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Perturbations of operators similar to contractions and the commutator equation

by

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Abstract. Let T and V be two Hilbert space contractions and let X be a linear bounded operator. It was proved by C. Foiaş and J. P. Williams that in certain cases the operator block matrix $R(X;T,V)$ (equation (1.1) below) is similar to a contraction if and only if the commutator equation $X = TZ - ZV$ has a bounded solution Z. We characterize here the similarity to contractions of some operator matrices $R(X;T,V)$ in terms of growth conditions or of perturbations of $R(0;T,V) = T \oplus V$.

1. Introduction. A bounded linear operator $T \in \mathcal{B}(\mathcal{H})$, acting on a Hilbert space $\mathcal{H}$, is said to be *polynomially bounded* if there exists a constant M such that the inequality

$$\|p(T)\| \le M\|p\|_\infty =: M \sup\{|p(z)| : |z| = 1\}$$

holds for all polynomials $p \in \mathbb{C}[z]$. It is said to be *power bounded* if the same inequality holds for all monomials $p_n(z) = z^n$.

The von Neumann inequality implies that every operator T similar to a contraction is polynomially bounded. Recall that T is said to be *similar to a contraction* if there exists an invertible operator $L \in \mathcal{B}(\mathcal{H})$ such that $\|L^{-1}TL\| \le 1$.

There are power bounded operators which are not polynomially bounded (Foguel [F]; see also [Da, Pe, Bo]), as well as polynomially bounded operators which are not similar to a contraction (Pisier [Pi]; see also [DaPa]). Both Foguel's and Pisier's counterexamples are operators of the following type:

$$R(X;V^*,V) = \begin{bmatrix} V^* & X \\ 0 & V \end{bmatrix} \in \mathcal{B}(\mathcal{K} \oplus \mathcal{K}),$$

where $V \in \mathcal{B}(\mathcal{K})$ is a pure isometry (i.e. a unilateral shift), V^* is its adjoint and X is an operator in $\mathcal{B}(\mathcal{K})$. In Foguel's counterexample, $\mathcal{K} = \ell^2$ and X is a suitable diagonal projection onto a subspace of ℓ^2, while in Pisier's counterexample, $\mathcal{K} = \ell^2(\mathcal{H})$, V is the unilateral shift of infinite multiplicity

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$\dim \mathcal{H}$ and X is a suitable Hankel operator with suitable operator-valued entries.

Let $T \in \mathcal{B}(\mathcal{K})$, $V \in \mathcal{B}(\mathcal{H})$ and $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be three Hilbert space operators. In this paper we will consider operators of the form

$$R(X) = R(X; T, V) = \begin{bmatrix} T & X \\ 0 & V \end{bmatrix} \in \mathcal{B}(\mathcal{K} \oplus \mathcal{H}). \tag{1.1}$$

Suppose that T is a coisometry (i.e. its adjoint T^* is an isometry) and V is a contraction or that T is a contraction and V is an isometry. It was proved by C. Foiaş and J. P. Williams [FW] (cf. [CCFW, Pa2, Cl]) that in these cases $R(X)$ is similar to a contraction if and only if the (generalized) commutator equation

$$X = TZ - ZV \tag{1.2}$$

has a bounded solution $Z \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. This implies that $R(X)$ is similar to a contraction if and only if $R(X)$ is similar to $R(0) = T \oplus V$, if and only if the commutator equation has a bounded solution. This follows from the matrix identity

$$\begin{bmatrix} I & -Z \\ 0 & I \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix} = \begin{bmatrix} T & TZ - ZV \\ 0 & V \end{bmatrix}.$$

Here I denotes the identity operator, possibly on different spaces.

Note that, given T and V, the equivalence between the similarity of $R(X)$ to $R(0)$ and the solvability of the commutator equation (1.2) holds for all operators T and V if $\mathcal{H}$ and $\mathcal{K}$ are finite-dimensional (Roth's theorem [Ro]). It also holds for some, but not all, pairs (T, V) in the infinite-dimensional situation [Ros], [BhRo]. We also note that equation (1.2) is sometimes called in the literature the Sylvester equation, while the operator $Z \mapsto TZ - ZV$ is concurrently called the Rosenblum operator, an elementary operator, a generalized commutator or a generalized derivation. We refer to the survey paper [BhRo] by Bhatia and Rosenthal for more information concerning the commutator equation (1.2).

The aim of the present paper is to give different characterizations of operators $R(X)$ similar to contractions. In all cases the Foiaş–Williams theorem will be applicable, and so these results can be viewed as criteria for the solvability of the commutator equation. All characterizations will be applicable to the case when T is a coisometry and V is an isometry, not necessarily acting on the same space.

In the first such result (Theorem 2.4), T is supposed to be a right invertible contraction and V an isometry or T a coisometry and V a left invertible contraction. The result says that in both cases $R(X)$ is similar to a contraction if and only if X can be decomposed as the sum of two bounded operators satisfying growth conditions or product identities.

A different characterization for operators $R(X;T,V)$, with T a coisometry and V a weighted unilateral shift of arbitrary multiplicity, is given in Theorem 5.1. This time the necessary and sufficient condition is that $R(X)$ is power bounded and it can be written as a zero-product perturbation by a nilpotent of an operator which is near (in a certain sense) to $R(0)$. Basically the same characterization holds (Theorem 5.2) if T is a right invertible contraction and V is an isometry. These results are related to a previous result of Apostol [A2] concerning the commutator equation. Note also that other variations are possible.

Starting with work of K. O. Friedrichs (cf. [DS, Part III, XX.2.2]), several results of the type "a perturbation of an operator C is similar to C" exist in the literature. The characterizations described above can be interpreted as results of the following type: "perturbations of operators near $R(0) = T \oplus V$ are similar to $R(0)$".

The paper is organized as follows. In the next section we study the commutator equation and Theorem 2.4 is proved. The zero-product perturbations are introduced in Section 3 and the stability of the class of operators similar to contractions under such perturbations is studied. In Section 4 we introduce the notion of β-quadratically near operators modulo subspaces, extending [Ba, Definition 2.5]. This notion is used in the main results of Section 5.

2. Growth conditions and the commutator equation. The proofs of the following result, and that of Theorem 2.3 below, are similar to the proof of [BaPa, Theorem 4.1]. We refer to this paper for more references and to [BhRo, p. 9] for the significance of the condition (2.1).

2.1. THEOREM. *Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry, let $T \in \mathcal{B}(\mathcal{K})$ be a power bounded operator and let $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$. Then the commutator equation*

$$X = TZ - ZV$$

has a bounded solution $Z \in \mathcal{B}(\mathcal{H},\mathcal{K})$ satisfying $Z(I-VV^*) = 0$ if and only if

$$\sup_n \Big\| \sum_{j=0}^{n} T^j X V^{*j+1} \Big\| < \infty. \tag{2.1}$$

Proof. Suppose the equation $X = TZ - ZV$ has a bounded solution $Z \in \mathcal{B}(\mathcal{H},\mathcal{K})$ satisfying $Z = ZVV^*$. Then

$$\begin{aligned}\sum_{j=0}^{n} T^j X V^{*j+1} &= \sum_{j=0}^{n} T^j (TZ - ZV) V^{*j+1} \\ &= \sum_{j=0}^{n} T^{j+1} Z V^{*j+1} - \sum_{j=0}^{n} T^j Z V V^* V^{*j}\end{aligned}$$

$$= \sum_{j=0}^{n} T^{j+1} Z V^{*j+1} - \sum_{j=0}^{n} T^{j} Z V^{*j}$$
$$= T^{n+1} Z V^{*n+1} - Z.$$

Therefore, $\|T^{n+1}\| \le M$ implies

$$\sup_n \Big\| \sum_{j=0}^{n} T^j X V^{*j+1} \Big\| \le (M+1)\|Z\|.$$

Suppose now that (2.1) holds. Let $\mathcal{L}$ be a Banach limit [DS, Part I, p. 73], that is, a bounded linear functional on $\ell_\infty(\mathbb{C})$ such that $1 = \mathcal{L}(\mathbf{1}) = \|\mathcal{L}\|$ and $\mathcal{L}((x_{n+1})_{n\ge 0}) = \mathcal{L}((x_n)_{n\ge 0})$ for every $(x_n)_{n\ge 0} \in \ell_\infty(\mathbb{C})$. Here $\mathbf{1} = (1, 1, \ldots)$.

Consider the linear operator $Z : \mathcal{H} \to \mathcal{K}$ given by

$$\langle Zh, k\rangle = -\mathcal{L}\Big(\Big\langle \sum_{j=0}^{n} T^j X V^{*j+1} h, k \Big\rangle\Big).$$

Then (2.1) shows that Z is well defined and bounded. We have

$$\begin{aligned}\langle (TZ - ZV)h, k\rangle &= \langle Zh, T^* k\rangle - \langle ZVh, k\rangle \\ &= \mathcal{L}\Big(\Big\langle \sum_{j=0}^{n} T^j X V^{*j} h, k\Big\rangle - \Big\langle \sum_{j=0}^{n} T^{j+1} X V^{*j+1} h, k\Big\rangle\Big) \\ &= \langle Xh, k\rangle - \mathcal{L}(\langle T^{n+1} X V^{*n+1} h, k\rangle).\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle T^{n+1} X V^{*n+1} h, k\rangle &= \langle T^{n+1} X V^{*n+2} V h, k\rangle \\ &= \Big\langle \sum_{j=0}^{n+1} T^j X V^{*j+1} V h, k\Big\rangle - \Big\langle \sum_{j=0}^{n} T^j X V^{*j+1} V h, k\Big\rangle.\end{aligned}$$

Therefore $\mathcal{L}(\langle T^{n+1} X V^{*n+1} h, k\rangle) = 0$ and thus $X = TZ - ZV$. We also have

$$\langle ZVV^* h, k\rangle = -\mathcal{L}\Big(\Big\langle \sum_{j=0}^{n} T^j X V^{*j+1} V V^* h, k\Big\rangle\Big) = \langle Zh, k\rangle. \ \blacksquare$$

We obtain the following known result (cf. [FW, Wu, CMS, CCFW]).

2.2. Corollary. *Let $U \in \mathcal{B}(\mathcal{H})$ be unitary, let $T \in \mathcal{B}(\mathcal{K})$ be a power bounded operator and let $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then the operator*

$$R(X) = R(X; T, U) = \begin{bmatrix} T & X \\ 0 & U \end{bmatrix} \in \mathcal{B}(\mathcal{K} \oplus \mathcal{H})$$

is similar to $R(0) = T \oplus U$ if and only if $R(X)$ is power bounded.

Proof. If $R(X)$ is similar to $R(0) = T \oplus U$, then $R(X)$ is power bounded since $R(0)$ is.

Suppose that $R(X)$ is power bounded. We have

$$R^n = \begin{bmatrix} T^n & \sum_{j=0}^{n-1} T^j X U^{n-j-1} \\ 0 & U^n \end{bmatrix}$$

and thus

$$\Big\| \sum_{j=0}^{n} T^j X U^{n-j} \Big\| \le M$$

for a suitable positive constant M. This yields

$$\begin{aligned} \Big\| \sum_{j=0}^{n} T^j X U^{*j+1} \Big\| &= \Big\| \sum_{j=0}^{n} T^j X U^{n-j} U^{*n-j} U^{*j+1} \Big\| \\ &\le \Big\| \sum_{j=0}^{n} T^j X U^{n-j} \Big\| \cdot \|U^{*n+1}\| \le M. \end{aligned}$$

By Theorem 2.1, there is a bounded operator Z such that $X = TZ - ZU$. Therefore $R(X)$ is similar to $R(0) = T \oplus U$. ∎

It follows from [Ca] that if U is a coisometry and T is a contraction, then $R(X)$ is similar to a contraction if and only if $R(X)$ is power bounded. However, $R(X)$ is not necessarily similar to $R(0)$ as [Wu, Example 2.15] shows.

The following result is a counterpart of Theorem 2.1 (see also [BaPa, Th. 4.1]). Its proof will be omitted.

2.3. Theorem. *Let $T \in \mathcal{B}(\mathcal{K})$ be a coisometry (i.e. $TT^* = I$), let $V \in \mathcal{B}(\mathcal{H})$ be a power bounded operator and let $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then the commutator equation*

$$X = TZ - ZV$$

has a bounded solution $Z \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ with $(I - T^*T)Z = 0$ if and only if

$$\sup_n \Big\| \sum_{j=0}^{n} T^{*j+1} X V^j \Big\| < \infty. \tag{2.2}$$

The following result is the first characterization of operators $R(X)$ similar to contractions.

2.4. Theorem (growth condition). (a) *Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry, let $T \in \mathcal{B}(\mathcal{K})$ be a right invertible contraction and let $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then the operator*

$$R(X) = \begin{bmatrix} T & X \\ 0 & V \end{bmatrix} \in \mathcal{B}(\mathcal{K} \oplus \mathcal{H})$$

is similar to a contraction if and only if $X = A + F$, with $A, F \in \mathcal{B}(\mathcal{H}, \mathcal{K})$

satisfying

$$\sup_n \Big\| \sum_{j=0}^{n} T^j A V^{*j+1} \Big\| < \infty \quad \textit{and} \quad FV = 0.$$

(b) *Let $T \in \mathcal{B}(\mathcal{K})$ be a coisometry, let $V \in \mathcal{B}(\mathcal{H})$ be a left invertible contraction and let $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then the operator*

$$R(X) = \begin{bmatrix} T & X \\ 0 & V \end{bmatrix} \in \mathcal{B}(\mathcal{K} \oplus \mathcal{H})$$

is similar to a contraction if and only if $X = A + F$ with $A, F \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ satisfying

$$\sup_n \Big\| \sum_{j=0}^{n} T^{*j+1} A V^j \Big\| < \infty \quad \textit{and} \quad TF = 0.$$

Proof. We give the proof only for the second part. By Foiaş–Williams' [CCFW] theorem, $R(X)$ is similar to a contraction if and only if the commutator equation $X = TZ - ZV$ has a bounded solution $Z \in \mathcal{B}(\mathcal{H}, \mathcal{K})$.

Suppose the commutator equation has a bounded solution $Z \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Set

$$F = -(I - T^*T)ZV \quad \text{and} \quad A = X - F = TZ - T^*TZV.$$

Then $X = A + F$, $TF = 0$ and $A = TD - DV$ with $D = T^*TZ$ satisfying $(I - T^*T)D = 0$. Apply Theorem 2.3.

Suppose now that X has a decomposition $X = A + F$ as required. By Theorem 2.3 there exists a bounded operator D such that $A = TD - DV$ and $(I - TT^*)D = 0$. Let L be a left inverse for V. Set $Z = D - FL$. Then

$$TZ - ZV = TD - DV - TFL + FLV = A + F = X. \ \blacksquare$$

We also mention the following result.

2.5. Proposition. *Let $V \in \mathcal{B}(\mathcal{H})$ be a unilateral shift (i.e. a pure isometry). Let $X \in \mathcal{B}(\mathcal{H})$ be an operator such that*

$$\sup_n \Big\| \sum_{j=0}^{n} (V^{j+1} X V^j - V^{*j} X V^{*j+1}) \Big\| < \infty.$$

Then the commutator equation $X = V^*Z - ZV$ has a bounded solution $Z \in \mathcal{B}(\mathcal{H})$.

Proof. Consider $Z \in \mathcal{B}(\mathcal{H})$ given by

$$\langle Zh, k \rangle = \mathcal{L}\bigg(\Big\langle \sum_{j=0}^{n} \frac{V^{j+1} X V^j - V^{*j} X V^{*j+1}}{2} h, k \Big\rangle \bigg).$$

Then

$$\langle (V^*Z - ZV)h, k \rangle = \langle Xh, k \rangle - \tfrac{1}{2} \mathcal{L}(\langle (V^{n+1} X V^{n+1} + V^{*n+1} X V^{*n+1})h, k \rangle).$$

We have

$$|\langle V^{n+1}XV^{n+1}h,k\rangle| = |\langle XV^{n+1}h, V^{*n+1}k\rangle| \leq \|X\|\cdot\|h\|\cdot\|V^{*n+1}k\|$$

and the last term tends to zero since V^{*n} tends strongly to 0 as n tends to ∞. We conclude that Z is a bounded solution of the commutator equation. ■

3. Zero-product perturbations

3.1. DEFINITION. The operator $T \in \mathcal{B}(\mathcal{H})$ is said to be a *zero-product perturbation* of $C \in \mathcal{B}(\mathcal{H})$ by $E \in \mathcal{B}(\mathcal{H})$ if $T = C + E$ and $EC = 0$.

The following result shows that the class of operators similar to contractions is stable under zero-product perturbations by operators of spectral radius smaller than one, in particular by nilpotent operators.

3.2. THEOREM. *A zero-product perturbation of an operator similar to a contraction by an operator of spectral radius smaller than* 1 *is similar to a contraction.*

Proof. Let $C \in \mathcal{B}(\mathcal{H})$ be an operator similar to a contraction and thus completely polynomially bounded [Pa1]. This means that there exists a universal constant $K > 0$ such that

$$\|P(C)\| \leq K\|P\|_\infty = \sup_{|z|=1} \|P(z)\|_{M_p(\mathbb{C})} \tag{3.1}$$

for each polynomial P with matrix coefficients in $M_p(\mathbb{C})$ for any p. Recall that $P(C)$ is identified with an operator acting on the direct sum of p copies of $\mathcal{H}$ in a natural way.

Let $E \in \mathcal{B}(\mathcal{H})$ be an operator of spectral radius $r(E)$ smaller than 1 such that $EC = 0$. Set $T = C + E$. We have

$$T^2 = (C+E)^2 = C^2 + CE + E^2$$

and, by recurrence,

$$T^n = C^n + C^{n-1}E + \ldots + CE^{n-1} + E^n \tag{3.2}$$

for every n.

Let $p \geq 1$ be fixed. Let

$$P(z) = \sum_{j=0}^{d} A_j z^j$$

be a polynomial of degree d with $p \times p$ matrix coefficients $A_j \in M_p(\mathbb{C})$.

Define $P_{(0)} = P$,

$$P_{(1)}(z) = \sum_{j=1}^{d} A_j z^{j-1} = \frac{P(z) - P(0)}{z}$$

and, recursively,

$$P_{(n)}(z) = \frac{P_{n-1}(z) - P_{n-1}(0)}{z}.$$

The representation (3.2) implies

(3.3) $P(T) = P(C) + P_{(1)}(C)E + P_{(2)}(C)E^2 + \ldots + P_{(d-1)}(C)E^{d-1}.$

We can estimate the norm

$$\|P_{(n)}\|_\infty = \sup_{|z|=1} \|P_{(n)}(z)\|_{M_p(\mathbb{C})}$$

as follows:

$$\|P_{(n)}\|_\infty = \|z^n P_{(n)}\|_\infty = \|P - D_{n-1} * P\|_\infty.$$

Here

$$D_n(t) = \sum_{|j|\le n} e^{ijt} = \frac{\sin((n+1/2)t)}{\sin(t/2)}$$

is the Dirichlet kernel and the convolution $D_{n-1} * P$ with the polynomial P with matrix coefficients has an obvious meaning.

The L^1-norm of the Dirichlet kernel grows like $\log n$ [Z, II.(12.1)]. Therefore there exists a positive constant A such that

$$\|P_{(n)}\|_\infty \le A \log(n+2) \|P\|_\infty \tag{3.4}$$

for every $n \ge 0$ and every p.

Combining now equations (3.1), (3.3) and (3.4), we obtain

$$\|P(T)\|_\infty \le AK\Big(\sum_{n=0}^{\infty} \log(n+2)\|E^n\|\Big)\|P\|_\infty.$$

Since $r(E) < r < 1$, there exists a constant C such that

$$\|E^n\| \le Cr^n$$

for all n and thus

$$\sum_{n=0}^{\infty} \log(n+2)\|E^n\| \le C\sum_{n=0}^{\infty} r^n \log(n+2)$$

is convergent. Therefore T is completely polynomially bounded and thus similar to a contraction by Paulsen's [Pa1] criterion. ∎

3.3. Remarks. (a) The series $\sum_{n=0}^{\infty} \log(n+2)\|E^n\|$ is convergent if and only if $r(E) < 1$. Indeed, the convergence of the series with the logarithm implies the convergence of $\sum_{n=0}^{\infty} \|E^n\|$ and thus $r(E) < 1$.

(b) For $C = 0$ we obtain the classical theorem of Rota [R] stating that an operator T with $r(T) < 1$ is similar to a contraction.

(c) By applying Theorem 3.2 to $T^* = C^* + E^*$ we obtain a similar statement for perturbations satisfying the reversed zero-product condition $CE = 0$.

In what follows we will only use the following corollary for zero-product perturbations by nilpotents of order two.

3.4. Corollary. *Let $C \in \mathcal{B}(\mathcal{H})$ be an operator similar to a contraction and let $E \in \mathcal{B}(\mathcal{H})$ be a nilpotent operator such that $EC = 0$. Then $T = C+E$ is similar to a contraction.*

3.5. Remark. The zero-product condition $EC = 0$ (or $CE = 0$) is necessary in Theorem 3.2 and in Corollary 3.4. Indeed, if $\mathcal{H}$ is the Euclidean space $\mathbb{C}^2$,

$$C = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

then $E^2 = 0$ and $EC = CE = E$. However,

$$T = C + E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

is not similar to a contraction. Indeed,

$$T^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

is not power bounded.

Moreover, there exists a contraction C and an operator E such that $E^2 = 0$, $EC = CE$, $T = C + E$ is a polynomially bounded operator but T is not similar to a contraction.

For a counterexample, let

$$T = \begin{bmatrix} S^* & \Gamma \\ 0 & S \end{bmatrix} \in \mathcal{B}(\ell^2(\mathcal{H}) \oplus \ell^2(\mathcal{H}))$$

be the polynomially bounded operator not similar to a contraction constructed by G. Pisier [Pi]. Here S is the unilateral shift of (infinite) multiplicity $\dim \mathcal{H}$ and Γ is a suitable [Pi] operator-valued Hankel operator, thus satisfying $S^*\Gamma = \Gamma S$. Then T is the sum of the contraction

$$C = \begin{bmatrix} S^* & 0 \\ 0 & S \end{bmatrix}$$

and the operator

$$E = \begin{bmatrix} 0 & \Gamma \\ 0 & 0 \end{bmatrix}$$

satisfying $E^2 = 0$ and $EC = CE$.

4. Quadratically near operators modulo subspaces. The following definition was introduced in [Ba, Definition 2.5].

4.1. Definition. Let $\beta : \mathbb{Z}_+ \to \mathbb{R}_+^*$. Two operators $T \in \mathcal{B}(\mathcal{H})$ and $C \in \mathcal{B}(\mathcal{H})$ are said to be *β-quadratically near* if

$$s := \left[\sup_{N \geq 0} \left\| \sum_{n=0}^{N} \frac{1}{\beta(n)^2} (T^n - C^n)(T^n - C^n)^* \right\| \right]^{1/2} < \infty.$$

T and C are simply called *quadratically near* if this condition holds with $\beta(n) = 1$ for each n. We denote s in the above definition by $\mathrm{near}(T, C, \beta)$. If $\beta(n) = 1$ for each n, we call $s = \mathrm{near}(T, C)$ the *nearness* (or 2-nearness) between T and C.

The following result gives equivalent definitions.

4.2. Lemma. *Let $\beta : \mathbb{Z}_+ \to \mathbb{R}_+^*$. Two operators T and C in $\mathcal{B}(\mathcal{H})$ are β-quadratically near with* $\mathrm{near}(T, C, \beta) \le s$

- *if and only if*

$$\sum_{n=0}^{+\infty} \frac{1}{\beta(n)^2} \|(T^n - C^n)^* y\|^2 \le s^2 \|y\|^2, \tag{4.1}$$

for all $y \in \mathcal{H}$,

- *if and only if, for every $N \in \mathbb{Z}_+$ and all $x_0, \ldots, x_N \in \mathcal{H}$, we have*

$$\left\| \sum_{n=0}^{N} \frac{1}{\beta(n)} (T^n - C^n) x_n \right\| \le s \sqrt{\sum_{n=0}^{N} \|x_n\|^2}. \tag{4.2}$$

Proof. The first equivalence was remarked in [Ba, Lemma 2.6] and follows from the fact that the numerical radius of A equals $\|A\|$ for normal operators A. The second equivalence follows from the fact that Definition 4.1 and condition (4.2) are both saying that

$$\sup_N \|R_N\| \le s,$$

where R_N is the row operator

$$R_N = \begin{bmatrix} 0 & \dfrac{T - C}{\beta(1)} & \ldots & \dfrac{T^N - C^N}{\beta(n)} \end{bmatrix}$$

acting on column vectors of $\ell_N^2(\mathcal{H})$. ∎

4.3. Remark. If

$$\limsup_{n \to \infty} \|T^n - C^n\|^{1/n} < 1,$$

or if

$$\sum_{n=0}^{\infty} \|T^n - C^n\|^2 < \infty,$$

then T and C are quadratically near. Operators satisfying

$$\limsup_{n \to \infty} \|T^n - C^n\|^{1/n} = 0$$

are called in [A1] *asymptotically equivalent* operators.

It follows from the more general result proved in [Ba] that if C is similar to a contraction and if $\mathrm{near}(T, C) < \infty$, then T is similar to a contraction. In particular, Rota's theorem is obtained for $C = 0$ since every operator of spectral radius smaller than 1 is quadratically near the null operator. We refer to [Ba] for consequences of the conditon $\mathrm{near}(T, C, \beta) < \infty$.

We introduce the following definition.

4.4. DEFINITION. Let $\mathcal{H}_0$ be a subspace of $\mathcal{H}$. Two operators T and C in $\mathcal{B}(\mathcal{H})$ are said to be *β-quadratically near modulo $\mathcal{H}_0$* if for every $N \in \mathbb{Z}_+$ and for all $x_0, \ldots, x_N \in \mathcal{H}_0$ we have

$$\Big\| \sum_{n=0}^{N} \frac{1}{\beta(n)} (T^n - C^n) x_n \Big\| \le s \sqrt{\sum_{n=0}^{N} \|x_n\|^2}. \tag{4.3}$$

For $\beta(n) \equiv 1$, we say that T and C are *quadratically near modulo $\mathcal{H}_0$*.

Quadratically near operators correspond to $\mathcal{H}_0 \equiv \mathcal{H}$. We will identify $\mathcal{H}$ with the subspace $\mathcal{H} \oplus \{0\} \oplus \{0\} \oplus \ldots$ in $\ell^2(\mathcal{H})$.

4.5. EXAMPLE. *Let $(\omega_k)_{k\ge 0}$ be a sequence of strictly positive weights. Denote by S_ω the unilateral weighted shift operator*

$$S_\omega : \ell^2(\mathcal{H}) \ni (x_0, x_1, \ldots) \to (0, \omega_0 x_0, \omega_1 x_1, \ldots) \in \ell^2(\mathcal{H})$$

on $\ell^2(\mathcal{H})$. Define

$$\beta(n) = \omega_0 \ldots \omega_{n-1} \quad \textit{for } n \ge 1 \quad \textit{and} \quad \beta(0) = 1.$$

Then S_ω is β-quadratically near 0 (*the null operator*) *modulo $\mathcal{H}$. If*

$$\sup_{n\ge 0} \beta(n) < \infty,$$

then S_ω is quadratically near 0 *modulo $\mathcal{H}$.*

Proof. Let $x_n = (x_n, 0, 0, \ldots) \in \mathcal{H}$, $0 \le n \le N$. Since

$$\sum_{n=0}^{N} S_\omega^n (x_n, 0, 0, \ldots) = (x_0, \beta(1) x_1, \ldots, \beta(n) x_n, \ldots),$$

we have

$$\Big\| \sum_{n=0}^{N} \frac{1}{\beta(n)} S_\omega^n x_n \Big\|^2 = \sum_{n=0}^{N} \|x_n\|^2$$

and

$$\Big\| \sum_{n=0}^{N} S_\omega^n x_n \Big\|^2 = \sum_{n=0}^{N} \beta(n)^2 \|x_n\|^2 \le [\sup_{k\ge 0} \beta(k)]^2 \sum_{n=0}^{N} \|x_n\|^2. \ \blacksquare$$

Another example in the same vein is the following.

4.6. EXAMPLE. *Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry. Then V is quadratically near* 0 (*the null operator*) *modulo* $\ker V^*$.

Proof. According to the Wold decomposition, V is the direct sum of a unitary operator U and a pure isometry S. Since $\ker V^* = \{0\} \oplus \ker S^*$, and the unilateral shift S is quadratically near 0 modulo $\ker S^*$ by the preceding example, $U \oplus S$ is quadratically near 0 modulo $\ker V^*$. $\blacksquare$

4.7. THEOREM. *Let* $(\omega_k)_{k\geq 0}$ *be a sequence of strictly positive weights and let* $\beta(0) = 1$ *and*

$$\beta(n) = \omega_0 \dots \omega_{n-1} \quad (n \geq 1).$$

Suppose that

$$\sup_{n,k\geq 0} \frac{\beta(n+k)}{\beta(n)} < \infty. \tag{4.4}$$

Consider $S_\omega \in \mathcal{B}(\ell^2(\mathcal{H}))$, *the unilateral weighted shift with weights* ω_k. *Let* $T \in \mathcal{B}(\mathcal{K})$ *be an operator similar to a contraction and let* $X \in \mathcal{B}(\ell^2(\mathcal{H}), \mathcal{K})$. *Suppose the operator*

$$R(X) = \begin{bmatrix} T & X \\ 0 & S_\omega \end{bmatrix} \in \mathcal{B}(\mathcal{K} \oplus \ell^2(\mathcal{H}))$$

is β*-quadratically near* $R(0) = T \oplus S_\omega$ *modulo* $\mathcal{H}$. *Then* $R(X)$ *is similar to a contraction.*

Proof. Condition (4.4) means that S_ω is power bounded, and thus S_ω is similar to a contraction [Sh]. Without loss of any generality we can assume that T and S_ω are two contractions. In order to prove the similarity of $R(X)$ to a contraction we will construct an equivalent *Hilbertian* norm on $\mathcal{B}(\mathcal{K} \oplus \ell^2(\mathcal{H}))$ such that $R(X)$, with respect to this norm, is a contraction. The construction of this new norm is similar to constructions in [Ho] and [Ba].

Define

$$X_n = \sum_{j=0}^{n-1} T^j X S_\omega^{n-j-1}.$$

Since

$$R(X)^n = \begin{bmatrix} T^n & X_n \\ 0 & S_\omega^n \end{bmatrix},$$

we have, using the β-quadratic nearness condition,

$$\left\| \sum_{n=0}^{N} \frac{1}{\beta(n)} X_n u_n \right\| \leq C \sqrt{\sum_{n=0}^{N} \|u_n\|^2} \tag{4.5}$$

for all $u_n \in \mathcal{H}$.

Every element $h = (h_0, h_1, \dots) \in \ell^2(\mathcal{H})$ can be (uniquely) written as

$$h = \sum_{n=0}^{\infty} \frac{1}{\beta(n)} S_\omega^n h_n \tag{4.6}$$

with $h_n = (h_n, 0, 0 \dots) \in \mathcal{H}$.

Consider decompositions of elements (k, h) of $\mathcal{K} \times \ell^2(\mathcal{H})$ of the following type: decompose $h = (h_0, h_1, \dots)$ as in (4.6); then decompose $k \in \mathcal{K}$ as follows:

$$k = \sum_{n=0}^{\infty} T^n k_n + \sum_{n=0}^{\infty} \frac{1}{\beta(n)} X_n h_n. \tag{4.7}$$

Several remarks are in order about this decomposition. Firstly, the above series $\sum_{n=0}^{\infty}(1/\beta(n))X_n h_n$ converges for all h. Indeed, we have

$$\Big\| \sum_{n=m}^{m+p} \frac{1}{\beta(n)} X_n h_n \Big\|^2 \leq C^2 \sum_{n=m}^{m+p} \|h_n\|^2$$

and the last sum is bounded by the tail of a convergent series. Secondly, we suppose that only a finite number of the k_n's are non-zero, that is, we consider only finite sums in the first part of the decomposition (4.7).

Such decompositions of $(k,h) \in \mathcal{K} \times \ell^2(\mathcal{H})$ always exist. Indeed, given the unique decomposition of h as in (4.6), there is at least one finite decomposition of $k \in \mathcal{K}$ as in (4.7): take for instance

$$k_0 = k - \sum_{n=0}^{\infty} \frac{1}{\beta(n)} X_n h_n \quad \text{and} \quad k_n = 0 \quad \text{for } n \geq 1. \tag{4.8}$$

We define a new norm $|(\cdot,*)|$ on $\mathcal{K} \times \ell^2(\mathcal{H})$ by setting

$$|(k,h)|^2 = \inf \Big\{ \Big\| \sum_{n\geq 0} T^n k_n \Big\|^2 + \|h\|^2 + \sum_{n=0}^{\infty} \|k_n\|^2 \Big\}, \tag{4.9}$$

where the infimum is taken over all decompositions of (k,h) described above. Note that $\sum_{n=0}^{\infty} \|k_n\|^2$ is also a finite sum.

We prove now that $|(\cdot,*)|$ is a Hilbertian norm on $\mathcal{K} \times \ell^2(\mathcal{H})$, equivalent to the usual ℓ^2 norm on $\mathcal{K} \oplus \ell^2(\mathcal{H})$ and that $R(X)$ is a contraction with respect to this new norm.

1. $|(\cdot,*)|$ *is a seminorm.* Take two elements $(k,h), (k',h')$ in $\mathcal{K} \times \ell^2(\mathcal{H})$ with their corresponding decompositions as above given by two sequences (k_n,h_n) and (k'_n,h'_n) in $\mathcal{K} \times \mathcal{H}$. By adding eventually zeros, we may assume that both decompositions have the same (finite) number of k's. Then $(k+k', h+h')$ is decomposed using the sequence $(k'_n + k_n, h'_n + h_n)$, $n \geq 0$.

Using the triangle inequality $\|a+b\| \leq \|a\| + \|b\|$ for

$$a = \Big(\sum_{n\geq 0} T^n k_n, h, k_0, k_1, \dots \Big) \quad \text{and} \quad b = \Big(\sum_{n\geq 0} T^n k'_n, h', k'_0, k'_1, \dots \Big)$$

and taking the infimum over all representations of (k,h) and (k',h'), we get the triangle inequality for the new norm.

The proofs of the inequality $|\lambda(k,h)| \leq |\lambda| \cdot |(k,h)|$ and its converse are left to the reader.

2. $|(\cdot,*)|$ *is an equivalent norm.* Let $(k,h) \in \mathcal{K} \times \ell^2(\mathcal{H})$ and consider a decomposition as above. Then, using (4.7) and (4.5), we obtain

$$\begin{aligned} \|k\|^2 + \|h\|^2 &= \Big\| \sum_{n\geq 0} T^n k_n + \sum_{n=0}^{\infty} \frac{1}{\beta(n)} X_n h_n \Big\|^2 + \|h\|^2 \\ &\leq 2\Big\| \sum_{n\geq 0} T^n k_n \Big\|^2 + 2\Big\| \sum_{n=0}^{\infty} \frac{1}{\beta(n)} X_n h_n \Big\|^2 + \|h\|^2 \\ &\leq 2\Big\| \sum_{n\geq 0} T^n k_n \Big\|^2 + (2C^2+1)\|h\|^2. \end{aligned}$$

Taking the infimum over all representations of (k,h), we see that $\|k\|^2+\|h\|^2$, which is the norm of (k,h) in $\mathcal{K} \oplus \ell^2(\mathcal{H})$, is no greater than a constant times the new norm $|(k,h)|^2$.

For the reverse inequality, consider the representation of h as in (4.6) and of k as in (4.8). Then, using (4.5), we have

$$\begin{aligned} |(k,h)|^2 &\leq 2\Big\| k - \sum_{n=0}^{\infty} \frac{1}{\beta(n)} X_n h_n \Big\|^2 + \|h\|^2 \\ &\leq 4\|k\|^2 + 4\Big\| \sum_{n=0}^{\infty} \frac{1}{\beta(n)} X_n h_n \Big\|^2 + \|h\|^2 \\ &\leq M[\|k\|^2 + \|h\|^2] \end{aligned}$$

for a suitable constant M.

3. $|(\cdot,*)|$ *is Hilbertian.* Let $(k,h),(k',h') \in \mathcal{K} \times \ell^2(\mathcal{H})$ with their corresponding decompositions as above given by $(k_n,h_n),(k'_n,h'_n) \in \mathcal{K} \times \mathcal{H}$. Then $(k \pm k', h \pm h')$ are decomposed by $(k_n \pm k'_n, h_n \pm h'_n)$. Let $\Sigma = |(k+k',h+h')|^2 + |(k-k',h-h')|^2$. Using the parallelogram law in $\mathcal{K}$ and $\mathcal{H}$ we obtain

$$\begin{aligned} \Sigma &\leq \Big\| \sum_{n\geq 0} T^n (k_n + k'_n) \Big\|^2 + \|h+h'\|^2 + \sum_{n=0}^{\infty} \|k_n + k'_n\|^2 \\ &\quad + \Big\| \sum_{n\geq 0} T^n (k_n - k'_n) \Big\|^2 + \|h-h'\|^2 + \sum_{n=0}^{\infty} \|k_n - k'_n\|^2 \\ &= 2\Big[\Big\| \sum_{n\geq 0} T^n k_n \Big\|^2 + \|h\|^2 + \sum_{n=0}^{\infty} \|k_n\|^2\Big] \\ &\quad + 2\Big[\Big\| \sum_{n\geq 0} T^n k'_n \Big\|^2 + \|h'\|^2 + \sum_{n=0}^{\infty} \|k'_n\|^2\Big]. \end{aligned}$$

Taking the infimum over all representations of (k,h) and (k',h'), we obtain the parallelogram (in)equality for $|(\cdot,*)|$. This implies [Am] that the norm comes from a scalar product.

4. *The operator $R(X)$ with respect to $|(\cdot,*)|$.* Let $(k,h)\in\mathcal{K}\times\ell^2(\mathcal{H})$ and decompose k and h as in (4.6) and (4.7). Then

$$S_\omega h=(0,\omega_0h_0,\omega_1h_1,\ldots)=\sum_{n\geq 0}\frac{1}{\beta(n+1)}S_\omega^{n+1}\frac{\beta(n+1)}{\beta(n)}h_n$$

and

$$\begin{aligned}Tk+Xh&=\sum_{n\geq 0}T^{n+1}k_n+\sum_{n=0}^{\infty}\frac{1}{\beta(n)}TX_nh_n+\sum_{n\geq 0}\frac{1}{\beta(n)}XS_\omega^n h_n\\&=\sum_{n\geq 0}T^{n+1}k_n+\sum_{n=0}^{\infty}\frac{1}{\beta(n)}X_{n+1}h_n\\&=\sum_{n\geq 0}T^{n+1}k_n+\sum_{n=0}^{\infty}\frac{1}{\beta(n+1)}X_{n+1}\frac{\beta(n+1)}{\beta(n)}h_n.\end{aligned}$$

Therefore, the second and the first component of

$$R(X)(k,h)=\begin{bmatrix}T & X\\0 & S_\omega\end{bmatrix}\begin{bmatrix}k\\h\end{bmatrix}\in\mathcal{K}\oplus\ell^2(\mathcal{H})$$

are decomposed by $(0,(\beta(1)/\beta(0))h_0,(\beta(2)/\beta(1))h_1,\ldots)$ and, respectively, $(0,k_0,k_1,\ldots)$. Then

$$\begin{aligned}|R(X)(k,h)|^2&\leq\Big\|\sum_{n\geq 0}T^{n+1}k_n\Big\|^2+\|S_\omega h\|^2+\sum_{n=0}^{\infty}\|k_n\|^2\\&\leq\Big\|\sum_{n\geq 0}T^{n}k_n\Big\|^2+\|h\|^2+\sum_{n=0}^{\infty}\|k_n\|^2,\end{aligned}$$

since T and S_ω are supposed to be contractions. We deduce that $R(X)$ is a contraction in the new norm $|(\cdot,*)|$. Therefore $R(X)$ is similar to a contraction. ∎

5. Perturbations of $R(0)$

5.1. Theorem (perturbation of $R(0)$). *Let $(\omega_k)_{k\geq 0}$ be a sequence of strictly positive weights and set $\beta(0)=1$ and $\beta(n)=\omega_0\ldots\omega_{n-1}$ for $n\geq 1$. Consider $S_\omega\in\mathcal{B}(\ell^2(\mathcal{H}))$, the unilateral weighted shift with weights ω_k. Let $T\in\mathcal{B}(\mathcal{K})$ be a coisometry and let $X\in\mathcal{B}(\ell^2(\mathcal{H}),\mathcal{K})$. Then*

$$R(X)=\begin{bmatrix}T & X\\0 & S_\omega\end{bmatrix}\in\mathcal{B}(\mathcal{K}\oplus\ell^2(\mathcal{H}))$$

is similar to a contraction if and only if $R(X)$ is power bounded and it is the zero-product perturbation by a nilpotent (of order two) of an operator $R(A)$ which is β-quadratically near $R(0)=T\oplus S_\omega$ modulo $\mathcal{H}$.

Proof. Suppose that $R(X)$ is similar to a contraction. Then $R(X)$ and S_ω are power bounded. Since T is a coisometry, the Foiaş–Williams' [CCFW] theorem implies that the commutator equation $X = TZ - ZS_\omega$ has a bounded solution $Z \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Let L denote the operator

$$L : \ell^2(\mathcal{H}) \ni (y_0, y_1, \ldots) \mapsto \left(\frac{1}{\omega_0} y_1, \frac{1}{\omega_1} y_2, \ldots\right) \in \ell^2(\mathcal{H}),$$

which is a left inverse of S_ω. Set

$$F = TZ(I - S_\omega L) \quad \text{and} \quad A = X - F = TZS_\omega L - ZS_\omega.$$

Then $FS_\omega = 0$ and $A = TD - DS_\omega$ with $D = ZS_\omega L$. We have

$$R(X) = \begin{bmatrix} T & A \\ 0 & S_\omega \end{bmatrix} + \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix} =: R(A) + E.$$

The second operator is nilpotent of order two and also

$$ER(A) = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T & A \\ 0 & S_\omega \end{bmatrix} = 0.$$

Thus $R(X)$ is a zero-product perturbation by a nilpotent of order 2 of

$$R(A) = \begin{bmatrix} T & A \\ 0 & S_\omega \end{bmatrix}.$$

We show now that $R(A)$ is β-quadratically near to $R(0) = T \oplus S_\omega$ modulo $\mathcal{H}$. Recall that $A = TD - DV$ with $D = ZS_\omega L$. Consider a sequence of elements $x_n = (x_n, 0, 0 \ldots)$ in $\mathcal{H}$. Then $Dx_n = DS_\omega L x_n = 0$. Define

$$A_n = \sum_{j=0}^{n-1} T^j A S_\omega^{n-j-1}.$$

The powers of $R(A)$ are given by

$$R(A)^n = \begin{bmatrix} T^n & A_n \\ 0 & S_\omega^n \end{bmatrix}.$$

We obtain

$$\begin{aligned}
\left\| \sum_{n=1}^{N} \frac{R(A)^n - R(0)^n}{\beta(n)} x_n \right\| &= \left\| \sum_{n=1}^{N} \frac{1}{\beta(n)} A_n x_n \right\| \\
&= \left\| \sum_{n=1}^{N} \frac{1}{\beta(n)} \sum_{j=0}^{n-1} T^j (TD - DS_\omega) S_\omega^{n-j-1} x_n \right\| \\
&= \left\| \sum_{n=1}^{N} \frac{1}{\beta(n)} (T^n D - DS_\omega^n) x_n \right\| \\
&= \left\| -D \sum_{n=1}^{N} \frac{1}{\beta(n)} S_\omega^n x_n \right\| \\
&\leq \|D\| \Big(\sum_{n=1}^{N} \|x_n\|^2 \Big)^{1/2}.
\end{aligned}$$

In the last line we have used Example 4.5. Thus $R(A)$ is β-quadratically near $R(0)$ modulo $\mathcal{H}$.

For the converse implication, suppose that $R(X) = R(A) + E$ can be decomposed as in the theorem. We obtain $X = A + F$ with $FS_\omega = 0$. Note that the power boundedness of $R(X)$ implies that of S_ω. By Theorem 4.7, the operator

$$R(A) = \begin{bmatrix} T & A \\ 0 & S_\omega \end{bmatrix} \in \mathcal{B}(\mathcal{K} \oplus \mathcal{H})$$

is similar to a contraction. By Corollary 3.4, the zero-product perturbation $R(A) + E$ is also similar to a contraction. ∎

5.2. THEOREM (perturbation of $R(0)$ again). *Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry, let $T \in \mathcal{B}(\mathcal{K})$ be a right invertible contraction and let $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then the operator*

$$R(X) = \begin{bmatrix} T & X \\ 0 & V \end{bmatrix} \in \mathcal{B}(\mathcal{K} \oplus \mathcal{H})$$

is similar to a contraction if and only if $R(X)$ is power bounded and it is the zero-product perturbation by a nilpotent of an operator $R(A)$ which is quadratically near $R(0) = T \oplus V$ modulo $\ker V^*$.

Proof. Suppose that $R(X)$ is similar to a contraction. Since V is an isometry, the commutator equation $X = TZ - ZV$ has [CCFW] a bounded solution $Z \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Set

$$F = TZ(I - VV^*) \quad \text{and} \quad A = X - F = TZVV^* - ZV.$$

Then $FV = 0$ and $A = TD - DV$ with $D = ZVV^*$. The proof that the decomposition

$$R(X) = \begin{bmatrix} T & A \\ 0 & V \end{bmatrix} + \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$$

satisfies all the requirements is similar to that given above.

For the converse implication, suppose that $R(X) = R(A) + E$ can be decomposed as in the theorem. In particular $E^2 = 0$ and $X = A + F$ with $FV = 0$. By Corollary 3.4 it is sufficient to show that

$$R(A) = \begin{bmatrix} T & A \\ 0 & V \end{bmatrix} \in \mathcal{B}(\mathcal{K} \oplus \mathcal{H})$$

is similar to a contraction.

Now $R(A) = R(X) - E$ and $ER(X) = 0$ since $FV = 0$. This implies that $R(A)^n = R(X)^n - R(X)^{n-1}E$ and thus $R(A)$ is power bounded since $R(X)$ is.

Writing the Wold decomposition for V, we get the orthogonal sum $\mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_u$ in which $\mathcal{H}_s$ and $\mathcal{H}_u$ reduce V. Denote by U the part of V on $\mathcal{H}_u$ (U is unitary) and by S the part of V on $\mathcal{H}_s$ which is a pure isometry (a unilateral shift).

With respect to the decomposition $\mathcal{K}\oplus\mathcal{H}=\mathcal{K}\oplus\mathcal{H}_s\oplus\mathcal{H}_u$, the matrix of $R(A)$ is given by

$$R(A)=\begin{bmatrix} T & A_s & A_u \\ 0 & S & 0 \\ 0 & 0 & U \end{bmatrix}.$$

Recall that U is unitary and $R(A)$ is power bounded. By Corollary 2.2, $R(A)$ is similar to $C\oplus U$, where

$$C=\begin{bmatrix} T & X_s \\ 0 & S \end{bmatrix}\in\mathcal{B}(\mathcal{K}\oplus\mathcal{H}_s).$$

Now $R(A)$ quadratically near $R(0)$ modulo $\ker V^*$ implies that C is quadratically near $T\oplus S$ modulo $\ker S^*$ and thus C is similar to a contraction by Theorem 4.7. ∎

5.3. Remark. Another proof of Theorem 5.2 can be obtained by using a result due to C. Apostol [A2]. With our terminology, Apostol proved that if T and X are arbitrary and V is a unilateral shift, then the commutator equation $X=TZ-ZV$ has a bounded solution Z satisfying $Z(I-VV^*)=0$ if and only if $R(X)$ is quadratically near $R(0)$ modulo $\ker V^*$. Apostol's result, together with Theorem 2.1, shows that if V is a unilateral shift and T is power bounded, then $R(X)$ is quadratically near $R(0)$ modulo $\ker V^*$ if and only if the growth condition (2.1) holds. It seems that Apostol's proof does not generalize to weighted shifts and this explains our proofs of Theorems 4.7 and 5.1.

The conclusion of Theorem 5.2 can be strengthened if the isometry V is supposed to be a unilateral shift. Indeed, combining the result of C. Foiaş and J. P. Williams [FW], the result of Apostol [A2], and our previous results we obtain the following characterization.

5.4. Theorem (nearness plus admissible perturbations). *Let $V\in\mathcal{B}(\mathcal{H})$ be a pure isometry (a unilateral shift), let $T\in\mathcal{B}(\mathcal{K})$ be a right invertible contraction and let $X\in\mathcal{B}(\mathcal{H},\mathcal{K})$. Then the operator*

$$R(X)=\begin{bmatrix} T & X \\ 0 & V \end{bmatrix}\in\mathcal{B}(\mathcal{K}\oplus\mathcal{H})$$

is similar to a contraction if and only if $R(X)$ is the zero-product perturbation by a nilpotent of an operator $R(A)$ which is quadratically near $R(0)=T\oplus V$ modulo $\ker V^*$.

The difference between this characterization and Theorem 5.2 is that the condition of power boundedness is now missing.

6. Concluding remarks. Sometimes the nearness conditions suffice in the characterization given by Theorem 5.4. We will show that this happens for the class of operators studied by Pisier [Pi] and Davidson–Paulsen [DaPa].

Let Λ be a function from $\mathcal{H}$ into $\mathcal{B}(\mathcal{H})$ satisfying the CAR, *canonical anticommutation relations*: for all $u, v \in H$,

$$\Lambda(u)\Lambda(v)+\Lambda(v)\Lambda(u) = 0 \quad \text{and} \quad \Lambda(u)\Lambda(v)^*+\Lambda(v)^*\Lambda(u) = (u,v)I. \tag{6.1}$$

The range of Λ is isometric to Hilbert space. Let $\{e_n\}_{n\ge 0}$ be an orthonormal basis for $\mathcal{H}$, and let $C_n = \Lambda(e_n)$ for $n \ge 0$. For an arbitrary sequence $\alpha = (\alpha_0, \alpha_1, \ldots)$ in ℓ^2, let

$$\Gamma_\alpha = [\alpha_{i+j}C_{i+j}]$$

be a CAR-valued Hankel operator. Let

$$R(\Gamma_\alpha) = R(S^*, S; \Gamma_\alpha) = \begin{bmatrix} S^* & \Gamma_\alpha \\ 0 & S \end{bmatrix} \in \mathcal{B}(\ell^2(\mathcal{H}) \oplus \ell^2(\mathcal{H}))$$

be the corresponding CAR-valued Foguel–Hankel operator [Pi, DaPa]. Here $S \in \mathcal{B}(\ell^2(\mathcal{H}))$ denotes the unilateral forward shift of multiplicity $\dim \mathcal{H}$.

For a fixed sequence $\alpha = (\alpha_0, \alpha_1, \ldots) \in \ell^2$, let

$$A(\alpha) = \sup_{k\ge 0}(k+1)^2 \sum_{i\ge k} |\alpha_i|^2 \quad \text{and} \quad B(\alpha) = \sum_{k\ge 0}(k+1)^2|\alpha_k|^2.$$

The operator $R(\Gamma_\alpha)$ is [Pi, DaPa] polynomially bounded if and only if $A(\alpha)$ is finite. On the other hand, $R(\Gamma_\alpha)$ is [Pi, DaPa, Ri] similar to a contraction if and only if $B(\alpha)$ is finite (see also [Ba, BaPa]).

6.1. Theorem. *Let $\alpha = (\alpha_0, \alpha_1, \ldots) \in \ell^2$. The operator $R(\Gamma_\alpha)$ is similar to a contraction if and only if it is quadratically near $R(0) = S^* \oplus S$ modulo $\mathcal{H}$.*

Proof. By Theorem 4.7, the nearness condition implies similarity to a contraction. For the converse implication, note that $S^*\Gamma_\alpha = \Gamma_\alpha S$. This implies that

$$R(\Gamma_\alpha)^n = \begin{bmatrix} S^{*n} & \Gamma_n \\ 0 & S^n \end{bmatrix},$$

where

$$\Gamma_n = \sum_{j=0}^{n-1} S^{*j}\Gamma_\alpha S^{n-j-1} = n\Gamma_\alpha S^{n-1}.$$

We have

$$\begin{aligned}\sum_{n=1}^{N}[R(\Gamma_\alpha)^n - R(0)^n](h_n, 0, 0, \ldots) &= \sum_{n=1}^{N} n\Gamma_\alpha S^{n-1}(h_n, 0, 0, \ldots) \\ &= [(j+1)\alpha_{i+j}C_{i+j}]_{i,j\ge 0} \begin{bmatrix} h_1 \\ \vdots \\ h_N \end{bmatrix}\end{aligned}$$

If $R(\Gamma_\alpha)$ is similar to a contraction, then $B(\alpha)$ is finite [DaPa], and thus [Ri, Proposition 1] the norm of the matrix $[(j+1)\alpha_{i+j}C_{i+j}]_{i,j\geq 0}$ is bounded by $B(\alpha)^{1/2}$. We obtain

$$\Big\|\sum_{n=1}^{N}[R(\Gamma_\alpha)^n - R(0)^n]x_n\Big\| \leq B(\alpha)^{1/2}\sqrt{\sum_{n=1}^{N}\|x_n\|^2}$$

for all $x_n \in \mathcal{H}$. Therefore, $R(\Gamma_\alpha)$ is quadratically near $R(0) = S^* \oplus S$ modulo $\mathcal{H}$. ■

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