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Well-posedness of second order degenerate differential equations in vector-valued function spaces

by

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Abstract. Using known results on operator-valued Fourier multipliers on vector-valued function spaces, we give necessary or sufficient conditions for the well-posedness of the second order degenerate equations (P_2): $\frac{d}{dt}(Mu')(t) = Au(t) + f(t)$ $(0 \le t \le 2\pi)$ with periodic boundary conditions $u(0) = u(2\pi)$, $(Mu')(0) = (Mu')(2\pi)$, in Lebesgue–Bochner spaces $L^p(\mathbb{T}, X)$, periodic Besov spaces $B^s_{p,q}(\mathbb{T}, X)$ and periodic Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{T}, X)$, where A and M are closed operators in a Banach space X satisfying $D(A) \subset D(M)$. Our results generalize the previous results of W. Arendt and S. Q. Bu when $M = I_X$.

1. Introduction. In this paper, we consider the second order degenerate equations

$$(P_2): \quad \frac{d}{dt}(Mu')(t) = Au(t) + f(t) \quad (0 \le t \le 2\pi)$$

with periodic boundary conditions $u(0) = u(2\pi)$, $(Mu')(0) = (Mu')(2\pi)$, where A and M are closed operators in a Banach space X satisfying $D(A) \subset D(M)$; f is an X-valued function.

The class of equations (P_2) arises as models for nonlinear heat conduction in materials of fading memory type and in population dynamics. Much literature has been devoted to such problems [1, 2, 5]. W. Arendt and S. Q. Bu have considered the L^p-well-posedness of (P_2) when $M = I_X$ is the identity operator of X; they have shown that when X is a UMD Banach space and $1 < p < \infty$, (P_2) is L^p-well-posed if and only if $-\mathbb{Z}^2 \subset \rho(A)$ and the set $\{k^2(k^2 + A)^{-1} : k \in \mathbb{Z}\}$ is Rademacher bounded [1, Theorem 6.1]. They have also studied the well-posedness of (P_2) in periodic Besov spaces $B^s_{p,q}(\mathbb{T}, X)$ and showed that, when $M = I_X$, (P_2) is $B^s_{p,q}$-well-posed if and only if $-\mathbb{Z}^2 \subset \rho(A)$ and the set $\{k^2(k^2 + A)^{-1} : k \in \mathbb{Z}\}$ is norm bounded

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[2, Theorem 5.3]. A similar characterization of the well-posedness of (P_2) in periodic Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{T}, X)$ has been obtained by S. Q. Bu and J. M. Kim when $M = I_X$ [5, Theorem 4.2].

We notice that a detailed study of linear abstract degenerate differential equations using semigroup theory and the extension of the operational method of G. Da Prato and P. Grisvard has been treated in the monograph [6], where second order degenerate equations have been systematically studied in Chapter VI, mainly focusing on the existence and uniqueness of classical solutions of (P_2). We also notice that similar second order equations with delay have also been extensively studied (see e.g. [4, 8]).

Similar first order degenerate equations

$$(P_1): \quad \frac{d}{dt}(Mu)(t) = Au(t) + f(t) \quad (0 \le t \le 2\pi),$$

with periodic boundary conditions $Mu(0) = Mu(2\pi)$, have recently been studied by C. Lizama and R. Ponce [10]; under suitable assumptions on the R-boundedness of the modified resolvent operator determined by (P_1), they gave necessary and sufficient conditions for the well-posedness of (P_1) in Lebesgue–Bochner spaces $L^p(\mathbb{T}, X)$, Besov spaces $B^s_{p,q}(\mathbb{T}, X)$ and Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{T}, X)$.

In this paper, using suitable assumptions on the growth of the modified resolvent operator determined by (P_2), we give necessary or sufficient conditions for (P_2) to be L^p-well-posed (resp. $B^s_{p,q}$-well-posed and $F^s_{p,q}$-well-posed). Since A is not necessarily the generator of a semigroup in our situation, semigroup theory is no longer applicable. Our main tool is the operator-valued Fourier multiplier theorems obtained by W. Arendt and S. Q. Bu [1, 2] on $L^p(\mathbb{T}, X)$ and $B^s_{p,q}(\mathbb{T}, X)$, and by S. Q. Bu and J. M. Kim [5] on $F^s_{p,q}(\mathbb{T}, X)$. In fact we will transform the well-posedness of (P_2) to an operator-valued Fourier multiplier problem in the corresponding vector-valued function spaces. Our results concerning the L^p-well-posedness of (P_2) involve UMD Banach spaces and the R-boundedness for sets of bounded linear operators on Banach spaces, which are not too restrictive conditions for applications. Our results recover the known results of [1, 2, 5] in the case $M = I_X$.

This paper is organized as follows: in the second section, we study the L^p-well-posedness for (P_2), while the last section is devoted to the well-posedness of (P_2) in Besov spaces $B^s_{p,q}(\mathbb{T}, X)$ and Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{T}, X)$.

2. The L^p-well-posedness of (P_2). We first recall some notions and notation. Throughout, X will be a complex Banach space and $\mathbb{T} := [0, 2\pi]$. For $1 \le p < \infty$, we let $L^p(\mathbb{T}, X)$ be the space of all X-valued measurable functions f defined on $\mathbb{T}$ satisfying

$$\|f\|_{L^p} := \Big(\int_0^{2\pi} \|f(t)\|^p \,\frac{dt}{2\pi}\Big)^{1/p} < \infty.$$

For $f \in L^1(\mathbb{T}, X)$, we denote by

$$\hat{f}(k) := \frac{1}{2\pi}\int_0^{2\pi} e_{-k}(t)f(t)\,dt$$

the kth Fourier coefficient of f, where $k \in \mathbb{Z}$ and $e_k(t) = e^{ikt}$ for $t \in \mathbb{T}$.

X is said to be a *UMD Banach space* if the Riesz projection

$$Rf := \sum_{k\geq 0} \hat{f}(k)e_k$$

is bounded on $L^p(\mathbb{T}, X)$ for some (equivalently for all) $1 < p < \infty$ [3]. The scalar L^p-spaces, Sobolev spaces $W^{k,p}$ and Schatten class S_p are UMD Banach spaces when $1 < p < \infty$. If Y is another Banach space, we denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators from X to Y. If $X = Y$, we simply write $\mathcal{L}(X)$.

For results about R-boundedness, we refer to J. Bourgain [3], L. Weis [11, 12] and W. Arendt and S. Q. Bu [1]. We merely recall the definition and some basic properties. Let r_j be the jth Rademacher function on $[0, 1]$ given by $r_j(t) = \operatorname{sgn}(\sin(2^{j-1}t))$ when $j \geq 1$. For $x \in X$, we denote by $r_j \otimes x$ the vector-valued function $t \mapsto r_j(t)x$ on $[0, 1]$.

Definition 2.1. Let X and Y be Banach spaces. A set $\mathbf{T} \subset \mathcal{L}(X, Y)$ is said to be *Rademacher bounded* (*R-bounded*, for short) if there exists $C \geq 0$ such that

$$\Big\|\sum_{j=1}^{n} r_j \otimes T_j x_j\Big\|_{L^1([0,1],Y)} \leq C\Big\|\sum_{j=1}^{n} r_j \otimes x_j\Big\|_{L^1([0,1],X)} \tag{2.1}$$

for all $T_1, \ldots, T_n \in \mathbf{T}$, $x_1, \ldots, x_n \in X$ and $n \in \mathbb{N}$.

Remark 2.2. Let $\mathbf{S}, \mathbf{T} \subset \mathcal{L}(X)$ be R-bounded sets. Then it is clear from the definition that $\mathbf{ST} := \{ST : S \in \mathbf{S}, T \in \mathbf{T}\}$ and $\mathbf{S} + \mathbf{T} := \{S + T : S \in \mathbf{S}, T \in \mathbf{T}\}$ are still R-bounded. It is also clear that if $\Omega \subset \mathbb{C}$ is bounded, then the set $\{\lambda I_X : \lambda \in \Omega\}$ is R-bounded [9, Theorem 4.4].

The main tools in our study of the L^p-well-posedness of (P_2) are operator-valued L^p-Fourier multipliers which were studied in [1].

Definition 2.3. For $1 \leq p < \infty$, we say that $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$ is an *L^p-Fourier multiplier* if for each $f \in L^p(\mathbb{T}, X)$, there exists a unique $u \in L^p(\mathbb{T}, Y)$ such that $\hat{u}(k) = M_k\hat{f}(k)$ for all $k \in \mathbb{Z}$.

It is known that when $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$ is an L^p-Fourier multiplier, then the set $\{M_k : k \in \mathbb{Z}\}$ is R-bounded [1, Proposition 1.11]. The following theorem, due to W. Arendt and S. Q. Bu [1, Theorem 1.3], plays an important role in our investigations.

THEOREM 2.4. *Let X, Y be UMD Banach spaces and $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$. If the sets $\{M_k : k \in \mathbb{Z}\}$ and $\{k(M_{k+1} - M_k) : k \in \mathbb{Z}\}$ are R-bounded, then $(M_k)_{k\in\mathbb{Z}}$ defines an L^p-Fourier multiplier whenever $1 < p < \infty$.*

REMARKS 2.5. (i) If $(M_k)_{k\in\mathbb{Z}}$ and $(N_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers, then so is the product sequence $(M_kN_k)_{k\in\mathbb{Z}}$.

(ii) Let $c_k = 1/k$ when $k \neq 0$ and $c_0 = 1$, and let $f = \sum_{k\in\mathbb{Z}} c_k e_k$. Then $f \in L^1([0, 2\pi])$ by [7]. Thus if $1 \le p < \infty$, then $(c_k I_X)_{k\in\mathbb{Z}}$ is an L^p-Fourier multiplier by Young's inequality [7].

For $1 \le p < \infty$, we define the first order periodic "Sobolev" spaces [1] by

$$W^{1,p}_{\mathrm{per}}(\mathbb{T}, X) := \{u \in L^p(\mathbb{T}, X) : \text{there exists } v \in L^p(\mathbb{T}, X) \text{ such that } \hat{v}(k) = ik\hat{u}(k) \text{ for all } k \in \mathbb{Z}\}.$$

Let $u \in L^p(\mathbb{T}, X)$. Then $u \in W^{1,p}_{\mathrm{per}}(\mathbb{T}, X)$ if and only if u is differentiable a.e. on $\mathbb{T}$ and $u' \in L^p(\mathbb{T}, X)$ (in fact u' is just the function v in the definition of $W^{1,p}_{\mathrm{per}}(\mathbb{T}, X)$); in this case u is actually continuous and $u(0) = u(2\pi)$ [1, Lemma 2.1].

$W^{1,p}_{\mathrm{per}}(\mathbb{T}, X)$ is a Banach space under the norm

$$\|u\|_{W^{1,p}_{\mathrm{per}}} := \|u\|_{L^p} + \|u'\|_{L^p}.$$

We consider the second order degenerate equations

$$(P_2)\colon \quad \frac{d}{dt}(Mu')(t) = Au(t) + f(t) \qquad (0 \le t \le 2\pi)$$

with periodic boundary conditions $u(0) = u(2\pi)$, $(Mu')(0) = (Mu')(2\pi)$, where A and M are closed operators in X satisfying $D(A) \subset D(M)$, and f is an X-valued function defined on $\mathbb{T}$.

Let $1 \le p < \infty$. We define the L^p-well-posedness solution space for (P_2) by

$$S_p(A, M) := \{u \in W^{1,p}_{\mathrm{per}}(\mathbb{T}, X) \cap L^p(\mathbb{T}, D(A)) : u' \in L^p(\mathbb{T}, D(M)),\ Mu' \in W^{1,p}_{\mathrm{per}}(\mathbb{T}, X)\};$$

here we consider $D(A)$ and $D(M)$ as Banach spaces equipped with their graph norms. $S_p(A, M)$ is a Banach space with the norm

$$\|u\|_{S_p(A,M)} := \|u\|_{L^p} + \|u'\|_{L^p} + \|Au\|_{L^p} + \|Mu'\|_{L^p} + \|(Mu')'\|_{L^p}.$$

By [1, Lemma 2.1], if $u \in S_p(A, M)$, then u and Mu' are X-valued continuous functions on $\mathbb{T}$, and $u(0) = u(2\pi)$, $(Mu')(0) = (Mu')(2\pi)$.

DEFINITION 2.6. Let $1 \le p < \infty$ and $f \in L^p(\mathbb{T}, X)$. Then $u \in S_p(A, M)$ is called a *strong* L^p*-solution* of (P_2) if (P_2) is satisfied a.e. on $\mathbb{T}$. We say that (P_2) is L^p*-well-posed* if for each $f \in L^p(\mathbb{T}, X)$, there exists a unique strong L^p-solution of (P_2).

When (P_2) is L^p-well-posed, there exists a constant $C > 0$ such that for each $f \in L^p(\mathbb{T}, X)$, if $u \in S_p(A, M)$ is the unique strong L^p-solution of (P_2), then

$$\|u\|_{L^p} + \|u'\|_{L^p} + \|Au\|_{L^p} + \|Mu'\|_{L^p} + \|(Mu')'\|_{L^p} \le C\|f\|_{L^p}. \tag{2.2}$$

This is an easy consequence of the Closed Graph Theorem.

Now we introduce the M-resolvent set of A. We recall that under the assumption that $D(A) \subset D(M)$, for any $\lambda \in \mathbb{C}$, the sum operator $\lambda M - A$ is a linear operator from $D(A)$ into X. We call

$$\rho_M(A) := \{\lambda \in \mathbb{C} : \lambda^2 M - A : D(A) \to X \text{ is invertible and } (\lambda^2 M - A)^{-1} \in \mathcal{L}(X)\}$$

the M*-resolvent set* of A. If $\lambda \in \rho_M(A)$, then the operator $M(\lambda^2 M - A)^{-1}$ is well defined by the assumption $D(A) \subset D(M)$, and $M(\lambda^2 M - A)^{-1} \in \mathcal{L}(X)$ by the closedness of M and boundedness of $(\lambda^2 M - A)^{-1}$.

Now we are able to state our first result which gives a necessary condition for the L^p-well-posedness of (P_2).

THEOREM 2.7. *Let X be a Banach space, $1 \le p < \infty$ and let A, M be closed linear operators in X satisfying $D(A) \subset D(M)$. Assume that (P_2) is L^p-well-posed. Then $i\mathbb{Z} \subset \rho_M(A)$ and $\{N_k : k \in \mathbb{Z}\}$ is R-bounded, where $N_k = k^2 M(k^2 M + A)^{-1}$ for $k \in \mathbb{Z}$.*

Proof. Let $k \in \mathbb{Z}$ and $y \in X$ be fixed. We define $f(t) = e^{ikt}y$ $(t \in \mathbb{T})$. Then $f \in L^p(\mathbb{T}, X)$, $\hat{f}(k) = y$ and $\hat{f}(n) = 0$ for $n \ne k$. Since (P_2) is L^p-well-posed, there exists a unique $u \in S_p(A, M)$ satisfying

$$(Mu')'(t) = Au(t) + f(t) \tag{2.3}$$

a.e. on $\mathbb{T}$. We have $\hat{u}(n) \in D(A)$ for $n \in \mathbb{Z}$ by [1, Lemma 3.1]. We remark that f, Au, u', Mu' and $(Mu')'$ are elements of $L^p(\mathbb{T}, X)$, so their Fourier transforms make sense. Thus

$$[(Mu')']^\wedge(n) = in[Mu']^\wedge(n) = inM(u')^\wedge(n) = -n^2 M\hat{u}(n) \tag{2.4}$$

for $n \in \mathbb{Z}$ by the closedness of M and [1, Lemma 3.1]. Taking Fourier transforms on both sides of (2.3), we obtain

$$(-k^2 M - A)\hat{u}(k) = y \tag{2.5}$$

and $(-n^2M - A)\hat{u}(n) = 0$ when $n \neq k$. Thus $-k^2M - A$ is surjective. To show that it is also injective, take $x \in D(A)$ such that $(-k^2M - A)x = 0$, so $-k^2Mx = Ax$. Let $u(t) = e^{ikt}x$ for $t \in \mathbb{T}$. Then clearly $u \in S_p(A, M)$ and $(Mu')'(t) = Au(t)$ a.e. on $\mathbb{T}$. Thus u is a strong L^p-solution of (P_2) with $f = 0$. Hence $x = 0$ by the uniqueness assumption. We have shown that $-k^2M - A$ is injective. Therefore $-k^2M - A$ is bijective from $D(A)$ onto X.

Next, we show that $(-k^2M - A)^{-1} \in \mathcal{L}(X)$. For $f(t) = e^{ikt}y$, we let $u \in S_p(A, M)$ be the unique strong L^p-solution of (P_2). Then by (2.5),

$$\hat{u}(n) = \begin{cases} 0, & n \neq k, \\ (-k^2M - A)^{-1}y, & n = k. \end{cases}$$

This implies that $u(t) = e^{ikt}(-k^2M - A)^{-1}y$. By (2.2), there exists a constant $C > 0$, independent of y and k, such that

$$\|u\|_{L^p} + \|u'\|_{L^p} + \|Au\|_{L^p} + \|Mu'\|_{L^p} + \|(Mu')'\|_{L^p} \le C\|f\|_{L^p}.$$

Hence $\|(k^2M + A)^{-1}y\| \le C\|y\|$ for all $y \in X$. Therefore $\|(k^2M + A)^{-1}\| \le C$. We have shown that $ik \in \rho_M(A)$. Thus $i\mathbb{Z} \subset \rho_M(A)$.

Finally, we show that if $N_k = k^2M(k^2M + A)^{-1}$ for $k \in \mathbb{Z}$, then $(N_k)_{k\in\mathbb{Z}}$ defines an L^p-Fourier multiplier. Let $f \in L^p(\mathbb{T}, X)$. Then by assumption there exists a strong L^p-solution $u \in S_p(A, M)$ of (P_2). Taking Fourier transforms on both sides of (P_2), by using (2.4) we find that $\hat{u}(k) \in D(A)$ for all $k \in \mathbb{Z}$, and

$$(-k^2M - A)\hat{u}(k) = \hat{f}(k) \quad (k \in \mathbb{Z}).$$

Since $-k^2M - A$ is invertible, we have $\hat{u}(k) = (-k^2M - A)^{-1}\hat{f}(k)$. Since $u \in S_p(A, M)$, we have $[(Mu')']^\wedge(k) = -k^2\hat{u}(k)$ by (2.4) and so

$$[(Mu')']^\wedge(k)) = -k^2M\hat{u}(k) = -N_k\hat{f}(k) \quad (k \in \mathbb{Z}).$$

We conclude that $(N_k)_{k\in\mathbb{Z}}$ defines an L^p-Fourier multiplier as $(Mu')' \in L^p(\mathbb{T}, X)$. Now the result follows immediately from [1, Proposition 1.11]. ∎

Our next result gives a sufficient condition for the L^p-well-posedness of (P_2) when X is a UMD Banach space and $1 < p < \infty$.

THEOREM 2.8. *Let X be a UMD Banach space, $1 < p < \infty$ and let A, M be closed linear operators in X satisfying $D(A) \subset D(M)$. Assume that $i\mathbb{Z} \subset \rho_M(A)$ and the sets $\{k(k^2M+A)^{-1} : k \in \mathbb{Z}\}$ and $\{k^2M(k^2M+A)^{-1} : k \in \mathbb{Z}\}$ are R-bounded. Then (P_2) is L^p-well-posed.*

Proof. Let $N_k = k^2M(k^2M + A)^{-1}$ and $S_k = k(k^2M + A)^{-1}$ for $k \in \mathbb{Z}$. Then $\{N_k : k \in \mathbb{Z}\}$ and $\{S_k : k \in \mathbb{Z}\}$ are R-bounded by assumption. We claim that $(N_k)_{k\in\mathbb{Z}}$ defines an L^p-Fourier multiplier. In view of Theorem 2.4, it is sufficient to show that $\{k(N_{k+1} - N_k) : k \in \mathbb{Z}\}$ is R-bounded. Let

$H_k = (k^2M + A)^{-1} \in \mathcal{L}(X)$, so $N_k = k^2MH_k$. We have

$$\text{(2.6)} \qquad \begin{aligned} H_{k+1} - H_k &= H_k[(k^2M + A) - ((k+1)^2M + A)]H_{k+1} \\ &= -(2k+1)H_kMH_{k+1}. \end{aligned}$$

Using this equality, we obtain

$$\text{(2.7)} \quad \begin{aligned} k(N_{k+1} - N_k) &= kM[(k+1)^2H_{k+1} - k^2H_k] \\ &= kM[(k+1)^2 - k^2]H_{k+1} + k^3M[H_{k+1} - H_k] \\ &= k(2k+1)MH_{k+1} - k^3(2k+1)MH_kMH_{k+1} \\ &= \frac{k(2k+1)}{(k+1)^2}N_{k+1} - \frac{k(2k+1)}{(k+1)^2}N_kN_{k+1}. \end{aligned}$$

This implies that $\{k(N_{k+1} - N_k) : k \in \mathbb{Z}\}$ is R-bounded by Remark 2.2.

Now, we are going to show that $(S_k)_{k\in\mathbb{Z}}$ also defines an L^p-Fourier multiplier. By Theorem 2.4 it is sufficient to show that $\{k(S_{k+1} - S_k) : k \in \mathbb{Z}\}$ is R-bounded. Using (2.6), we have

$$\begin{aligned} k(S_{k+1} - S_k) &= kH_{k+1} + k^2[H_{k+1} - H_k] \\ &= \frac{k}{k+1}S_{k+1} - k^2(2k+1)H_{k+1}MH_k \\ &= \frac{k}{k+1}S_{k+1} - \frac{2k+1}{k+1}S_{k+1}N_k. \end{aligned}$$

Therefore $\{k(S_{k+1} - S_k) : k \in \mathbb{Z}\}$ is R-bounded by Remark 2.2. We have shown that $(N_k)_{k\in\mathbb{Z}}$ and $(S_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers.

Let $T_k = -ikM(k^2M + A)^{-1}$ for $k \in \mathbb{Z}$. Then $(T_k)_{k\in\mathbb{Z}}$ is also an L^p-Fourier multiplier. This follows easily from the fact that $(N_k)_{k\in\mathbb{Z}}$ is an L^p-Fourier multiplier and Remarks 2.5.

Now, we are going to show that (P_2) is L^p-well-posed. Since $(N_k)_{k\in\mathbb{Z}}$ is an L^p-Fourier multiplier, for all $f \in L^p(\mathbb{T}, X)$ there exists $u \in L^p(\mathbb{T}, X)$ satisfying $\hat{u}(k) = N_k\hat{f}(k)$ for all $k \in \mathbb{Z}$. Using the identity

$$I_X = k^2M(k^2M + A)^{-1} + A(k^2M + A)^{-1},$$

we have

$$\hat{u}(k) = k^2M(k^2M + A)^{-1}\hat{f}(k) = (I_X - A(k^2M + A)^{-1})\hat{f}(k).$$

Thus $(u-f)^\wedge(k) = -A(k^2M+A)^{-1}\hat{f}(k)$. Let $v = u - f$. Then $v \in L^p(\mathbb{T}, X)$ and $\hat{v}(k) = -A(k^2M + A)^{-1}\hat{f}(k)$. We notice that A^{-1} is an isomorphism from X onto $D(A)$ as $0 \in \rho_M(A)$ by assumption; here we consider $D(A)$ as a Banach space equipped with its graph norm. It follows that $A^{-1}\hat{v}(k) = -(k^2M + A)^{-1}\hat{f}(k)$. Put $w = A^{-1}v$. Then $w \in L^p(\mathbb{T}, D(A))$, $\hat{w}(k) \in D(A)$ and $\hat{w}(k) = -(k^2M + A)^{-1}\hat{f}(k)$. We have

$$ik\hat{w}(k) = -ik(k^2M + A)^{-1}\hat{f}(k) = -iS_k\hat{f}(k),$$

so $w \in W^{1,p}_{\mathrm{per}}(\mathbb{T}, X)$ as $(S_k)_{k\in\mathbb{Z}}$ is an L^p-Fourier multiplier. We have shown that $w \in W^{1,p}_{\mathrm{per}}(\mathbb{T}, X) \cap L^p(\mathbb{T}, D(A))$ and $(w')^\wedge(k) = -ik(k^2M + A)^{-1}\hat{f}(k)$. Now,

$$M(w')^\wedge(k) = -ikM(k^2M + A)^{-1}\hat{f}(k) = T_k\hat{f}(k).$$

Thus $w' \in L^p(\mathbb{T}, D(M))$ as $(T_k)_{k\in\mathbb{Z}}$ is an L^p-Fourier multiplier. Hence

$$ik(Mw')^\wedge(k) = -k^2M(k^2M + A)^{-1}\hat{f}(k) = -N_k\hat{f}(k).$$

Thus $Mw' \in W^{1,p}_{\mathrm{per}}(\mathbb{T}, X)$ and $[(Mw')']^\wedge(k) = -N_k\hat{f}(k)$. Here we have used the fact that $(N_k)_{k\in\mathbb{Z}}$ is an L^p-Fourier multiplier. We have shown that $w \in S_p(A, M)$. Now,

$$[(Mw')']^\wedge(k) - A\hat{w}(k) = (k^2M + A)(k^2M + A)^{-1}\hat{f}(k) = \hat{f}(k)$$

for $k \in \mathbb{Z}$. It follows that $(Mw')'(t) = Aw(t) + f(t)$ a.e. on $\mathbb{T}$ by the uniqueness theorem [1]. Thus w is a strong L^p-solution of (P_2). This shows the existence.

To show the uniqueness, we let $u \in S_p(A, M)$ satisfy $(Mu')'(t) = Au(t)$ a.e. on $\mathbb{T}$. Taking Fourier transforms on both sides, one sees by (2.4) that $-k^2M\hat{u}(k) = A\hat{u}(k)$ for all $k \in \mathbb{Z}$. It follows that $(k^2M + A)^{-1}\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$. Therefore $u = 0$ as $ik \in \rho_M(A)$ for all $k \in \mathbb{Z}$. Thus (P_2) is L^p-well-posed. ■

Remark 2.9. It was shown in [1, Theorem 6.1] that if X is a UMD Banach space, $1 < p < \infty$ and $M = I_X$, then (P_2) is L^p-well-posed if and only if $i\mathbb{Z} \subset \rho(A)$ and the set $\{k^2(k^2 + A)^{-1} : k \in \mathbb{Z}\}$ is R-bounded. When $M = I_X$, we have actually $k^2(k^2M + A)^{-1} = k^2M(k^2M + A)^{-1}$. Thus under the only assumption that $\{k^2M(k^2M + A)^{-1} : k \in \mathbb{Z}\}$ is R-bounded, we easily deduce the R-boundedness of $\{k(k^2M + A)^{-1} : k \in \mathbb{Z}\}$. Thus Theorems 2.7 and 2.8 together recover the result of W. Arendt and S. Q. Bu [1, Theorem 6.1] in the special case $M = I_X$.

3. The $B^s_{p,q}$-well-posedness of (P_2). In this section, we study the $B^s_{p,q}$-well-posedness of (P_2). First, we briefly recall the definition of periodic Besov spaces $B^s_{p,q}(\mathbb{T}, X)$ in the vector-valued case, introduced in [2]. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of all rapidly decreasing smooth functions on $\mathbb{R}$. Let $\mathcal{D}(\mathbb{T})$ be the space of all infinitely differentiable functions on $\mathbb{T}$ equipped with the locally convex topology given by the seminorms $\|f\|_\alpha = \sup_{x\in\mathbb{T}} |f^{(\alpha)}(x)|$ for $\alpha \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $\mathcal{D}'(\mathbb{T}, X) := \mathcal{L}(\mathcal{D}(\mathbb{T}), X)$ be the space of all bounded linear operators from $\mathcal{D}(\mathbb{T})$ to X. In order to define Besov spaces, we consider the dyadic-like subsets of $\mathbb{R}$:

$$I_0 = \{t \in \mathbb{R} : |t| \le 2\}, \qquad I_k = \{t \in \mathbb{R} : 2^{k-1} < |t| \le 2^{k+1}\}$$

for $k \in \mathbb{N}$. Let $\phi(\mathbb{R})$ be the set of all systems $\phi = (\phi_k)_{k\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R})$ satisfying $\mathrm{supp}(\phi_k) \subset \bar{I_k}$ for each $k \in \mathbb{N}_0$,

$$\sum_{k\in\mathbb{N}_0} \phi_k(x) = 1 \quad \text{for } x \in \mathbb{R},$$

and for each $\alpha \in \mathbb{N}_0$,

$$\sup_{\substack{x\in\mathbb{R}\\ k\in\mathbb{N}_0}} 2^{k\alpha} |\phi_k^{(\alpha)}(x)| < \infty.$$

Let $\phi = (\phi_k)_{k\in\mathbb{N}_0} \in \phi(\mathbb{R})$ be fixed. For $1 \le p, q \le \infty$, $s \in \mathbb{R}$, the X-valued periodic Besov space is defined by

$$B^s_{p,q}(\mathbb{T}, X) = \Big\{ f \in \mathcal{D}'(\mathbb{T}, X) : \\ \|f\|_{B^s_{p,q}} := \Big(\sum_{j\ge 0} 2^{sjq} \Big\| \sum_{k\in\mathbb{Z}} e_k \phi_j(k) \hat{f}(k) \Big\|_p^q \Big)^{1/q} < \infty \Big\}$$

with the usual modification if $q = \infty$.

The set $B^s_{p,q}(\mathbb{T}, X)$ is independent of the choice of ϕ, and different choices of ϕ lead to equivalent norms $\|\cdot\|_{B^s_{p,q}}$. Equipped with the norm $\|\cdot\|_{B^s_{p,q}}$, $B^s_{p,q}(\mathbb{T}, X)$ is a Banach space.

It is known that if $s_1 \le s_2$, then $B^{s_2}_{p,q}(\mathbb{T}, X) \subset B^{s_1}_{p,q}(\mathbb{T}, X)$ and the embedding is continuous [2]. When $s > 0$, it is shown in [2] that $B^s_{p,q}(\mathbb{T}, X) \subset L^p(\mathbb{T}, X)$ and the embedding is continuous; moreover, $f \in B^{s+1}_{p,q}(\mathbb{T}, X)$ if and only f is differentiable a.e. on $\mathbb{T}$ and $f' \in B^s_{p,q}(\mathbb{T}, X)$. This implies that if $u \in B^s_{p,q}(\mathbb{T}, X)$ is such that there exists $v \in B^s_{p,q}(\mathbb{T}, X)$ satisfying $\hat{v}(k) = ik\hat{u}(k)$ for all $k \in \mathbb{Z}$, then $u \in B^{s+1}_{p,q}(\mathbb{T}, X)$ and $u' = v$. See [2, Section 2] for more information about the space $B^s_{p,q}(\mathbb{T}, X)$.

For $1 \le p, q \le \infty$, $s > 0$, we define the $B^s_{p,q}$-well-posedness solution space for (P_2) by

$$S_{p,q,s}(A, M) := \{u \in B^{s+1}_{p,q}(\mathbb{T}, X) \cap B^s_{p,q}(\mathbb{T}, D(A)) : \\ u' \in B^s_{p,q}(\mathbb{T}, D(M)),\ Mu' \in B^{s+1}_{p,q}(\mathbb{T}, X), X)\},$$

where we consider $D(A)$ and $D(M)$ as Banach spaces equipped with their graph norms. $S_{p,q,s}(A, M)$ is a Banach space under the norm

$$\|u\|_{S_{p,q,s}(A,M)} := \|u\|_{B^s_{p,q}} + \|u'\|_{B^s_{p,q}} + \|Au\|_{B^s_{p,q}} + \|Mu'\|_{B^s_{p,q}} + \|(Mu')'\|_{B^s_{p,q}}.$$

By [1, Lemma 2.1], if $u \in S_{p,q,s}(A, M)$, then u and Mu' are continuous functions on $\mathbb{T}$, and $u(0) = u(2\pi)$, $(Mu')(0) = (Mu')(2\pi)$. Now we give the definition of the $B^s_{p,q}$-well-posedness of the problem (P_2).

Definition 3.1. Let $1 \le p, q \le \infty$, $s > 0$ and $f \in B^s_{p,q}(\mathbb{T}, X)$. Then $u \in S_{p,q,s}(A, M)$ is called a *strong $B^s_{p,q}$-solution* of (P_2) if (P_2) is satisfied a.e. on $\mathbb{T}$. We say that (P_2) is $B^s_{p,q}$-*well-posed* if for each $f \in B^s_{p,q}(\mathbb{T}, X)$, (P_2) has a unique strong $B^s_{p,q}$-solution.

If (P_2) is $B^s_{p,q}$-well-posed, then there exists a constant $C > 0$, such that for all $f \in B^s_{p,q}(\mathbb{T}, X)$, if u is the unique strong $B^s_{p,q}$-solution of (P_2), then

$$\|u\|_{B^s_{p,q}} + \|u'\|_{B^s_{p,q}} + \|Mu'\|_{B^s_{p,q}} + \|(Mu')'\|_{B^s_{p,q}} + \|Au\|_{B^s_{p,q}} \le C\|f\|_{B^s_{p,q}}. \tag{3.1}$$

This can be easily obtained by the closedness of the operators A, M and the Closed Graph Theorem.

The main tool in studying the $B^s_{p,q}$-well-posedness of (P_2) is the operator-valued $B^s_{p,q}$-Fourier multiplier technique.

Definition 3.2. Let X, Y be Banach spaces, $1 \le p, q \le \infty$, $s \in \mathbb{R}$ and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$. We say that $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-*Fourier multiplier* if for each $f \in B^s_{p,q}(\mathbb{T}, X)$, there exists (a unique) $u \in B^s_{p,q}(\mathbb{T}, Y)$ such that $\hat{u}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$.

It is easy to see that if $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier, then the set $\{M_k : k \in \mathbb{Z}\}$ must be bounded. The following result of [2] gives a sufficient condition for an operator-valued sequence to be a $B^s_{p,q}$-Fourier multiplier:

Theorem 3.3. *Let X, Y be Banach spaces, $1 \le p, q \le \infty$, $s \in \mathbb{R}$ and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$. Assume that*

$$\sup_{k\in\mathbb{Z}} (\|M_k\| + \|k(M_{k+1} - M_k)\|) < \infty, \tag{3.2}$$

$$\sup_{k\in\mathbb{Z}} \|k^2(M_{k+2} - 2M_{k+1} + M_k)\| < \infty. \tag{3.3}$$

Then $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier.

Remarks 3.4. (i) If $(M_k)_{k\in\mathbb{Z}}$ and $(N_k)_{k\in Z}$ are $B^s_{p,q}$-Fourier multipliers, then so is $(M_k N_k)_{k\in\mathbb{Z}}$.

(ii) If $c_k = 1/k$ when $k \neq 0$ and $c_0 = 1$, then $(c_k I_X)_{k\in\mathbb{Z}}$ satisfies the sufficient conditions (3.2) and (3.3) in Theorem 3.3. Thus $(c_k I_X)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier. This can also be deduced from Young's inequality [7].

Now we are able to state a necessary condition for the $B^s_{p,q}$-well-posedness of (P_2).

Theorem 3.5. *Let X be a Banach space, $1 \le p, q \le \infty$, $s > 0$ and A, M be closed linear operators in X satisfying $D(A) \subset D(M)$. Assume that (P_2) is $B^s_{p,q}$-well-posed. Then $i\mathbb{Z} \subset \rho_M(A)$ and $(N_k)_{k\in\mathbb{Z}}$ defines a $B^s_{p,q}$-Fourier multiplier, where $N_k = k^2M(k^2M + A)^{-1}$ for $k \in \mathbb{Z}$. In particular, the set $\{N_k : k \in \mathbb{Z}\}$ is bounded.*

Proof. Fix $k \in \mathbb{Z}$ and $y \in X$ and let $f(t) = e^{ikt}y$ for $t \in \mathbb{T}$. Then $f \in B^s_{p,q}(\mathbb{T}, X)$, $\hat{f}(k) = y$ and $\hat{f}(n) = 0$ when $n \neq k$. There exists $u \in$

$S_{p,q,s}(A,M)$ such that

$$(Mu')'(t) = Au(t) + f(t) \tag{3.4}$$

for almost all $t \in \mathbb{T}$ by assumption. We remark that f, Au, u, u', Mu' and $(Mu')'$ are elements of $L^p(\mathbb{T},X)$ as $B^s_{p,q}(\mathbb{T},X) \subset L^p(\mathbb{T},X)$ (here we have used the assumption that $s > 0$), so their Fourier transforms make sense. We have $\hat{u}(n) \in D(A)$ for all $n \in \mathbb{Z}$ by [1, Lemma 3.1]. Therefore

$$[(Mu')']^\wedge(n) = in[Mu']^\wedge(n) = inM(u')^\wedge(n) = -n^2 M\hat{u}(n) \tag{3.5}$$

for $n \in \mathbb{Z}$ by the closedness of M and [1, Lemma 3.1]. Taking Fourier transforms on both sides of (3.4), and using the closedness of A and (3.5), we obtain

$$(-k^2 M - A)\hat{u}(k) = y \tag{3.6}$$

and $(-n^2 M - A)\hat{u}(n) = 0$ when $n \neq k$. Thus $-k^2M - A$ is surjective. We are going to show that it is also injective. Assume that $x \in D(A)$ satisfies $(-k^2M - A)x = 0$, so $-k^2Mx = Ax$. Let $u(t) = e^{ikt}x$ for $t \in \mathbb{T}$. Then $u \in S_{p,q,s}(A,M)$ as $x \in D(A) \subset D(M)$. It is clear that $(Mu')'(t) = Au(t)$ for almost all $t \in \mathbb{T}$. Hence $x = 0$ by the uniqueness assumption. We have shown that $-k^2M - A$ is bijective from $D(A)$ to X.

Next, we show that $(-k^2M - A)^{-1} \in \mathcal{L}(X)$. For $f(t) = e^{ikt}y$, we let $u \in S_{p,q,s}(A,M)$ be the unique strong $B^s_{p,q}$-solution of (P_2). Then by (3.6),

$$\hat{u}(n) = \begin{cases} 0, & n \neq k, \\ (-k^2M - A)^{-1}y, & n = k. \end{cases}$$

This implies that $u(t) = e^{ikt}(-k^2M - A)^{-1}y$. By (3.1), there exists a constant $C > 0$, independent of y and k, such that

$$\|u\|_{B^s_{p,q}} + \|u'\|_{B^s_{p,q}} + \|Mu'\|_{B^s_{p,q}} + \|(Mu')'\|_{B^s_{p,q}} + \|Au\|_{B^s_{p,q}} \le C\|f\|_{B^s_{p,q}}.$$

This implies that $\|(k^2M + A)^{-1}\| \le C$. We have shown that $ik \in \rho_M(A)$ whenever $k \in \mathbb{Z}$. Thus $i\mathbb{Z} \subset \rho_M(A)$.

Finally, we show that if $N_k = k^2M(k^2M + A)^{-1}$ for $k \in \mathbb{Z}$, then $(N_k)_{k\in\mathbb{Z}}$ defines a $B^s_{p,q}$-Fourier multiplier. Let $f \in B^s_{p,q}(\mathbb{T},X)$. Then by assumption there exists $u \in S_{p,q,s}(A,M)$ which is the unique strong $B^s_{p,q}$-solution of (P_2). Taking Fourier transforms on both sides of (P_2), we find that $\hat{u}(k) \in D(A)$ for all $k \in \mathbb{Z}$ by [1, Lemma 3.1], and

$$(-k^2M - A)\hat{u}(k) = \hat{f}(k) \quad (k \in \mathbb{Z})$$

by (3.5). Since $-k^2M - A$ is invertible, we have $\hat{u}(k) = (-k^2M - A)^{-1}\hat{f}(k)$. By (3.5),

$$((Mu')')^\wedge(k) = -k^2M\hat{u}(k) = -N_k\hat{f}(k) \quad (k \in \mathbb{Z}).$$

Hence $(N_k)_{k\in\mathbb{Z}}$ defines a $B^s_{p,q}$-Fourier multiplier as $(Mu')' \in B^s_{p,q}(\mathbb{T},X)$. ∎

The following result gives a sufficient condition for (P_2) to be $B^s_{p,q}$-well-posed.

THEOREM 3.6. *Let X be a Banach space, $1 \le p, q \le \infty$, $s > 0$ and A, M be closed linear operators in X satisfying $D(A) \subset D(M)$. Assume that $i\mathbb{Z} \subset \rho_M(A)$ and the sets $\{k(k^2M+A)^{-1} : k \in \mathbb{Z}\}$ and $\{k^2M(k^2M+A)^{-1} : k \in \mathbb{Z}\}$ are bounded. Then (P_2) is $B^s_{p,q}$-well-posed.*

Proof. Let $N_k = k^2M(k^2M + A)^{-1}$ and $S_k = k(k^2M + A)^{-1}$ for $k \in \mathbb{Z}$. Then $\{N_k : k \in \mathbb{Z}\}$ and $\{S_k : k \in \mathbb{Z}\}$ are bounded by assumption. We claim that $(N_k)_{k\in\mathbb{Z}}$ defines a $B^s_{p,q}$-Fourier multiplier. In view of Theorem 3.3, it is sufficient to show that $\{k(N_{k+1}-N_k) : k \in \mathbb{Z}\}$ and $\{k^2(N_{k+2}-2N_{k+1}+N_k) : k \in \mathbb{Z}\}$ are bounded. The proof that the former set is bounded is the same as in the proof of Theorem 2.8.

Let $H_k = (k^2M + A)^{-1} \in \mathcal{L}(X)$. Then $N_k = k^2MH_k$ and $S_k = kH_k$. We again have the equality

$$H_{k+1} - H_k = -(2k + 1)H_kMH_{k+1}. \tag{3.7}$$

By the proof of Theorem 2.8,

$$N_{k+1} - N_k = (2k + 1)MH_{k+1} - k^2(2k + 1)MH_kMH_{k+1} =: A_k - B_k.$$

To show that $\{k^2(N_{k+2} - 2N_{k+1} + N_k) : k \in \mathbb{Z}\}$ is bounded, it will suffice to show that both $\{k^2(A_{k+1} - A_k) : k \in \mathbb{Z}\}$ and $\{k^2(B_{k+1} - B_k) : k \in \mathbb{Z}\}$ are bounded. By (3.7) we have

$$\begin{aligned} k^2(A_{k+1} - A_k) &= k^2(2k + 3)MH_{k+2} - k^2(2k + 1)MH_{k+1} \\ &= 2k^2MH_{k+2} + k^2(2k + 1)M(H_{k+2} - H_{k+1}) \\ &= \frac{2k^2}{(k + 2)^2}N_{k+2} - \frac{k^2(2k + 1)(2k + 3)}{(k + 1)^2(k + 2)^2}N_{k+1}N_{k+2}, \end{aligned}$$

which is clearly uniformly bounded in $k \in \mathbb{Z}$ by assumption. On the other hand, again by (3.7),

$$\begin{aligned} k^2(B_{k+1} - B_k) &= k^2(k + 1)^2(2k + 3)MH_{k+1}MH_{k+2} \\ &\quad - k^4(2k + 1)MH_kMH_{k+1} \\ &= k^2(6k^2 + 8k + 3)MH_{k+1}MH_{k+2} \\ &\quad + k^4(2k + 1)M[H_{k+1}MH_{k+2} - H_kMH_{k+1}]. \end{aligned} \tag{3.8}$$

The first term is just $\frac{k^2(6k^2+8k+3)}{(k+1)^2(k+2)^2}N_{k+1}N_{k+2}$, so it is uniformly bounded in $k \in \mathbb{Z}$ by assumption. To estimate the second term, by (3.7) we have

$$\begin{aligned} H_{k+1}MH_{k+2} - H_kMH_{k+1} &= H_{k+1}M(H_{k+2} - H_{k+1}) + (H_{k+1} - H_k)MH_{k+1} \\ &= -(2k + 3)H_{k+1}MH_{k+1}MH_{k+2} \\ &\quad - (2k + 1)H_kMH_{k+1}MH_{k+1}. \end{aligned}$$

Therefore, the second term in (3.8) is just

$$-\frac{k^4(2k+3)(2k+1)}{(k+1)^4(k+2)^2}N_{k+1}^2N_{k+2}-\frac{k^2(2k+1)^2}{(k+1)^4}N_kN_{k+1}^2,$$

which is uniformly bounded in $k\in\mathbb{Z}$ by assumption. It follows that the set $\{k^2(N_{k+2}-2N_{k+1}+N_k) : k\in\mathbb{Z}\}$ is bounded. Thus $(N_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier by Theorem 3.3.

Now, we show that $(S_k)_{k\in\mathbb{Z}}$ is also a $B^s_{p,q}$-Fourier multiplier. It is sufficient to show that $\{k(S_{k+1}-S_k) : k\in\mathbb{Z}\}$ and $\{k^2(S_{k+2}-2S_{k+1}+S_k) : k\in\mathbb{Z}\}$ are bounded by Theorem 3.3. The proof for the former set is the same as in the proof of Theorem 2.8. We know from that proof that

$$S_{k+1}-S_k=H_{k+1}-k(2k+1)H_{k+1}MH_k$$

for all $k\in\mathbb{Z}$. By (3.7) we have

$$\begin{aligned}k^2(H_{k+2}-H_{k+1})&=-k^2(2k+3)H_{k+1}MH_{k+2}\\&=-\frac{k^2(2k+3)}{(k+2)^2(k+1)}S_{k+1}N_{k+2},\end{aligned}$$

which is uniformly bounded in $k\in\mathbb{Z}$. On the other hand, again by (3.7),

$$\begin{aligned}&k^2[(k+1)(2k+3)H_{k+2}MH_{k+1}-k(2k+1)H_{k+1}MH_k]\\&\quad=k^2(4k+3)H_{k+2}MH_{k+1}+k^3(2k+1)[H_{k+2}MH_{k+1}-H_{k+1}MH_k]\\&\quad=k^2(4k+3)H_{k+2}MH_{k+1}+k^3(2k+1)(H_{k+2}-H_{k+1})MH_{k+1}\\&\qquad+k^3(2k+1)H_{k+1}M(H_{k+1}-H_k)\\&\quad=k^2(4k+3)H_{k+2}MH_{k+1}-k^3(2k+1)(2k+3)H_{k+1}MH_{k+2}MH_{k+1}\\&\qquad-k^3(2k+1)^2H_{k+1}MH_kMH_{k+1}\\&\quad=\frac{k^2(4k+3)}{(k+2)(k+1)^2}S_{k+2}N_{k+1}-\frac{k^3(2k+1)(2k+3)}{(k+1)^3(k+2)^2}S_{k+1}N_{k+2}N_{k+1}\\&\qquad-\frac{k(2k+1)^2}{(k+1)^3}S_{k+1}N_kN_{k+1},\end{aligned}$$

which is uniformly bounded in $k\in\mathbb{Z}$ by assumption. We have shown that the set $\{k^2(S_{k+2}-2S_{k+1}+S_k) : k\in\mathbb{Z}\}$ is bounded. Thus $(S_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier by Theorem 3.3.

Let $T_k=-ikM(k^2M+A)^{-1}$ for $k\in\mathbb{Z}$. Then $(T_k)_{k\in\mathbb{Z}}$ is also a $B^s_{p,q}$-Fourier multiplier. This follows easily from the fact that $(N_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier and from Remarks 3.4.

Now we are going to show that (P_2) is $B^s_{p,q}$-well-posed. Since $(N_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier, for all $f\in B^s_{p,q}(\mathbb{T},X)$ there exists $u\in B^s_{p,q}(\mathbb{T},X)$ satisfying $\hat{u}(k)=N_k\hat{f}(k)$ for all $k\in\mathbb{Z}$. Using the identity

$$I_X=k^2M(k^2M+A)^{-1}+A(k^2M+A)^{-1},$$

we deduce that

$$\hat{u}(k) = k^2M(k^2M + A)^{-1}\hat{f}(k) = (I_X - A(k^2M + A)^{-1})\hat{f}(k).$$

Thus $(u-f)^\wedge(k) = -A(k^2M+A)^{-1}\hat{f}(k)$. Let $v = u-f$. Then $v \in B^s_{p,q}(\mathbb{T}, X)$ and $\hat{v}(k) = -A(k^2M + A)^{-1}\hat{f}(k)$. We notice that A^{-1} is an isomorphism from X onto $D(A)$ as $0 \in \rho_M(A)$ by assumption; here we consider $D(A)$ as a Banach space equipped with its graph norm. It follows that $A^{-1}\hat{v}(k) = -(k^2M + A)^{-1}\hat{f}(k)$. Put $w = A^{-1}v$. Then $w \in B^s_{p,q}(\mathbb{T}, D(A))$, $\hat{w}(k) \in D(A)$ and $\hat{w}(k) = -(k^2M + A)^{-1}\hat{f}(k)$. We have

$$ik\hat{w}(k) = -ik(k^2M + A)^{-1}\hat{f}(k) = -iS_k\hat{f}(k),$$

so $w \in B^{s+1}_{p,q}(\mathbb{T}, X)$ as $(S_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier. We have shown that $w \in B^{s+1}_{p,q}(\mathbb{T}, X) \cap B^s_{p,q}(\mathbb{T}, D(A))$ and $(w')^\wedge(k) = -ik(k^2M + A)^{-1}\hat{f}(k)$. Now,

$$M(w')^\wedge(k) = -ikM(k^2M + A)^{-1}\hat{f}(k) = T_k\hat{f}(k).$$

Thus $w' \in B^s_{p,q}(\mathbb{T}, D(M))$ or equivalently $Mw' \in B^s_{p,q}(\mathbb{T}, X)$ as $(T_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier. From this we deduce that

$$ik(Mw')^\wedge(k) = -k^2M(k^2M + A)^{-1}\hat{f}(k) = -N_k\hat{f}(k).$$

Thus $Mw' \in B^{s+1}_{p,q}(\mathbb{T}, X)$ and $[(Mw')']^\wedge(k) = -N_k\hat{f}(k)$. Here we have used (3.5) and the fact that $(N_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier. We have shown that $w \in S_{p,q,s}(A, M)$ and

$$[(Mw')']^\wedge(k) - A\hat{w}(k) = -(k^2M + A)(k^2M + A)^{-1}\hat{f}(k) = \hat{f}(k)$$

for $k \in \mathbb{Z}$. It follows that $(Mw')'(t) = Aw(t) + f(t)$ a.e. on $\mathbb{T}$ by [1]. Thus w is a strong $B^s_{p,q}$-solution of (P_2). This shows the existence.

To show the uniqueness, we let $u \in S_{p,q,s}(A, M)$ satisfy $(Mu')'(t) = Au(t)$ a.e. on $\mathbb{T}$. Taking Fourier transforms on both sides, one sees by (3.5) that $-k^2M\hat{u}(k) = A\hat{u}(k)$ for all $k \in \mathbb{Z}$, so $(k^2M+A)^{-1}\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$. Therefore $u = 0$ as $ik \in \rho_M(A)$ for all $k \in \mathbb{Z}$. Thus (P_2) is $B^s_{p,q}$-well-posed. The proof is complete. ■

Remark 3.7. It was shown in [2, Theorem 5.3] that if X is a Banach space, $1 \le p, q \le \infty$, $s > 0$ and $M = I_X$, then (P_2) is $B^s_{p,q}$-well-posed if and only if $i\mathbb{Z} \subset \rho(A)$ and the set $\{k^2(k^2 + A)^{-1} : k \in \mathbb{Z}\}$ is bounded. When $M = I_X$, we actually have $k^2(k^2M + A)^{-1} = k^2M(k^2M + A)^{-1}$. Thus under the only assumption that $\{k^2M(k^2M + A)^{-1} : k \in \mathbb{Z}\}$ is bounded, we easily deduce the boundedness of $\{k(k^2M + A)^{-1} : k \in \mathbb{Z}\}$. Hence Theorems 3.5 and 3.6 together recover the result of W. Arendt and S. Q. Bu [2, Theorem 5.3].

The periodic Hölder continuous function space is a particular case of the periodic Besov space $B^s_{p,q}(\mathbb{T}, X)$. From [2, Theorem 3.1], we have

$B^\alpha_{\infty,\infty}(\mathbb{T},X) = C^\alpha_{\mathrm{per}}(\mathbb{T},X)$ whenever $0<\alpha<1$, where $C^\alpha_{\mathrm{per}}(\mathbb{T},X)$ is the space of all X-valued functions f defined on $\mathbb{T}$ satisfying $f(0)=f(2\pi)$ and

$$\sup_{x\neq y}\frac{\|f(x)-f(y)\|}{|x-y|^\alpha}<\infty.$$

Moreover the norm

$$\|f\|_{C^\alpha_{\mathrm{per}}} := \max_{t\in\mathbb{T}}\|f(t)\| + \sup_{x\neq y}\frac{\|f(x)-f(y)\|}{|x-y|^\alpha}$$

on $C^\alpha_{\mathrm{per}}(\mathbb{T},X)$ is an equivalent norm on $B^\alpha_{\infty,\infty}(\mathbb{T},X)$. If $0<\alpha<1$, we say that (P_2) is C^α_{per}-*well-posed* if for every $f\in C^\alpha_{\mathrm{per}}(\mathbb{T},X)$, there exists a unique $u\in C^{\alpha+1}(\mathbb{T},X)\cap C^\alpha_{\mathrm{per}}(\mathbb{T},D(A))$ such that $u'\in C^\alpha(\mathbb{T},X)$, $Mu'\in C^{\alpha+1}(\mathbb{T},X)$ and (P_2) holds true for all $t\in[0,2\pi]$. Here $C^{\alpha+1}(\mathbb{T},X)$ is the space of all $u\in C^1(\mathbb{T},X)$ such that $u,u'\in C^\alpha_{\mathrm{per}}(\mathbb{T},X)$. Theorem 3.6 has the following corollary.

COROLLARY 3.8. *Let X be a Banach space, $0<\alpha<1$, and A, M be closed linear operators in X satisfying $D(A)\subset D(M)$. Assume that $i\mathbb{Z}\subset\rho_M(A)$ and the sets $\{k(k^2M+A)^{-1}:k\in\mathbb{Z}\}$ are $\{k^2M(k^2M+A)^{-1}:k\in\mathbb{Z}\}$ are bounded. Then (P_2) is C^α-well-posed.*

REMARK 3.9. We can introduce the notion of well-posedness of (P_2) in periodic Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{T},X)$ in a similar way. Using operator-valued Fourier multiplier results on vector-valued periodic Triebel–Lizorkin spaces established in [5], one can prove a result analogous to Theorem 3.6 for the scale of Triebel–Lizorkin spaces.

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