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Composition of $(E,2)$-summing operators

by

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To Professor Aleksander Pełczyński on his 70th birthday

Abstract. The Banach operator ideal of $(q,2)$-summing operators plays a fundamental role within the theory of s-number and eigenvalue distribution of Riesz operators in Banach spaces. A key result in this context is a composition formula for such operators due to H. König, J. R. Retherford and N. Tomczak-Jaegermann. Based on abstract interpolation theory, we prove a variant of this result for $(E,2)$-summing operators, E a symmetric Banach sequence space.

1. Introduction and preliminaries. The theory of $(q,2)$-summing operators today is considered to be at the heart of modern Banach space theory with many deep applications to various parts of analysis. For a Banach sequence space E containing ℓ_2, the Banach operator ideal of $(E,2)$-summing operators consists of all (bounded linear) operators T between Banach spaces for which $\{\|T(x_n)\|\} \in E$ for all weakly 2-summable sequences $\{x_n\}$. Mainly basing on interpolation theory, several key results within the theory of $(q,2)$-summing operators and its applications have recently been extended to the more general case of $(E,2)$-summing operators (see [4]–[6]).

In this article we present a variant for $(E,2)$-summing operators of a striking composition formula due to H. König, J. R. Retherford and N. Tomczak-Jaegermann [9], which in its original formulation says that the composition $T_N \circ \ldots \circ T_1$ of $(q_k,2)$-summing operators T_k between Banach spaces is 2-summing provided that $1/q_1 + \ldots + 1/q_N > 1/2$. By completely different methods a special case of this multiplication formula was obtained earlier by B. Maurey and A. Pełczyński [14].

A. Pietsch [17] recovered the composition formula by taking advantage of the intimate relationship between $(p,2)$-summing norms and approximation numbers of operators in Banach spaces (for elaborations of this proof see [18, 2.7.7] and [8, 2.a.12]). Our approach to a variant for $(E,2)$-summing

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operators, where E is a symmetric Banach sequence which is an interpolation space with respect to the couple (ℓ_2, ℓ_∞), combines Pietsch's ideas with abstract interpolation theory.

Before we sketch the content of our paper in more detail we fix some basic preliminaries about sequence spaces, s-numbers, and interpolation.

We shall use standard notation and notions from Banach space theory, as presented, e.g. in [7], [11] and [12]. In particular, for all information needed on p-convexity and p-concavity of Banach lattices/sequence spaces we refer to [11] and [12] (for the notion of 2-convexity see also end of Section 2). If a quasi-normed space $(E, \|\cdot\|_E)$ is a vector subspace of the space $\omega := \mathbb{R}^{\mathbb{N}}$ of all real sequences, and its quasi-norm satisfies (i) if $x \in E$, $y \in \omega$, $|y| \le |x|$, then $y \in E$ and $\|y\|_E \le \|x\|_E$, then E is said to be a *quasi-normed sequence space*. If E also satisfies (ii) if $x = \{x_n\} \in E$, then $x^* \in E$ and $\|x\|_E = \|x^*\|_E$, where $x^* = \{x_n^*\}$ is the non-increasing rearrangement of x, then E is called a *symmetric quasi-normed sequence space* (resp., a *symmetric* (*quasi-*) *Banach sequence space*, whenever E is a (quasi-)Banach space). Note that if E is a separable Banach sequence space, then the sequences e_n form an unconditional basis in E; here and throughout the paper, $\{e_n\}$ denotes the standard unit vector basis in c_0.

The *Köthe dual* E' of a Banach sequence space E is as usual defined by

$$E' := \Big\{x = \{x_n\} \in \omega;\ \sum_{n=1}^{\infty} |x_n y_n| < \infty \text{ for all } y = \{y_n\} \in E\Big\},$$

which equipped with the norm $\|x\| := \sup\{\sum_{n=1}^{\infty} |x_n y_n|;\ \|y\|_E \le 1\}$ forms a Banach sequence space (symmetric provided so is E). The *fundamental function* λ_E of a quasi-Banach sequence space E is given by

$$\lambda_E(n) := \Big\|\sum_{k=1}^{n} e_k\Big\|_E, \qquad n \in \mathbb{N}.$$

For two Banach sequence spaces E and F, the space $M(E,F)$ of *multipliers* from E into F consists of all scalar sequences $x = \{x_n\}$ such that the associated multiplication operator $\{y_n\} \mapsto \{x_n y_n\}$ is defined and bounded from E into F. The space $M(E,F)$ is a Banach sequence space (symmetric provided so are E and F) equipped with the norm

$$\|x\| := \sup\{\|xy\|_F;\ \|y\|_E \le 1\}.$$

We note that if E is a Banach sequence space, then $M(E,\ell_1) = E'$ isometrically.

For all information on Banach operator ideals and s-numbers we refer to [7], [8], [16], [18] and [20]. As usual $\mathcal{L}(X,Y)$ denotes the Banach space of all (bounded linear) operators from X into Y endowed with the operator norm.

Recall that for an operator $T \in \mathcal{L}(X,Y)$ the nth *approximation number* is

$$a_n(T) := \inf\{\|T-R\|;\ R \in \mathcal{L}(X,Y),\ \operatorname{rank} R < n\},$$

and the nth *Weyl number* is

$$x_n(T) := \sup\{a_n(TS);\ S \in \mathcal{L}(\ell_2,X) \text{ with } \|S\| \le 1\}.$$

Clearly, these sequences are non-increasing with $x_n(T) \le a_n(T)$ and $x_1(T) = a_1(T) = \|T\|$, and they are equal whenever T is defined on a Hilbert space. If E is a quasi-normed sequence space contained in ℓ_∞, and if s stands for the approximation numbers or Weyl numbers, or more generally if s is an s-number function (for the definition see, e.g., [18, 2.2.1]), then $\mathcal{L}_E^s(X,Y)$ denotes the set of all operators $T \in \mathcal{L}(X,Y)$ such that $\{s_n(T)\} \in E$. It can be easily seen that the functional

$$s_E(T) := \|\{s_n(T)\}\|_E \quad \text{for } T \in \mathcal{L}_E^s(X,Y)$$

defines a quasi-norm under which $\mathcal{L}_E^s(X,Y)$ is a quasi-Banach space whenever E is a maximal symmetric quasi-Banach sequence space (i.e., the unit ball $B_E := \{x;\ \|x\|_E \le 1\}$ of E is a closed subset of ω equipped with the topology of pointwise convergence).

Finally, for details and basic results on interpolation theory we refer to [1] or [2]. We recall that a mapping $\mathcal{F}$ from the category of all couples of Banach spaces into the category of all Banach spaces is said to be an *interpolation functor* if for every Banach couple $\overline{X} := (X_0, X_1)$, the Banach space $\mathcal{F}(\overline{X})$ is intermediate with respect to $\overline{X}$ (which means $X_0 \cap X_1 \hookrightarrow \mathcal{F}(\overline{X}) \hookrightarrow X_0 + X_1$), and moreover $T : \mathcal{F}(\overline{X}) \to \mathcal{F}(\overline{Y})$ is bounded for all operators $T : \overline{X} \to \overline{Y}$ between couples (meaning that $T : X_0 + X_1 \to Y_0 + Y_1$ is linear with bounded restrictions from X_j into Y_j). By the closed graph theorem, for any couples $\overline{X}$ and $\overline{Y}$,

$$\|T\|_{\mathcal{F}(X_0,X_1)\to\mathcal{F}(Y_0,Y_1)} \le C\|T\|_{\overline{X}\to\overline{Y}} := C\max\{\|T\|_{X_0\to Y_0}, \|T\|_{X_1\to Y_1}\}.$$

If C may be chosen independently of $\overline{X}$ and $\overline{Y}$, we say that $\mathcal{F}$ is a C-*exact* interpolation functor; and $\mathcal{F}$ is said to be *exact* whenever $C = 1$. An intermediate space X of the couple $\overline{X}$ is said to be an *interpolation space* with respect to this couple whenever X can be realized as $\mathcal{F}(\overline{X})$ for some interpolation functor $\mathcal{F}$. The definition of C-exact and exact interpolation spaces is obvious.

As already mentioned we follow Pietsch's route to prove our extension of the composition formula of König, Retherford and Tomczak-Jaegermann, which can be divided into three parts of independent interest, and each of our three sections is devoted to one of these parts. In Section 2 we estimate the $(E,2)$-summing norm of finite rank operators between Banach spaces under the assumption that E is a symmetric Banach sequence which is an interpolation space with respect to the couple (ℓ_2, ℓ_∞). It is shown that there exists a constant $C > 0$ such that $\pi_{E,2}(\mathrm{id}_X) \le C\lambda_E(n)$ for every Banach space X with $\dim X = n$. A key result in the theory of s-numbers states that for $2 \le p < \infty$ each $(p,2)$-summing operator T has its Weyl numbers in the Lorentz space $\ell_{p,1}$, and conversely each operator with Weyl numbers in the Marcinkiewicz space $\ell_{p,\infty}$ is $(p,2)$-summing. In Section 3 the estimate from Section 2 is used to prove an analogue of these inclusions for $(E,2)$-summing operators. Finally, in Section 4 we combine this relationship with a straightforward multiplication formula for s-number ideals in order to obtain the desired composition theorem for $(E,2)$-summing operators in terms of appropriate indices of the sequence space E.

2. $(E,2)$-summing norms of finite rank operators. The following definition is a natural extension of the notion of absolutely (q,p)-summing operators: Let E be a Banach sequence space which contains ℓ_p for some $1 \le p \le \infty$. Then an operator $T : X \to Y$ between Banach spaces X and Y is called (E,p)-*summing* if there exists a constant $C > 0$ such that for any weakly p-summable sequence $\{x_n\}$ (i.e., the scalar sequences $\{x^*(x_n)\}$ are in ℓ_p for every $x^* \in X^*$)

$$\|\{\|Tx_n\|_Y\}\|_E \le C \sup_{\|x^*\|_{X^*}\le 1} \Big(\sum_{n=1}^{\infty} |x^*(x_n)|^p\Big)^{1/p}.$$

We write $\pi_{E,p}(T)$ for the smallest constant C with the above property. The Banach space of all (E,p)-summing operators between Banach spaces X and Y is denoted by $\mathit{\Pi}_{E,p}(X,Y)$. If $\|e_n\|_E = 1$ for all n, then $(\mathit{\Pi}_{E,p}, \pi_{E,p})$ is a Banach operator ideal, in particular for $E = \ell_q$ $(q \ge p)$ we obtain the well known ideal $(\mathit{\Pi}_{q,p}, \pi_{q,p})$ of all (q,p)-summing operators. For an Orlicz sequence space ℓ_φ, we write $\mathit{\Pi}_{\varphi,p}$ and $\pi_{\varphi,p}$ instead of $\mathit{\Pi}_{\ell_\varphi,p}$ and $\pi_{\ell_\varphi,p}$, respectively (for the theory of Orlicz functions and Orlicz spaces we refer to [11] and [12]). Recall that if φ is an *Orlicz function* (i.e., $\varphi : [0,\infty) \to [0,\infty)$ is a continuous and convex function with $\varphi^{-1}(\{0\}) = \{0\}$), then the *Orlicz sequence space* ℓ_φ consists of all real sequences $x = \{x_n\} \in \omega$ such that $\sum_{n=1}^{\infty} \varphi(|x_n|/\varepsilon) < \infty$ for some $\varepsilon > 0$. It is well known that ℓ_φ equipped with the norm

$$\|x\| := \inf\Big\{\varepsilon > 0;\ \sum_{n=1}^{\infty} \varphi(|x_n|/\varepsilon) \le 1\Big\}$$

is a symmetric Banach sequence space.

We note an obvious fact, useful in what follows, that an operator $T : X \to Y$ is $(E,2)$-summing if and only if

$$\pi_{E,2}(T) = \sup\{\pi_{E,2}(TS);\ S \in \mathcal{L}(\ell_2, X),\ \|S\| \le 1\}.$$

We refer to [4] and [5] for non-trivial examples of (E,p)-summing operators. We present new simple examples involving symmetric Banach function spaces on finite non-atomic measure spaces (for the theory of these spaces see, e.g., [10] and [12]). We recall that the *fundamental function* ψ_X of a symmetric Banach function space X on a non-atomic measure space $(\Omega,\mu) := (\Omega,\Sigma,\mu)$ is defined by $\psi_X(t) := \|\chi_A\|_X$ for any $0 \le t < \mu(\Omega)$ with $t = \mu(A)$, $A \in \Sigma$.

PROPOSITION 2.1. *Let X be a symmetric Banach function space defined on a non-atomic finite measure space (Ω,μ) which does not coincide with $L_\infty := L_\infty(\mu)$. Then the inclusion map* $\mathrm{id} : L_\infty \hookrightarrow X$ *is $(\ell_\varphi,1)$-summing, where φ is any Orlicz function such that $\varphi^{-1} \asymp \psi_X$ on $(0,\mu(\Omega))$. In particular*, $\mathrm{id} : L_\infty \hookrightarrow X$ *is $(M(\ell_q,\ell_\varphi),p)$-summing with $1/p+1/q=1$.*

Proof. It is well known that for any symmetric Banach function space X we have $\Lambda(X) \hookrightarrow X$ (with norm ≤ 1), where $\Lambda(X)$ is the Lorentz symmetric function space equipped with the norm

$$\|x\|_{\Lambda(X)} := \int_0^{\mu(\Omega)} x^*(s)\, d\psi_X(s).$$

We put $\psi := \psi_X$. It is well known (see [10, formula, (5.4), p. 111]) that

$$\|x\|_{\Lambda(X)} = \int_0^\infty \psi(\mu_x(s))\, ds,$$

where $\mu_x(s) := \mu(\{\omega \in \Omega; |x(\omega)| > s\})$ for $s > 0$. Let $x_1,\ldots,x_n \in L_\infty$. Then (see, e.g., [7, p. 41])

$$\sup_{\|x^*\|_{L_\infty^*} \le 1} \sum_{k=1}^n |x^*(x_k)| = \Big\| \sum_{k=1}^n |x_k| \Big\|_{L_\infty}.$$

Assume that $\big\| \sum_{k=1}^n |x_k| \big\|_{L_\infty} \le 1$. Hence $\mu_{x_k}(s) = 0$ for any $s > 1$. Combining the above remarks with Jensen's inequality and the fact that $\varphi(\psi(t)) \asymp t$ on $(0,\mu(\Omega))$ shows that there exists a constant $C > 0$ such that

$$\begin{aligned}\sum_{k=1}^n \varphi(\|x_k\|_X) &\le \sum_{k=1}^n \varphi(\|x_k\|_{\Lambda(X)}) = \sum_{k=1}^n \varphi\Big(\int_0^1 \psi(\mu_{x_k}(s))\, ds\Big)\\ &\le \sum_{k=1}^n \int_0^1 \varphi(\psi(\mu_{x_k}(s)))\, ds \le C \sum_{k=1}^n \int_0^1 \mu_{x_k}(s)\, ds\\ &= C \int_\Omega \sum_{k=1}^n |x_k|\, d\mu \le C\mu(\Omega) \Big\| \sum_{k=1}^n |x_k| \Big\|_{L_\infty}.\end{aligned}$$

This yields as desired $\mathrm{id} \in \Pi_{\varphi,1}(L_\infty, X)$ with $\pi_{\varphi,1}(\mathrm{id}) \le \max\{1, C\mu(\Omega)\}$. To conclude the proof it is enough to apply [4, Lemma 3.4]. ∎

The following interpolative estimate for the $(E,2)$-summing norm of finite rank operators is crucial in what follows.

THEOREM 2.2. *Assume that E is a symmetric Banach sequence space, and moreover E is a C-exact interpolation space with respect to the couple (E_1, E_2), where both E_j are symmetric Banach sequence spaces which contain ℓ_2. Then for every operator T between Banach spaces with* $\operatorname{rank} T \le n$,

$$\pi_{E,2}(T) \le C\lambda_E(n)\|T\|,$$

provided $\pi_{E_j,2}(T) \le \lambda_{E_j}(n)\|T\|$ for $j = 1, 2$.

The proof of this result needs two lemmas, both based on interpolation. Following [15] the *characteristic function* $\varphi_{\mathcal{F}} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ of an exact interpolation functor $\mathcal{F}$ is defined by

$$\varphi_{\mathcal{F}}(s,t)\mathbb{R} := \mathcal{F}(s\mathbb{R}, t\mathbb{R}), \qquad s,t > 0,$$

where $\alpha\mathbb{R}$ with $\alpha > 0$ is $\mathbb{R}$ equipped with the norm $\|x\|_{\alpha\mathbb{R}} := \alpha\|x\|$. Note that $\varphi_{\mathcal{F}}$ is homogeneous of degree one, that is, $\varphi_{\mathcal{F}}(\alpha s, \alpha t) = \alpha\varphi_{\mathcal{F}}(s,t)$ for any $\alpha > 0$, and $\varphi_{\mathcal{F}}(\cdot,\cdot)$ is non-decreasing in each variable. Further for any Banach couple $\overline{X} = (X_0, X_1)$ one has

$$\|x\|_{\mathcal{F}(\overline{X})} \le \varphi_{\mathcal{F}}(\|x\|_{X_0}, \|x\|_{X_1}) \tag{1}$$

for every $0 \ne x \in X_0 \cap X_1$. For a fixed Banach space A which is intermediate with respect to the Banach couple $\overline{A}$, define the exact interpolation functor $H_A^{\overline{A}} : \overline{X} \mapsto (H_A^{\overline{A}}(\overline{X}), \|\cdot\|)$, where the space $H_A^{\overline{A}}(\overline{X})$ consists of all $x \in X_0 + X_1$ such that

$$\|x\| := \sup\{\|Tx\|_A;\ \|T\|_{\overline{X}\to\overline{A}} \le 1\} < \infty.$$

We need the well known fact (see, e.g., [1, 2.5.1]) that for any C-exact interpolation space A with respect to $\overline{A} = (A_0, A_1)$ the following continuous inclusions hold (with norms C and 1, respectively):

$$A \overset{C}{\hookrightarrow} H_A^{\overline{A}}(A_0, A_1) \overset{1}{\hookrightarrow} A. \tag{2}$$

LEMMA 2.3. *Let E be a Banach space which is intermediate with respect to the Banach couple $\overline{E} = (E_0, E_1)$.*

(i) *If $\varphi_{\mathcal{F}}$ is the characteristic function of the maximal functor $\mathcal{F} = H_E^{\overline{E}}$, then for any $s,t > 0$,*

$$\varphi_{\mathcal{F}}(s,t) = \psi_E(s,t) := \sup\{\|x\|_E;\ x \in E_0 \cap E_1,\ \|x\|_{E_0} \le s,\ \|x\|_{E_1} \le t\}.$$

(ii) *If $\overline{E}$ is a couple of symmetric Banach sequence spaces and E is a symmetric Banach sequence space which is a C-exact interpolation space with respect $\overline{E}$, then for each n,*

$$\lambda_E(n) \le \psi_E(\lambda_{E_0}(n), \lambda_{E_1}(n)) \le C\lambda_E(n).$$

Proof. (i) Fix $s,t>0$ and put $\overline{X} := (s\mathbb{R}, t\mathbb{R})$. Note first that for $T:\overline{X}\to\overline{E}$ and $x\in E_0\cap E_1$ with $T1=x$, we get $\|T\|_{\overline{X}\to\overline{E}} = \max\{\|x\|_{E_0}/s, \|x\|_{E_1}/t\}$. Hence as desired

$$\begin{aligned}\varphi_{\mathcal{F}}(s,t) = \|1\|_{\mathcal{F}(\overline{X})} &= \sup\{\|T1\|_E;\ \|T\|_{\overline{X}\to\overline{E}}\le 1\}\\ &= \sup\{\|x\|_E;\ x\in E_0\cap E_1,\ \|x\|_{E_0}\le s,\ \|x\|_{E_1}\le t\}.\end{aligned}$$

The proof of (ii) is a minor modification of a similar result for the case of symmetric spaces on $\mathbb{R}_+$ presented in [13]. Put $\mathcal{F} := H_E^{\overline{E}}$. Combining (i) with (1) and (2), we infer for every n that

$$\lambda_E(n)\le\psi_E(\lambda_{E_0}(n),\lambda_{E_1}(n)).$$

To prove the reverse inequality fix $n\in\mathbb{N}$ and $y\in E_0\cap E_1$, and define the rank one operator $T:\overline{E}\to\overline{E}$ by

$$T := x^*\otimes y,$$

where $x^*(x) := \sum_{k=1}^n \xi_k$ for $x=\{\xi_k\}\in E_0+E_1$. Clearly, $\|T\|_{E\to E} = \lambda_{E'}(n)\|y\|_E$ and $\|T\|_{E_j\to E_j} = \lambda_{E_j'}(n)\|y\|_{E_j}$, $j=0,1$. This yields, by the interpolation property,

$$\lambda_{E'}(n)\|y\|_E\le C\max\{\lambda_{E_0'}(n)\|y\|_{E_0},\lambda_{E_1'}(n)\|y\|_{E_1}\},$$

which, since $\lambda_E(n)\lambda_{E'}(n) = n$ (see [11, 3.a.6]), gives $\|y\|_E\le C\lambda_E(n)/n$ whenever $\|y\|_{E_j}\le\lambda_{E_j}(n)/n$, $j=0,1$. Hence

$$\psi_E(\lambda_{E_0}(n)/n,\lambda_{E_1}(n)/n)\le C\lambda_E(n)/n,$$

which completes the proof since ψ_E is positively homogeneous. ∎

In the classical ℓ_p-case the second lemma needed is due to König [8]; for its extension to $(E,2)$-summing operators see [4, 6.2]. Henceforth, $\hookrightarrow$ denotes a continuous inclusion.

Lemma 2.4. *Let $\mathcal{F}$ be an exact interpolation functor and let (E_0,E_1) be a couple of Banach sequence spaces which both contain ℓ_2. Then for any Banach spaces X,Y,*

$$\mathcal{F}(\Pi_{E_0,2}(X,Y),\Pi_{E_1,2}(X,Y))\hookrightarrow\Pi_{\mathcal{F}(E_0,E_1),2}(X,Y).$$

Proof of Theorem 2.2. Let $T\in\mathcal{L}(X,Y)$ be of rank n. Put $\mathcal{F} := H_E^{(E_1,E_2)}$. Since E is a C-exact interpolation space with respect to (E_1,E_2), by (2) we have $E\overset{C}{\hookrightarrow}\mathcal{F}(E_1,E_2)\overset{1}{\hookrightarrow}E$. Hence by applying Lemma 2.3, estimate (1) and Lemma 2.2, we obtain

$$\begin{aligned}\pi_{E,2}(T)\le\pi_{\mathcal{F}(E_1,E_2),2}(T) &\le \|T\|_{\mathcal{F}(\Pi_{E_1,2}(X,Y),\Pi_{E_2,2}(X,Y))}\\ &\le\varphi_{\mathcal{F}}(\lambda_{E_1}(n),\lambda_{E_2}(n))\|T\| = \psi_E(\lambda_{E_1}(n),\lambda_{E_2}(n))\|T\|\\ &\le C\lambda_E(n)\|T\|. \ ∎\end{aligned}$$

The following estimate will be of special interest.

COROLLARY 2.5. *Let E be a symmetric Banach sequence space which is an interpolation space with respect to (ℓ_2, ℓ_∞). Then there is a constant C depending on E such that*

(i) *For any operator T with* $\operatorname{rank} T \le n$,

$$\pi_{E,2}(T) \le C\lambda_E(n)\|T\|.$$

(ii) *For the identity map* id_X *on an n-dimensional Banach space X*,

$$C^{-1}\lambda_E(n) \le \pi_{E,2}(\mathrm{id}_X) \le C\lambda_E(n).$$

Proof. (i) Take T of rank n. Then it is well known that $\pi_{\ell_2,2}(\mathrm{id}_X) = \pi_2(\mathrm{id}_X) \le n^{1/2}\|\mathrm{id}_X\| = \lambda_{\ell_2}(n)\|\mathrm{id}_X\|$ (see, e.g., [8, 2.a.2 1.11]) and trivially $\pi_{\ell_\infty,2}(\mathrm{id}_X) = \|\mathrm{id}_X\| = \lambda_{\ell_\infty}(n)\|\mathrm{id}_X\|$. Hence the desired result is an immediate consequence of the preceding theorem and the fact that there exists a constant $C > 0$ such that E is a C-exact interpolation space with respect to (ℓ_2, ℓ_∞).

(ii) The upper estimate in the second inequality follows by (i). For the lower bound we may assume that $n > 1$. Take $k \in \mathbb{N}$ so that $2k \le n \le 2k+1$. Then by the Dvoretzky–Rogers Lemma (see [7, Lemma 1.3]) there exist k vectors $x_1, \ldots, x_k \in X$ in the unit ball of X, each of norm $\|x_j\|_X \ge 1/2$, such that

$$\sup_{\|x^*\|_{X^*} \le 1} \Big(\sum_{j=1}^k |x^*(x_j)|^2\Big)^{1/2} \le 1.$$

Hence

$$\frac{1}{2}\,\lambda_E(k) \le \Big\|\sum_{j=1}^k \|\mathrm{id}_X(x_j)\|_X e_j\Big\|_E \le \pi_{E,2}(\mathrm{id}_X).$$

Since $k \mapsto \lambda_E(k)/k$ is a non-increasing function, $\lambda_E(n)/6 \le \pi_{E,2}(\mathrm{id}_X)$. ∎

Implicitly and in a quite different way this result has been proved in [4, 6.4]—however, in [4] it was stated for the smaller class of all E's which are 2-convex although an analysis of the proof shows that in fact the formulation given here holds true. Recall that a Banach sequence space E is 2-*convex* (or equivalently, the dual of E has cotype 2, see [12, 1.f.16]) if there is a constant $C > 0$ such that for each choice of finitely many $x_1, \ldots, x_n \in E$,

$$\Big\|\Big(\sum_{k=1}^n |x_k|^2\Big)^{1/2}\Big\| \le C\Big(\sum_{k=1}^n \|x_k\|^2\Big)^{1/2}.$$

It was shown in [4, Lemma 4.3] that each 2-convex maximal and symmetric Banach sequence space E is an exact interpolation space with respect to the couple (ℓ_2, ℓ_∞). We note that there is a quite large class of symmetric sequence spaces which are exact interpolation spaces with respect to (ℓ_2, ℓ_∞), but fail to be 2-convex. For example the real interpolation spaces

$(\ell_2,\ell_\infty)_{\theta,q} = \ell_{p,q}$ with $1/p = (1-\theta)/2$, $0<\theta<1$, $1\le q<\infty$ contain order isomorphic copies of ℓ_q (see, e.g., [11, Prop. 4.e.3]). Thus these spaces are not 2-convex for any $1\le q<2$.

3. $(E,2)$-summing operators and Weyl numbers. Let $w=\{w_n\}$ be a weight sequence (i.e., $w_n>0$ for all n). For $0<p<\infty$ the *Lorentz space* $d(w,p)$ is given by

$$d(w,p) := \Big\{x=\{x_n\}\in\omega;\ \|x\| := \Big(\sum_{n=1}^{\infty}(x_n^*)^p w_n\Big)^{1/p}<\infty\Big\}.$$

If $0<p,q<\infty$ and $w=\{n^{1/p-1/q}\}$, then as usual $d(w,p)$ is denoted by $\ell_{p,q}$. It is well known that if w is a non-increasing sequence, then $d(w,p)$ is a maximal symmetric quasi-Banach sequence space (Banach whenever $1\le p<\infty$). If E is a symmetric Banach sequence space with the fundamental function λ_E, then the Lorentz space $d(w,1)$ with $w=\{\lambda_E(n)/n\}$ is denoted by $\lambda(E)$. For a positive non-decreasing function $\psi:\mathbb{N}\to\mathbb{R}_+$ satisfying $\psi(2n)\le C\psi(n)$ for some $C>0$ and all $n\in\mathbb{N}$, m_ψ stands for the *Marcinkiewicz sequence space* of all real sequences x for which $\{\psi(n)x_n^*\}\in\ell_\infty$; equipped with $\|x\| := \sup_{n\ge1}\psi(n)x_n^*$, it forms a maximal symmetric quasi-Banach sequence space. In the case when $\psi(n)=n^{1/p}$, $0<p<\infty$, we write $\ell_{p,\infty}$ for m_ψ. If E is a symmetric Banach sequence space with the fundamental function λ_E, then we write $m(E)$ for the Marcinkiewicz space m_ψ defined by $\psi=\lambda_E$.

As pointed out in the introduction we now prove an extension of one of the key results of the theory of asymptotic s-number/eigenvalue distribution of power compact operators in Banach spaces, namely the inclusions $\mathcal{L}^x_{p,1}\hookrightarrow \Pi_{p,2}\hookrightarrow\mathcal{L}^x_{p,\infty}$, $p\ge2$ (see, e.g., [18, 2.7.4] and [8, 2.a.11]).

THEOREM 3.1. *Let E be a symmetric Banach sequence such that $\ell_2\hookrightarrow E$.*

(i) *If H is a Hilbert space, then $\Pi_{E,2}(H,Y)\hookrightarrow\mathcal{L}^a_E(H,Y)$ for every Banach space Y. In particular, $\Pi_{E,2}\hookrightarrow\mathcal{L}^x_{m(E)}$.*

(ii) *If E is an interpolation space with respect to the couple (ℓ_2,ℓ_∞), then*

$$\mathcal{L}^x_{\lambda(E)}\hookrightarrow\Pi_{E,2}.$$

Proof. Generalizing an inequality of König (see, e.g., [8, 2.a.3]) it was stated in [4, 3.6] (and proved in [6, Proposition 1]) that (i) holds.

(ii) Assume without loss of generality that the fundamental function $\lambda:=\lambda_E$ of E satisfies $\lambda(1)=1$, and note that λ is non-decreasing and $\lambda(2n)\le2\lambda(n)$ for all $n\in\mathbb{N}$. Then for $x=\{x_n\}\in\lambda(E)$,

$$\|x\|_{\lambda(E)} = \sum_{n=1}^{\infty}x_n^*\,\frac{\lambda(n)}{n} = x_1^* + \sum_{k=0}^{\infty}\Big(\sum_{2^k<n\le2^{k+1}}x_n^*\,\frac{\lambda(n)}{n}\Big)$$

$$\geq x_1^* + \sum_{k=0}^{\infty} \lambda(2^k) x_{2^{k+1}}^* \bigg(\sum_{2^k < n \leq 2^{k+1}} \frac{1}{n} \bigg)$$
$$\geq x_1^* + 4^{-1} \sum_{k=0}^{\infty} x_{2^{k+1}}^* \lambda(2^{k+1}) \geq 4^{-1} \sum_{k=1}^{\infty} x_{2^k}^* \lambda(2^k).$$

Fix $T \in \mathcal{L}^a_{\lambda(E)}(X,Y)$ and choose $S_k : X \to Y$ with $\operatorname{rank} S_k < 2^k$ such that $\|T - S_k\| \leq 2a_{2^k}(T)$, $k = 0, 1, \ldots$, where $T_0 := 0$. Put $T_k = S_{k+1} - S_k$. Combining the above inequalities, we obtain

$$T = \sum_{k=0}^{\infty} T_k \quad \text{with } \operatorname{rank} T_k \leq 2^k \text{ and } \{\lambda(2^k)\|T_k\|\} \in \ell_1.$$

Now by applying Theorem 2.1, we get $\{\pi_{E,2}(T_k)\} \in \ell_1$. In consequence $T \in \mathit{\Pi}_{E,2}(X,Y)$, by the completeness of $\mathit{\Pi}_{E,2}(X,Y)$. Finally, if $T \in \mathcal{L}^x_{\lambda(E)}(X,Y)$, then for any $S \in \mathcal{L}(\ell_2, X)$, we have

$$TS \in \mathcal{L}^x_{\lambda(E)}(\ell_2, Y) = \mathcal{L}^a_{\lambda(E)}(\ell_2, Y) \hookrightarrow \mathit{\Pi}_{E,2}(\ell_2, Y).$$

Since

$$\pi_{E,2}(T) = \sup\{\pi_{E,2}(TS);\, S \in \mathcal{L}(\ell_2, X),\, \|S\| \leq 1\},$$

we obtain $T \in \mathit{\Pi}_{E,2}(X,Y)$. ∎

Remark 3.2. We note that for a large class of symmetric Banach sequence spaces E which are interpolation spaces with respect to (ℓ_2, ℓ_∞) the continuous inclusion $\mathit{\Pi}_{E,2} \hookrightarrow \mathcal{L}^x_{m(E)}$ is optimal. In fact, let F be any 2-concave symmetric Banach sequence space and let $E := M(\ell_2, F)$ be a symmetric Banach sequence space of all multipliers from ℓ_2 into F. Then E is 2-convex, and thus an interpolation space with respect to (ℓ_2, ℓ_∞) (see [4, Lemma 4.3]). Let $\mathrm{id} : F \hookrightarrow \ell_2$ denote the inclusion map. Applying [4, 3.2 and 4.1] we conclude that $\mathrm{id} \in \mathit{\Pi}_{E,2}(F, \ell_2)$. Further, we have $\lambda_E(n) \asymp \lambda_F(n)/\sqrt{n}$ by [5, Proposition 3.5]. This easily implies that

$$x_n(\mathrm{id} : F \hookrightarrow \ell_2) \asymp 1/\lambda_E(n).$$

4. Composition theorem for $(E,2)$-summing operators. In this section we present the composition theorem for $(E,2)$-summing operators. Recall that an operator $T \in \mathcal{L}(X,Y)$ belongs to the product $\mathcal{B} \circ \mathcal{A}$ of two quasi-Banach operator ideals $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ if T can be written in the form $T = VU$, where $U \in \mathcal{A}(X,Z)$ and $V \in \mathcal{B}(Z,Y)$ with a suitable Banach space Z. Clearly, $\mathcal{B} \circ \mathcal{A}$ is an operator ideal which, if endowed with the quasi-norm

$$\|T\|_{\mathcal{B}\circ\mathcal{A}} := \inf\{\alpha(U)\beta(V);\, T = VU,\, U \in \mathcal{A}(X,Z),\, V \in \mathcal{B}(Z,Y)\},$$

forms a quasi-normed Banach operator ideal (see [16, 7.1.2]). In what follows, if E, F and G are symmetric quasi-Banach sequence spaces, and the map B given by $B(x,y) := \{x_n^* y_n^*\}$ is defined and bounded from $E \times F$ into G (i.e., $\|B(x,y)\|_G \le C\|x\|_E\|y\|_F$ for some $C > 0$), then we write $E * F \hookrightarrow G$. Recall that an s-function s is called *multiplicative* if $s_{m+n-1}(VU) \le s_m(U)s_n(V)$ for any $U \in \mathcal{L}(X,Z)$, $V \in \mathcal{L}(Z,Y)$ and $m, n \in \mathbb{N}$. We need the following easy technical observation.

PROPOSITION 4.1. *Let E, F and G be maximal symmetric quasi-Banach sequence spaces such that $E * F \hookrightarrow G$, and let s be any multiplicative function. Then $\mathcal{L}_E^s \circ \mathcal{L}_F^s \hookrightarrow \mathcal{L}_G^s$.*

Proof. Let us first remark the following: If $x \in \omega$ is such that $\{x_{2n-1}^*\} \in G$, then $x \in G$ and $\|x\|_G \le 2C_G\|\{x_{2n-1}^*\}\|_G$, where C_G is the constant from the quasi-triangle inequality of G. Indeed, we have $x^* = y + z$ with $y = \sum_n x_{2n-1}^* e_{2n-1}$ and $z = \sum_n x_{2n}^* e_{2n}$ in ω. Since $y^* = \{x_{2n-1}^*\} \in G$ and $z^* \le y^*$, we get $z^* \in G$ and also $z \in G$ with $\|z\|_G \le \|y\|_G$. But then $x \in G$ and

$$\|x\|_G = \|x^*\|_G \le 2C_G\|y\|_G = 2C_G\|y^*\|_G = \|\{x_{2n-1}^*\}\|_G.$$

Let now $T \in \mathcal{L}_E^s \circ \mathcal{L}_F^s(X,Y)$, hence T has a factorization $T = VU$ through a Banach space Z with $U \in \mathcal{L}_F^s(X,Z)$ and $V \in \mathcal{L}_E^s(Z,Y)$. According to our remark we now show that $\{s_{2n-1}(T)\} \in G$. Since $s_{2n-1}(VU) \le s_n(U)s_n(V)$ for any n, by the multiplicity of s, and by assumption $E * F \hookrightarrow G$ with constant C, we obtain $\{s_{2n-1}(T)\} \in G$ and

$$s_G(T) \le 2C_G\|\{s_{2n-1}(T)\}\|_G \le 2C_G C\|\{s_n(U)\}\|_E\|\{s_n(V)\}\|_F.$$

The proof is completed by taking the infimum over all possible factorizations of T. ∎

As announced we now prove several composition formulas for $(E,2)$-summing operators.

PROPOSITION 4.2. *Suppose that E, F and G are symmetric Banach sequence spaces such that E and F both contain ℓ_2, and G is an interpolation space with respect to (ℓ_2, ℓ_∞). If $\{\lambda_G(2^n)/\lambda_E(2^n) \cdot \lambda_F(2^n)\} \in \ell_1$, then*

$$\Pi_{E,2} \circ \Pi_{F,2} \hookrightarrow \Pi_{G,2}.$$

Proof. Since $m(E) * m(F) \hookrightarrow m_\psi$ with $\psi(n) = \lambda_E(n)\lambda_F(n)$ for $n \in \mathbb{N}$, it follows by Theorem 3.1 and the preceding proposition that

$$\Pi_{E,2} \circ \Pi_{F,2} \hookrightarrow \mathcal{L}_{m(E)}^x \circ \mathcal{L}_{m(F)}^x \hookrightarrow \mathcal{L}_{m_\psi}^x.$$

It is easy to see that $\{x_n\} \in \lambda(G)$ with $\|\{x_n\}\|_{\lambda(G)} \prec \|\{x_{2^n}^* \lambda_G(2^n)\}\|_{\ell_1}$ whenever $\{x_{2^n}^* \lambda_G(2^n)\} \in \ell_1$. Combining this with the assumption, we conclude that $\mathcal{L}_{m_\psi}^x \hookrightarrow \mathcal{L}_{\lambda(G)}^x$. In consequence, Theorem 3.1 applies and the proof is complete. ∎

In order to formulate a more comfortable version we introduce two types of indices for symmetric Banach sequence spaces. For a symmetric Banach sequence space E with fundamental function λ_E define the indices α_E and β_E as follows:

$$\alpha_E := \sup\left\{\alpha > 0;\ \inf_{n\in\mathbb{N}} \frac{\lambda_E(n)}{n^\alpha} > 0\right\}, \quad \beta_E := \inf\left\{\beta > 0; \sup_{n\in\mathbb{N}} \frac{\lambda_E(n)}{n^\beta} < \infty\right\}.$$

The following is our main result.

Theorem 4.3. *Assume that $E_1, \ldots, E_N$ and F are symmetric Banach sequence spaces such that all E_j contain ℓ_2, and F is an interpolation space with respect to (ℓ_2, ℓ_∞). If $\alpha_{E_1} + \ldots + \alpha_{E_N} > \beta_F$, then*

$$\Pi_{E_N,2} \circ \ldots \circ \Pi_{E_1,2} \hookrightarrow \Pi_{F,2}.$$

Proof. By the preceding proposition we check that

$$\{\lambda_F(2^n)/\lambda_{E_1}(2^n) \cdot \ldots \cdot \lambda_{E_N}(2^n)\} \in \ell_1.$$

To see this take $\varepsilon > 0$ such that $\delta := \alpha_{E_1} + \ldots + \alpha_{E_N} - \beta_F - 2\varepsilon N > 0$. By the definition of α_{E_j} and β_F we have, for all $j = 1, \ldots, N$,

$$\lambda_F(n) \prec n^{\beta_F + N\varepsilon}, \qquad n^{\alpha_{E_j} - \varepsilon} \prec \lambda_{E_j}(n).$$

Altogether we conclude as desired that

$$\sum_n \frac{\lambda_F(2^n)}{\lambda_{E_1}(2^n) \cdot \ldots \cdot \lambda_{E_N}(2^n)} \prec \sum_n \frac{1}{2^{\delta n}} < \infty. \ \blacksquare$$

Obviously, the preceding result recovers the classical ℓ_p-case, which is due to König, Retherford and Tomczak-Jaegermann. To show that our result really leads to new applications we add the following

Example 4.4. Let $0 < \alpha < 1$, $0 < \beta < \infty$, $1 \le p < \infty$,

$$w = \{n^{-\alpha}(1 + \log n)^{-\beta}\},$$

and let φ be an Orlicz function. Then

(i) $\lambda_E(n) \asymp n^{(1-\alpha)/p}/(1 + \log n)^{\beta/p}$, *and hence* $\alpha_E = (1 - \alpha)/p$ *with* $E := d(w, p)$.

(ii) *If* $2 \le p < \infty$, *then* $d(w, p)$ *is* 2-*convex, and hence an exact interpolation space with respect to the couple* (ℓ_2, ℓ_∞).

(iii) $\lambda_{\ell_\varphi}(n) = 1/\varphi^{-1}(1/n)$ *and* ℓ_φ *is* 2-*convex provided that the function* $t \mapsto \varphi(\sqrt{t})$ *is equivalent to a convex function.*

Proof. (i) Approximating the Riemann sums of $\int_1^n t^{-\alpha}(1+\log t)^{-\beta}\,dt$, we conclude that

$$n^{1-\alpha}(1 + \log n)^{-\beta} \asymp \sum_{k=1}^n k^{-\alpha}(1 + \log k)^{-\beta}.$$

In particular, this yields $\lambda_E(n) \asymp n^{(1-\alpha)/p}/(1 + \log n)^{\beta/p}$.

(ii) It is easy to see that any Lorentz space $d(w,p)$ is p-convex, and thus it is also 2-convex whenever $2 \le p < \infty$. Finally (iii) is an easy exercise. ∎

In the case $F = \ell_2$ the theorem allows the following improvement.

COROLLARY 4.5. *Assume that $E_1, \ldots, E_N$ are symmetric sequence Banach spaces which all contain ℓ_2, and let $T_j : X_{j-1} \to X_j$ be $(E_j, 2)$-summing operators between Banach spaces, $j = 1, \ldots, N$. Then $T_N \circ \ldots \circ T_1$ is a 2-summing compact operator whenever $\alpha_{E_1} + \ldots + \alpha_{E_N} > 1/2$.*

Proof. For $F = \ell_2$, we have $\beta_F = 1/2$. Thus $T := T_N \circ \ldots \circ T_1 \in \Pi_2(X_0, X_N)$ by Theorem 4.3. To prove that T is compact we need only show that

$$\lim_{n\to\infty} n^{1/2} x_n(T) = 0$$

(see [18, Theorem 2.10.8]). To see this note that similarly to the proof of Proposition 4.1, we obtain

$$\Pi_{E_N,2} \circ \ldots \circ \Pi_{E_1,2} \hookrightarrow \mathcal{L}^x_{m_\psi}$$

with $\psi(n) := \lambda_{E_1}(n) \cdot \ldots \cdot \lambda_{E_N}(n)$ for any $n \in \mathbb{N}$, hence $x_n(T) \prec 1/\psi(n)$. Now fix $\varepsilon > 0$ so that

$$\delta := \alpha_{E_1} + \ldots + \alpha_{E_N} - \varepsilon N - 1/2 > 0,$$

and note that $n^{\alpha_{E_j} - \varepsilon} \prec \lambda_{E_j}(n)$ for all $j = 1, \ldots, N$. Combining these inequalities yields $n^{1/2} x_n(T) \prec 1/n^\delta$, which clearly gives our conclusion. ∎

We finish with a result on powers of the Banach operator ideal $\Pi_{E,2}$ (cf. [17, Theorem 7] for $E = \ell_p$ with $2 < p < \infty$). Recall that an operator $T \in \mathcal{L}(X,Y)$ is *nuclear*, $T \in \mathcal{N}(X,Y)$, if there are sequences $\{x_n^*\} \subset X^*$, $\{y_n\} \subset Y$ with

$$T = \sum_{n=1}^{\infty} x_n^* \otimes y_n, \qquad \sum_{n=1}^{\infty} \|x_n^*\|\, \|y_n\| < \infty.$$

COROLLARY 4.6. *Assume that E is a symmetric Banach sequence space which contains ℓ_2, and $\alpha_E > 1/2n$ for a positive integer n. Then $\Pi_{E,2}^{2n} \hookrightarrow \mathcal{N}$, where $\mathcal{N}$ denotes the Banach ideal of nuclear operators.*

Proof. The previous corollary yields $\Pi_{E,2}^n \hookrightarrow \Pi_2$. To finish the proof we need only recall the well known fact that the composition of two 2-summing operators is nuclear (see, e.g., [7, Theorem 5.31]). ∎

It is well known that each nuclear operator has 2-summable eigenvalues. We note that it is an open problem whether the eigenvalues of nuclear operators on Banach spaces of nontrivial type are better than 2-summable. It is known (see [8, p. 110]) that if X is a Banach space such that

$$\mathcal{L}(\ell_1, X) = \Pi_{s,2}(\ell_1, X)$$

for some $2 < s < \infty$, then for each nuclear operator $T : X \to X$ the sequence of its eigenvalues belongs to ℓ_r for some $1 < r < 2$.

Combining the following proposition with the inequality connecting eigenvalues and Weyl numbers due to Weyl (for $E = \ell_p$) and König (general case, see [8, 2a.8]) yields further information on eigenvalues of nuclear operators. We leave the details to the interested reader.

PROPOSITION 4.7. *Let X be a Banach space such that $\mathcal{L}(\ell_1, X) = \Pi_{E,2}(\ell_1, X)$ for some symmetric Banach sequence space E with $\ell_2 \hookrightarrow E$. Then the sequence of Weyl numbers of any nuclear operator T on X satisfies $\{n^{1/2}x_n(T)\} \in E$.*

Proof. It is well known that any nuclear operator $T : X \to X$ has a factorization

$$T : X \xrightarrow{R} \ell_\infty \xrightarrow{D} \ell_1 \xrightarrow{S} X$$

with D being a diagonal operator. Clearly, D has a factorization

$$D : \ell_\infty \xrightarrow{D_1} \ell_2 \xrightarrow{D_2} \ell_1,$$

where both D_1 and D_2 are diagonal operators. Since D_1 is 2-summing, $T = UV$ with $V \in \Pi_2(X, \ell_2)$ and $U \in \Pi_{E,2}(\ell_2, X)$. Combining these remarks with Theorem 3.1 and Proposition 4.1 we obtain the desired result. ∎

We conclude the paper with the following corollary:

COROLLARY 4.8. *Let φ be an Orlicz function such that the function $t \mapsto \varphi(\sqrt{t})$ is equivalent to a concave function on $\mathbb{R}_+$ and let φ be supermultiplicative (i.e., there exists $C > 0$ such that $\varphi(st) \geq C\varphi(s)\varphi(t)$ for all $s, t > 0$). Then the sequence of Weyl numbers of any nuclear operator T on the Orlicz space ℓ_φ satisfies the condition*

$$\{n^{1/2}x_n(T)\} \in M(\ell_2, \ell_\phi),$$

where ϕ is any Orlicz function such that $\phi^{-1}(t) \asymp t^{3/2}\varphi^{-1}(1/t)$.

Proof. It is shown in [3] that under the above assumptions we have $\mathcal{L}(\ell_1, \ell_\varphi) = \Pi_{\phi,1}(\ell_1, \ell_\varphi)$. By applying [5, Lemma 3.4], we obtain

$$\mathcal{L}(\ell_1, \ell_\varphi) = \Pi_{E,2}(\ell_1, \ell_\varphi)$$

with $E = M(\ell_2, \ell_\phi)$. The claim now follows by Proposition 4.7. ∎

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