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The spectrally bounded linear maps on operator algebras

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Abstract. We show that every spectrally bounded linear map Φ from a Banach algebra onto a standard operator algebra acting on a complex Banach space is square-zero preserving. This result is used to show that if Φ_2 is spectrally bounded, then Φ is a homomorphism multiplied by a nonzero complex number. As another application to the Hilbert space case, a classification theorem is obtained which states that every spectrally bounded linear bijection Φ from $\mathcal{B}(H)$ onto $\mathcal{B}(K)$, where H and K are infinite-dimensional complex Hilbert spaces, is either an isomorphism or an anti-isomorphism multiplied by a nonzero complex number. If Φ is not injective, then Φ vanishes at all compact operators.

1. Introduction. Over the past decades, there has been a considerable interest in the study of linear maps on operator algebras that preserve certain properties of operators. In particular, a problem how to characterize linear maps that preserve the spectrum of each operator has attracted the attention of many mathematicians. In [11], Jafarian and Sourour proved that a surjective linear map preserving spectrum from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$, where X and Y are Banach spaces, is either an isomorphism or an anti-isomorphism. Aupetit and Mouton [3] extended this result to primitive Banach algebras with minimal ideals. It is shown in [13] that every point spectrum preserving and surjective linear map on $\mathcal{B}(X)$ is an automorphism. Brešar and Šemrl [5] proved that a linear surjective map preserving spectral radius on $\mathcal{B}(X)$ is either an automorphism or an anti-automorphism multiplied by a scalar with modulus 1. For some other papers concerning this type of linear preservers, see [1, 2, 4, 6–8, 10–11, 13, 16–18].

A natural and interesting question is to ask how to classify the surjective linear maps Φ on $\mathcal{B}(X)$ which are *spectrally bounded*, i.e., there exists a positive constant M such that $r(\Phi(T)) \leq Mr(T)$ for every $T \in \mathcal{B}(X)$, where $r(T)$ denotes the spectral radius of T. In the case that $M = 1$ we say that Φ

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is *spectral radius nonincreasing*. Note that the study of spectrally bounded linear maps can be reduced to that of the spectral radius nonincreasing ones. Indeed, let $\Psi = (1/M)\Phi$; then $r(\Psi(T)) = (1/M)r(\Phi(T)) \leq r(T)$, that is, Ψ is spectral radius nonincreasing.

A related reference is [14], where Šemrl proved that a unital bijective linear map on $\mathcal{B}(H)$ with H infinite-dimensional is spectrally bounded if and only if it is either an automorphism or an anti-automorphism. One of our main purposes in this note is to improve Šemrl's result above by omitting the assumption that Φ is unital.

When the Hilbert space H is finite-dimensional, together with the discussion of Šemrl [14, Remark 4], one can easily check that a bijective linear map Φ on $\mathcal{B}(H)$ is spectrally bounded if and only if Φ has the form $\Phi(T) = c\varphi(T) + (\Phi(I) - cI)\operatorname{tr} T/n$ for every $T \in \mathcal{B}(H)$, where φ is either an automorphism or an anti-automorphism of $\mathcal{B}(H)$, c is a nonzero complex number and $n = \dim H$. So when we discuss the spectrally bounded linear maps, we may always assume that the Hilbert space H is infinite-dimensional.

Now we describe the main results of this note. In Section 2, we prove in a quite general framework that a spectrally bounded linear map from a unital complex Banach algebra onto a standard operator algebra acting on a complex Banach space preserves square-zero elements (see Theorem 2.1), which generalizes the first step of the proof of Šemrl's result in [14] mentioned above. This allows us to prove that a two-fold spectrally bounded surjective linear map from a unital complex Banach algebra onto a standard operator algebra acting on a complex Banach space is a homomorphism multiplied by a nonzero complex number (Theorem 2.2). Here a linear map $\Phi : \mathcal{A} \to \mathcal{B}$ is said to be *two-fold spectrally bounded* if $\Phi_2 : \mathcal{A} \otimes M_2(\mathbb{C}) \to \mathcal{B} \otimes M_2(\mathbb{C})$ defined by $\Phi_2((T_{ij})) = (\Phi(T_{ij}))$ is spectrally bounded. Section 3 concerns applications of Theorem 2.1 to the Hilbert space case. We show that a linear bijection Φ from $\mathcal{B}(H)$ onto $\mathcal{B}(K)$, where H and K are infinite-dimensional complex Hilbert spaces, is spectrally bounded if and only if there exist a nonzero complex number d and an invertible operator $A \in \mathcal{B}(H,K)$ such that either $\Phi(T) = dATA^{-1}$ for all $T \in \mathcal{B}(H)$ or $\Phi(T) = dAT^{\mathrm{tr}}A^{-1}$ for all $T \in \mathcal{B}(H)$ (Theorem 3.1). If the injectivity assumption on Φ is omitted, then either Φ has one of the above forms or Φ vanishes at every compact operator (Theorem 3.3). In particular, if H is separable, then Φ is surjective and spectrally bounded if and only if it is either an isomorphism or an anti-isomorphism multiplied by a nonzero scalar. From our arguments, we also answer affirmatively a question raised by Šemrl in [15] where he gave a characterization of unital bijections on $\mathcal{B}(H)$ which preserve square-zero operators and asked whether or not the "unital" assumption can be omitted.

Let us fix some notations. Let $\mathcal{B}(X)$ and $\mathcal{F}(X)$ be the sets of all bounded linear operators and of all finite rank bounded linear operators on the Banach space X, respectively. A subalgebra $\mathcal{B}$ in $\mathcal{B}(X)$ is called a *standard operator algebra* if $\mathcal{B}$ is closed and contains the identity operator and $\mathcal{F}(X)$. For $T \in \mathcal{B}(X)$, we denote by $\mathcal{R}(T)$ and $\ker(T)$ the range and kernel of T, respectively. If $T^2 = T$, we say T is an *idempotent operator*. Throughout this paper, we denote by $x \otimes f$ the bounded linear operator on X defined for any $x \in X$ and $f \in X^*$ by $(x \otimes f)(z) = \langle z, f\rangle x$ for every $z \in X$, where $\langle z, f\rangle$ is the value of f at z. Note that this operator is of rank one whenever both x and f are nonzero, and that every operator of rank one can be written in this form. By a *projection* we mean a self-adjoint idempotent in $\mathcal{B}(H)$, where H is a Hilbert space.

2. General results. In this section we consider the general case of spectrally bounded linear maps from a Banach algebra onto a standard operator algebra on a complex Banach space. The following is our main result.

THEOREM 2.1. *Let $\mathcal{A}$ be a unital complex Banach algebra and $\mathcal{B}$ be a standard operator algebra on a complex Banach space X. Assume that $\Phi : \mathcal{A} \to \mathcal{B}$ is a surjective linear map. If Φ is spectrally bounded, then Φ preserves square-zero elements.*

Proof. By the discussion in the introduction, we may assume that Φ is spectral radius nonincreasing. We divide the proof into two steps. We mention that the idea of the proof of Step 1 is the same as an idea used in [5].

STEP 1. *Let $A \in \mathcal{A}$ be such that $A^k = 0$ for some $k \geq 2$. If $B \in \mathcal{B}$ satisfies $BQ^iB = 0$, $i = 0, 1, \ldots, k-1$, where $Q = \Phi(A)$, then*

$$r(\lambda Q^k + BQ^{k-1} + QBQ^{k-2} + \ldots + Q^{k-1}B) = 0$$

for every complex number λ.

Let $B_1 = BQ^{k-1} + QBQ^{k-2} + \ldots + Q^{k-1}B$ and $B_2 = Q^k$. Since $r(B + \lambda Q)^k = r[(B + \lambda Q)^k]$ and $B^2 = BQB = BQ^2B = \ldots = BQ^{k-2}B = 0$, it follows that

$$r(B + \lambda Q)^k = |\lambda|^{k-1} r(B_1 + \lambda B_2).$$

As Φ is surjective, there exists $C \in \mathcal{A}$ such that $\Phi(C) = B$. Moreover, since Φ is spectral radius nonincreasing, we have

$$r(B + \lambda Q)^k = r(\Phi(C + \lambda A))^k \leq r(C + \lambda A)^k.$$

As $A^k = 0$, it follows that

$$r(C + \lambda A)^k = r[(C + \lambda A)^k] = r(C_0 + \lambda C_1 + \ldots + \lambda^{k-1} C_{k-1}),$$

where $C_0 = C^k$, $C_1 = C^{k-1}A + \ldots + AC^{k-1}$, $\ldots$, $C_{k-1} = A^{k-1}C + \ldots + CA^{k-1}$. Thus we have shown that

$$|\lambda|^{k-1} r(B_1 + \lambda B_2) \le r(C_0 + \lambda C_1 + \ldots + \lambda^{k-1} C_{k-1}).$$

Therefore, for any complex λ satisfying $|\lambda| \ge 1$, we have

$$\begin{aligned} r(B_1 + \lambda B_2) &\le |\lambda|^{-k+1} r(C_0 + \lambda C_1 + \ldots + \lambda^{k-1} C_{k-1}) \\ &\le \|C_0\| + \|C_1\| + \ldots + \|C_{k-1}\|. \end{aligned}$$

On the other hand, for every complex λ satisfying $|\lambda| \le 1$, one gets

$$r(B_1 + \lambda B_2) \le \|B_1 + \lambda B_2\| \le \|B_1\| + \|B_2\|.$$

Thus the function $\lambda \mapsto r(B_1 + \lambda B_2)$ is bounded on $\mathbb{C}$. As it is subharmonic, the Liouville theorem for subharmonic functions [3] shows that $r(B_1 + \lambda B_2) = r(B_1)$ for every complex λ. Observing that $B_1 B = 0$ it is easy to see that $B_1^2 = D_{k-2} B Q^{k-2} + D_{k-3} B Q^{k-3} + \ldots + D_0 B$ for some $D_i \in \mathcal{B}$ and so $B_1^2 Q B = B_1^2 B = 0$. This further implies that B_1^3 has the form $B_1^3 = E_{k-3} B Q^{k-3} + E_{k-2} B Q^{k-2} + \ldots + E_0 B$ for some $E_i \in \mathcal{B}$, and consequently, $B_1^3 Q^2 B = B_1^3 Q B = B_1^3 B = 0$. Repeating this procedure one shows that $B_1^{k+1} = 0$. So $r(B_1 + \lambda B_2) = 0$ for every complex λ, as desired.

Step 2. *If $A \in \mathcal{A}$ and $A^2 = 0$, then $\Phi(A)^2 = 0$.*

Based on Step 1, an argument similar to that in Step 5 of the proof of the main results in [5], where $\mathcal{B}(X)$ is replaced by $\mathcal{A}$ or $\mathcal{B}$, shows that, for every $k \ge 2$ and every $A \in \mathcal{A}$ with $A^k = 0$, we have $\Phi(A)^{2k-1} = 0$. So for every $A \in \mathcal{A}$ satisfying $A^2 = 0$, we have $\Phi(A)^3 = 0$. Assume that there exists some $A \in \mathcal{A}$ with $A^2 = 0$ but $\Phi(A)^2 \ne 0$. Let $\Phi(A) = Q$. It follows that $p(Q) \ne 0$ for every complex polynomial p of degree not exceeding 2. Kaplansky's theorem on local algebraic operators tells us that there is $u \in X$ such that the vectors u, Qu and Q^2u are linearly independent. Therefore, $u \notin M = \operatorname{span}\{Qu, Q^2u - u\}$. Hence there exists a linear functional $f \in X^*$ such that $f(u) = f(Q^2u) = 1$ and $f(Qu) = 0$. Let $B = (Q^2u - u) \otimes f$. Then a straightforward computation shows that $B^2 = BQB = 0$. So, by Step 1, we have $r(Q^2 + BQ + QB) = 0$. On the other hand, one can easily check

$$(Q^2 + BQ + QB)(u - Qu) = u - Qu,$$

so $r(Q^2 + BQ + QB) \ge 1$, which is a contradiction. Hence Φ preserves square-zero elements. ∎

Applying Theorem 2.1, we can prove the following results.

Theorem 2.2. *Let $\mathcal{A}$ be a unital complex Banach algebra and $\mathcal{B}$ be a standard operator algebra on a complex Banach space. Assume that $\Phi : \mathcal{A} \to \mathcal{B}$ is a surjective linear map. Then $\Phi_2 : \mathcal{A} \otimes M_2(\mathbb{C}) \to \mathcal{B} \otimes M_2(\mathbb{C})$ is spectrally bounded if and only if Φ is a homomorphism multiplied by a nonzero complex number.*

Proof. The sufficiency is clear. Now let us check the necessity. Assume that $\Phi_2 : \mathcal{A} \otimes M_2(\mathbb{C}) \to \mathcal{B} \otimes M_2(\mathbb{C})$ is spectrally bounded. By Theorem 2.1, Φ_2 preserves square-zero elements. Let $C \in \mathcal{A}$ be invertible. For any $A \in \mathcal{A}$, since

$$\begin{pmatrix} A & C \\ -C^{-1}A^2 & -C^{-1}AC \end{pmatrix}^2 = 0,$$

we have

$$\begin{pmatrix} \Phi(A) & \Phi(C) \\ -\Phi(C^{-1}A^2) & -\Phi(C^{-1}AC) \end{pmatrix}^2 = 0.$$

So

$$\Phi(A)^2 - \Phi(C)\Phi(C^{-1}A^2) = 0 \tag{2.1}$$

and

$$\Phi(A)\Phi(C) - \Phi(C)\Phi(C^{-1}AC) = 0. \tag{2.2}$$

Letting $C = I$ in (2.2) gives $\Phi(I)\Phi(A) = \Phi(A)\Phi(I)$. Since Φ is surjective and also spectral radius nonincreasing, and since $\mathcal{B}$ is a standard operator algebra, we must have $\Phi(I) = cI$ for some complex number c with $|c| \le 1$. We claim $c \neq 0$. Otherwise, taking $C = I$ in (2.1) gives $\Phi(A)^2 = 0$ for every $A \in \mathcal{A}$, which contradicts the surjectivity of Φ. So, without loss of generality, we may assume that $\Phi(I) = I$. Now it is clear that $\Phi(A^2) = \Phi(A)^2$, that is, Φ is a Jordan homomorphism. By taking $A = I$ in (2.1), we see that Φ also preserves invertibility. So (2.2) implies

$$\Phi(C^{-1}AC) = \Phi(C)^{-1}\Phi(A)\Phi(C).$$

Since Φ is Jordan, we have $\Phi(CAC) = \Phi(C)\Phi(A)\Phi(C)$ for any $A, C \in \mathcal{A}$. Hence

$$\Phi(AC^2) = \Phi(C^{-1}CACC) = \Phi(C)^{-1}\Phi(CAC)\Phi(C) = \Phi(A)\Phi(C)^2. \tag{2.3}$$

Next, choosing any nonzero $\lambda \in \mathbb{C}$ so that $\lambda - C$ is invertible, we get, replacing C by $\lambda - C$ in (2.3),

$$\Phi(AC) = \Phi(A)\Phi(C)$$

for all A and invertible C in $\mathcal{A}$. When C is not invertible, take $\lambda \in \mathbb{C}$ so that $\lambda - C$ is invertible; then $\Phi(A(\lambda - C)) = \Phi(A)(\lambda - \Phi(C))$, which again implies that $\Phi(AC) = \Phi(A)\Phi(C)$. Therefore, Φ is a homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. ∎

COROLLARY 2.3. *Let $\mathcal{A}$ be a unital complex Banach algebra and $\mathcal{B}$ be a standard operator algebra on a complex Banach space. Assume that $\Phi : \mathcal{A} \to \mathcal{B}$ is a bijective linear map. Then $\Phi_2 : \mathcal{A} \otimes M_2(\mathbb{C}) \to \mathcal{B} \otimes M_2(\mathbb{C})$ is spectrally bounded if and only if Φ is an isomorphism multiplied by a nonzero complex number.*

If $\mathcal{A}$ is a standard operator algebra, then Φ has a more concrete characterization.

COROLLARY 2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be two standard operator algebras on a complex Banach space X. Assume that $\Phi : \mathcal{A} \to \mathcal{B}$ is a bijective linear map. Then $\Phi_2 : \mathcal{A} \otimes M_2(\mathbb{C}) \to \mathcal{B} \otimes M_2(\mathbb{C})$ is spectrally bounded if and only if there exists a complex number c and an invertible operator $A \in \mathcal{B}(X)$ such that $\Phi(T) = cATA^{-1}$ for every operator $T \in \mathcal{A}$.*

Proof. It is clear that we need only check the necessity. Assume that Φ_2 is spectrally bounded. By Theorem 2.2, Φ is a scalar multiple of an isomorphism. Since every isomorphism between standard operator algebras is spatial, there exists an invertible operator $A \in \mathcal{B}(X)$ such that $\Phi(T) = cATA^{-1}$ for every $T \in \mathcal{A}$. ∎

3. Application to Hilbert space case. Let H and K be two infinite-dimensional complex Hilbert spaces. Applying the results of Section 2, we can get a complete classification of the spectrally bounded linear maps from $\mathcal{B}(H)$ onto $\mathcal{B}(K)$ without the assumption that $\Phi(I) = I$. The following theorem is our main result in this section.

THEOREM 3.1. *Let H and K be two infinite-dimensional Hilbert spaces. Assume that $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a bijective linear map. Then the following conditions are equivalent.*

(1) *Φ is spectrally bounded.*

(2) *There exists a nonzero complex number d and a bounded linear map $\Psi : \mathcal{B}(H) \to \mathcal{B}(K)$ preserving idempotents such that $\Phi = d\Psi$.*

(3) *Φ is a Jordan isomorphism multiplied by a nonzero complex number.*

(4) *Φ is either an isomorphism or an anti-isomorphism multiplied by a nonzero complex number.*

(5) *There exists a nonzero complex number d and an invertible operator $A \in \mathcal{B}(H, K)$ such that either $\Phi(T) = dATA^{-1}$ for all $T \in \mathcal{B}(H)$ or $\Phi(T) = dAT^{\mathrm{tr}}A^{-1}$ for all $T \in \mathcal{B}(H)$, where T^{tr} denotes the transpose of T relative to a fixed but arbitrary orthonormal basis of H.*

To prove this theorem, the following lemma is needed.

LEMMA 3.2. *Suppose that $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a surjective linear map which preserves square-zero operators. Then*

$$\Phi(R)^2\Phi(I) = \Phi(I)\Phi(R)^2 \tag{3.1}$$

for all idempotents $R \in \mathcal{B}(H)$.

Proof. Let H be a direct sum of two closed infinite-dimensional linear subspaces H_1 and H_2 (note that we do not assume H_1 and H_2 are orthogonal). Let P and $Q = I - P$ be the idempotents corresponding to this direct sum decomposition, that is, $\mathcal{R}(P) = H_1$ and $\ker(P) = H_2$. Assume that operators $A, B \in B(H)$ satisfy $PAP = A$ and $QBQ = B$. It follows from [12, Theorem 2] that A and B can be written as sums of five operators with square zero. Say $A = A_1+A_2+A_3+A_4+A_5$ and $B = B_1+B_2+B_3+B_4+B_5$, with $PA_iP = A_i$ and $QB_iQ = B_i$ $(i = 1,\dots,5)$. Clearly, $(A_i + B_j)^2 = 0$. Consequently, we have $\Phi(A_i)\Phi(B_j) + \Phi(B_j)\Phi(A_i) = 0$, which further yields

$$\Phi(A)\Phi(B) + \Phi(B)\Phi(A) = 0.$$

In other words, we have

$$\Phi(PAP)\Phi((I-P)B(I-P)) + \Phi((I-P)B(I-P))\Phi(PAP) = 0 \tag{3.2}$$

for every $A, B \in \mathcal{B}(H)$.

We claim that

$$\Phi(R)\Phi(I) + \Phi(I)\Phi(R) = 2\Phi(R)^2 \quad \text{for all idempotents } R. \tag{3.3}$$

If $R \in \mathcal{B}(H)$ is an idempotent such that both its range and kernel are infinite-dimensional, then by (3.2) with $A = B = I$ we get

$$\Phi(R)\Phi(I-R) + \Phi(I-R)\Phi(R) = 0,$$

which implies (3.3) immediately.

If R has a finite-dimensional image, take an idempotent P_1 with both range and kernel infinite-dimensional such that $P_1 \perp R$. Then

$$\begin{aligned}&\Phi(P_1)\Phi(R) + \Phi(R)\Phi(P_1)\\ &= \Phi(P_1P_1P_1)\Phi((I-P_1)R(I-P_1)) + \Phi((I-P_1)R(I-P_1))\Phi(P_1P_1P_1) = 0.\end{aligned}$$

Therefore,

$$\Phi(P_1+R)^2 = \Phi(P_1)^2 + \Phi(R)^2$$

and

$$\begin{aligned}\Phi(R)\Phi(I) &+ \Phi(I)\Phi(R)\\ &= \Phi(P_1+R-P_1)\Phi(I) + \Phi(I)\Phi(P_1+R-P_1)\\ &= \Phi(P_1+R)\Phi(I) + \Phi(I)\Phi(P_1+R) - \Phi(P_1)\Phi(I) - \Phi(I)\Phi(P_1)\\ &= 2\Phi(P_1+R)^2 - 2\Phi(P_1)^2 = 2\Phi(R)^2.\end{aligned}$$

If R has finite-dimensional kernel, then

$$\Phi(I-R)\Phi(I) + \Phi(I)\Phi(I-R) = 2\Phi(I-R)^2,$$

and hence

$$\Phi(R)\Phi(I) + \Phi(I)\Phi(R) = 2\Phi(R)^2,$$

completing the proof of (3.3). Thus, for any idempotent $R \in \mathcal{B}(H)$, we have

$$\Phi(R)^2\Phi(I) + \Phi(R)\Phi(I)\Phi(R) = 2\Phi(R)^3$$

and

$$\Phi(I)\Phi(R)^2 + \Phi(R)\Phi(I)\Phi(R) = 2\Phi(R)^3,$$

which implies that $\Phi(R)^2\Phi(I) = \Phi(I)\Phi(R)^2$. ■

THEOREM 3.3. *Let H and K be two infinite-dimensional Hilbert spaces. Assume that $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a surjective linear map. If Φ is spectrally bounded, then either $\Phi(T) = 0$ for all compact operators $T \in \mathcal{B}(H)$ or Φ is injective. In the last case, there exists a nonzero complex number d and an invertible operator $A \in \mathcal{B}(H,K)$ such that either $\Phi(T) = dATA^{-1}$ for all $T \in \mathcal{B}(H)$ or $\Phi(T) = dAT^{\mathrm{tr}}A^{-1}$ for all $T \in \mathcal{B}(H)$, where T^{tr} denotes the transpose of T relative to a fixed but arbitrary orthonormal basis of H.*

Proof. Assume that Φ is spectrally bounded. By Theorem 2.1, Φ preserves square-zero operators. We may assume that Φ is spectral radius nonincreasing by the discussion in the introduction.

CLAIM 1. *For any orthogonal idempotents P_1 and P_2 in $\mathcal{B}(H)$, we have*

$$\Phi(P_1 + P_2)^2 = \Phi(P_1)^2 + \Phi(P_2)^2. \tag{3.4}$$

It is easily seen from the proof of Lemma 3.2 that (3.4) is true if $\dim \mathcal{R}(P_i) = \dim \mathcal{R}(I - P_i) = \infty$ for $i = 1$ or 2. There are four cases left to check:

(i) $\dim \mathcal{R}(P_1) < \infty$, $\dim \mathcal{R}(P_2) < \infty$;
(ii) $\dim \mathcal{R}(P_1) < \infty$, $\dim \mathcal{R}(I - P_2) < \infty$;
(iii) $\dim \mathcal{R}(I - P_1) < \infty$, $\dim \mathcal{R}(P_2) < \infty$;
(iv) $\dim \mathcal{R}(I - P_1) < \infty$, $\dim \mathcal{R}(I - P_2) < \infty$.

In case (i), we can find an idempotent P_3 orthogonal to $P_1 + P_2$ with $\dim \mathcal{R}(P_3) = \dim \mathcal{R}(I - P_3) = \infty$. Thus

$$\begin{aligned}\Phi(P_1 + P_2)^2 + \Phi(P_3)^2 = \Phi(P_1 + P_2 + P_3)^2 &= \Phi(P_1)^2 + \Phi(P_2 + P_3)^2 \\ &= \Phi(P_1)^2 + \Phi(P_2)^2 + \Phi(P_3)^2,\end{aligned}$$

so $\Phi(P_1 + P_2)^2 = \Phi(P_1)^2 + \Phi(P_2)^2$. The remaining cases are similar.

CLAIM 2. *$\Phi(I) = cI$ for some nonzero complex number c.*

We first prove that Φ is bounded. Since Φ is a spectral radius nonincreasing surjection and $\mathcal{B}(K)$ is semisimple, it follows from Aupetit [2] that Φ is bounded.

Let $C \in \mathcal{B}(H)$ be a linear combination of orthogonal projections, that is, $C = \sum_{i=1}^n \alpha_i P_i$, where $\{P_i\}_{i=1}^n$ is an orthogonal set of projections. Then $\Phi(C)^2 = \sum_{i=1}^n \alpha_i^2 \Phi(P_i)^2$ by (3.4). It follows that $\Phi(C)^2\Phi(I) = \Phi(I)\Phi(C)^2$ by Lemma 3.2. Now suppose $D \in \mathcal{B}(H)$ is self-adjoint; then D is a limit of linear combinations of orthogonal projections. Since Φ is bounded and linear, we have $\Phi(D)^2\Phi(I) = \Phi(I)\Phi(D)^2$. Let $C, D \in \mathcal{B}(H)$ be self-adjoint. Then $\Phi(C + D)^2\Phi(I) = \Phi(I)\Phi(C + D)^2$, which yields

$$(\Phi(C)\Phi(D) + \Phi(D)\Phi(C))\Phi(I) = \Phi(I)(\Phi(C)\Phi(D) + \Phi(D)\Phi(C)).$$

For any $T \in \mathcal{B}(H)$, there exist self-adjoint operators C and D such that $T = C + iD$. Because

$$\begin{aligned}\Phi(C+iD)^2\Phi(I) \\ &= \Phi(C)^2\Phi(I) - \Phi(D)^2\Phi(I) + i(\Phi(C)\Phi(D) + \Phi(D)\Phi(C))\Phi(I) \\ &= \Phi(I)\Phi(C+iD)^2,\end{aligned}$$

we see that

$$\Phi(T)^2\Phi(I) = \Phi(I)\Phi(T)^2$$

for every $T \in \mathcal{B}(H)$, and consequently $S^2\Phi(I) = \Phi(I)S^2$ for all $S \in \mathcal{B}(K)$ by the surjectivity of Φ. This implies that $\Phi(I) = cI$ for some $c \in \mathbb{C}$. Furthermore, $c \neq 0$. Indeed, if $c = 0$, then $\Phi(I) = 0$. Thus by (3.3) we have $\Phi(R)^2 = 0$ for all idempotents R, which implies that $\Phi(C)^2 = 0$ for all self-adjoint operators C by the boundedness of Φ. Hence $\Phi(T)^2 = 0$ for all $T \in \mathcal{B}(H)$, which contradicts the surjectivity of Φ.

Therefore, with no loss of generality, we may assume $\Phi(I) = I$. Thus, by (3.3) again, Φ is idempotent preserving. Let $A \in \mathcal{B}(H)$ be self-adjoint and $A = \sum_{i=1}^n t_i P_i$ where $t_i \in \mathbb{R}$ and P_i are pairwise orthogonal projections. Since Φ maps mutually orthogonal projections to mutually orthogonal idempotents, $\Phi(A^2) = \Phi(A)^2$. Now, because the set of self-adjoint elements that are finite real linear combinations of orthogonal projections is dense in the set of all self-adjoint elements in $\mathcal{B}(H)$, we see that $\Phi(A^2) = \Phi(A)^2$ for all self-adjoint A in $\mathcal{B}(H)$ by the boundedness of Φ. Replacing A by $A+B$ where both A and B are self-adjoint, we get $\Phi(AB+BA) = \Phi(A)\Phi(B)+\Phi(B)\Phi(A)$. Since every $T \in \mathcal{B}(H)$ can be written in the form $T = A + iB$ with A and B self-adjoint, the last relations imply that $\Phi(T^2) = \Phi(T)^2$. So Φ is Jordan.

Since $\mathcal{B}(K)$ is a prime ring, by [9, Thm. 3.1], Φ is either a homomorphism or an anti-homomorphism. If Φ is not injective, then $\ker\Phi$ is a nonzero closed ideal in $\mathcal{B}(H)$. Since the smallest nontrivial closed ideal of $\mathcal{B}(H)$ is the ideal $\mathcal{K}(H)$ of compact operators, we have $\ker\Phi \supseteq \mathcal{K}(H)$. Hence $\Phi(T) = 0$ for all compact operators $T \in \mathcal{B}(H)$. If Φ is injective, then Φ is either an isomorphism or an anti-isomorphism. Since every isomorphism or anti-isomorphism from $\mathcal{B}(H)$ onto $\mathcal{B}(K)$ is spatial, there exists an invertible operator $A \in \mathcal{B}(H,K)$ such that $\Phi(T) = ATA^{-1}$ for all $T \in \mathcal{B}(H)$ or $\Phi(T) = AT^{\mathrm{tr}}A^{-1}$ for all $T \in \mathcal{B}(H)$, where T^{tr} denotes the transpose of T relative to an orthonormal basis in H. ∎

REMARK 3.4. Claim 2 in the proof of Theorem 3.3 also answers affirmatively a question due to Šemrl [15], who showed that a unital linear bijection on $\mathcal{B}(H)$ is square-zero preserving if and only if it is either an automorphism or an anti-automorphism and he asked whether or not the unital assumption may be omitted. So we find that a linear bijection on

$\mathcal{B}(H)$ is square-zero preserving if and only if it is either an automorphism or an anti-automorphism multiplied by a nonzero scalar.

Proof of Theorem 3.1. (2)⇒(3)⇒(4)⇒(5)⇒(1) are obvious. (1)⇒(2) follows from Theorem 3.3. ■

Note that if the Hilbert space H is separable, then Φ is injective in Theorem 3.3. In fact, if Φ is not injective, then $\mathcal{B}(H)$ is isomorphic to the quotient algebra $\mathcal{B}(H)/\mathcal{K}(H)$, which contradicts the fact that $\mathcal{B}(H)/\mathcal{K}(H)$ is simple. So we have the following corollary which generalizes [14, Thm. 2] by omitting the unital assumption.

COROLLARY 3.5. *Let H and K be two infinite-dimensional Hilbert spaces with H separable. Assume that $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a surjective linear map. Then Φ is spectrally bounded if and only if there exists a nonzero complex number d and an invertible operator $A \in \mathcal{B}(H,K)$ such that either $\Phi(T) = dATA^{-1}$ for all $T \in \mathcal{B}(H)$ or $\Phi(T) = dAT^{\mathrm{tr}}A^{-1}$ for all $T \in \mathcal{B}(H)$, where T^{tr} denotes the transpose of T relative to a fixed but arbitrary orthonormal basis of H.*

REMARK 3.6. Our proofs still work if $\mathcal{B}(K)$ is replaced by a standard operator algebra acting on a complex Banach space. So the results in this section also hold true when Φ is a linear map from $\mathcal{B}(H)$ onto a standard operator algebra $\mathcal{B}$.

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