The spectrally bounded linear maps on operator algebras

by

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Abstract. We show that every spectrally bounded linear map $\Phi$ from a Banach algebra onto a standard operator algebra acting on a complex Banach space is square-zero preserving. This result is used to show that if $\Phi_2$ is spectrally bounded, then $\Phi$ is a homomorphism multiplied by a nonzero complex number. As another application to the Hilbert space case, a classification theorem is obtained which states that every spectrally bounded linear bijection $\Phi$ from $\mathcal{B}(H)$ onto $\mathcal{B}(K)$, where $H$ and $K$ are infinite-dimensional complex Hilbert spaces, is either an isomorphism or an anti-isomorphism multiplied by a nonzero complex number. If $\Phi$ is not injective, then $\Phi$ vanishes at all compact operators.

1. Introduction. Over the past decades, there has been a considerable interest in the study of linear maps on operator algebras that preserve certain properties of operators. In particular, a problem how to characterize linear maps that preserve the spectrum of each operator has attracted the attention of many mathematicians. In [11], Jafarian and Sourour proved that a surjective linear map preserving spectrum from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$, where $X$ and $Y$ are Banach spaces, is either an isomorphism or an anti-isomorphism. Aupetit and Mouton [3] extended this result to primitive Banach algebras with minimal ideals. It is shown in [13] that every point spectrum preserving and surjective linear map on $\mathcal{B}(X)$ is an automorphism. Brešar and Šemrl [5] proved that a linear surjective map preserving spectral radius on $\mathcal{B}(X)$ is either an automorphism or an anti-automorphism multiplied by a scalar with modulus 1. For some other papers concerning this type of linear preservers, see [1, 2, 4, 6–8, 10–11, 13, 16–18].

A natural and interesting question is to ask how to classify the surjective linear maps $\Phi$ on $\mathcal{B}(X)$ which are spectrally bounded, i.e., there exists a positive constant $M$ such that $r(\Phi(T)) \leq Mr(T)$ for every $T \in \mathcal{B}(X)$, where $r(T)$ denotes the spectral radius of $T$. In the case that $M = 1$ we say that $\Phi$
is spectral radius nonincreasing. Note that the study of spectrally bounded linear maps can be reduced to that of the spectral radius nonincreasing ones. Indeed, let \( \Psi = (1/M)\Phi \); then \( r(\Psi(T)) = (1/M)r(\Phi(T)) \leq r(T) \), that is, \( \Psi \) is spectral radius nonincreasing.

A related reference is [14], where Šemrl proved that a unital bijective linear map on \( \mathcal{B}(H) \) with \( H \) infinite-dimensional is spectrally bounded if and only if it is either an automorphism or an anti-automorphism. One of our main purposes in this note is to improve Šemrl’s result above by omitting the assumption that \( \Phi \) is unital.

When the Hilbert space \( H \) is finite-dimensional, together with the discussion of Šemrl [14, Remark 4], one can easily check that a bijective linear map \( \Phi \) on \( \mathcal{B}(H) \) is spectrally bounded if and only if \( \Phi \) has the form \( \Phi(T) = c\varphi(T) + (\Phi(I) - cI) \text{tr} T/n \) for every \( T \in \mathcal{B}(H) \), where \( \varphi \) is either an automorphism or an anti-automorphism of \( \mathcal{B}(H) \), \( c \) is a nonzero complex number and \( n = \text{dim} \, H \). So when we discuss the spectrally bounded linear maps, we may always assume that the Hilbert space \( H \) is infinite-dimensional.

Now we describe the main results of this note. In Section 2, we prove in a quite general framework that a spectrally bounded linear map from a unital complex Banach algebra onto a standard operator algebra acting on a complex Banach space preserves square-zero elements (see Theorem 2.1), which generalizes the first step of the proof of Šemrl’s result in [14] mentioned above. This allows us to prove that a two-fold spectrally bounded surjective linear map from a unital complex Banach algebra onto a standard operator algebra acting on a complex Banach space is a homomorphism multiplied by a nonzero complex number (Theorem 2.2). Here a linear map \( \Phi : \mathcal{A} \to \mathcal{B} \) is said to be two-fold spectrally bounded if \( \Phi_2 : \mathcal{A} \otimes M_2(\mathbb{C}) \to \mathcal{B} \otimes M_2(\mathbb{C}) \) defined by \( \Phi_2((T_{ij})) = (\Phi(T_{ij})) \) is spectrally bounded. Section 3 concerns applications of Theorem 2.1 to the Hilbert space case. We show that a linear bijection \( \Phi \) from \( \mathcal{B}(H) \) onto \( \mathcal{B}(K) \), where \( H \) and \( K \) are infinite-dimensional complex Hilbert spaces, is spectrally bounded if and only if there exist a nonzero complex number \( d \) and an invertible operator \( A \in \mathcal{B}(H, K) \) such that either \( \Phi(T) = dATA^{-1} \) for all \( T \in \mathcal{B}(H) \) or \( \Phi(T) = dAT^tA^{-1} \) for all \( T \in \mathcal{B}(H) \) (Theorem 3.1). If the injectivity assumption on \( \Phi \) is omitted, then either \( \Phi \) has one of the above forms or \( \Phi \) vanishes at every compact operator (Theorem 3.3). In particular, if \( H \) is separable, then \( \Phi \) is surjective and spectrally bounded if and only if it is either an isomorphism or an anti-isomorphism multiplied by a nonzero scalar. From our arguments, we also answer affirmatively a question raised by Šemrl in [15] where he gave a characterization of unital bijections on \( \mathcal{B}(H) \) which preserve square-zero operators and asked whether or not the “unital” assumption can be omitted.
Let us fix some notations. Let $\mathcal{B}(X)$ and $\mathcal{F}(X)$ be the sets of all bounded linear operators and of all finite rank bounded linear operators on the Banach space $X$, respectively. A subalgebra $\mathcal{B}$ in $\mathcal{B}(X)$ is called a **standard operator algebra** if $\mathcal{B}$ is closed and contains the identity operator and $\mathcal{F}(X)$.

For $T \in \mathcal{B}(X)$, we denote by $\mathcal{R}(T)$ and $\ker(T)$ the range and kernel of $T$, respectively. If $T^2 = T$, we say $T$ is an **idempotent operator**. Throughout this paper, we denote by $x \otimes f$ the bounded linear operator on $X$ defined for any $x \in X$ and $f \in X^*$ by $(x \otimes f)(z) = \langle z, f \rangle x$ for every $z \in X$, where $\langle z, f \rangle$ is the value of $f$ at $z$. Note that this operator is of rank one whenever both $x$ and $f$ are nonzero, and that every operator of rank one can be written in this form. By a **projection** we mean a self-adjoint idempotent in $\mathcal{B}(H)$, where $H$ is a Hilbert space.

### 2. General results.

In this section we consider the general case of spectrally bounded linear maps from a Banach algebra onto a standard operator algebra on a complex Banach space. The following is our main result.

**Theorem 2.1.** Let $\mathcal{A}$ be a unital complex Banach algebra and $\mathcal{B}$ be a standard operator algebra on a complex Banach space $X$. Assume that $\Phi : \mathcal{A} \to \mathcal{B}$ is a surjective linear map. If $\Phi$ is spectrally bounded, then $\Phi$ preserves square-zero elements.

**Proof.** By the discussion in the introduction, we may assume that $\Phi$ is spectral radius nonincreasing. We divide the proof into two steps. We mention that the idea of the proof of Step 1 is the same as an idea used in [5].

**Step 1.** Let $A \in \mathcal{A}$ be such that $A^k = 0$ for some $k \geq 2$. If $B \in \mathcal{B}$ satisfies $BQ^i B = 0$, $i = 0, 1, \ldots, k-1$, where $Q = \Phi(A)$, then

$$r(\lambda Q^k + BQ^{k-1} + QBQ^{k-2} + \ldots + Q^{k-1}B) = 0$$

for every complex number $\lambda$.

Let $B_1 = BQ^{k-1} + QBQ^{k-2} + \ldots + Q^{k-1}B$ and $B_2 = Q^k$. Since $r(B + \lambda Q)^k = r[(B + \lambda Q)^k]$ and $B^2 = BQB = BQ^2B = \ldots = BQ^{k-2}B = 0$, it follows that

$$r(B + \lambda Q)^k = |\lambda|^{k-1}r(B_1 + \lambda B_2).$$

As $\Phi$ is surjective, there exists $C \in \mathcal{A}$ such that $\Phi(C) = B$. Moreover, since $\Phi$ is spectral radius nonincreasing, we have

$$r(B + \lambda Q)^k = r(\Phi(C + \lambda A))^k \leq r(C + \lambda A)^k.$$

As $A^k = 0$, it follows that

$$r(C + \lambda A)^k = r[(C + \lambda A)^k] = r(C_0 + \lambda C_1 + \ldots + \lambda^{k-1}C_{k-1}),$$
where \( C_0 = C^k, C_1 = C^{k-1}A + \ldots + AC^{k-1}, \ldots, C_{k-1} = A^{k-1}C + \ldots + CA^{k-1} \). Thus we have shown that
\[
|\lambda|^{-k+1}r(B_1 + \lambda B_2) \leq r(C_0 + \lambda C_1 + \ldots + \lambda^{k-1}C_{k-1}).
\]
Therefore, for any complex \( \lambda \) satisfying \( |\lambda| \geq 1 \), we have
\[
r(B_1 + \lambda B_2) \leq |\lambda|^{-k+1}r(C_0 + \lambda C_1 + \ldots + \lambda^{k-1}C_{k-1}) \\
\leq \|C_0\| + \|C_1\| + \ldots + \|C_{k-1}\|.
\]
On the other hand, for every complex \( \lambda \) satisfying \( |\lambda| \leq 1 \), one gets
\[
r(B_1 + \lambda B_2) \leq \|B_1 + \lambda B_2\| \leq \|B_1\| + \|B_2\|.
\]
Thus the function \( \lambda \mapsto r(B_1 + \lambda B_2) \) is bounded on \( \mathbb{C} \). As it is subharmonic, the Liouville theorem for subharmonic functions [3] shows that \( r(B_1 + \lambda B_2) = r(B_1) \) for every complex \( \lambda \). Observing that \( B_1B = 0 \) it is easy to see that \( B_2^2 = D_{k-2}BQ^{k-2} + D_{k-3}BQ^{k-3} + \ldots + D_0B \) for some \( D_i \in \mathcal{B} \) and so \( B_1^3QB = B_2^2B = 0 \). This further implies that \( B_1^3 \) has the form \( B_1^3 = E_{k-3}BQ^{k-3} + E_{k-2}BQ^{k-2} + \ldots + E_0B \) for some \( E_i \in \mathcal{B} \), and consequently, \( B_1^3Q^2B = B_1^3QB = B_1^3B = 0 \). Repeating this procedure one shows that \( B_1^{k+1} = 0 \). So \( r(B_1 + \lambda B_2) = 0 \) for every complex \( \lambda \), as desired.

**Step 2.** If \( A \in \mathcal{A} \) and \( A^2 = 0 \), then \( \Phi(A)^2 = 0 \).

Based on Step 1, an argument similar to that in Step 5 of the proof of the main results in [5], where \( \mathcal{B}(X) \) is replaced by \( \mathcal{A} \) or \( \mathcal{B} \), shows that, for every \( k \geq 2 \) and every \( A \in \mathcal{A} \) with \( A^k = 0 \), we have \( \Phi(A)^{2k-1} = 0 \). So for every \( A \in \mathcal{A} \) satisfying \( A^2 = 0 \), we have \( \Phi(A)^3 = 0 \). Assume that there exists some \( A \in \mathcal{A} \) with \( A^2 = 0 \) but \( \Phi(A)^2 \neq 0 \). Let \( \Phi(A) = Q \). It follows that \( p(Q) \neq 0 \) for every complex polynomial \( p \) of degree not exceeding 2. Kaplansky’s theorem on local algebraic operators tells us that there is \( u \in X \) such that the vectors \( u, Qu \) and \( Q^2u \) are linearly independent. Therefore, \( u \notin M = \text{span}\{Qu, Q^2u - u\} \). Hence there exists a linear functional \( f \in X^* \) such that \( f(u) = f(Q^2u) = 1 \) and \( f(Qu) = 0 \). Let \( B = (Q^2u - u) \otimes f \). Then a straightforward computation shows that \( B^2 = BQB = 0 \). So, by Step 1, we have \( r(Q^2 + BQ + QB) = 0 \). On the other hand, one can easily check
\[
(Q^2 + BQ + QB)(u - Qu) = u - Qu,
\]
so \( r(Q^2 + BQ + QB) \geq 1 \), which is a contradiction. Hence \( \Phi \) preserves square-zero elements.

Applying Theorem 2.1, we can prove the following results.

**Theorem 2.2.** Let \( \mathcal{A} \) be a unital complex Banach algebra and \( \mathcal{B} \) be a standard operator algebra on a complex Banach space. Assume that \( \Phi : \mathcal{A} \to \mathcal{B} \) is a surjective linear map. Then \( \Phi_2 : \mathcal{A} \otimes M_2(\mathbb{C}) \to \mathcal{B} \otimes M_2(\mathbb{C}) \) is spectrally bounded if and only if \( \Phi \) is a homomorphism multiplied by a nonzero complex number.
Proof. The sufficiency is clear. Now let us check the necessity. Assume that $\Phi_2 : \mathcal{A} \otimes M_2(\mathbb{C}) \rightarrow \mathcal{B} \otimes M_2(\mathbb{C})$ is spectrally bounded. By Theorem 2.1, $\Phi_2$ preserves square-zero elements. Let $C \in \mathcal{A}$ be invertible. For any $A \in \mathcal{A}$, since

\[
\begin{pmatrix}
A & C \\
-C^{-1}A^2 & -C^{-1}AC
\end{pmatrix}^2 = 0,
\]

we have

\[
\begin{pmatrix}
\Phi(A) & \Phi(C) \\
-\Phi(C^{-1}A^2) & -\Phi(C^{-1}AC)
\end{pmatrix}^2 = 0.
\]

So

(2.1) \quad \Phi(A)^2 - \Phi(C)\Phi(C^{-1}A^2) = 0

and

(2.2) \quad \Phi(A)\Phi(C) - \Phi(C)\Phi(C^{-1}AC) = 0.

Letting $C = I$ in (2.2) gives $\Phi(I)\Phi(A) = \Phi(A)\Phi(I)$. Since $\Phi$ is surjective and also spectral radius nonincreasing, and since $\mathcal{B}$ is a standard operator algebra, we must have $\Phi(I) = cI$ for some complex number $c$ with $|c| \leq 1$. We claim $c \neq 0$. Otherwise, taking $C = I$ in (2.1) gives $\Phi(A)^2 = 0$ for every $A \in \mathcal{A}$, which contradicts the surjectivity of $\Phi$. So, without loss of generality, we may assume that $\Phi(I) = I$. Now it is clear that $\Phi(A^2) = \Phi(A)^2$, that is, $\Phi$ is a Jordan homomorphism. By taking $A = I$ in (2.1), we see that $\Phi$ also preserves invertibility. So (2.2) implies

\[
\Phi(C^{-1}AC) = \Phi(C)^{-1}\Phi(A)\Phi(C).
\]

Since $\Phi$ is Jordan, we have $\Phi(CAC) = \Phi(C)\Phi(A)\Phi(C)$ for any $A, C \in \mathcal{A}$. Hence

(2.3) \quad \Phi(AC^2) = \Phi(C^{-1}CAC) = \Phi(C)^{-1}\Phi(CAC)\Phi(C) = \Phi(A)\Phi(C)^2.

Next, choosing any nonzero $\lambda \in \mathbb{C}$ so that $\lambda - C$ is invertible, we get, replacing $C$ by $\lambda - C$ in (2.3),

\[
\Phi(AC) = \Phi(A)\Phi(C)
\]

for all $A$ and invertible $C$ in $\mathcal{A}$. When $C$ is not invertible, take $\lambda \in \mathbb{C}$ so that $\lambda - C$ is invertible; then $\Phi(A(\lambda - C)) = \Phi(A)(\lambda - \Phi(C))$, which again implies that $\Phi(AC) = \Phi(A)\Phi(C)$. Therefore, $\Phi$ is a homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. \qed

Corollary 2.3. Let $\mathcal{A}$ be a unital complex Banach algebra and $\mathcal{B}$ be a standard operator algebra on a complex Banach space. Assume that $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ is a bijective linear map. Then $\Phi_2 : \mathcal{A} \otimes M_2(\mathbb{C}) \rightarrow \mathcal{B} \otimes M_2(\mathbb{C})$ is spectrally bounded if and only if $\Phi$ is an isomorphism multiplied by a nonzero complex number.
If $\mathcal{A}$ is a standard operator algebra, then $\Phi$ has a more concrete characterization.

**Corollary 2.4.** Let $\mathcal{A}$ and $\mathcal{B}$ be two standard operator algebras on a complex Banach space $X$. Assume that $\Phi : \mathcal{A} \to \mathcal{B}$ is a bijective linear map. Then $\Phi_2 : \mathcal{A} \otimes M_2(\mathbb{C}) \to \mathcal{B} \otimes M_2(\mathbb{C})$ is spectrally bounded if and only if there exists a complex number $c$ and an invertible operator $A \in \mathcal{B}(X)$ such that $\Phi(T) = cATA^{-1}$ for every operator $T \in \mathcal{A}$.

**Proof.** It is clear that we need only check the necessity. Assume that $\Phi_2$ is spectrally bounded. By Theorem 2.2, $\Phi$ is a scalar multiple of an isomorphism. Since every isomorphism between standard operator algebras is spatial, there exists an invertible operator $A \in \mathcal{B}(X)$ such that $\Phi(T) = cATA^{-1}$ for every $T \in \mathcal{A}$. $\blacksquare$

### 3. Application to Hilbert space case.

Let $H$ and $K$ be two infinite-dimensional complex Hilbert spaces. Applying the results of Section 2, we can get a complete classification of the spectrally bounded linear maps from $\mathcal{B}(H)$ onto $\mathcal{B}(K)$ without the assumption that $\Phi(I) = I$. The following theorem is our main result in this section.

**Theorem 3.1.** Let $H$ and $K$ be two infinite-dimensional Hilbert spaces. Assume that $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a bijective linear map. Then the following conditions are equivalent.

1. $\Phi$ is spectrally bounded.
2. There exists a nonzero complex number $d$ and a bounded linear map $\Psi : \mathcal{B}(H) \to \mathcal{B}(K)$ preserving idempotents such that $\Phi = d\Psi$.
3. $\Phi$ is a Jordan isomorphism multiplied by a nonzero complex number.
4. $\Phi$ is either an isomorphism or an anti-isomorphism multiplied by a nonzero complex number.
5. There exists a nonzero complex number $d$ and an invertible operator $A \in \mathcal{B}(H, K)$ such that either $\Phi(T) = dATA^{-1}$ for all $T \in \mathcal{B}(H)$ or $\Phi(T) = dAT^{\text{tr}}A^{-1}$ for all $T \in \mathcal{B}(H)$, where $T^{\text{tr}}$ denotes the transpose of $T$ relative to a fixed but arbitrary orthonormal basis of $H$.

To prove this theorem, the following lemma is needed.

**Lemma 3.2.** Suppose that $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a surjective linear map which preserves square-zero operators. Then

$$\Phi(R)^2\Phi(I) = \Phi(I)\Phi(R)^2$$

for all idempotents $R \in \mathcal{B}(H)$.

**Proof.** Let $H$ be a direct sum of two closed infinite-dimensional linear subspaces $H_1$ and $H_2$ (note that we do not assume $H_1$ and $H_2$ are orthogonal). Let $P$ and $Q = I - P$ be the idempotents corresponding to this direct
sum decomposition, that is, $\mathcal{R}(P) = H_1$ and $\text{ker}(P) = H_2$. Assume that operators $A, B \in B(H)$ satisfy $PAP = A$ and $QBP = B$. It follows from [12, Theorem 2] that $A$ and $B$ can be written as sums of five operators with square zero. Say $A = A_1 + A_2 + A_3 + A_4 + A_5$ and $B = B_1 + B_2 + B_3 + B_4 + B_5$, with $PA_i P = A_i$ and $QB_i Q = B_i$ ($i = 1, \ldots, 5$). Clearly, $(A_i + B_j)^2 = 0$. Consequently, we have $\Phi(A_i)\Phi(B_j) + \Phi(B_j)\Phi(A_i) = 0$, which further yields $\Phi(A)\Phi(B) + \Phi(B)\Phi(A) = 0$.

In other words, we have

$$\Phi(PAP)\Phi((I - P)B(I - P)) + \Phi((I - P)B(I - P))\Phi(PAP) = 0$$

for every $A, B \in B(H)$.

We claim that

$$\Phi(R)\Phi(I) + \Phi(I)\Phi(R) = 2\Phi(R)^2 \quad \text{for all idempotents } R.$$ 

If $R \in B(H)$ is an idempotent such that both its range and kernel are infinite-dimensional, then by (3.2) with $A = B = I$ we get

$$\Phi(R)\Phi(I - R) + \Phi(I - R)\Phi(R) = 0,$$

which implies (3.3) immediately.

If $R$ has a finite-dimensional image, take an idempotent $P_1$ with both range and kernel infinite-dimensional such that $P_1 \perp R$. Then

$$\Phi(P_1)\Phi(R) + \Phi(R)\Phi(P_1)$$

$$= \Phi(P_1 P_1 P_1)\Phi((I - P_1)R(I - P_1)) + \Phi((I - P_1)R(I - P_1))\Phi(P_1 P_1 P_1) = 0.$$ 

Therefore,

$$\Phi(P_1 + R)^2 = \Phi(P_1)^2 + \Phi(R)^2$$

and

$$\Phi(R)\Phi(I) + \Phi(I)\Phi(R)$$

$$= \Phi(P_1 + R - P_1)\Phi(I) + \Phi(I)\Phi(P_1 + R - P_1)$$

$$= \Phi(P_1 + R)\Phi(I) + \Phi(I)\Phi(P_1 + R) - \Phi(P_1)\Phi(I) - \Phi(I)\Phi(P_1)$$

$$= 2\Phi(P_1 + R)^2 - 2\Phi(P_1)^2 = 2\Phi(R)^2.$$ 

If $R$ has finite-dimensional kernel, then

$$\Phi(I - R)\Phi(I) + \Phi(I)\Phi(I - R) = 2\Phi(I - R)^2,$$

and hence

$$\Phi(R)\Phi(I) + \Phi(I)\Phi(R) = 2\Phi(R)^2,$$

completing the proof of (3.3). Thus, for any idempotent $R \in B(H)$, we have

$$\Phi(R)^2\Phi(I) + \Phi(R)\Phi(I)\Phi(R) = 2\Phi(R)^3$$

and
\[ \Phi(I)\Phi(R)^2 + \Phi(R)\Phi(I)\Phi(R) = 2\Phi(R)^3, \]

which implies that \( \Phi(R)^2\Phi(I) = \Phi(I)\Phi(R)^2. \)

**Theorem 3.3.** Let \( H \) and \( K \) be two infinite-dimensional Hilbert spaces. Assume that \( \Phi : \mathcal{B}(H) \to \mathcal{B}(K) \) is a surjective linear map. If \( \Phi \) is spectrally bounded, then either \( \Phi(T) = 0 \) for all compact operators \( T \in \mathcal{B}(H) \) or \( \Phi \) is injective. In the last case, there exists a nonzero complex number \( d \) and an invertible operator \( A \in \mathcal{B}(H, K) \) such that either \( \Phi(T) = dAT A^{-1} \) for all \( T \in \mathcal{B}(H) \) or \( \Phi(T) = dA^{tr} A^{-1} \) for all \( T \in \mathcal{B}(H) \), where \( T^{tr} \) denotes the transpose of \( T \) relative to a fixed but arbitrary orthonormal basis of \( H \).

**Proof.** Assume that \( \Phi \) is spectrally bounded. By Theorem 2.1, \( \Phi \) preserves square-zero operators. We may assume that \( \Phi \) is spectral radius nonincreasing by the discussion in the introduction.

**Claim 1.** For any orthogonal idempotents \( P_1 \) and \( P_2 \) in \( \mathcal{B}(H) \), we have

\[ \Phi(P_1 + P_2)^2 = \Phi(P_1)^2 + \Phi(P_2)^2. \]

It is easily seen from the proof of Lemma 3.2 that (3.4) is true if \( \dim \mathcal{R}(P_i) = \dim \mathcal{R}(I - P_i) = \infty \) for \( i = 1 \) or 2. There are four cases left to check:

(i) \( \dim \mathcal{R}(P_1) < \infty, \dim \mathcal{R}(P_2) < \infty; \)
(ii) \( \dim \mathcal{R}(P_1) < \infty, \dim \mathcal{R}(I - P_2) < \infty; \)
(iii) \( \dim \mathcal{R}(I - P_1) < \infty, \dim \mathcal{R}(P_2) < \infty; \)
(iv) \( \dim \mathcal{R}(I - P_1) < \infty, \dim \mathcal{R}(I - P_2) < \infty. \)

In case (i), we can find an idempotent \( P_3 \) orthogonal to \( P_1 + P_2 \) with \( \dim \mathcal{R}(P_3) = \dim \mathcal{R}(I - P_3) = \infty. \) Thus

\[ \Phi(P_1 + P_2)^2 + \Phi(P_3)^2 = \Phi(P_1 + P_2 + P_3)^2 = \Phi(P_1)^2 + \Phi(P_2 + P_3)^2 = \Phi(P_1)^2 + \Phi(P_2)^2 + \Phi(P_3)^2, \]

so \( \Phi(P_1 + P_2)^2 = \Phi(P_1)^2 + \Phi(P_2)^2. \) The remaining cases are similar.

**Claim 2.** \( \Phi(I) = cI \) for some nonzero complex number \( c. \)

We first prove that \( \Phi \) is bounded. Since \( \Phi \) is a spectral radius nonincreasing surjection and \( \mathcal{B}(K) \) is semisimple, it follows from Aupetit [2] that \( \Phi \) is bounded.

Let \( C \in \mathcal{B}(H) \) be a linear combination of orthogonal projections, that is, \( C = \sum_{i=1}^{n} \alpha_i P_i, \) where \( \{P_i\}_{i=1}^{n} \) is an orthogonal set of projections. Then \( \Phi(C)^2 = \sum_{i=1}^{n} \alpha_i^2 \Phi(P_i)^2 \) by (3.4). It follows that \( \Phi(C)^2 \Phi(I) = \Phi(I)\Phi(C)^2 \) by Lemma 3.2. Now suppose \( D \in \mathcal{B}(H) \) is self-adjoint; then \( D \) is a limit of linear combinations of orthogonal projections. Since \( \Phi \) is bounded and linear, we have \( \Phi(D)^2 \Phi(I) = \Phi(I)\Phi(D)^2. \) Let \( C, D \in \mathcal{B}(H) \) be self-adjoint. Then \( \Phi(C + D)^2 \Phi(I) = \Phi(I)\Phi(C + D)^2, \) which yields

\[ (\Phi(C)\Phi(D) + \Phi(D)\Phi(C))\Phi(I) = \Phi(I)(\Phi(C)\Phi(D) + \Phi(D)\Phi(C)). \]
For any $T \in \mathcal{B}(H)$, there exist self-adjoint operators $C$ and $D$ such that $T = C + iD$. Because
\begin{align*}
\Phi(C + iD)^2\Phi(I) &= \Phi(C)^2\Phi(I) - \Phi(D)^2\Phi(I) + i(\Phi(C)\Phi(D) + \Phi(D)\Phi(C))\Phi(I) \\
&= \Phi(I)\Phi(C + iD)^2,
\end{align*}
we see that
\[ \Phi(T)^2\Phi(I) = \Phi(I)\Phi(T)^2 \]
for every $T \in \mathcal{B}(H)$, and consequently $S^2\Phi(I) = \Phi(I)S^2$ for all $S \in \mathcal{B}(K)$ by the surjectivity of $\Phi$. This implies that $\Phi(I) = cI$ for some $c \in \mathbb{C}$. Furthermore, $c \neq 0$. Indeed, if $c = 0$, then $\Phi(I) = 0$. Thus by (3.3) we have $\Phi(R)^2 = 0$ for all idempotents $R$, which implies that $\Phi(C)^2 = 0$ for all self-adjoint operators $C$ by the boundedness of $\Phi$. Hence $\Phi(T)^2 = 0$ for all $T \in \mathcal{B}(H)$, which contradicts the surjectivity of $\Phi$.

Therefore, with no loss of generality, we may assume $\Phi(I) = I$. Thus, by (3.3) again, $\Phi$ is idempotent preserving. Let $A \in \mathcal{B}(H)$ be self-adjoint and $A = \sum_{i=1}^{n} t_i P_i$ where $t_i \in \mathbb{R}$ and $P_i$ are pairwise orthogonal projections. Since $\Phi$ maps mutually orthogonal projections to mutually orthogonal idempotents, $\Phi(A^2) = \Phi(A)^2$. Now, because the set of self-adjoint elements that are finite real linear combinations of orthogonal projections is dense in the set of all self-adjoint elements in $\mathcal{B}(H)$, we see that $\Phi(A^2) = \Phi(A)^2$ for all self-adjoint $A$ in $\mathcal{B}(H)$ by the boundedness of $\Phi$. Replacing $A$ by $A + B$ where both $A$ and $B$ are self-adjoint, we get $\Phi(AB + BA) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A)$. Since every $T \in \mathcal{B}(H)$ can be written in the form $T = A + iB$ with $A$ and $B$ self-adjoint, the last relations imply that $\Phi(T^2) = \Phi(T)^2$. So $\Phi$ is Jordan.

Since $\mathcal{B}(K)$ is a prime ring, by [9, Thm. 3.1], $\Phi$ is either a homomorphism or an anti-homomorphism. If $\Phi$ is not injective, then $\ker \Phi$ is a nonzero closed ideal in $\mathcal{B}(H)$. Since the smallest nontrivial closed ideal of $\mathcal{B}(H)$ is the ideal $\mathcal{K}(H)$ of compact operators, we have $\ker \Phi \supseteq \mathcal{K}(H)$. Hence $\Phi(T) = 0$ for all compact operators $T \in \mathcal{B}(H)$. If $\Phi$ is injective, then $\Phi$ is either an isomorphism or an anti-isomorphism. Since every isomorphism or anti-isomorphism from $\mathcal{B}(H)$ onto $\mathcal{B}(K)$ is spatial, there exists an invertible operator $A \in \mathcal{B}(H, K)$ such that $\Phi(T) = ATA^{-1}$ for all $T \in \mathcal{B}(H)$ or $\Phi(T) = AT^\text{tr}A^{-1}$ for all $T \in \mathcal{B}(H)$, where $T^\text{tr}$ denotes the transpose of $T$ relative to an orthonormal basis in $H$.

**Remark 3.4.** Claim 2 in the proof of Theorem 3.3 also answers affirmatively a question due to Šemrl [15], who showed that a unital linear bijection on $\mathcal{B}(H)$ is square-zero preserving if and only if it is either an automorphism or an anti-automorphism and he asked whether or not the unital assumption may be omitted. So we find that a linear bijection on
$\mathcal{B}(H)$ is square-zero preserving if and only if it is either an automorphism or an anti-automorphism multiplied by a nonzero scalar.

**Proof of Theorem 3.1.** (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1) are obvious. (1) $\Rightarrow$ (2) follows from Theorem 3.3. ■

Note that if the Hilbert space $H$ is separable, then $\Phi$ is injective in Theorem 3.3. In fact, if $\Phi$ is not injective, then $\mathcal{B}(H)$ is isomorphic to the quotient algebra $\mathcal{B}(H)/\mathcal{K}(H)$, which contradicts the fact that $\mathcal{B}(H)/\mathcal{K}(H)$ is simple. So we have the following corollary which generalizes [14, Thm. 2] by omitting the unital assumption.

**Corollary 3.5.** Let $H$ and $K$ be two infinite-dimensional Hilbert spaces with $H$ separable. Assume that $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a surjective linear map. Then $\Phi$ is spectrally bounded if and only if there exists a nonzero complex number $d$ and an invertible operator $A \in \mathcal{B}(H, K)$ such that either $\Phi(T) = dATA^{-1}$ for all $T \in \mathcal{B}(H)$ or $\Phi(T) = dAT^\text{tr}A^{-1}$ for all $T \in \mathcal{B}(H)$, where $T^\text{tr}$ denotes the transpose of $T$ relative to a fixed but arbitrary orthonormal basis of $H$.

**Remark 3.6.** Our proofs still work if $\mathcal{B}(K)$ is replaced by a standard operator algebra acting on a complex Banach space. So the results in this section also hold true when $\Phi$ is a linear map from $\mathcal{B}(H)$ onto a standard operator algebra $\mathcal{B}$.

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**References**


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