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Supercyclicity in the operator algebra

by

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Abstract. We prove a Supercyclicity Criterion for a continuous linear mapping that is defined on the operator algebra of a separable Banach space $\mathcal{B}$. Our result extends a recent result on hypercyclicity on the operator algebra of a Hilbert space. This kind of result is a powerful tool to analyze the structure of supercyclic vectors of a supercyclic operator that is defined on $\mathcal{B}$. For instance, as a consequence of the main result, we give a very simple proof of the recently established fact that certain supercyclic operators defined on a Banach space have an infinite-dimensional closed subspace of supercyclic vectors.

1. Introduction. A bounded linear operator T on a Banach space $\mathcal{B}$ is called *supercyclic* if there is a vector $x \in \mathcal{B}$ such that the projective orbit

$$\{\lambda T^n x : \lambda \in \mathbb{C},\ n = 0, 1, 2, \ldots\}$$

is dense in $\mathcal{B}$. Such an x is also called a *supercyclic vector*. When the orbit itself is dense, without the help of the scalar multiples, the operator as well as the vector are called *hypercyclic*. Clearly, separability of the underlying space is a necessary condition for supercyclicity, and hypercyclicity is a stronger form of supercyclicity.

The notion of supercyclicity was introduced by Hilden and Wallen in the early seventies [HW]. However, the first example of a supercyclic operator in the Banach space setting was due to Rolewicz in the late sixties [Ro]. He proved that the backward shift defined on ℓ^p, $1 \le p < \infty$, or c_0 is supercyclic.

Recall that the operator T is *cyclic* if there is a vector $x \in \mathcal{B}$, called a *cyclic vector*, such that the linear span of the orbit $\{T^n x\}_{n\ge 0}$ is dense in $\mathcal{B}$. For instance, it is easy to see that the forward shift defined on ℓ^p or c_0 is cyclic with cyclic vector e_0. Cyclicity has been studied for a long time due to its relationship with invariant subspaces. Supercyclicity is a stronger form of

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cyclicity. Thus supercyclic operators have many more properties than cyclic ones. For instance, any supercyclic operator always has a residual set of supercyclic vectors and this, of course, is not true for the forward shift. In this context, it is worth to mention Herrero's work [He1] in which it is shown that there are different situations in which the set of noncyclic vectors of a cyclic operator has nonempty interior.

Suppose that $\mathcal{H}$ is a separable Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the operator algebra on $\mathcal{H}$, that is, the space of all bounded operators on $\mathcal{H}$. Since this space is not separable when considered with its operator norm topology, the study of supercyclicity or hypercyclicity of a continuous linear mapping defined on $\mathcal{L}(\mathcal{H})$ does not make sense. However, Kit Chan [Ch1] had the nice idea of considering $\mathcal{L}(\mathcal{H})$ endowed with the strong operator topology; with this new topology, $\mathcal{L}(\mathcal{H})$ becomes a separable space. Thus, Chan gave sense to considering hypercyclic operators on $\mathcal{L}(\mathcal{H})$. The extension of the hypercyclicity concept to the operator algebra is interesting due to its applications to hypercyclic operators on Hilbert spaces (see also [Ch2]). For instance, Chan gave a new and very simple proof (in the Hilbert space setting) of Theorem 2.2 in [Mo] that provides sufficient conditions on an operator to have an infinite-dimensional closed subspace of hypercyclic vectors. However, some of the techniques of [Ch1] depend strongly on the Hilbert space structure and do not always generalize to the Banach space setting. For the sake of completeness, even if Chan's techniques do work for some statements given below, proofs are also included.

The aim of this work is twofold: On the one hand, we can remove the hypothesis that the operator algebra lies on the Hilbert space and, on the other hand, we will prove the results for the more general concept of supercyclicity on $\mathcal{L}(\mathcal{B})$, the operator algebra of all bounded operators on a Banach space $\mathcal{B}$. Since hypercyclicity is a particular case of supercyclicity, we will recover Chan's results for Hilbert spaces. Actually, even in the Hilbert space case we obtain some improvements.

In Section 2, we construct a strong operator topology dense subset in the operator algebra $\mathcal{L}(\mathcal{B})$ on a separable Banach space, which will be used in later parts of this work. The fact that the operator algebra $\mathcal{L}(\mathcal{B})$ is separable with respect to the strong operator topology may be known to specialists. Therefore, the supercyclicity concept also makes sense for continuous linear mappings defined on $\mathcal{L}(\mathcal{B})$. Then we will prove a Supercyclicity Criterion for a continuous linear mapping $\Lambda : \mathcal{L}(\mathcal{B}) \to \mathcal{L}(\mathcal{B})$. This condition is similar to the Supercyclicity Criterion stated by Salas and the first named author [MS, Thm. 2.2], which follows from the Universality Criterion for complete metric spaces of Gethner and Shapiro (see [GS, Theorem 2.2 and remarks following it]). We stress here that $\mathcal{L}(\mathcal{B})$ endowed with the strong operator topology is not a complete metric space.

In Section 3, multiplication operators on $\mathcal{L}(\mathcal{B})$ are considered. If $T:\mathcal{B}\to\mathcal{B}$ is a bounded operator, then the multiplication operator Λ_T that assigns to each $V \in \mathcal{L}(\mathcal{B})$ the operator $\Lambda_T = TV$, is continuous and linear. We prove that if Λ_T is supercyclic, then so is T. The converse is also true provided that T satisfies the Supercyclicity Criterion. We close that section by showing a theorem that guarantees the existence of an infinite-dimensional closed subspace of supercyclic vectors. This latter result has been first stated by Héctor N. Salas and the first named author [MS, Thm. 3.2], and (at least) for weighted shifts it furnishes a necessary and sufficient condition (see [MS, Sections 5 and 6]). As a particular case, for hypercyclicity we obtain

THEOREM 1.1. *Let $\mathcal{B}$ be a separable Banach space and $T \in \mathcal{L}(\mathcal{B})$. Suppose that there is a strictly increasing sequence $\{n_k\}$ of positive integers for which there are*

(a) *a dense subset X of $\mathcal{B}$ and a right inverse $S : X \to X$ (possibly discontinuous) with TS = identity on X such that $\|T^{n_k}x\| \to 0$ and $\|S^{n_k}x\| \to 0$ for every $x \in X$.*

(b) *an infinite-dimensional Banach subspace $\mathcal{B}_0 \subset \mathcal{B}$ such that $\|T^{n_k}e\| \to 0$ for every $e \in \mathcal{B}_0$.*

Then there is an infinite-dimensional Banach subspace $\mathcal{B}_1 \subset \mathcal{B}$ such that each $z \in \mathcal{B}_1 \setminus \{0\}$ is hypercyclic for T.

The above theorem was first obtained in [Mo, Thm. 2.2] (see remarks following it) and characterizes the operators satisfying the Hypercyclicity Criterion that have an infinite-dimensional closed subspace of hypercyclic vectors (see [LM1], [LM2] and [GLM]). The proof of Theorem 1.1 we present here is slightly simpler than that of Chan for the Hilbert space case. Hypothesis (a) in Theorem 1.1 is one of the more usual forms of the Hypercyclicty Criterion. Salas [Sa1-2] proved the existence of hypercyclic operators that did not satisfy the Hypercyclicity Criterion for the *whole sequence* of positive integers and some people thought for some time that there were counterexamples to the criterion (see [He2, p. 189], for instance). In [LM1] and [LM2] it is shown that the operators constructed by Salas do satisfy the Hypercyclicity Criterion for *subsequences* of positive integers. Thus the question of whether every hypercyclic operator satisfied the criterion for subsequences arose again (see [LM1, p. 544]). Bès [Be] (see also [BP]) took over this question and he gave a refinement of the original criterion and proved that satisfying the criterion is equivalent to being hereditarily hypercyclic. So far, it is not known if every hypercyclic operator satisfies the Hypercyclicity Criterion.

The supercyclicity/hypercyclicity of multiplication operators in the operator algebra of a separable Banach space turns out to be a very useful tool to handle large subspaces of supercyclic/hypercyclic vectors. For instance, we will see that if there is a $T \in \mathcal{L}(\mathcal{B})$ which is supercyclic/hypercyclic for Λ_T, then every nonzero vector is supercyclic/hypercyclic for T. The same is true if there is an onto operator V which is supercyclic/hypercyclic for Λ_T. Thus, in any of these cases, we would obtain an operator that has no invariant closed subspace/subset. Therefore, the invariant subspace/subset problem would be solved. As is well known, Enflo [En] solved the invariant subspace problem in the Banach space setting and Read [Re] solved the invariant subset problem in ℓ^1. But both problems remain open in the Hilbert space setting.

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2. Supercyclic operators on the operator algebra

The Supercyclicity Criterion. Our starting point will be the following Supercyclicity Criterion stated in [MS, Thm. 2.2].

THEOREM 2.1. *Let T be a bounded linear operator on a separable Banach space $\mathcal{B}$. Suppose that there exist a strictly increasing sequence $\{n_k\}$ of positive integers and a sequence $\{\lambda_{n_k}\} \subset \mathbb{C} \setminus \{0\}$ for which there are a dense subset X in $\mathcal{B}$ and a right inverse $S : X \to X$ with $TS =$ identity on X such that $\|\lambda_{n_k} T^{n_k} x\| \to 0$ and $\|(1/\lambda_{n_k}) S^{n_k} x\| \to 0$ for every $x \in X$. Then there is a vector x such that $\{\lambda_{n_k} T^{n_k} x\}$ is dense in $\mathcal{B}$. In particular, T is supercyclic.*

Proposition 2.5 in [MS] states that the above Criterion is equivalent to another one proved by Salas [Sa3, Lemma 2.6] and used by him to characterize the supercyclic bilateral weighted shifts. In particular, Salas showed that there are operators satisfying the Supercyclicity Criterion that do not satisfy the Hypercyclicity Criterion. If we take, in Theorem 2.1, $\lambda_{n_k} = 1$ for every k, we recover the Hypercyclicity Criterion. In fact, our point of view will be to study supercyclicity and treat hypercyclicity as a special case. There are good reasons to do so. First, supercyclic operators are divided into two classes: those whose set of normal eigenvalues is empty (as for hypercyclic operators) and those whose set of normal eigenvalues is a one-point set $\{\alpha\}$ with $\alpha \neq 0$ (see [He2, Prop. 3.1]). The former share many properties with hypercyclic operators (see [MS, Sections 2–4]): for instance, it is not known if there is a supercyclic operator whose point spectrum is empty and which does not satisfy the Supercyclicity Criterion. If T satisfies the Supercyclicity Criterion, then the set of normal eigenvalues is empty [MS, Props. 4.3 and 4.6]. Finally, we point out here that these supercyclic operators are completely understood in terms of hypercyclic operators [GLM, Thm. 5.2].

Separability of the operator algebra. Next, we will see that it is possible to extend Theorem 2.1 to the operator algebra $\mathcal{L}(\mathcal{B})$ endowed with the strong operator topology. The *strong operator topology* (SOT) on $\mathcal{L}(\mathcal{B})$ is defined by the family $\{p_x : x \in \mathcal{B}\}$ of seminorms, where $p_x(T) = \|Tx\|$ for each $T \in \mathcal{L}(\mathcal{B})$. A basic open neighborhood of the origin is given by

$$\mathcal{U}(\varepsilon; y_1, \ldots, y_n) = \{T \in \mathcal{L}(\mathcal{B}) : \|Ty_i\| < \varepsilon \text{ for every } i = 1, \ldots, n\},$$

where $\varepsilon > 0$ and $y_1, \ldots, y_n$ are vectors in $\mathcal{B}$.

Clearly, $\mathcal{L}(\mathcal{B})$ endowed with the strong operator topology becomes a locally convex topological vector space, but, as an easy exercise shows, it is not metrizable. Along with the SOT on $\mathcal{L}(\mathcal{B})$ we will use the *operator norm topology* that comes from the usual norm in $\mathcal{L}(\mathcal{B})$ and makes it a Banach space. We agree that, in what follows, only those topological terms with the prefix "SOT" refer to the strong operator topology; otherwise, they refer to the operator norm topology.

The following result may be known to specialists in Operator Theory on general Banach spaces, but we have not found a reference in the literature. Since it is the key point in the proofs of Theorems 2.3 and 3.1, we include a proof.

Theorem 2.2. *If $\mathcal{B}$ is a separable Banach space, then $\mathcal{L}(\mathcal{B})$ is separable for the strong operator topology.*

Proof. First of all, there is a basis of SOT neighborhoods $\mathcal{U}(\varepsilon; y_1, \ldots, y_n)$ of zero, where $\varepsilon > 0$ and y_i, $1 \le i \le n$, are linearly independent. For if $y_1, \ldots, y_n$, not all zero, are linearly dependent, then we choose a basis $F = \{y_{n_1}, \ldots, y_{n_k}\}$ with $1 \le k \le n$ of $\operatorname{span}\{y_i : 1 \le i \le n\}$. Then the linear dependence of y_i, $1 \le i \le n$, with respect to F shows that if we choose δ small enough, then $\mathcal{U}(\delta; y_{n_1}, \ldots, y_{n_k}) \subset \mathcal{U}(\varepsilon; y_1, \ldots, y_n)$.

Now, let $T \in \mathcal{U}(\varepsilon; y_1, \ldots, y_n)$ where $\varepsilon > 0$ and y_i, $1 \le i \le n$, are linearly independent. As $\operatorname{span}\{y_1, \ldots, y_n\}$ is finite-dimensional, it is complemented in $\mathcal{B}$. Therefore, we can take a projection P from $\mathcal{B}$ onto $\operatorname{span}\{y_1, \ldots, y_n\}$. The finite rank operator $H = TP$ belongs to $\mathcal{U}(\varepsilon; y_1, \ldots, y_n)$ since $Hy_i = Ty_i$ for $1 \le i \le n$. Since the SOT is invariant under translations, we find that the finite rank operators are SOT dense in $\mathcal{L}(\mathcal{B})$.

Finally, since $\mathcal{B}$ is separable, so is $\mathcal{B}^*$ with respect to the weak star topology. Let Y be a denumerable weak star dense subset of $\mathcal{B}^*$ and let X be a denumerable dense subset of $\mathcal{B}$. Then the set $\mathcal{D}(X)$ defined to be the linear span over the rational numbers of $\{x \mapsto y(x)x_0 : (y, x_0) \in Y \times X\}$ is a denumerable dense subset in the finite rank operators and the result follows. ■

The Supercyclicity Criterion in the operator algebra. Now, once Theorem 2.2 is proved, as in [Ch1] for hypercyclicity, we can say that a continuous linear mapping $\Lambda : \mathcal{L}(\mathcal{B}) \to \mathcal{L}(\mathcal{B})$ is *supercyclic* if there is an operator T in $\mathcal{L}(\mathcal{B})$ such that $\{\lambda \Lambda^n T : \lambda \in \mathbb{C},\ n = 0, 1, 2, \ldots\}$ is SOT dense in $\mathcal{L}(\mathcal{B})$. In such a case, the operator T is said to be a *supercyclic vector* for Λ. The hypercyclicity of a map Λ on $\mathcal{L}(\mathcal{B})$ can be defined in the obvious way.

We next give a Supercyclicity Criterion on $\mathcal{L}(\mathcal{B})$. As a particular case we obtain the Hypercyclicity Criterion proved by Chan in the Hilbert space setting. Unlike the corresponding result of Chan, we do not require linearity nor continuity of the inverse map.

THEOREM 2.3. *Let $\Lambda : \mathcal{L}(\mathcal{B}) \to \mathcal{L}(\mathcal{B})$ be a continuous linear mapping. Suppose that there exist a strictly increasing sequence $\{n_k\}$ of positive integers and a sequence $\{\lambda_{n_k}\} \subset \mathbb{C} \setminus \{0\}$ for which there are a denumerable SOT dense subset $\mathcal{D}$ in $\mathcal{L}(\mathcal{B})$ and a right inverse $\Theta : \mathcal{D} \to \mathcal{D}$ with $\Lambda\Theta =$ identity on $\mathcal{D}$ such that*

$$\|\lambda_{n_k} \Lambda^{n_k} V\| \to 0 \quad \textit{and} \quad \|(1/\lambda_{n_k}) \Theta^{n_k} V\| \to 0 \quad \textit{for each } V \in \mathcal{D}.$$

Then there is an operator $T \in \mathcal{L}(\mathcal{B})$ such that $\{\lambda_{n_k} \Lambda^{n_k} T\}$ is SOT dense in $\mathcal{L}(\mathcal{B})$. In particular, T is a supercyclic vector for Λ. Furthermore, if $\lambda_{n_k} = 1$ for every k, then T is a hypercyclic vector for Λ.

Proof. The hypercyclic part of the statement is a trivial consequence of the supercyclic part and it needs no proof. To simplify the notation we will prove the theorem in the case that the hypotheses are satisfied for the whole sequence $\{n\}$ of positive integers. One easily checks that the same proof works for a subsequence. Suppose that $\mathcal{D} = \{T_k : k \geq 1\}$. We construct a sequence $\{n_k : k \geq 1\}$ of integers as follows. Let n_1 be a positive integer such that

$$\left\| \frac{1}{\lambda_{n_1}}\, \Theta^{n_1} T_1 \right\| < \frac{1}{2}.$$

We choose a positive integer n_2 large enough to have

$$\left\| \frac{1}{\lambda_{n_2}}\, \Theta^{n_2} T_2 \right\| < \frac{1}{2^2},$$

$$\left\| \lambda_{n_1} \Lambda^{n_1} \frac{1}{\lambda_{n_2}}\, \Theta^{n_2} T_2 \right\| = \left\| \frac{\lambda_{n_1}}{\lambda_{n_2}}\, \Theta^{n_2 - n_1} T_2 \right\| < \frac{1}{2^2},$$

$$\left\| \lambda_{n_2} \Lambda^{n_2} \frac{1}{\lambda_{n_1}}\, \Theta^{n_1} T_1 \right\| = \left\| \frac{\lambda_{n_2}}{\lambda_{n_1}} \Lambda^{n_2 - n_1} T_1 \right\| < \frac{1}{2^2}.$$

Once n_{k-1} is chosen, we can take n_k large enough to have

$$\left\| \frac{1}{\lambda_{n_k}}\, \Theta^{n_k} T_k \right\| < \frac{1}{2^k}, \tag{1}$$

$$\left\| \lambda_{n_j} \Lambda^{n_j} \frac{1}{\lambda_{n_k}}\, \Theta^{n_k} T_k \right\| = \left\| \frac{\lambda_{n_j}}{\lambda_{n_k}}\, \Theta^{n_k - n_j} T_k \right\| < \frac{1}{2^k} \quad \text{for } 1 \leq j \leq k-1, \tag{2}$$

$$\left\|\lambda_{n_k} \varLambda^{n_k} \sum_{j=1}^{k-1} \frac{1}{\lambda_{n_j}}\, \varTheta^{n_j} T_j\right\| = \left\| \sum_{j=1}^{k-1} \frac{\lambda_{n_k}}{\lambda_{n_j}} \varLambda^{n_k-n_j} T_j\right\| < \frac{1}{2^k}. \tag{3}$$

Now, condition (1) ensures that the series

$$T = \sum_{j=1}^{\infty} \frac{1}{\lambda_{n_j}}\, \varTheta^{n_j} T_j$$

defines an operator that belongs to $\mathcal{L}(\mathcal{B})$. We now show that T is a supercyclic vector for $\varLambda$. For every integer k we can write

$$\begin{aligned}\lambda_{n_k} \varLambda^{n_k} T &= \sum_{j=1}^{\infty} \frac{\lambda_{n_k}}{\lambda_{n_j}} \varLambda^{n_k}\, \varTheta^{n_j} T_j \\ &= \sum_{j=1}^{k-1} \frac{\lambda_{n_k}}{\lambda_{n_j}} \varLambda^{n_k} \varTheta^{n_j} T_j + T_k + \sum_{j=k+1}^{\infty} \frac{\lambda_{n_k}}{\lambda_{n_j}} \varLambda^{n_k} \varTheta^{n_j} T_j.\end{aligned}$$

Therefore, applying (3) and (2) in the second inequality below, we have

$$\begin{aligned}\|\lambda_{n_k} \varLambda^{n_k} T - T_k\| &\le \left\| \sum_{j=1}^{k-1} \frac{\lambda_{n_k}}{\lambda_{n_j}} \varLambda^{n_k-n_j} T_j\right\| + \sum_{j=k+1}^{\infty} \left\| \frac{\lambda_{n_k}}{\lambda_{n_j}}\, \varTheta^{n_j-n_k} T_j\right\| \\ &< \frac{1}{2^k} + \sum_{j=k+1}^{\infty} \frac{1}{2^j} = \frac{1}{2^{k-1}}.\end{aligned}$$

Therefore,

$$\|\lambda_{n_k} \varLambda^{n_k} T - T_k\| \to 0 \quad \text{as } k\to\infty. \tag{4}$$

Let V_0 be any element in $\mathcal{L}(\mathcal{B})$ and let $\mathcal{U} = \mathcal{U}(\varepsilon; x_1, \ldots, x_N)$ be any basic SOT neighborhood of the origin, where $\varepsilon > 0$ and $x_1, \ldots, x_N$ are vectors in $\mathcal{B}$. By (4), there is an integer k_0 such that for $k \ge k_0$ and for all x_i, $1 \le i \le N$, we have

$$\|\lambda_{n_k} \varLambda^{n_k} T x_i - T_k x_i\| \le \|\lambda_{n_k} \varLambda^{n_k} T - T_k\| \cdot \|x_i\| < \varepsilon/2.$$

On the other hand, since $\{T_k : k \ge k_0\}$ is SOT dense in $\mathcal{L}(\mathcal{B})$, we can choose an integer $j \ge k_0$ such that $T_j - V_0 \in \mathcal{U}(\varepsilon/2; x_1, \ldots, x_N)$. Hence, for $1 \le i \le N$, we have

$$\|\lambda_{n_j} \varLambda^{n_j} T x_i - V_0 x_i\| \le \|\lambda_{n_j} \varLambda^{n_j} T x_i - T_j x_i\| + \|T_j x_i - V_0 x_i\| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

which proves that $\lambda_{n_j} \varLambda^{n_j} T - V_0 \in \mathcal{U}$ and shows that $\varLambda$ is supercyclic. ■

Remark. If we require linearity of the inverse map (as in [Ch1]) and that $\varLambda D \subset \mathcal{D}$ an easier proof can be given. For in this case, the hypotheses of the theorem imply that the Banach subspace

$$\mathcal{B}_1 = \overline{\operatorname{span}}\{V : V \in \mathcal{D}\}$$

of $\mathcal{L}(\mathcal{B})$ is separable and the restriction of Λ to $\mathcal{B}_1$ satisfies all the hypotheses of the Supercyclicity Criterion. In fact, in Theorem 2.1 we take

$$X = \operatorname{span}\{V : V \in \mathcal{D}\}$$

as a dense set in $\mathcal{B}_1$, define the right inverse by $SV = \Theta V$ if $V \in \mathcal{D}$ and then extend by linearity to X. Therefore, there is $V \in \mathcal{B}_1$ such that $\{\lambda_{n_k} \Lambda^{n_k} V\}$ is dense in $\mathcal{B}_1$. Since $\mathcal{B}_1$ is SOT dense in $\mathcal{L}(\mathcal{B})$, the result follows. This proof also works in more general situations but it gets more complicated.

It is obvious that a necessary condition for a continuous linear mapping $\Lambda : \mathcal{L}(\mathcal{B}) \to \mathcal{L}(\mathcal{B})$ to be supercyclic is that Λ has SOT dense range. In particular, the same is true for hypercyclicity. In the latter case more is true: for each complex number λ the mapping $\Lambda - \lambda$ has SOT dense range [Ch1, Lem. 3]. For hypercyclic operators on Banach spaces this result was proved by Kitai [Ki, Cor. 2.4]. In [MS, Prop. 4.3], it was shown that if an operator satisfies the Supercyclicity Criterion (Theorem 2.1), then $T - \lambda$ always has dense range. The next proposition shows that this result also extends to the operator algebra. The proof is similar to that of Proposition 4.3 in [MS].

PROPOSITION 2.4. *Suppose that $\Lambda : \mathcal{L}(\mathcal{B}) \to \mathcal{L}(\mathcal{B})$ is a continuous linear mapping satisfying the Supercyclicity Criterion. Then for each complex number λ the mapping $\Lambda - \lambda$ has dense range for the strong operator topology.*

Proof. We denote by Λ^* the adjoint of Λ. Assume that there exist $\lambda \in \mathbb{C}$ and a nonzero SOT functional S such that $\Lambda^* S = \lambda S$. We set $\lambda = |\lambda| e^{i\alpha}$. Since Λ satisfies the Supercyclicity Criterion, so does $e^{-i\alpha}\Lambda$. Without loss of generality, we may suppose that the sequence $\{\lambda_{n_k}\}$ for which the Supercyclicity Criterion is satisfied is a sequence of positive numbers. Now, if T is a supercyclic vector for Λ such that $\{\lambda_{n_k} e^{-in_k\alpha} \Lambda^{n_k} T\}$ is SOT dense in $\mathcal{L}(\mathcal{B})$, then the set of complex numbers $\langle \lambda_{n_k} e^{-in_k\alpha} \Lambda^{n_k} T, S\rangle$ is also dense in the complex plane. But we have

$$\langle \lambda_{n_k} e^{-in_k\alpha} \Lambda^{n_k} T, S\rangle = \lambda_{n_k} \langle T, e^{-in_k\alpha} \Lambda^{*n_k} S\rangle = \lambda_{n_k} |\lambda|^{n_k} \langle T, S\rangle$$

and it is obvious that the set of complex numbers defined by the right side of the above display, as k ranges through the positive integers, is at most dense in a straight line through the origin, a contradiction. Therefore, if Λ satisfies the Supercyclicity Criterion, and S is a nonzero SOT continuous linear functional on $\mathcal{L}(\mathcal{B})$, then there is an operator $T \in \mathcal{L}(\mathcal{B})$ such that $\langle (\Lambda - \lambda)T, S\rangle \neq 0$. Thus $(\Lambda - \lambda)\mathcal{L}(\mathcal{B})$ is SOT dense. ■

If a continuous linear mapping $\Lambda : \mathcal{L}(\mathcal{B}) \to \mathcal{L}(\mathcal{B})$ is supercyclic, then it has a SOT dense set of supercyclic vectors. This follows immediately from the fact that if $\{\lambda_n \Lambda^n T\}_{n\ge 0}$ is SOT dense in $\mathcal{L}(\mathcal{B})$ for some T, then so is

$\{\lambda_n \Lambda^n T\}_{n\geq k} = \{\lambda_{n+k}\Lambda^n \Lambda^k T\}_{n\geq 0}$. Thus if T is supercyclic for Λ, then so is $\Lambda^k T$ for each positive integer k.

Now, suppose that Λ satisfies the Supercyclicity Criterion. Proposition 2.4, as in [Ch1, Prop. 4] for hypercyclic operators, can be used to prove that

$$\mathcal{M} = \{p(\Lambda)T : p \text{ is a polynomial}\}$$

is an invariant linear manifold of supercyclic vectors for Λ. This result for hypercyclic operators on Banach spaces was proved, independently, by Herrero [He2] and Bourdon [Bo]. The analogous fact for operators satisfying the Supercyclicity Criterion in the Banach space setting was noticed in [MS, Section 2].

We close this section by stating the following theorem which is the best that one can obtain using the proof of Theorem 2.3. It should be compared with the comparison principle for hypercyclic or supercyclic operators (see [Sh, p. 111]).

THEOREM 2.5. *Let $\mathcal{F}$ be an F-space, that is, a complete metric vector space. Let d denote the metric on $\mathcal{F}$. Let τ be a topology on $\mathcal{F}$ weaker than that induced by d and that makes $\mathcal{F}$ a separable topological vector space. Let $\Lambda : \mathcal{F} \to \mathcal{F}$ be a continuous linear mapping. Suppose that there exist a strictly increasing sequence $\{n_k\}$ of positive integers and a sequence $\{\lambda_{n_k}\} \subset \mathbb{C}\setminus\{0\}$ for which there is a τ-dense subset $\mathcal{D}$ in $\mathcal{F}$ and a right inverse $\Theta : \mathcal{D} \to \mathcal{D}$ with $\Lambda\Theta =$ identity on $\mathcal{D}$ such that*

$$d(0, \lambda_{n_k}\Lambda^{n_k} x) \to 0 \quad \textit{and} \quad d(0, (1/\lambda_{n_k})\Theta^{n_k} x) \to 0 \quad \textit{for each } x \in \mathcal{D}.$$

Then there is $x \in \mathcal{F}$ such that $\{\lambda_{n_k}\Lambda^{n_k} x\}$ is τ-dense in $\mathcal{F}$. In particular, x is a τ-supercyclic vector for T. Furthermore, if $\lambda_{n_k} = 1$ for every k, then x is τ-hypercyclic for T.

For instance, as a corollary of Theorem 2.5, we find that in Theorem 2.3 the strong operator topology can be replaced by the weak operator topology.

3. Multiplication operators on the operator algebra. In this section we consider multiplication operators on $\mathcal{L}(\mathcal{B})$. Corresponding to any operator $T \in \mathcal{L}(\mathcal{B})$, we consider the (left) multiplication operator $\Lambda_T : \mathcal{L}(\mathcal{B}) \to \mathcal{L}(\mathcal{B})$ that assigns to each $V \in \mathcal{L}(\mathcal{B})$ the operator $\Lambda_T V = TV$. Again, in the next theorem we do not require the linearity or continuity of the inverse maps, which makes its proof different from that of [Ch1, Prop. 6].

THEOREM 3.1. *Let T be a bounded operator on a separable Banach space $\mathcal{B}$. If T satisfies the Supercyclicity Criterion with respect to a sequence $\{\lambda_{n_k}\}$, then so does $\Lambda_T : \mathcal{L}(\mathcal{B}) \to \mathcal{L}(\mathcal{B})$ with respect to the same sequence. In particular, if T satisfies the Hypercyclicity Criterion, then so does Λ_T.*

Proof. To save some notation, we assume that the Supercyclicity Criterion is satisfied for $\{\lambda_n\}_{n\geq 1}$. One easily checks that the same proof works for $\{\lambda_{n_k}\}_{k\geq 1}$. It is easy to check that there is no loss of generality if we suppose that the set X in the hypotheses of Theorem 2.1 is denumerable. Since X is dense in $\mathcal{B}$, we may consider the set $\mathcal{D}(X)$ furnished by the proof of Theorem 2.2. Let $V \in \mathcal{D}(X)$. Recall that V can be written for all $x \in \mathcal{B}$ as $Vx = y_1^*(x)x_1 + \ldots + y_m^*(x)x_m$ where $y_i^* \in \mathcal{B}^*$ and $x_i \in X$. In addition, we may assume that the set $\{x_1, \ldots, x_m\}$ is linearly independent and still contained in X. We fix one such representation for each $V \in \mathcal{D}(X)$. Now, for each nonnegative integer j we set

$$\Theta_j Vx = \sum_{i=1}^{m} y_i^*(x) S^j x_i \quad (x \in \mathcal{B}). \tag{1}$$

Since $x_i \in X$, $1 \leq i \leq m$, and $S : X \to X$, we find that $\Theta_j V$ defines a finite rank operator from $\mathcal{B}$ into $\operatorname{span}(X)$. Therefore, Θ_j defines a map from $\mathcal{D}(X)$ into $\mathcal{L}(\mathcal{B})$. Note that S^j may not even be defined on $\operatorname{span}\{x_i : 1 \leq i \leq m\}$ and if it is defined on some element of this subspace, the hypotheses of Theorem 2.1 do not require S to be linear. In particular, this implies that, in general, the definition of $\Theta_j V$ depends on the fixed set $\{x_1, \ldots, x_m\}$. Using the fact that S^n tends pointwise to zero on X and that $\{x_1, \ldots, x_m\}$ is a linearly independent set, it is not difficult to show that $\Theta_j V \neq \Theta_i V$ whenever $V \neq 0$ and $i \neq j$. This will allow us to define the map Θ below.

Now, consider the set

$$\mathcal{E} = \bigcup_{j=0}^{\infty} \Theta_j \mathcal{D}(X)$$

which is denumerable and SOT dense in $\mathcal{L}(\mathcal{B})$. To define the inverse map Θ we set $\mathcal{D}(X) = \{V_k\}_{k\geq 1}$. We may suppose that 0 is not one of the V_k's. Now, by induction, for $k = 1$ we define Θ on $\mathcal{V}_1 = \{\Theta_j V_1 : j \geq 0\}$ as $\Theta\Theta_j V_1 = \Theta_{j+1} V_1$. Next we consider $\{V_k\} \setminus \mathcal{V}_1$; if this set is void, we are finished. If not, let k_0 be the first positive integer such that $V_{k_0} \in \{V_k\} \setminus \mathcal{V}_1$ and then define Θ on $\mathcal{V}_{k_0} = \{\Theta_j V_{k_0} : j \geq 0\} \setminus \mathcal{V}_1$ as $\Theta\Theta_j V_{k_0} = \Theta_{j+1} V_{k_0}$. Note that the iterates $\Theta^n \Theta_j V_{k_0}$ may be eventually equal to $\Theta^{n+m} V_1$, where m is an integer. Next we consider $\{V_k\} \setminus (\mathcal{V}_1 \cup \mathcal{V}_{k_0})$. It is clear that we can continue in this way to get a well defined map $\Theta : \mathcal{E} \to \mathcal{E}$. Since Λ_T is linear and $TS =$ identity on X, it is easy to see that $\Lambda_T \Theta =$ identity on $\mathcal{E}$.

Now, we prove that $(1/\lambda_n)\Theta^n$ tends to zero pointwise on $\mathcal{E}$. Any element in $\mathcal{E}$ is of the form $\Theta_j V$ for some $V \in \mathcal{D}(X)$ and some nonnegative integer j, and the iterates $\Theta^n \Theta_j V$ are eventually of the form $\Theta^{n+m} W$ for some $W \in \mathcal{D}(X)$, where m is an integer. We have $Wx = y_1^*(x)x_1 + \ldots + y_m^*(x)x_m$ where $y_i^* \in \mathcal{B}^*$, $x_i \in X$ and $x \in \mathcal{B}$. If $m \geq 0$, for n large enough we have

$$\begin{aligned}\|(1/\lambda_n)\Theta^{n+m}W\| &= \sup_{\|x\|=1}\|(1/\lambda_n)\Theta_{n+m}Wx\|\\ &= \sup_{\|x\|=1}\Big\|\sum_{i=1}^{m} y_i^*(x)(1/\lambda_n)S^nS^m x_i\Big\|\\ &\le \max_{1\le i\le m}\|y_i^*\|\sum_{i=1}^{m}\|(1/\lambda_n)S^nS^m x_i\|.\end{aligned}$$

Since $S^m x_i$, $1\le i\le m$, belongs to X, we find that $(1/\lambda_n)S^nS^m x_i$ goes to zero pointwise on X and, therefore, so does the last display. If $m<0$, then for $n>-m$ we have $(1/\lambda_n)\Theta^{n+m}W = \Lambda_T^{-m}(1/\lambda_n)\Theta^n W$, which also goes to zero because T is bounded.

A similar argument also shows that $\lambda_n\Lambda_T^n$ goes to zero pointwise on $\mathcal{E}$ and, therefore, the hypotheses of Theorem 2.3 are fulfilled. The proof is complete. ■

Theorem 3.1 states that, under the assumption that T satisfies the Supercyclicity Criterion, the supercyclicity of T implies that of Λ_T. The converse is always true, even if Λ_T does not satisfy the Supercyclicity Criterion (compare with the proof given in [Ch1]):

THEOREM 3.2. *Let $\mathcal{B}$ be a separable Banach space and suppose that $\Lambda_T:\mathcal{L}(\mathcal{B})\to\mathcal{L}(\mathcal{B})$ has a supercyclic* (*resp. hypercyclic*) *vector V and $x_0\in\mathcal{B}$ is a nonzero vector. Then Vx_0 is a supercyclic* (*resp. hypercyclic*) *vector for T. In particular*, *if Λ_T is supercyclic* (*resp. hypercyclic*), *then so is T.*

Proof. We only prove the supercyclic part. We take $x_0^*\in\mathcal{B}^*$ such that $x_0^*(x_0)=1$. Now for each vector $y\in\mathcal{B}$, we define the one-dimensional operator $S_y\in\mathcal{L}(\mathcal{B})$ by $S_yx = x_0^*(x)y$. Since V is a supercyclic vector for Λ_T, there exists a sequence $\{n_k\}$ of integers and a sequence $\{\lambda_{n_k}\}$ of scalars such that $\lambda_{n_k}\Lambda_T^{n_k}Vx_0\to S_yx_0=y$ in $\mathcal{B}$. Thus, $\{\lambda T^nVx_0:\lambda\in\mathbb{C},\ n=0,1,2,\ldots\}$ is dense in $\mathcal{B}$. ■

It would be interesting to know if the converse of Theorem 3.2 holds. Theorem 3.2 has the following intriguing corollary.

COROLLARY 3.3. *If T is a supercyclic* (*resp. hypercyclic*) *vector for Λ_T, then every nonzero vector in $\mathcal{B}$ is supercyclic* (*resp. hypercyclic*) *for T. The same is true if V is onto and supercyclic* (*resp. hypercyclic*) *for Λ_T. In other words*, *in either case T has no nontrivial*, *closed invariant subspace* (*resp. subset*).

Since 0 cannot be a supercyclic vector of T, the supercyclic vectors V of Λ_T are (at least) one-to-one and have dense range.

Finally, we have the following theorem whose proof is slightly simpler than that of Chan for hypercyclic operators in the Hilbert space setting.

Theorem 3.4. *Let $\mathcal{B}$ be a separable Banach space. Suppose that there exist a strictly increasing sequence $\{n_k\}$ of positive integers and a sequence $\{\lambda_{n_k}\} \subset \mathbb{C} \setminus \{0\}$ for which there are*

(a) *a dense subset X in $\mathcal{B}$ and a right inverse $S : X \to X$ with $TS =$ identity on X such that $\|\lambda_{n_k} T^{n_k}\| \to 0$ and $\|(1/\lambda_{n_k}) T^{n_k}\| \to 0$,*

(b) *an infinite-dimensional Banach subspace $\mathcal{B}_0 \subset \mathcal{B}$ such that $\lambda_{n_k} T^{n_k} e \to 0$ for each $e \in \mathcal{B}_0$.*

Then there is an infinite-dimensional closed subspace $\mathcal{B}_1 \subset \mathcal{B}$ such that for each $z \in \mathcal{B}_1$ the sequence $\{\lambda_{n_k} T^{n_k} z\}$ is dense in $\mathcal{B}$. In particular, $\mathcal{B}_1$ is an infinite-dimensional closed subspace consisting, except for zero, of supercyclic vectors for T.

Proof. By Theorem 3.1, Λ_T satisfies the Supercyclicity Criterion with respect to $\{\lambda_{n_k}\}$. Therefore, there is an operator $V \in \mathcal{L}(\mathcal{B})$ such that $\{\lambda_{n_k} \Lambda_T^{n_k} V\}$ is SOT dense in $\mathcal{L}(\mathcal{B})$. The proof of Theorem 3.2 implies that for any nonzero x in $\mathcal{B}$ the sequence $\{\lambda_{n_k} T^{n_k} V x\}$ is dense in $\mathcal{B}$. Without loss of generality we may assume that $\|V\| < 1$. Consequently, the operator $I + V$ is invertible and, therefore, $\mathcal{B}_1 = (I + V)\mathcal{B}_0$ is a closed infinite-dimensional subspace of $\mathcal{B}$. Take any $z \in \mathcal{B}_1$; then $z = e + y$ with $y = Ve$. Now, the result follows because

$$\lambda_{n_k} T^{n_k}(e + y) - \lambda_{n_k} T^{n_k} y = \lambda_{n_k} T^{n_k} e \to 0$$

as $k \to \infty$ and $\{\lambda_{n_k} T^{n_k} y\}$ is dense in $\mathcal{B}$. ∎

Remark. The above theorem was first stated in [MS, Prop. 4.1]. It is a particular case of the remarks following Theorem 2.2 in [Mo]. By just taking λ_{n_k} equal to 1 for every k we obtain Theorem 1.1 as a corollary. Actually, we have obtained a slight improvement of Theorem 2.2 in [Mo]. In fact, by taking V to be a supercyclic vector for Λ_T with $0 < \|V\| < \varepsilon$ for $\varepsilon \in (0, 1)$ we see that $\mathcal{B}_1$ is as close to $\mathcal{B}_0$ as desired. The proof of Theorem 2.2 in [Mo] only shows that $\mathcal{B}_1$ is as close as desired to a subspace of $\mathcal{B}_0$ with a Schauder basis.

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