assume that for every $A \in \mathcal{G}_{\exp}(X)$,*

$$\|V^n(A)\| \leq M_A g(n), \qquad n \in \mathbb{N}, \tag{3.2}$$

where g is a non-decreasing function, not depending on A, i.e. $0 < g(\alpha) \leq g(\beta)$ for all $0 \leq \alpha \leq \beta$, and M_A is a constant not depending on n. Then for any $A \in \mathcal{G}_{\exp}(X)$ the semigroup $(e^{A^{-1}t})_{t\geq 0}$ satisfies a similar estimate:

$$\|e^{A^{-1}t}\| \leq \tilde{M}_A g(2et/\omega), \qquad t \geq 0. \tag{3.3}$$

Here the (positive) constant ω is such that $-\omega$ is larger than the growth bound of A, i.e., there exists an $M > 0$ such that $\|e^{At}\| \leq Me^{-\omega t}$, $t \geq 0$.

Proof. By Lemma 2.1 we may assume that $g(n) \leq c_0(1 + \sqrt[4]{n})$. For ω as in the statement, it is easy to see that

$$A_0 = 2\omega^{-1}A + I \in \mathcal{G}_{\exp}(X).$$

Furthermore, $V(A_0) = I + \omega A^{-1}$. Hence

$$\|e^{\omega A^{-1}t}\| = e^{-t}\|e^{V(A_0)t}\|, \quad t \ge 0. \tag{3.4}$$

Since $A_0 \in \mathcal{G}_{\exp}(X)$, the powers of $V(A_0)$ satisfy the estimate (3.2). From this and Stirling's estimate, we have, for $t \ge 1$,

$$\begin{aligned} \|e^{V(A_0)t}\| &\le \sum_{n=0}^{\infty} \frac{t^n \|V^n(A_0)\|}{n!} \le M_{A_0} \sum_{n=0}^{\infty} \frac{g(n)t^n}{n!} \\ &\le c_1 \Big[\sum_{n \ge 2et} \Big(\frac{et}{n}\Big)^n \frac{g(n)}{\sqrt{n}} + \sum_{n \le 2et} \frac{g(n)t^n}{n!} \Big] \\ &\le c_1 \Big[\sum_{n=1}^{\infty} \frac{g(n)}{2^n \sqrt{n}} + g(2et) \sum_{n=0}^{\infty} \frac{t^n}{n!} \Big] = c_2 + c_1 g(2et) e^t. \end{aligned} \tag{3.5}$$

Here we have used the assumption $g(n) \le c_0(1 + \sqrt[4]{n})$. From (3.5) and (3.4) we obtain the estimate (3.3). ∎

There are several consequences of this result. We start with the relation between power boundedness of $V(A)$ and the uniform boundedness of $(e^{A^{-1}t})_{t \ge 0}$.

COROLLARY 3.2. *Suppose that on the Banach space X there exists $A \in \mathcal{G}_{\exp}(X)$ such that $A^{-1} \notin \mathcal{G}(X)$. Then there exists $A_0 \in \mathcal{G}_{\exp}(X)$ such that $V(A_0)$ is not power bounded.*

Proof. If $V(A_0)$ is power bounded for every $A_0 \in \mathcal{G}_{\exp}(X)$, then we can choose $g(n) \equiv 1$ (see (3.2)). Thus by Theorem 3.1, $A^{-1} \in \mathcal{G}(X)$ whenever $A \in \mathcal{G}_{\exp}(X)$. This contradicts the assumptions. ∎

If X is finite-dimensional, then the function g in Theorem 3.1 can always be chosen to be a constant. However, this constant may depend on the dimension of X: see e.g. equation (35) in [10]. Hence from the theorem we see that the dependence on the dimension of the supremum of $\|e^{A^{-1}t}\|$, $t \ge 0$, and $\|V^n(A)\|$, $n \in \mathbb{N}$, is the same.

Using this theorem, we obtain different proofs for some estimates found in the literature. The first result, which can be found in [17], follows from the previous theorem and Lemma 2.1.

COROLLARY 3.3. *Let X be a Banach space and $A \in \mathcal{G}_{\exp}(X)$. Then*

$$\|e^{A^{-1}t}\| \le 1 + M_0 t^{1/4}, \quad t \ge 0,$$

where M_0 does not depend on t.

For Hilbert spaces it was shown by Gomilko [9] that g in (3.2) can be chosen to be $\log(n+2)$. Combining this with the theorem gives the estimate found in [18].

Corollary 3.4. *Let X be a Hilbert space and $A \in \mathcal{G}_{\exp}(X)$. Then*

$$\|e^{A^{-1}t}\| \le M_0 \log(t+2), \qquad t \ge 0,$$

with M_0 independent of t.

The following theorem can be seen as the converse of Theorem 3.1.

Theorem 3.5. *Assume that for every $A \in \mathcal{G}_{\exp}(X)$,*

$$\|e^{A^{-1}t}\| \le \tilde{M}_A g(t),$$

where g is a non-decreasing function, not depending on A, i.e., $0 < g(\alpha) \le g(\beta)$ for all $0 \le \alpha \le \beta$, and $\tilde{M}_A$ is a constant not depending on t. Then for every $A \in \mathcal{G}_{\exp}(X)$ satisfying $\|e^{At}\| \le Me^{-\omega t}$ for $\omega > 1$, there exists for all $\alpha > 1$ an $M_{\alpha,A}$ such that

$$\|V^n(A)\| \le M_{\alpha,A}\, g(\alpha n).$$

Proof. By Corollary 3.3 we may assume that $g(t) \le c_0(1 + t^{1/4})$. Let $\alpha > 1$. Since $\alpha - 1 - \log(\alpha) > 0$, there exists an $\varepsilon \in (0,1)$ such that

$$\alpha\varepsilon < \alpha - 1 - \log(\alpha). \tag{3.6}$$

Secondly, the function $e^{-(1-\varepsilon)t}t^{n-1}$, $t > 0$, has a maximum at $\tau = \frac{n-1}{1-\varepsilon}$ and is decreasing for $t > \tau$. We now choose

$$t_1 = \alpha(1-\varepsilon)\tau = \alpha(n-1). \tag{3.7}$$

Since $\alpha > 1$, and since (3.6) holds, we have $t_1 > \tau$ for $n \ge 2$.

We define $A_1 = \frac{1}{2}(A + I)$. By the assumption on A, we have $A_1 \in \mathcal{G}_{\exp}(X)$. Furthermore,

$$V(A) = (A+I)(A-I)^{-1} = 2A_1(2A_1 - 2I)^{-1} = (I - A_1^{-1})^{-1}.$$

Hence

$$\begin{aligned} \|V^n(A)\| = \|(I - A_1^{-1})^{-n}\| &\le \frac{1}{(n-1)!}\int_0^\infty e^{-t}\|e^{A_1^{-1}t}\|t^{n-1}\,dt \\ &\le \frac{\tilde{M}_{A_1}}{(n-1)!}\int_0^\infty e^{-t}g(t)t^{n-1}\,dt \\ &= \frac{\tilde{M}_{A_1}}{(n-1)!}\int_0^{t_1} e^{-t}g(t)t^{n-1}\,dt + \frac{\tilde{M}_{A_1}}{(n-1)!}\int_{t_1}^\infty e^{-t}g(t)t^{n-1}\,dt. \end{aligned} \tag{3.8}$$

On the time interval from 0 to t_1, since g is non-decreasing, we have

$$\frac{\tilde{M}_{A_1}}{(n-1)!}\int_0^{t_1} e^{-t}g(t)t^{n-1}\,dt \le \frac{\tilde{M}_{A_1}g(t_1)}{(n-1)!}\int_0^\infty e^{-t}t^{n-1}\,dt = \tilde{M}_{A_1}g(t_1). \tag{3.9}$$

On the time interval from t_1 to ∞ we use two observations. First,

$$n! \ge (n/e)^n, \qquad n \in \mathbb{N},$$

and secondly, by the choice of t_1 (see (3.7)),

$$e^{-(1-\varepsilon)t}t^{n-1} \le e^{-(1-\varepsilon)\alpha(n-1)}(\alpha(n-1))^{n-1} \qquad \text{for } t \ge t_1,\ n \ge 2.$$

Combining these, we find that for $n \ge 2$,

$$\begin{aligned}\frac{\tilde{M}_{A_1}}{(n-1)!}&\int_{t_1}^{\infty} e^{-t}g(t)t^{n-1}\,dt \\ &\le \tilde{M}_{A_1}\left(\frac{e}{n-1}\right)^{n-1}\int_{t_1}^{\infty} e^{-(1-\varepsilon)t}t^{n-1}e^{-\varepsilon t}g(t)\,dt \\ &\le \tilde{M}_{A_1}\left(\frac{e}{n-1}\right)^{n-1} e^{-(1-\varepsilon)\alpha(n-1)}(\alpha(n-1))^{n-1}\int_{t_1}^{\infty} e^{-\varepsilon t}g(t)\,dt \\ &= \tilde{M}_{A_1}(e^{-(1-\varepsilon)\alpha+1}\alpha)^{n-1}\int_{t_1}^{\infty} e^{-\varepsilon t}g(t)\,dt.\end{aligned}$$

Using (3.6) we see that $e^{-(1-\varepsilon)\alpha+1}\alpha < 1$. Thus

$$\frac{\tilde{M}_{A_1}}{(n-1)!}\int_{t_1}^{\infty} e^{-t}g(t)t^{n-1}\,dt \le \tilde{M}_{A_1}\int_{t_1}^{\infty} e^{-\varepsilon t}g(t)\,dt = M_\alpha \tag{3.10}$$

with M_α independent of n. Combining (3.8)–(3.10) gives

$$\|V^n(A)\| \le \tilde{M}_{A_1}g(t_1) + M_\alpha.$$

Since $t_1 = \alpha(n-1)$ and since g is non-decreasing, we can find a constant $M_{\alpha,A}$ such that $\|V^n(A)\| \le \tilde{M}_{\alpha,A}g(\alpha n)$, which proves the result. ∎

4. Growth of $V^n(A)$ and $(e^{A^{-1}t})_{t\ge0}$ in Hilbert spaces. In the previous section, we have investigated the growth of $V(A)$ and $(e^{A^{-1}t})_{t\ge0}$ under the condition that $A \in \mathcal{G}(X)$, with X a Banach space. In the following, we take for X a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$, and to clearly distinguish the results from those of the previous sections we denote our space by H.

It is known (see [11], [9], [2]) that if a bounded operator on a Hilbert space is the generator of a uniformly bounded semigroup, then its Cayley transform is a power bounded operator. We state this in a lemma for future references.

LEMMA 4.1. *Let $A \in \mathcal{L}(H)$ be the generator of a uniformly bounded semigroup. Then $V(A)$ is power bounded.*

If A generates a uniformly bounded semigroup, but A is unbounded, then it is unknown whether the above result holds. However, if A and A^{-1} each generate a uniformly bounded semigroup, i.e., $A \in \mathcal{G}(H)$ and $A^{-1} \in \mathcal{G}(H)$, then $V(A)$ is power bounded. We present a new proof of this theorem; for that we need some facts on Lyapunov equations. For the proof we refer to [6, Exercise 4.29] or [7, Section II.6].

LEMMA 4.2. *Let Q be a bounded operator on H. If there exists a non-negative $P \in \mathcal{L}(H)$ which is a solution of the Lyapunov equation*

$$Q^*PQ - P = -I, \tag{4.1}$$

then $\sum_{n=0}^{\infty} \|Q^n x\|^2$ is finite for all $x \in H$.

Conversely, if $\sum_{n=0}^{\infty} \|Q^n x\|^2$ is finite for every $x \in H$, then there exists a unique solution of (4.1). *Furthermore, this solution is positive, and*

$$\langle Px, x\rangle = \sum_{n=0}^{\infty} \|Q^n x\|^2. \tag{4.2}$$

Under the assumption that $\lambda \geq 1$ and λ, λ^{-1} are in the resolvent set $\rho(A)$ of A, we consider the following two Lyapunov equations:

$$(\lambda^2 - 1)(\lambda I - A^*)^{-1} P_1 (\lambda I - A)^{-1} - P_1 = -I, \tag{4.3}$$

$$(\lambda^2 - 1)(\lambda I - A^{-1})^{-*} P_2 (\lambda I - A^{-1})^{-1} - P_2 = -I, \tag{4.4}$$

where Q^{-*} indicates $(Q^*)^{-1}$, or equivalently $(Q^{-1})^*$.

For the Cayley transform we consider the following Lyapunov equation:

$$\frac{\lambda - 1}{\lambda + 1} V(A)^* P_V V(A) - P_V = -I. \tag{4.5}$$

In the following lemma we will manipulate these Lyapunov equations. Every Lyapunov equation can be written using the inner product. For instance, (4.3) is the same as

$$(\lambda^2-1)\langle P_1(\lambda I-A)^{-1}x_1, (\lambda I-A)^{-1}x_2\rangle - \langle P_1 x_1, x_2\rangle = -\langle x_1, x_2\rangle, \quad x_1, x_2 \in H.$$

Choosing $z_k = (\lambda I - A)^{-1} x_k \in D(A)$, $k = 1, 2$, we rewrite this as

$$(\lambda^2 - 1)\langle P_1 z_1, z_2\rangle - \langle P_1(\lambda I - A)z_1, (\lambda I - A)z_2\rangle = -\langle(\lambda I - A)z_1, (\lambda I - A)z_2\rangle.$$

Simple algebraic manipulations imply that this is the same as

$$\begin{aligned} - \langle P_1 A z_1, A z_2\rangle + \lambda\langle P_1 z_1, A z_2\rangle + \lambda\langle P_1 A z_1, z_2\rangle - \langle P_1 z_1, z_2\rangle \\ = -\langle(\lambda I - A)z_1, (\lambda I - A)z_2\rangle. \end{aligned}$$

To simplify notation, we introduce for $A \in \mathcal{E}(H)$ and $P \in \mathcal{L}(H)$ the bilinear form $B_\lambda[A, P]$ on $D(A)$ as

$$\begin{aligned} B_\lambda[A, P](z_1, z_2) := \lambda\langle PAz_1, z_2\rangle + \lambda\langle Pz_1, Az_2\rangle \\ - \langle PAz_1, Az_2\rangle - \langle Pz_1, z_2\rangle, \quad z_1, z_2 \in D(A). \end{aligned} \tag{4.6}$$

Using it, we see that (4.3) reads

$$B_\lambda[A, P_1](z_1, z_2) = -\langle(\lambda I - A)z_1, (\lambda I - A)z_2\rangle, \quad \forall z_1, z_2 \in D(A). \tag{4.7}$$

In an analogous way, (4.4) is equivalent to

$$B_\lambda[A,P_2](z_1,z_2) = -\langle (I-\lambda A)z_1,(I-\lambda A)z_2\rangle, \qquad \forall z_1,z_2\in D(A), \tag{4.8}$$

and (4.5) is equivalent to

$$B_\lambda[A,P_V](z_1,z_2) = -\frac{\lambda+1}{2}\langle (A-I)z_1,(A-I)z_2\rangle, \qquad \forall z_1,z_2\in D(A). \tag{4.9}$$

In the rest of this section we use the notation $R(A,\lambda)=(\lambda I-A)^{-1}$.

LEMMA 4.3. *Let $\lambda\in(1,\infty)$ be such that λ,λ^{-1} are in the resolvent set $\rho(A)$. Furthermore, assume that $1\in\rho(A)$.*

(1) *The Lyapunov equation* (4.3) *has a bounded solution if and only if* (4.4) *has a bounded solution. Furthermore, the solutions are related via*

$$P_2 = (I-\lambda A)^*(\lambda I-A)^{-*}P_1(I-\lambda A)(\lambda I-A)^{-1}. \tag{4.10}$$

(2) *If* (4.3) *has a bounded solution, then a bounded solution of* (4.5) *is given by*

$$P_V = \frac{1}{2\lambda}(P_1+P_2+\lambda I-I). \tag{4.11}$$

Proof. (1) Let $P_1\in\mathcal{L}(H)$ be the solution of (4.3) (or (4.7)) and let Q denote the bounded operator

$$Q := (I-\lambda A^*)R^*(A,\lambda)P_1(I-\lambda A)R(A,\lambda). \tag{4.12}$$

Then, using the relations

$$[(I-\lambda A^*)R(A^*,\lambda)]^* = (I-\lambda A)R(A,\lambda),$$

and

$$(I-\lambda A)R(A,\lambda)Az = AR(A,\lambda)(I-\lambda A)z, \quad z\in D(A),$$

we find that

$$\begin{aligned} B_\lambda[A,Q](z_1,z_2) &= B_\lambda[A,P_1](R(A,\lambda)(I-\lambda A)z_1, R(A,\lambda)(I-\lambda A)z_2)\\ &= -\langle (\lambda I-A)R(A,\lambda)(I-\lambda A)z_1,(\lambda-A)R(A,\lambda)(I-\lambda A)z_2\rangle\\ &= -\langle (I-\lambda A)z_1,(I-\lambda A)z_2\rangle, \qquad \forall z_1,z_2\in D(A). \end{aligned}$$

So, the operator $P_2=Q$ is the solution of (4.4) and (4.8).

(2) Let P_1 and P_2 be the solutions of (4.7) and (4.8), respectively, and let $\tilde{Q}\in\mathcal{L}(H)$ be defined as

$$\tilde{Q} = P_1+P_2+(\lambda-1)I.$$

Then

$$\begin{aligned}B_\lambda[A,\tilde{Q}](z_1,z_2) &= B_\lambda[A,P_1](z_1,z_2)+B_\lambda[A,P_2](z_1,z_2)\\ &\quad +(\lambda-1)B_\lambda[A,I](z_1,z_2)\\ &= -\langle(\lambda I-A)z_1,(\lambda I-A)z_2\rangle-\langle(I-\lambda A)z_1,(I-\lambda A)z_2\rangle\\ &\quad +(\lambda-1)[\lambda\langle Az_1,z_2\rangle+\lambda\langle z_1,Az_2\rangle-\langle Az_1,Az_2\rangle-\langle z_1,z_2\rangle]\\ &= \lambda(\lambda+1)[\langle Az_1,z_2\rangle+\langle z_1,Az_2\rangle-\langle Az_1,Az_2\rangle-\langle z_1,z_2\rangle]\\ &= -\lambda(\lambda+1)\langle(A-I)z_1,(A-I)z_2\rangle,\qquad \forall z_1,z_2\in D(A).\end{aligned}$$

Thus if we choose $P_V=(2\lambda)^{-1}\tilde{Q}$, then we obtain

$$B_\lambda[A,P_V](z_1,z_2)=-\frac{\lambda+1}{2}\langle(A-I)z_1,(A-I)z_2\rangle,\qquad \forall z_1,z_2\in D(A)$$

and we see that P_V is the solution of (4.9) and thus of (4.5). ■

Remark 4.4. Looking at the proof of the above lemma we can make some remarks.

(1) In the above proof we did not use the fact that $\lambda>1$, and so Lemma 4.3 holds for all $\lambda\in\mathbb{R}$ such that $\lambda,\lambda^{-1}\in\rho(A)$. For $\lambda^2-1<0$ the equations (4.3) and (4.4) will no longer be Lyapunov equations, but Sylvester equations. Since we only need the relations for $\lambda>1$, we have presented the proof for this case only.

(2) The equality (4.10) can be written in a different form. First of all, for all $z_1,z_2\in D(A)$ it can be equivalently written as

$$\langle P_2(\lambda I-A)z_1,(\lambda I-A)z_2\rangle=\langle P_1(I-\lambda A)z_1,(I-\lambda A)z_2\rangle,$$

or

$$\begin{aligned}(\lambda^2-1)\langle P_2z_1,z_2\rangle-B_\lambda[A,P_2](z_1,z_2)\qquad\qquad&\\ =(\lambda^2-1)\langle P_1Az_1,Az_2\rangle-B_\lambda[A,P_1](z_1,z_2).&\end{aligned}\tag{4.13}$$

Substituting the Lyapunov equations (4.7) and (4.8) in (4.13), we find, after simple calculations, that

$$\langle(P_2-I)z_1,z_2\rangle=\langle(P_1-I)Az_1,Az_2\rangle,\qquad \forall z_1,z_2\in D(A).\tag{4.14}$$

Assuming that P_1 or P_2 is non-negative, we find by Lemma 4.2 and equations (4.3), (4.4) and (4.10) that

$$\sum_{n=0}^{\infty}(\lambda^2-1)^n\|(\lambda I-A^{-1})^{-n}(I-\lambda A)^{-1}x\|^2=\langle P_2(I-\lambda A)^{-1}x,(I-\lambda A)^{-1}x\rangle$$

$$=\langle P_1(\lambda I-A)^{-1}x,(\lambda I-A)^{-1}x\rangle=\sum_{n=0}^{\infty}(\lambda^2-1)^n\|(\lambda I-A)^{-n}(\lambda I-A)^{-1}x\|^2$$

for all $x\in H$.

From [11, Theorem 8.3] we have the following result.

LEMMA 4.5. *Let X be a Banach space and let $T \in \mathcal{L}(X)$. Denote the dual space by X^* and the dual operator of T by T^*. Suppose that for all $x \in X$ and $z \in X^*$,*

$$\sup_{r\in[0,1)} (1-r)\sum_{n=0}^{\infty} \|T^n x\|^2 r^{2n} \le M(T)\|x\|^2, \tag{4.15}$$

$$\sup_{r\in[0,1)} (1-r)\sum_{n=0}^{\infty} \|(T^*)^n z\|^2 r^{2n} \le M(T^*)\|z\|^2. \tag{4.16}$$

Then

$$\|T^n\| \le e\sqrt{M(T)M(T^*)}, \qquad n \in \mathbb{N}.$$

THEOREM 4.6. *Let $A \in \mathcal{E}(H)$ with $\sigma(A) \subset \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) \le 0\}$, and let $V = V(A)$. Suppose that the inverse of A exists and $A^{-1} \in \mathcal{E}$.*

(1) *For $x \in H$ and $\lambda > 1$ define*

$$G_A(x;\lambda) := \frac{1}{\lambda^2}\sum_{n=0}^{\infty}(\lambda^2-1)^n[\|R^n(A,\lambda)x\|^2 + \|R^n(A^{-1},\lambda)x\|^2]. \tag{4.17}$$

Then

$$G_A(x;\lambda) + \frac{\lambda-1}{\lambda^2}\|x\|^2 = \frac{2(1-r^2)}{1+r^2}\sum_{n=0}^{\infty}\|V^n x\|^2 r^{2n}, \quad x \in H, \tag{4.18}$$

where $\lambda = (1+r^2)/(1-r^2)$, $r \in (0,1)$.

(2) *If $A, A^{-1} \in \mathcal{G}$, then $V(A)$ is power bounded and*

$$\|V^n(A)\| \le \frac{e}{2}\left(\frac{1}{4} + M^2 + M_1^2\right), \tag{4.19}$$

where $M = \sup_{t>0}\|e^{At}\|$ and $M_1 = \sup_{t>0}\|e^{A^{-1}t}\|$.

Proof. The proof of the first item can be found in [9, Theorem 2]. However, we present a new proof using Lyapunov equations. Note that there is a typo in the corresponding theorem of [9].

Proof of (1). Note that $G_A(x;\lambda)$ is finite for any $x \in H$, because for $\lambda > 0$ the spectra of $\lambda R(A,\lambda)$ and $\lambda R(A^{-1},\lambda)$ are outside the disk $|\mu| \le 1$. Thus by Lemma 4.2 there exists a unique solution of the Lyapunov equation (4.3). Furthermore, this solution satisfies

$$\langle P_1 x, x\rangle = \sum_{n=0}^{\infty}(\lambda^2-1)^n\|R(A,\lambda)^n x\|^2. \tag{4.20}$$

Similarly there exists $P_2 \in \mathcal{L}(H)$, the unique solution of (4.4), satisfying

$$\langle P_2 x, x\rangle = \sum_{n=0}^{\infty} (\lambda^2 - 1)^n \|R(A^{-1},\lambda)^n x\|^2. \tag{4.21}$$

By Lemma 4.3, the operator P_V defined by (4.11) is the unique solution of (4.5). Since P_1, P_2 are positive and $\lambda > 1$, we find that $P_V > 0$, and so by Lemma 4.2,

$$\langle P_V x, x\rangle = \sum_{n=0}^{\infty} \left(\frac{\lambda-1}{\lambda+1}\right)^n \|V^n x\|^2. \tag{4.22}$$

Writing (4.11) as

$$\langle (P_1 + P_2 + \lambda I - I)x, x\rangle = 2\lambda \langle P_V x, x\rangle,$$

and substituting (4.20)–(4.22) we find that

$$\sum_{n=0}^{\infty} (\lambda^2-1)^n \|R^n(A,\lambda)x\|^2 + \sum_{n=0}^{\infty} (\lambda^2-1)^n \|R^n(A^{-1},\lambda)x\|^2 + (\lambda-1)\|x\|^2 = 2\lambda \sum_{n=0}^{\infty} \left(\frac{\lambda-1}{\lambda+1}\right)^n \|V^n x\|^2.$$

Dividing by λ^2 and substituting $\lambda = (1+r^2)/(1-r^2)$ on the right-hand side, we obtain (4.18).

Proof of (2). Since A and A^{-1} are in $\mathcal{G}$, the Hille–Yosida Theorem yields constants $M, M_1 \geq 1$ such that

$$\|R^n(A,\lambda)\| \leq M\lambda^{-n}, \quad \|R^n(A^{-1},\lambda)\| \leq M_1\lambda^{-n}, \quad \lambda > 0,\ n \in \mathbb{N}. \tag{4.23}$$

Substituting this in (4.17), we find that for any $x \in H$,

$$G_A(x;\lambda) \leq \frac{M^2 + M_1^2}{\lambda^2} \sum_{n=0}^{\infty} \left[\frac{\lambda^2-1}{\lambda^2}\right]^n \|x\|^2 = (M^2 + M_1^2)\|x\|^2. \tag{4.24}$$

Next using elementary calculus, we find that

$$r \in (0,1) \ \Leftrightarrow\ \lambda = \frac{1+r^2}{1-r^2} \in (1,\infty),\ r > 0,$$

and

$$\frac{\lambda-1}{\lambda^2} \leq \frac14, \quad \lambda > 1, \qquad \frac{2(1+r)}{1+r^2} \geq 2, \quad r \in (0,1).$$

Using these estimates, we find from (4.18) and (4.24) that

$$(1-r)\sum_{n=0}^{\infty} \|V^n x\|^2 r^{2n} \leq \frac{M^2 + M_1^2 + 1/4}{2}\|x\|^2, \qquad r \in (0,1),\ x \in H.$$

Since the norm of the adjoint equals the norm of the operator, we see that (4.23) also holds for A^* and $(A^{-1})^*$. Thus, using analogous considerations for $G_{A^*}(x;\lambda)$, we have

$$(1-r)\sum_{n=0}^{\infty}\|V^n(A^*)x\|^2r^{2n}\le\frac{M^2+M_1^2+1/4}{2}\|x\|^2,\qquad r\in(0,1),\ x\in H.$$

Since $V(A^*)=V^*(A)$, we conclude from Lemma 4.5 that (4.19) holds. ∎

In the previous theorem we have shown that if A and A^{-1} each generate a bounded semigroup on the Hilbert space H, then $V(A)$ is power bounded. In the next theorem we give some other sufficient conditions.

THEOREM 4.7. *Let $A\in\mathcal{E}(H)$ with $\sigma(A)\subset\{\lambda\in\mathbb{C}\mid\operatorname{Re}\lambda\le 0\}$. If either*

(1) *there exists an $s\in\mathbb{R}$ such that $is\in\rho(A)$ and $-R(A,is)\in\mathcal{G}(H)$, or*
(2) *there exists a non-zero $s\in\mathbb{R}$ such that $(A-isI)^{-1}$ and $(A+is^{-1}I)^{-1}$ are in $\mathcal{G}(H)$,*

then the Cayley transform $V(A)$ is a power bounded operator.

Proof. Assume that (1) holds. Equation (2.13) defines $A_s\in\mathcal{L}(H)$. Combining (1) with (2.16), we conclude that $A_s\in\mathcal{G}(H)$. Consequently, $V(A_s)$ is power bounded by Lemma 4.1. Now Lemma 2.2 yields the equalities

$$V(A_s)=\alpha(s)V(A),\qquad|\alpha(s)|=1,$$

and thus $V(A)$ is power bounded as well.

Assume next that (2) holds. Then we can define A_s and $A_{-s^{-1}}$ (see (2.13)). Furthermore, $A_s^{-1}=A_{-s^{-1}}$. Moreover, from (2) and (2.16) it follows that $A_s,A_{-s^{-1}}\in\mathcal{G}(H)$. Applying Theorem 4.6 we conclude that $V(A_s)$ is power bounded, and Lemma 2.2 shows that so is $V(A)$. ∎

We remark that condition (2) is not stronger than (1), since in (1) it is assumed that $(A-isI)^{-1}$ is a bounded operator, whereas in (2) this operator is assumed to be an infinitesimal generator.

The following theorem extends the second statement of Theorem 4.6.

THEOREM 4.8. *Let H be a Hilbert space and let $A\in\mathcal{G}(H)$. Assume further that the inverse of A exists and lies in $\mathcal{E}$. Then the following statements are equivalent:*

(1) *For all $\varepsilon>0$ the operator $-R(A,\varepsilon)$ is in $\mathcal{G}(H)$ and there exists a constant $M_1\ge 1$, not depending on $\varepsilon>0$, such that*

$$\|e^{-R(A,\varepsilon)t}\|\le M_1,\qquad t\ge 0.$$

(2) *$A^{-1}\in\mathcal{G}(H)$.*
(3) *For all $\delta>0$ the Cayley transform $V(\delta A)$ is power bounded and there exists a constant $C\ge 1$, not depending on δ, such that*

$$\|V^n(\delta A)\|\le C,\qquad n=0,1,2,\ldots. \tag{4.25}$$

Proof. The implication (1)⇒(2) follows from the second Trotter–Kato approximation theorem (see [8, Theorem III.4.9]). Moreover, $\|e^{A^{-1}t}\| \le M_1$, $t \ge 0$, where M_1 is the constant from (1).

(2)⇒(3). If A generates a bounded semigroup, so does δA. Furthermore,

$$\sup_{t\ge 0} \|e^{\delta A t}\| = \sup_{t\ge 0} \|e^{At}\| = M.$$

Similarly,

$$\sup_{t\ge 0} \|e^{(\delta A)^{-1}t}\| = \sup_{t\ge 0} \|e^{A^{-1}t}\| = M_1.$$

By Theorem 4.6(2), $V(\delta A)$ is power bounded, with bound independent of δ. Thus we have proved (4.25) with $C = \frac{e}{2}(M^2 + M_1^2 + 1/4)$.

(3)⇒(1). Using (4.25), we see that for any $\delta > 0$,

$$\|e^{V(\delta A)t}\| \le Ce^t, \qquad t \ge 0. \tag{4.26}$$

Let $\varepsilon > 0$. By (3.1), we have

$$e^{-2\varepsilon t R(A,\varepsilon)} = e^{2t(\varepsilon^{-1}A - I)^{-1}} = e^{-t} e^{V(\varepsilon^{-1}A)t}.$$

Choosing $\delta = \varepsilon^{-1}$ and combining this with (4.26), we find that $\|e^{tR(A,\varepsilon)}\| \le C$ for all $t \ge 0$. Thus we have obtained (1). ∎

In the above theorem we have shown that for $A \in \mathcal{G}$ the inverse lies in $\mathcal{G}$ if and only if $(A - \varepsilon I)^{-1} \in \mathcal{G}$ for some (or all) $\varepsilon > 0$. Furthermore, if $A^{-1} \in \mathcal{G}$, then $e^{(A-\varepsilon I)^{-1}t}$ is bounded by a constant independent of t and ε. We would like to extend this result to $(A - \lambda I)^{-1}$ for $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \ge 0$. This will be the subject of Theorem 4.10. In that theorem we show that $e^{(A-\lambda I)^{-1}t}$ is uniformly bounded in t, but the constant depends on λ. For the proof, we need the following relation between the semigroups generated by A and by A^{-1} (see [17]). For $A \in \mathcal{G}_{\exp}$ we have

$$e^{A^{-1}t}x = x - \sqrt{t} \int_0^\infty \frac{J_1(2\sqrt{ts})}{\sqrt{s}} e^{As}x\, ds, \qquad t > 0,\ x \in H. \tag{4.27}$$

Here J_1 is the Bessel function of the first kind and of the first order.

Besides equation (4.27), we need the following lemma.

Lemma 4.9. *Let $f : [0,\infty) \to H$ be a continuous function with exponential decay. Define*

$$\hat{f}(\tau) := \sqrt{\tau} \int_0^\infty \frac{J_1(2\sqrt{\tau t})}{\sqrt{t}} f(t)\, dt, \qquad \tau > 0. \tag{4.28}$$

For this transformation, the following Parseval identity holds:

$$\int_0^\infty \|f(t)\|^2 \frac{dt}{t} = \int_0^\infty \|\hat{f}(\tau)\|^2 \frac{d\tau}{\tau}. \tag{4.29}$$

Proof. The statement follows easily from the Parseval identity

$$\int_0^\infty |\mathcal{H}[f](\tau)|^2 \tau \, d\tau = \int_0^\infty |f(t)|^2 t \, dt$$

for the (first order) classical Hankel transformation

$$\mathcal{H}[f](\tau) := \int_0^\infty t f(t) J_1(\tau t) \, dt, \qquad f \in L_2((0,\infty); t\, dt),$$

combined with the relation

$$\hat{f}(\tau) = \sqrt{\tau}(\mathcal{H}(f_0))(\sqrt{\tau}),$$

where $f_0(t) = (1/t) f(t^2/4)$. ∎

Hence we can extend the transformation (4.28) to all functions for which the left-hand side of (4.29) is finite.

THEOREM 4.10. *Suppose $A, A^{-1} \in \mathcal{G}(H)$. Furthermore, choose M, M_1 such that*

$$\|e^{tA}\| \le M, \qquad \|e^{tA^{-1}}\| \le M_1, \qquad \forall t \ge 0.$$

Then for any $\lambda = \varepsilon + i\tau$ with $\varepsilon > 0$, the bounded operator $-R(A, \varepsilon + i\tau)$ is in $\mathcal{G}(H)$, and moreover for all $t \ge 0$,

$$\|e^{-R(A,\varepsilon+i\tau)t}\| \le 2\left[e^2(M^2 + M_1^2 + 1/4)^2 + M^2 \log\left(1 + \frac{\tau^2}{4\varepsilon^2}\right)\right]. \tag{4.30}$$

Proof. From Theorem 4.8 it follows that $-R(A,\varepsilon) \in \mathcal{G}(H)$ for any $\varepsilon > 0$, and moreover

$$\|e^{-R(A,\varepsilon)t}\| \le \frac{e}{2}(M^2 + M_1^2 + 1/4), \qquad t \ge 0. \tag{4.31}$$

Next, by (4.27), for any $x \in H$ we have

$$(e^{-R(A,\varepsilon)t} - e^{-R(A,\varepsilon+i\tau)t})x = \sqrt{t} \int_0^\infty \frac{J_1(2\sqrt{ts})}{\sqrt{s}} [e^{-i\tau s} - 1] e^{-\varepsilon s} e^{As} x \, ds, \quad t > 0,$$

and thus by Lemma 4.9,

$$\int_0^\infty \|(e^{-R(A,\varepsilon)t} - e^{-R(A,\varepsilon+i\tau)t})x\|^2 \frac{dt}{t} = \int_0^\infty \frac{|1 - e^{i\tau t}|^2}{t} e^{-2\varepsilon t} \|e^{At}x\|^2 \, dt. \tag{4.32}$$

Since

$$\int_0^\infty \frac{|1 - e^{i\tau t}|^2}{t} e^{-2\varepsilon t} \, dt = 4 \int_0^\infty \frac{\sin^2(\tau t/2)}{t} e^{-2\varepsilon t} \, dt = \log\left(1 + \frac{\tau^2}{4\varepsilon^2}\right),$$

by (4.32) for any $x \in H$ we obtain the estimate

$$\int_0^\infty \|(e^{-R(A,\varepsilon)t} - e^{-R(A,\varepsilon+i\tau)t})x\|^2 \frac{dt}{t} \le M^2 \log\left(1 + \frac{\tau^2}{4\varepsilon^2}\right) \|x\|^2. \tag{4.33}$$

For $t > 0$ and $x \in H$, we have

$$\begin{aligned}\frac{1}{t}\int_0^t \|e^{-sR(A,\varepsilon+i\tau)}x\|^2 ds &\\ &\le \frac{2}{t}\int_0^t \|(e^{-R(A,\varepsilon)s} - e^{-R(A,\varepsilon+i\tau)s})x\|^2\, ds + \frac{2}{t}\int_0^t \|e^{-R(A,\varepsilon)s}x\|^2\, ds \\ &\le 2\int_0^t \|(e^{-R(A,\varepsilon)s} - e^{-R(A,\varepsilon+i\tau)s})x\|^2\, \frac{ds}{s} + 2\|x\|^2 \sup_{t\ge 0} \|e^{-R(A,\varepsilon)t}\|^2 \\ &\le 2M^2 \log\bigg(1 + \frac{\tau^2}{4\varepsilon^2}\bigg)\|x\|^2 + \frac{e^2}{2}(M^2 + M_1^2 + 1/4)^2\|x\|^2,\end{aligned}$$

where we have used (4.33) and (4.31). Thus for any $x \in H$ and $t > 0$,

$$\frac{1}{t}\int_0^t \|e^{-sR(A,\varepsilon+i\tau)}x\|^2 ds \le \bigg[\frac{e^2}{2}(M^2 + M_1^2 + 1/4)^2 + 2M^2 \log\bigg(1 + \frac{\tau^2}{4\varepsilon^2}\bigg)\bigg]\|x\|^2.$$

The analogous estimate holds for the adjoint semigroup $(e^{-tR(A^*,\varepsilon-i\tau)t})_{t\ge 0}$. Then, by [4, Proposition 3.1], we obtain the statement (4.30). ∎

In the above theorem we have assumed that A and A^{-1} are in $\mathcal{G}(H)$. The same result holds if we assume that there exists a $\lambda \in \mathbb{C}$ satisfying $\mathrm{Re}(\lambda) \ge 0$, $\lambda \ne 0$, such that $A, (A - \lambda I)^{-1} \in \mathcal{G}(H)$. The proof is very similar.

In this section we have concentrated mainly on the implication $A \in \mathcal{G}(H) \Rightarrow \sup_n \|V^n(A)\| < \infty$. We have been able to prove it under extra conditions formulated in terms of A^{-1}. In the following theorem we show that a (partial) converse holds as well.

THEOREM 4.11. *Let A generate an exponentially stable semigroup on the Hilbert space H and let $V(A)$ be power bounded. Then A^{-1} generates a bounded C_0-semigroup.*

Proof. We define $A_1 = A - I$. Then it is clear that

$$V(A) = (A + I)(A - I)^{-1} = I + 2A_1^{-1}.$$

Thus

$$\|e^{2A_1^{-1}t}\| = e^{-t}\|e^{V(A)t}\| \le M_d, \quad M_d := \sup_{n\ge 0} \|V^n(A)\|,$$

so A_1^{-1} generates a bounded semigroup. Since A generates an exponentially stable semigroup, we have

$$\int_0^\infty \|(e^{At} - e^{A_1 t})x\|^2\, \frac{dt}{t} = \int_0^\infty (1 - e^{-t})^2 \|e^{At}x\|^2\, \frac{dt}{t} < \infty.$$

Combining this with (4.27) and Lemma 4.9, we find that for every $x \in H$,

$$\int_0^\infty \|e^{A^{-1}t}x - e^{A_1^{-1}t}x\|^2 \frac{dt}{t} < \infty.$$

Since A_1^{-1} generates a bounded semigroup on H, we conclude as in the proof of Theorem 4.10 that A^{-1} generates a bounded semigroup. ∎

In the previous two theorems, we have related the growth of a semigroup with a perturbed one. For the discrete counterparts of these results, we refer to [3]. Combining Theorem 4.6 and 4.11, we obtain the following corollary.

COROLLARY 4.12. *Let $A \in \mathcal{G}_{\exp}(H)$ and let $V(A)$ be power bounded. Then $V(\alpha A)$ is power bounded for all $\alpha > 0$.*

Proof. By Theorem 4.11 we know that A^{-1} generates a bounded semigroup. It follows directly that so does $(\alpha A)^{-1}$. Combining this with the fact that also αA generates a bounded semigroup, we conclude by Theorem 4.6 that $V(\alpha A)$ is power bounded. ∎

Acknowledgements. The authors would like to thank the referee for useful remarks and suggestions.

References

[1] R. Askey and S. Wainger, *Mean convergence of expansions in Laguerre and Hermite series*, Amer. J. Math. 87 (1965), 695–708.

[2] T. Ya. Azizov, A. I. Barsukov and A. Dijksma, *Decompositions of a Krein space in regular subspaces invariant under a uniformly bounded C_0-semigroup of bi-contractions*, J. Funct. Anal. 211 (2004), 324–354.

[3] N. Besseling and H. Zwart, *Stability analysis in continuous and discrete time, using the Cayley transform*, Integral Equations Operator Theory 68 (2010), 487–503.

[4] J. A. van Casteren, *Boundedness properties of resolvents and semigroup of operators*, in: Linear Operators, Banach Center Publ. 38, Inst. Math., Polish Acad. Sci., Warszawa, 1997, 59–74.

[5] A. J. Chorin, T. J. R. Hughes, M. F. McCracken and J. E. Marsden, *Product formulas and numerical algorithms*, Comm. Pure Appl. Math. 31 (1978), 205–256.

[6] R. F. Curtain and H. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, Springer, New York, 1995.

[7] T. Eisner, *Stability of Operators and Operator Semigroups*, Oper. Theory Adv. Appl. 209, Birkhäuser, Basel, 2010.

[8] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Grad. Texts in Math. 194, Springer, New York, 2000.

[9] A. M. Gomilko, *Cayley transform of the generator of a uniformly bounded C_0-semigroup of operators*, Ukrain. Mat. Zh. 56 (2004), 1018–1029 (in Russian); English transl.: Ukrainian Math. J. 56 (2004), 1212–1226.

[10] A. M. Gomilko, H. Zwart and Y. Tomilov, *On the inverse operator of the generator of a C_0-semigroup*, Mat. Sb. 198 (2007), no. 8, 35–50 (in Russian); English transl.: Sbornik Mat. 198 (2007), 1095–1110.

[11] B. Z. Guo and H. Zwart, *On the relation between stability of continuous- and discrete-time evolution equations via the Cayley transform*, J. Integral Equations Operator Theory 54 (2006), 349–383.
[12] L. V. Kantorovich and G. P. Akilov, *Functional Analysis*, 2nd ed., Pergamon Press, 1982.
[13] H. Komatsu, *Fractional powers of operators*, Pacific J. Math. 19 (1966), 285–346.
[14] R. deLaubenfels, *Inverses of generators*, Proc. Amer. Math. Soc. 104 (1988), 443–448.
[15] A. Montes-Rodriguez, J. Sánches-Álvarez and J. Zemánek, *Uniform Abel–Kreiss boundedness and the extremal behaviour of the Volterra operator*, Proc. London Math. Soc. (3) 91 (2005), 761–788.
[16] P. K. Suetin, *Classical Orthogonal Polynomials*, Nauka, Moscow, 1979 (in Russian).
[17] H. Zwart, *Is* A^{-1} *an infinitesimal generator?*, in: Perspectives in Operator Theory, Banach Center Publ. 75, Inst. Math., Polish Acad. Sci., Warszawa, 2007, 303–313.
[18] —, *Growth estimates for* $\exp(A^{-1}t)$ *on a Hilbert space*, Semigroup Forum 74 (2007), 487–494.

Alexander Gomilko
Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: alex@gomilko.com

Hans Zwart, Niels Besseling
Department of Applied Mathematics
University of Twente
P.O. 217
7500 AE, Enschede, The Netherlands
E-mail: h.j.zwart@math.utwente.nl
n.besseling@math.utwente.nl

Received June 22, 2011
Revised version September 20, 2011 (7232)