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Absolute convergence of multiple Fourier integrals

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Abstract. Various new sufficient conditions for representation of a function of several variables as an absolutely convergent Fourier integral are obtained. The results are given in terms of L_p integrability of the function and its partial derivatives, each with a different p. These p are subject to certain relations known earlier only for some particular cases. Sharpness and applications of the results obtained are also discussed.

1. Introduction. If

$$f(y) = \int_{\mathbb{R}^d} g(x) e^{i(x,y)}\,dx, \qquad g \in L_1(\mathbb{R}^d),$$

we write $f \in A(\mathbb{R}^d)$ with $\|f\|_A = \|g\|_{L_1(\mathbb{R}^d)}$. Here

$$x = (x_1, \dots, x_d),\ y = (y_1, \dots, y_d) \in \mathbb{R}^d, \qquad (x, y) = x_1 y_1 + \dots + x_d y_d.$$

The possibility to represent a function as an absolutely convergent Fourier integral has been studied by many mathematicians and is of importance in various problems of analysis. For example, belonging of a function $m(x)$ to $A(\mathbb{R}^d)$ makes it an $L_1 \to L_1$ Fourier multiplier (or, equivalently, $L_\infty \to L_\infty$ Fourier multiplier), written $m \in M_1$ ($m \in M_\infty$, respectively). One of such m's attracted much attention in the 50–80s (see, e.g., [W], [F], [St, Ch. 4, 7.4], [Mi], and references therein):

$$m(x) := m_{\alpha,\beta}(x) = \theta(x) \frac{e^{i|x|^\alpha}}{|x|^\beta}, \tag{1.1}$$

where θ is a C^∞ function on $\mathbb{R}^d$, which vanishes near zero, and equals 1 outside a bounded set, and $0 < \alpha < 1$, $\beta > 0$. Of course, $|x|^2 = x_1^2 + \dots + x_d^2$. It is known (see, e.g., [Mi]) that for $d \ge 2$:

(I) *If* $\beta/\alpha > d/2$, *then* $m \in M_1(M_\infty)$ (*or, equivalently,* $m \in A(\mathbb{R}^d)$).
(II) *If* $\beta/\alpha \le d/2$, *then* $m \notin M_1(M_\infty)$ (*or, equivalently,* $m \notin A(\mathbb{R}^d)$).

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The first assertion holds true for $d = 1$ as well, while the second one only when $\alpha \neq 1$; however, the case $\alpha = d = 1$ is obvious.

Various sufficient conditions for absolute convergence of Fourier integrals were obtained by Titchmarsh, Beurling, Carleman, Sz.-Nagy, and others. One can find a comprehensive survey of this problem in [LST].

Let us unite certain of the known one-dimensional results closely related to our study in the following theorem. First, each $f \in A(\mathbb{R})$ satisfies the condition

(N-1) *$f \in C_0(\mathbb{R})$, that is, $f \in C(\mathbb{R})$ and $\lim f(t) = 0$ as $|t| \to \infty$, and f is locally absolutely continuous on $\mathbb{R}$.*

It is natural to restrict ourselves only to such functions in dimension one.

THEOREM A-1. *Let f satisfy condition (N-1), $f \in L_p(\mathbb{R})$ with $1 \le p \le 2$, and $f' \in L_q(\mathbb{R})$ with $1 < q \le 2$. Then $f \in A(\mathbb{R})$.*

For the multivariate case, we need additional notations. Let η be a d-dimensional vector with entries 0 or 1. We set $\mathbf{0} = (0, \dots, 0)$ and $\mathbf{1} = (1, \dots, 1)$. Inequalities of vectors are meant coordinatewise.

We write

$$D^\eta f(x) = \left(\frac{\partial}{\partial x_1}\right)^{\eta_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\eta_d} f(x).$$

In particular $D^\eta f = f$ for $\eta = \mathbf{0}$.

The multidimensional results we are going to generalize in a sense (see [M] and [S], respectively) read as follows.

THEOREM A1-*d*. *Let $f \in L_2(\mathbb{R}^d)$. If all the (distributional) partial derivatives $\partial^{\beta_j} f/\partial x_j^{\beta_j}$, $j = 1, \dots, d$, are in $L_2(\mathbb{R}^d)$, where β_j are any positive integers such that $\sum_{j=1}^d 1/\beta_j < 2$, then $f \in A(\mathbb{R}^d)$.*

THEOREM A2-*d*. *Let $f \in L_1(\mathbb{R}^d)$. If all the (distributional) partial derivatives $D^\eta f$, $\eta \neq \mathbf{0}$, are in $L_p(\mathbb{R}^d)$, where $1 < p \le 2$, then $f \in A(\mathbb{R}^d)$.*

We obtain new sufficient conditions for belonging to $A(\mathbb{R}^d)$. They are given in terms of belonging of the function considered and its derivatives to L_{p_j} spaces, with different p_j. These p_j are related in a special way. For example, in dimension one some of the relations obtained turn into the condition from [L] that ensures the possibility to represent a function f as an absolutely convergent Fourier integral: $f \in L_{p_0}(\mathbb{R})$, $1 \le p_0 < \infty$, and $f' \in L_{p_1}(\mathbb{R})$, $1 < p_1 < \infty$, with $1/p_0 + 1/p_1 > 1$. One can see a disparity in strength with Theorem A-1 above. Correspondingly, the range of conditions ensuring belonging to $A(\mathbb{R}^d)$ is much wider than known earlier in the multidimensional case.

An outline of the paper is as follows. In the next section we formulate the results. In Section 3 we present the needed auxiliary results. Then, in Section 4 we concentrate on the one-dimensional version of our main results. In the last section we give multidimensional proofs; one-dimensional arguments from the preceding section will be intensively used.

We shall denote absolute positive constants by C; they may be different in different occurrences.

2. Statement of results. It turns out that in several dimensions there are a variety of results in terms of different combinations of derivatives. However, in any case, it is reasonable to start with the following d-dimensional analog of the above (N-1) condition, common to all of them.

(N-d) *$f \in C_0(\mathbb{R}^d)$, and f and its partial derivatives $D^\eta f$, for all $\eta \neq \mathbf{1}$, are locally absolutely continuous on $(\mathbb{R} \setminus \{0\})^d$ in each variable.*

We will also need a similar condition in which $r = (r_1, \dots, r_d)$ is a vector with entries $r_j \in \mathbb{N}$, $j = 1, \dots, d$:

(N-d,r) *$f \in C_0(\mathbb{R}^d)$, and f and its partial derivatives $\partial^{r_j - 1} f / \partial x_j^{r_j - 1}$, for all $j = 1, \dots, d$, are locally absolutely continuous on $(\mathbb{R} \setminus \{0\})^d$ in each variable.*

Of course, (N-1) stands for (N-1,1). However, in higher dimensions these two conditions are different in nature.

Our first main result reads as follows.

Theorem 2.1.

(a) *Let f satisfy (N-d). Let $f \in L_{p_\mathbf{0}}(\mathbb{R}^d)$, $1 \le p_\mathbf{0} < \infty$, and suppose each partial derivative $D^\eta f$, $\eta \neq \mathbf{0}$, belongs to $L_{p_\eta}(\mathbb{R}^d)$, where $1 < p_\eta < \infty$. If for all $\eta \neq \mathbf{0}$,*

$$\frac{1}{p_\mathbf{0}} + \frac{1}{p_\eta} > 1, \tag{2.1}$$

then $f \in A(\mathbb{R}^d)$.

(b) *Let $1 < p_\mathbf{0}, p_\mathbf{1} < \infty$. If*

$$\frac{1}{p_\mathbf{0}} + \frac{1}{p_\mathbf{1}} < 1,$$

then there is a function f satisfying (N-d) such that $f \in L_{p_\mathbf{0}}(\mathbb{R}^d)$, $D^\mathbf{1} f \in L_{p_\mathbf{1}}(\mathbb{R}^d)$, but $f \notin A(\mathbb{R}^d)$.

We can also obtain a result in which all the derivatives intervene.

Theorem 2.2.

(a) *Let f satisfy (N-d). Let $f \in L_{p_\mathbf{0}}(\mathbb{R}^d)$, $1 \le p_\mathbf{0} < \infty$, and suppose each partial derivative $D^\eta f$, $\eta \neq \mathbf{0}$, belongs to $L_{p_\eta}(\mathbb{R}^d)$, where*

$1 < p_\eta < \infty$. If

$$\sum_{\mathbf{0}\le\eta\le\mathbf{1}} \frac{1}{p_\eta} > 2^{d-1}, \tag{2.2}$$

then $f \in A(\mathbb{R}^d)$.

(b) *Suppose that there are $\alpha, \beta > 0$ with $\alpha \ne 1$ and $\beta > d(\alpha - 1)$ such that $p_\eta > d/(\beta - |\eta|(\alpha - 1))$ for all η. If*

$$\sum_{\mathbf{0}\le\eta\le\mathbf{1}} \frac{1}{p_\eta} < 2^{d-1},$$

then there is a function f satisfying (N-d) *such that $f \in L_{p_{\mathbf{0}}}(\mathbb{R}^d)$ and each partial derivative $D^\eta f$, $\eta \ne \mathbf{0}$, belongs to $L_{p_\eta}(\mathbb{R}^d)$, but $f \notin A(\mathbb{R}^d)$.*

Remark 2.3. Let us compare the last two theorems. If $2 \le p_{\mathbf{0}} < \infty$, inequalities (2.1) imply (2.2). Hence Theorem 2.1 is a consequence of Theorem 2.2. However, if $1 \le p_{\mathbf{0}} < 2$, it is easy to see that one can choose p_0 close enough to 1 and p_η, $\eta \ne \mathbf{0}$, large enough and such that inequalities (2.1) are valid, while (2.2) is not.

Observe also that in any case one can choose $p_\eta > 1$ such that inequality (2.2) is valid, while inequalities (2.1) are not.

Finally, we give a generalization of Theorem 2.2 to the case of higher derivatives.

Theorem 2.4. *Let f satisfy condition* (N-d,r) *for some vector $r = (r_1, \dots, r_d)$ such that $\sum_{j=1}^d 1/r_j < 2$ and $r_j \in \mathbb{N}$.*

(a) *Let $f \in L_p(\mathbb{R}^d)$, $1 \le p < p_0 < \infty$, $\partial^{r_j} f/\partial x_j^{r_j} \in L_{p_j}(\mathbb{R}^d)$, $1 < p_j < \infty$, $1 \le j \le d$. If*

$$\Big(2 - \sum_{j=1}^d \frac{1}{r_j}\Big)\frac{1}{p_0} + \sum_{j=1}^d \frac{1}{r_j p_j} = 1,$$

then $f \in A(\mathbb{R}^d)$.

(b) *Let $1 \le p < \infty$, $1 < q < \infty$, and $r > d/2$, $r \in \mathbb{N}$. If*

$$\Big(2 - \frac{d}{r}\Big)\frac{1}{p} + \frac{d}{rq} < 1,$$

then there is a function f satisfying (N-d,r) *for $r = r_1 = \cdots = r_d$, $f \in L_p(\mathbb{R}^d)$, $\partial^r f/\partial x_j^r \in L_p(\mathbb{R}^d)$, $1 \le j \le d$, but $f \notin A(\mathbb{R}^d)$.*

This is a direct extension of Theorem A1-d.

Remark 2.5. We can prove that the conditions of the above results are sharp only for certain p_η. The point is that we make use of $m_{\alpha,\beta}$ for which intermediate derivatives cannot be arbitrary.

The next corollaries give conditions upon which power decay of a function f and its derivatives ensures $f \in A(\mathbb{R}^d)$.

COROLLARY 2.6. *Let f satisfy* (N-d). *If for all η, $\mathbf{0} \le \eta \le \mathbf{1}$,*

$$|D^\eta f(x)| \le \frac{C}{(1+|x|)^{\gamma_\eta}}, \tag{2.3}$$

where $\gamma_\eta > 0$ and either

$$\gamma_0 + \gamma_\eta > d \quad \text{for all } \eta \ne \mathbf{0}, \tag{2.4}$$

or

$$\sum_{\mathbf{0} \le \eta \le \mathbf{1}} \gamma_\eta > d2^{d-1}, \tag{2.5}$$

then $f \in A(\mathbb{R}^d)$.

COROLLARY 2.7. *Let f satisfy* (N-d,r) *for some vector $r = (r_1, \ldots, r_d)$ such that $\sum_{j=1}^d 1/r_j < 2$, $r_j \in \mathbb{N}$. If for all j, $0 \le j \le d$,*

$$\left|\frac{\partial^{r_j}}{\partial x_j^{r_j}} f(x)\right| \le \frac{C}{(1+|x|)^{\gamma_j}},$$

where $\gamma_j > 0$ and

$$\left(2 - \sum_{j=1}^d \frac{1}{r_j}\right)\gamma_0 + \sum_{j=1}^d \frac{\gamma_j}{r_j} > d,$$

then $f \in A(\mathbb{R}^d)$.

3. Auxiliary results. One of the basic tools is the following lemma (see [T, Lemma 4] or [B, Theorem 3]), which is a natural extension of the celebrated Bernstein test for the absolute convergence of Fourier series (see [K, Ch. II, §6]). In fact, the proof of every result of this paper will use it. More precisely, each proof will use the assumptions of the corresponding statement to prove that the basic estimate (3.1), ensuring the membership of the relevant function in $A(\mathbb{R}^d)$, holds true.

In order to formulate the lemma in any dimension, we denote

$$\Delta_u^{\eta,r} f(x) = \Delta_{u_1,\ldots,u_d}^{\eta,r} f(x) = \prod_{j:\eta_j=1} \Delta_{u_j}^{e_j,r_j} f(x),$$

where η, u, and r are d-dimensional vectors, and $\Delta_{u_j}^{e_j,r_j} f$ is defined as

$$\Delta_{u_j}^{e_j,r_j} f(x) = \sum_{k=0}^{r_j} \binom{r_j}{k} (-1)^k f(x + (2k - r_j)u_j e_j), \qquad 1 \le j \le d.$$

Here e_j are unit basis vectors. Denote also

$$\Delta_u f(x) = \Delta_{u_1,\dots,u_d} f(x) = \Delta^{\mathbf{1},\mathbf{1}}_{u_1,\dots,u_d} f(x)$$

and

$$\Delta^\eta_u f(x) = \Delta^{\eta,\mathbf{1}}_u f(x).$$

LEMMA B. *Let $f \in C_0(\mathbb{R}^d)$. If for some vector $r = (r_1, \dots, r_d)$,*

$$\sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_d=-\infty}^{\infty} 2^{\frac{1}{2}\sum_{j=1}^d s_j} \|\Delta^{\mathbf{1},r}_{h(s_1),\dots,h(s_d)}(f)\|_2 < \infty, \tag{3.1}$$

where $h(s) = \pi 2^{-s}$ and the norm is that in $L_2(\mathbb{R}^d)$, then $f \in A(\mathbb{R}^d)$.

We will also make use of the following Hardy–Steklov type inequality (see [KP, Cor. 3.14]):

For $F \ge 0$ and $1 < q \le Q < \infty$,

$$\Big(\int_{\mathbb{R}} \Big[\int_{t-h}^{t+h} F(s)\,ds\Big]^Q dt\Big)^{1/Q} \le Ch^{1/Q+1/q'}\Big(\int_{\mathbb{R}} F^q(t)\,dt\Big)^{1/q}. \tag{3.2}$$

Here $1/q + 1/q' = 1$.

We need the following direct multivariate generalization of (3.2).

LEMMA 3.1. *For $F \ge 0$, $1 \le k < d$, and $1 < q \le Q < \infty$,*

$$\Big(\int_{\mathbb{R}^d} \Big[\int_{x_1-h_1}^{x_1+h_1} \cdots \int_{x_k-h_k}^{x_k+h_k} F(u_1, \dots, u_k, x_{k+1}, \dots, x_d)\,du_1 \dots du_k\Big]^Q dx\Big)^{1/Q} \tag{3.3}$$
$$\le C(h_1 \dots h_k)^{1/Q+1/q'}\Big(\int_{\mathbb{R}^{d-k}} \Big[\int_{\mathbb{R}^k} F^q(x)\,dx_1 \dots dx_k\Big]^{Q/q} dx_{k+1} \dots dx_d\Big)^{1/Q}.$$

If $k = d$,

$$\Big(\int_{\mathbb{R}^d} \Big[\int_{x_1-h_1}^{x_1+h_1} \cdots \int_{x_d-h_d}^{x_d+h_d} F(u)\,du\Big]^Q dx\Big)^{1/Q} \tag{3.4}$$
$$\le C(h_1 \dots h_d)^{1/Q+1/q'}\Big(\int_{\mathbb{R}^d} F^q(x)\,dx\Big)^{1/q}.$$

Of course, the first k variables are taken in (3.3) for simplicity, the result is true for any k variables.

Proof. The proof is inductive. For $d = 1$, the result holds true: see (3.2). Supposing that it is true for $d-1$, $d = 2, 3, \dots$, let us prove (3.3) with $k = d$.

Applying the inductive assumption for the first $d-1$ variables, we obtain

$$\Big(\int\limits_{\mathbb{R}^d}\Big[\int\limits_{x_1-h_1}^{x_1+h_1}\cdots\int\limits_{x_d-h_d}^{x_d+h_d}F(u_1,\ldots,u_d)\,du_1\ldots du_d\Big]^Q dx_1\ldots dx_d\Big)^{1/Q}$$
$$=\Big(\int\limits_{\mathbb{R}}\Big\{\int\limits_{\mathbb{R}^{d-1}}\Big[\int\limits_{x_1-h_1}^{x_1+h_1}\cdots\int\limits_{x_{d-1}-h_{d-1}}^{x_{d-1}+h_{d-1}}\int\limits_{x_d-h_d}^{x_d+h_d}F(u_1,\ldots,u_d)\,du_1\ldots du_d\Big]^Q$$
$$\cdot\,dx_1\ldots dx_{d-1}\Big\}^{Q/Q}dx_d\Big)^{1/Q}$$
$$\le C(h_1\ldots h_{d-1})^{1/Q+1/q'}\Big(\int\limits_{\mathbb{R}}\Big\{\int\limits_{\mathbb{R}^{d-1}}\Big[\int\limits_{x_d-h_d}^{x_d+h_d}F(x_1,\ldots,x_{d-1},u_d)\,du_d\Big]^q$$
$$\cdot\,dx_1\ldots dx_{d-1}\Big\}^{Q/q}dx_d\Big)^{\frac{q}{Q}\frac{1}{q}}.$$

Using now the generalized Minkowski inequality with exponent $Q/q\ge 1$, we bound the right-hand side by a constant times

$$(h_1\ldots h_{d-1})^{1/Q+1/q'}$$
$$\cdot\Big(\int\limits_{\mathbb{R}^{d-1}}\Big\{\int\limits_{\mathbb{R}}\Big[\int\limits_{x_d-h_d}^{x_d+h_d}F(x_1,\ldots,x_{d-1},u_d)\,du_d\Big]^Q dx_d\Big\}^{q/Q}dx_1\ldots dx_{d-1}\Big)^{1/q}.$$

To obtain (3.3), it remains again to make use of (3.2) for the dth variable.

If $k<d$, we just represent the integral considered as

$$\Big(\int\limits_{\mathbb{R}^{d-k}}\Big(\int\limits_{\mathbb{R}^k}\Big[\int\limits_{x_1-h_1}^{x_1+h_1}\cdots\int\limits_{x_k-h_k}^{x_k+h_k}F(u_1,\ldots,u_k,x_{k+1},\ldots,x_d)\,du_1\ldots du_k\Big]^Q$$
$$\cdot\,dx_1\ldots dx_k\Big)^{\frac{1}{Q}Q}dx_{k+1}\ldots dx_d\Big)^{1/Q}$$

and apply the proven version to the inner integral. ■

We will also apply the following simple result.

LEMMA 3.2. *Let f satisfy* (N-d) *and $D^{\mathbf{1}}f\in L_q(\mathbb{R}^d)$. Then*

$$\|\Delta_{h_1,\ldots,h_d}f\|_\infty\le 2^{d/q'}(h_1\ldots h_d)^{1/q'}\|D^{\mathbf{1}}f\|_q.$$

Proof. By Hölder's inequality,

$$\|\Delta_{h_1,\ldots,h_d}f\|_\infty\le\int\limits_{x_1-h_1}^{x_1+h_1}\ldots\int\limits_{x_d-h_d}^{x_d+h_d}|D^{\mathbf{1}}f(u_1,\ldots,u_d)|\,du_1\ldots du_d$$
$$\le\Big(\int\limits_{x_1-h_1}^{x_1+h_1}\ldots\int\limits_{x_d-h_d}^{x_d+h_d}du_1\ldots du_d\Big)^{1/q'}\|D^{\mathbf{1}}f\|_q,$$

as required. ■

4. One-dimensional result. We prove a more general assertion than that from [L]. The latter coincides with it when $r = 1$. However, even in that case the present proof differs from that in [L].

THEOREM 4.1. *Suppose a function f of one variable satisfies condition* (N-1,r) *for some $r \in \mathbb{N}$.*

(a) *Let $f \in L_p(\mathbb{R})$, $1 \le p < \infty$, and $f^{(r)} \in L_q(\mathbb{R})$, $1 < q < \infty$. If*
$$\frac{2r-1}{p} + \frac{1}{q} > r,$$
then $f \in A(\mathbb{R})$.

(b) *If*
$$\frac{2r-1}{p} + \frac{1}{q} < r,$$
then there exists a function f satisfying (N-1,r) *such that $f \in L_p(\mathbb{R})$ and $f^{(r)} \in L_q(\mathbb{R})$ but $f \notin A(\mathbb{R})$.*

Proof. The proof of (b) goes along the same lines as the proof of (b) of Theorem 2.2 below.

(a) By Lemma B, it suffices to prove that

$$\sum_{\nu=1}^{\infty} 2^{-\nu/2} \|\Delta^r_{h(-\nu)} f\|_2 + \sum_{\nu=0}^{\infty} 2^{\nu/2} \|\Delta^r_{h(\nu)} f\|_2 < \infty. \tag{4.1}$$

Let us start with the first sum in (4.1).

We choose $p^* \ge p$ and $q^* > q$ so that
$$\left(1 - \frac{1}{2r}\right)\frac{1}{p^*} + \frac{1}{2r}\,\frac{1}{q^*} = \frac{1}{2}.$$
Using Hölder's inequality, we get
$$\|\Delta^r_{h(-k)} f\|_2 \le \|\Delta^r_{h(-k)} f\|_{p^*}^{1-1/2r} \|\Delta^r_{h(-k)} f\|_{q^*}^{1/2r}.$$
It is obvious that for $h > 0$,
$$|\Delta^r_h f(t)| = \Big| \int_{t-h}^{t+h} du_1 \ldots \int_{u_{r-1}-h}^{u_{r-1}+h} f^{(r)}(u_r)\, du_r \Big|.$$
Thus, by Lemma 3.1,
$$\|\Delta^r_{h(-k)} f\|_{q^*} \le C \|f\|_\infty^{1-q/q^*} \|\Delta^r_{h(-k)} f\|_q^{q/q^*} \le C 2^{rkq/q^*} \|f\|_\infty^{1-q/q^*} \|f^{(r)}\|_q^{q/q^*}.$$
Since also
$$\|\Delta^r_{h(-k)} f\|_{p^*} \le C \|f\|_{p^*} \le C \|f\|_\infty^{1-p/p^*} \|f\|_p^{p/p^*},$$
we have
$$\|\Delta^r_{h(-k)} f\|_2 = O(2^{kq/2q^*}).$$

Therefore

$$\sum_{k=1}^{\infty} 2^{-k/2}\|\Delta^r_{h(-k)}f\|_2 = O(1)\sum_{k=1}^{\infty} 2^{-\frac{k}{2}(1-q/q^*)} < \infty.$$

We next prove that

$$\sum_{k=1}^{\infty} 2^{k/2}\|\Delta^r_{h(k)}f\|_2 < \infty\,. \tag{4.2}$$

Choosing $p^* > p$ such that

$$\left(1-\frac{1}{2r}\right)\frac{1}{p^*} + \frac{1}{2r}\,\frac{1}{q} = \frac{1}{2},$$

we obtain

$$\|\Delta^r_{h(k)}f\|_2 \le \|\Delta^r_{h(k)}f\|_{p^*}^{1-1/2r}\|\Delta^r_{h(k)}f\|_q^{1/2r}.$$

An obvious generalization of Lemma 3.2 for higher derivatives yields

$$\begin{aligned}\|\Delta^r_{h(k)}f\|_{p^*} &\le C\|\Delta^r_{h(k)}f\|_{\infty}^{1-p/p^*}\|f\|_p^{p/p^*}\\ &\le C2^{-\frac{r}{q'}(1-p/p^*)k}\|f^{(r)}\|_q^{1-p/p^*}\|f\|_p^{p/p^*}.\end{aligned}$$

Further, applying Lemma 3.1, we obtain

$$\|\Delta^r_{h(k)}f\|_q \le C2^{-rk}\|f^{(r)}\|_q\,.$$

Thus,

$$\|\Delta^r_{h(k)}f\|_2 = O(2^{-k((1-\frac{1}{2r})\frac{r}{q'}(1-\frac{p}{p^*})+\frac{1}{2})}).$$

This readily implies (4.2), which completes the proof. ∎

5. Proofs of multidimensional results

5.1. Proof of Theorem 2.1. (a) The proof is surprisingly similar to that in dimension one. The part of the sum from Lemma B with

$$\sum_{k_1=1}^{\infty}\cdots\sum_{k_d=1}^{\infty} 2^{-\frac{1}{2}\sum_{j=1}^d k_j}$$

is represented as

$$\begin{aligned}\sum_{k_1=1}^{\infty}\cdots\sum_{k_d=1}^{\infty} 2^{-\frac{1}{2}\sum_{j=1}^d k_j}\Big(\int\limits_{\mathbb{R}^d} &|\Delta_{h(-k_1),\ldots,h(-k_d)}f(x)|\\ &\cdot\Big|\int\limits_{x_1-h(-k_1)}^{x_1+h(-k_1)}\cdots\int\limits_{x_d-h(-k_d)}^{x_d+h(-k_d)} D^{\mathbf{1}}f(u)\,du\Big|\,dx\Big)^{1/2}\end{aligned}$$

and dealt with exactly as in the proof of the first sum in dimension one.

Further, for

$$\sum_{k_1=0}^{\infty}\cdots\sum_{k_d=0}^{\infty} 2^{\frac12\sum_{j=1}^d k_j},$$

we proceed as in the proof of the second sum in dimension one. In both cases, (3.4) from Lemma 3.1 is applied.

Finally, when dealing with

$$\sum_{i:\,\eta_i=0} 2^{-\frac12\sum_{j=1}^d k_j}\cdots\sum_{i:\,\eta_i=1} 2^{\frac12\sum_{j=1}^d k_j},$$

where $\eta\ne\mathbf{0}$ and $\eta\ne\mathbf{1}$, we do not need to treat the first sum at all: when the rest is bounded, the series corresponding to $\eta_i=0$ converges automatically. As for the second sum, we proceed as in the proof for the second sum in dimension one. Since Lemma 3.1 is always applied with $Q=q$, we see that the integral norm on the right-hand side of (3.3) becomes the usual L_q norm, where each time q is taken to be some p_η.

The proof of (b) is similar to the proof of (b) in Theorem 2.2. ∎

5.2. Proof of Theorem 2.2. The proof will be divided into several steps.

Step 1. We start with the sum

$$\sum_{k_1=1}^{\infty}\cdots\sum_{k_d=1}^{\infty} 2^{-\frac12(k_1+\dots+k_d)}\|\varDelta_{h(-k_1),\dots,h(-k_d)}f\|_2. \tag{5.1}$$

Choosing $p_\eta^*>p_\eta$ such that

$$\sum_{\mathbf{0}\le\eta\le\mathbf{1}}\frac{1}{p_\eta^*}=2^{d-1},$$

and applying Hölder's inequality for 2^d factors (see [HLP, p. 140]) and Lemma 3.1 with $Q=q$, we obtain

$$\begin{aligned}
\|\varDelta_{h(-k_1),\dots,h(-k_d)}f\|_2 &\le C\Big(\prod_{\mathbf{0}\le\eta\le\mathbf{1}}\|\varDelta^{\eta}_{h(-k_1),\dots,h(-k_d)}f\|_{p_\eta^*}\Big)^{1/2^d}\\
&\le C\Big(\prod_{\mathbf{0}\le\eta\le\mathbf{1}}\|f\|_\infty^{1-p_\eta/p_\eta^*}(h(-k_1)^{\eta_1}\dots h(-k_d)^{\eta_d})^{p_\eta/p_\eta^*}\|D^\eta f\|_{p_\eta}^{p_\eta/p_\eta^*}\Big)^{1/2^d}\\
&= C\prod_{j=1}^d 2^{\frac{1}{2^d}(\sum_{\mathbf{0}\le\eta\le\mathbf{1},\,\eta_j=1}p_\eta/p_\eta^*)k_j}\Big(\prod_{\mathbf{0}\le\eta\le\mathbf{1}}\|f\|_\infty^{1-p_\eta/p_\eta^*}\|D^\eta f\|_{p_\eta}^{p_\eta/p_\eta^*}\Big)^{1/2^d}.
\end{aligned}$$

The last inequality along with the obvious inequality

$$\sum_{\mathbf{0}\le\eta\le\mathbf{1},\,\eta_j=1}\frac{p_\eta}{p_\eta^*}<2^{d-1},\qquad j=1,\dots,d,$$

yields the convergence of the sum in (5.1).

STEP 2. Assume that

$$\sum_{\eta\neq\mathbf{0}} \frac{1}{p_\eta} < 2^{d-1}. \tag{5.2}$$

STEP 2.1. Let us show that

$$\sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} 2^{\frac{1}{2}(k_1+\cdots+k_d)} \|\Delta_{h(k_1),\dots,h(k_d)} f\|_2 < \infty. \tag{5.3}$$

Conditions (5.2) and (2.2) allow us to choose $p_{\mathbf{0}}^* > p_{\mathbf{0}}$ such that

$$\frac{1}{p_{\mathbf{0}}^*} + \sum_{\eta\neq\mathbf{0}} \frac{1}{p_\eta} = 2^{d-1}.$$

Applying then Hölder's inequality and Lemmas 3.2 and 3.1, we obtain

$$\begin{aligned}
&\|\Delta_{h(k_1),\dots,h(k_d)} f\|_2 \\
&\le C\Big(\|\Delta_{h(k_1),\dots,h(k_d)} f\|_\infty^{1-p_{\mathbf{0}}/p_{\mathbf{0}}^*} \|f\|_{p_{\mathbf{0}}}^{p_{\mathbf{0}}/p_{\mathbf{0}}^*} \prod_{\eta\neq\mathbf{0}} \|\Delta^\eta_{h(k_1),\dots,h(k_d)} f\|_{p_\eta}\Big)^{1/2^d} \\
&\le C\Big(2^{-(2^{d-1}+\frac{1}{p'_{\mathbf{1}}}(1-p_{\mathbf{0}}/p_{\mathbf{0}}^*))(k_1+\cdots+k_d)} \|D^{\mathbf{1}} f\|_{p_{\mathbf{1}}}^{1-p_{\mathbf{0}}/p_{\mathbf{0}}^*} \|f\|_{p_{\mathbf{0}}}^{p_{\mathbf{0}}/p_{\mathbf{0}}^*} \prod_{\eta\neq\mathbf{0}} \|D^\eta f\|_{p_\eta}\Big)^{1/2^d}.
\end{aligned}$$

The last inequality readily yields the convergence of (5.3).

STEP 2.2. We now prove the convergence of all series of the type

$$\sum_{k_1=0}^{\infty} \cdots \sum_{k_j=0}^{\infty} \sum_{l_{j+1}=1}^{\infty} \cdots \sum_{l_d=1}^{\infty} \frac{2^{\frac{1}{2}(k_1+\cdots+k_j)}}{2^{\frac{1}{2}(l_{j+1}+\cdots+l_d)}} \|\Delta_u f\|_2, \tag{5.4}$$

where $1 \le j \le d-1$ and $u = (\pi 2^{-k_1}, \dots, \pi 2^{-k_j}, \pi 2^{l_{j+1}}, \dots, \pi 2^{l_d})$. Choose $p_{\mathbf{0}}^* > p_{\mathbf{0}}$ and $p_{e_i}^* > p_{e_i}$, $j+1 \le i \le d$, such that

$$\frac{1}{p_{\mathbf{0}}^*} + \sum_{i=1}^{j} \frac{1}{p_{e_i}} + \sum_{i=j+1}^{d} \frac{1}{p_{e_i}^*} + \sum_{|\eta|>1} \frac{1}{p_\eta} = 2^{d-1}.$$

Now, Hölder's inequality yields

$$\|\Delta_u f\|_2 \le C(S_1 S_2 S_3 S_4)^{1/2^d}, \tag{5.5}$$

where

$$S_1 = \|\Delta_u f\|_{p_{\mathbf{0}}^*}, \qquad S_2 = \prod_{i=1}^{j} \|\Delta^{e_i}_{h(k_i)} f\|_{p_{e_i}},$$

$$S_3 = \prod_{i=j+1}^{d} \|\Delta^{e_i}_{h(-l_i)} f\|_{p_{e_i}^*}, \qquad S_4 = \prod_{|\eta|>1} \|\Delta^\eta_u f\|_{p_\eta}.$$

Applying Lemma 3.2 leads to

$$
\begin{aligned}
(5.6)\quad S_1 &\le C\|f\|_{p_{\mathbf 0}}^{p_{\mathbf 0}/p^*_{\mathbf 0}}\|\varDelta_u f\|_\infty^{1-p_{\mathbf 0}/p^*_{\mathbf 0}} \le \|f\|_{p_{\mathbf 0}}^{p_{\mathbf 0}/p^*_{\mathbf 0}}\|f\|_\infty^{1-p_{\mathbf 0}/p^*_{\mathbf 0}-\varepsilon}\|\varDelta_u f\|_\infty^{\varepsilon}\\
&\le C2^{-\frac{\varepsilon}{p'_{\mathbf 1}}(k_1+\dots+k_j-l_{j+1}-\dots-l_d)}\|f\|_{p_{\mathbf 0}}^{p_{\mathbf 0}/p^*_{\mathbf 0}}\|f\|_\infty^{1-p_{\mathbf 0}/p^*_{\mathbf 0}-\varepsilon}\|D^{\mathbf 1}f\|_{p_{\mathbf 1}}^{\varepsilon},
\end{aligned}
$$

where $\varepsilon\in(0,1-p_{\mathbf 0}/p^*_{\mathbf 0})$. Further, applying Lemma 3.1, we obtain

$$
(5.7)\qquad S_2\le C\prod_{i=1}^{j}2^{-k_i}\|D^{e_i}f\|_{p_{e_i}},
$$

$$
\begin{aligned}
(5.8)\qquad S_3 &\le C\prod_{i=j+1}^{d}\|\varDelta_{h(-l_i)}f\|_\infty^{1-p_{e_i}/p^*_{e_i}}\|\varDelta_{h(-l_i)}f\|_{p_{e_i}}^{p_{e_i}/p^*_{e_i}}\\
&\le C\prod_{i=j+1}^{d}\|f\|_\infty^{1-p_{e_i}/p^*_{e_i}}2^{l_ip_{e_i}/p^*_{e_i}}\|D^{e_i}f\|_{p_{e_i}}^{p_{e_i}/p^*_{e_i}},
\end{aligned}
$$

$$
(5.9)\qquad S_4\le C\prod_{|\eta|>1}2^{\eta_1k_1+\dots+\eta_jk_j-\eta_{j+1}l_{j+1}-\dots-\eta_dl_d}\|D^\eta f\|_{p_\eta}.
$$

Combining then (5.5) and (5.6)–(5.9), we arrive at

$$
\|\varDelta_u f\|_2=O\Big(\prod_{i=1}^{j}2^{-(\frac12+\frac{\varepsilon}{2^dp'_{\mathbf 1}})k_i}\prod_{i=j+1}^{d}2^{(\frac12+\frac1{2^d}[\frac{\varepsilon}{p'_{\mathbf 1}}-1+\frac{p_{e_i}}{p^*_{e_i}}])l_i}\Big).
$$

Choosing $\varepsilon\in(0,1-p_{\mathbf 0}/p^*_{\mathbf 0})$ such that

$$
\frac{\varepsilon}{p'_{\mathbf 1}}-1+\frac{p_{e_i}}{p^*_{e_i}}<0,\quad j+1\le i\le d,
$$

we deduce that (5.4) is finite. This completes the proof of (a) provided (5.2) holds true.

STEP 3. We now assume that

$$
(5.10)\qquad \sum_{\eta\ne\mathbf 0}\frac1{p_\eta}\ge 2^{d-1}.
$$

STEP 3.1. Let us show the convergence of (5.3). It is obvious that

$$
\sum_{|\eta|\ne0,\,d}\frac1{p_\eta}+\frac2{p_{\mathbf 1}}>2^{d-1}.
$$

Suppose that there exist $p^*_\eta>p_\eta$, $|\eta|=1$, such that

$$
(5.11)\qquad \sum_{|\eta|=1}\frac1{p^*_\eta}+\sum_{|\eta|\ne0,1,d}\frac1{p_\eta}+\frac2{p_{\mathbf 1}}=2^{d-1}.
$$

Applying Hölder's inequality and Lemma 3.1 as above, we get

$$\|\Delta_{h(k_1),\dots,h(k_d)} f\|_2 = O\Big(2^{-\sum_{j=1}^d k_j(\frac{1}{2}+\frac{1}{2^d}\frac{p_{e_j}}{p^*_{e_j}})}\Big).$$

This readily yields the convergence of (5.3).

If $p^*_\eta > p_\eta$ satisfying (5.11) do not exist, then

$$\sum_{|\eta|\neq 0,1,d} \frac{1}{p_\eta} + \frac{2}{p_{\mathbf{1}}} \geq 2^{d-1},$$

and hence

$$\sum_{|\eta|\neq 0,1,d} \frac{1}{p_\eta} + \frac{2+d}{p_{\mathbf{1}}} > 2^{d-1}.$$

Now, suppose that there exist $p^*_\eta > p_\eta$, $|\eta| = 2$, such that

$$\sum_{|\eta|=2} \frac{1}{p^*_\eta} + \sum_{|\eta|\neq 0,1,2,d} \frac{1}{p_\eta} + \frac{2+d}{p_{\mathbf{1}}} = 2^{d-1}. \tag{5.12}$$

Applying Hölder's inequality and Lemma 3.1, we get

$$\|\Delta_{h(k_1),\dots,h(k_d)} f\|_2 = O\Big(2^{-\sum_{j=1}^d k_j[\frac{1}{2}+\frac{1}{2^d}(1+\sum_{|\eta|=2,\,\eta_j=1}\frac{p_\eta}{p^*_\eta})]}\Big).$$

As above, this readily yields the convergence of (5.3).

We repeat this procedure as many times as needed. For the sake of simplicity, we consider in detail the final step only for d odd (of course, $d \geq 3$); it will then be explained how to treat the case of d even.

Suppose that there are no numbers $p^*_\eta > p_\eta$, $|\eta| = (d-1)/2$, which satisfy

$$\sum_{|\eta|=(d-1)/2} \frac{1}{p^*_\eta} + \sum_{|\eta|\neq 0,1,\dots,(d-1)/2,d} \frac{1}{p_\eta} + \Big(\sum_{j=0}^{(d-3)/2} \binom{d}{j} + 1\Big)\frac{1}{p_{\mathbf{1}}} = 2^{d-1}.$$

Then

$$\sum_{|\eta|\neq 0,1,\dots,(d-1)/2,d} \frac{1}{p_\eta} + \Big(\sum_{j=0}^{(d-3)/2} \binom{d}{j} + 1\Big)\frac{1}{p_{\mathbf{1}}} \geq 2^{d-1},$$

and hence

$$\sum_{|\eta|\neq 0,1,\dots,(d-1)/2,d} \frac{1}{p_\eta} + \frac{2^{d-1}+1}{p_{\mathbf{1}}} > 2^{d-1}.$$

Taking into account that $1/2p_{\mathbf{1}} < 1/2$, we can choose $p^*_{\mathbf{0}} > p_{\mathbf{0}}$, $p^*_\eta > p_\eta$ and $\varepsilon \in (0,1)$ such that

$$\frac{1-\varepsilon}{2^d p^*_{\mathbf{0}}} + \frac{1}{2^d} \sum_{|\eta|\neq 0,1,\dots,(d-1)/2,d} \frac{1}{p^*_\eta} + \Big(\frac{1}{2} + \frac{\varepsilon}{2^d}\Big)\frac{1}{p_{\mathbf{1}}} = \frac{1}{2}.$$

Again applying Hölder's inequality and Lemma 3.1, we get

$$\|\varDelta_{h(k_1),\dots,h(k_d)}f\|_2 = O\big(2^{-\sum_{j=1}^d k_j(1/2+\gamma_j)}\big),$$

where $\gamma_j>0$. This implies that (5.3) is finite.

If d is even, then we will have the strict inequality

$$\sum_{|\eta|\neq 0,\,1,\dots,(d-1)/2,d}\frac{1}{p_\eta}+\sum_{|\eta|=d/2,\,\eta\notin A}\frac{1}{p_\eta}+\frac{2^{d-1}+1}{p_{\mathbf{1}}}>2^{d-1}$$

on the final step, where $\operatorname{card} A=\frac12\binom{d}{d/2}$ and $\eta\in A$ if $|\eta|=d/2$. This allows us to repeat the previous arguments.

Step 3.2. To complete the proof of (a) of the theorem, it remains to show the convergence of all series of type (5.4) under condition (5.10). We will consider only two cases: $j=d-1$ and $j=1$. The intermediate cases are proved similarly.

Step 3.2.1. Let first $j=d-1$ and $u=(\pi 2^{-k_1},\dots,\pi 2^{-k_{d-1}},\pi 2^{l_d})$. Suppose that there exist $p_{\mathbf{0}}^*>p_{\mathbf{0}}$ and $p_{e_d}^*>p_{e_d}$ such that

$$\frac{1}{p_{\mathbf{0}}^*}+\frac{1}{p_{e_d}^*}+\sum_{\eta\neq\mathbf{0},e_d}\frac{1}{p_\eta}=2^{d-1}. \tag{5.13}$$

Applying then Hölder's inequality and Lemmas 3.2 and 3.1, we obtain

$$\begin{aligned}&\|\varDelta_u f\|_2\\ &\le C\Big(\|\varDelta_u f\|_\infty^{2-p_{\mathbf{0}}/p_{\mathbf{0}}^*-p_{e_d}/p_{e_d}^*}\|f\|_{p_{\mathbf{0}}}^{p_{\mathbf{0}}/p_{\mathbf{0}}^*}\|\varDelta_u^{e_d}f\|_\infty^{p_{e_d}/p_{e_d}^*}\prod_{\eta\neq\mathbf{0},e_d}\|\varDelta_u^\eta f\|_{p_\eta}\Big)^{1/2^d}\\ &=O\big(2^{(-k_1-\dots-k_{d-1}+l_d)\varepsilon(2-p_{\mathbf{0}}/p_{\mathbf{0}}^*-p_{e_d}/p_{e_d}^*)+l_d(p_{e_d}/p_{e_d}^*+2^{d-1}-1)-2^{d-1}\sum_{j=1}^{d-1}k_j}\big)^{1/2^d}.\end{aligned}$$

Choosing $\varepsilon\in(0,1)$ such that

$$\varepsilon\Big(2-\frac{p_{\mathbf{0}}}{p_{\mathbf{0}}^*}-\frac{p_{e_d}}{p_{e_d}^*}\Big)<1-\frac{p_{e_d}}{p_{e_d}^*},$$

we obtain the finiteness of (5.4). Obviously, in the case of $d=2$, the proof of Step 3.2.1 is complete. Therefore, let further $d\ge 3$.

If for any $p_{\mathbf{0}}^*$ and $p_{e_d}^*$ equality (5.13) does not hold, then

$$\sum_{\eta\neq\mathbf{0},e_d}\frac{1}{p_\eta}\ge 2^{d-1}$$

and hence

$$\sum_{\eta\neq\mathbf{0},e_d,\mathbf{1}}\frac{1}{p_\eta}+\frac{2}{p_{\mathbf{1}}}>2^{d-1}.$$

We denote $A_j=\{\eta:|\eta|=j,\ \eta_d=1\}$ and $A_0=\{\mathbf{0}\}$.

Suppose that there exist $p_{\mathbf{0}}^{*} > p_{\mathbf{0}}$ and $p_{\eta}^{*} > p_{\eta}$, $\eta \in A_2$, such that

$$\frac{1}{p_{\mathbf{0}}^{*}} + \sum_{\eta\in A_2} \frac{1}{p_{\eta}^{*}} + \sum_{\eta\notin\bigcup_{j=0}^{2} A_j\cup A_d} \frac{1}{p_\eta} + \frac{2}{p_{\mathbf{1}}} = 2^{d-1}. \tag{5.14}$$

Then, applying Hölder's inequality and Lemma 3.1, we obtain

$$\|\Delta_u f\|_2 = O\big(2^{l_d[\frac{1}{2}-\frac{1}{2^d}(d-1-\sum_{\eta\in A_2}\frac{p_\eta}{p_\eta^*})]} \cdot 2^{-\sum_{j=1}^{d-1} k_j(\frac{1}{2}+\frac{1}{2^d}\sum_{\eta\in A_2,\,\eta_j=1}\frac{p_\eta}{p_\eta^*})}\big). \tag{5.15}$$

Thus, taking into account that

$$\sum_{\eta\in A_2} \frac{p_\eta}{p_\eta^*} < d-1,$$

and using (5.15), we obtain the convergence of (5.4).

If for any $p_{\mathbf{0}}^{*} > p_{\mathbf{0}}$ and $p_{\eta}^{*} > p_{\eta}$, $\eta \in A_2$, equality (5.14) does not hold, then

$$\sum_{\eta\notin\bigcup_{j=0}^{2} A_j\cup A_d} \frac{1}{p_\eta} + \frac{2}{p_{\mathbf{1}}} \ge 2^{d-1}$$

and hence

$$\sum_{\eta\notin\bigcup_{j=0}^{2} A_j\cup A_d} \frac{1}{p_\eta} + \frac{2+\binom{d-2}{1}}{p_{\mathbf{1}}} > 2^{d-1}.$$

Suppose that there exist $p_{\mathbf{0}}^{*} > p_{\mathbf{0}}$ and $p_{\eta}^{*} > p_{\eta}$, $\eta \in A_3$, such that

$$\frac{2}{p_{\mathbf{0}}^{*}} + \sum_{\eta\in A_3} \frac{1}{p_{\eta}^{*}} + \sum_{\eta\notin\bigcup_{j=0}^{3} A_j\cup A_d} \frac{1}{p_\eta} + \frac{2+\binom{d-2}{1}}{p_{\mathbf{1}}} = 2^{d-1}.$$

Applying then Hölder's inequality and Lemma 3.1, we obtain

$$\|\Delta_u f\|_2 = O\big(2^{l_d[\frac{1}{2}-\frac{1}{2^d}(\binom{d-1}{1}-\binom{d-2}{1}+\binom{d-1}{2}-\sum_{\eta\in A_3}\frac{p_\eta}{p_\eta^*})]} \cdot 2^{-\sum_{j=1}^{d-1} k_j(\frac{1}{2}+\frac{1}{2^d}\sum_{\eta\in A_3,\,\eta_j=1}\frac{p_\eta}{p_\eta^*})}\big).$$

The last inequality readily yields the convergence of (5.4).

Repeating this procedure, we consider the last step, if needed. Suppose that there are no $p_{\mathbf{0}}^{*} > p_{\mathbf{0}}$ and $p_{\eta}^{*} > p_{\eta}$, $\eta \in A_{d-1}$, such that

$$\frac{1+\sum_{j=1}^{d-4}\binom{d-2}{j}}{p_{\mathbf{0}}^{*}} + \sum_{\eta\in A_{d-1}} \frac{1}{p_{\eta}^{*}} + \sum_{\eta\notin\bigcup_{j=0}^{d} A_j} \frac{1}{p_\eta} + \frac{\sum_{j=0}^{d-3}\binom{d-2}{j}+1}{p_{\mathbf{1}}} = 2^{d-1}.$$

Then

$$\sum_{\eta\notin\bigcup_{j=0}^{d} A_j} \frac{1}{p_\eta} + \frac{2^{d-2}}{p_{\mathbf{1}}} \ge 2^{d-1},$$

and hence we can choose small $\varepsilon \in (0,1/2)$ such that

$$\frac{1}{2^d}\sum_{\eta\notin\bigcup_{j=0}^d A_j}\frac{1}{p_\eta}+\Big(\frac12-\varepsilon\Big)\frac{1}{p_{\mathbf 1}}>\frac12.$$

This implies that there exist $p_{\mathbf 0}^*\ge p_{\mathbf 0}$ and $p_\eta^*\ge p_\eta$ such that

$$\frac{\varepsilon}{p_{\mathbf 0}^*}+\frac{1}{2^d}\sum_{\eta\notin\bigcup_{j=1}^d A_j}\frac{1}{p_\eta^*}+\Big(\frac12-\varepsilon\Big)\frac{1}{p_{\mathbf 1}}=\frac12.$$

Applying then Hölder's inequality and Lemma 3.1, we obtain

$$\|\varDelta_u f\|_2=O\big(2^{l_d(\frac12-\varepsilon)-\sum_{j=1}^{d-1}k_j(\frac12-\varepsilon+\frac{1}{2^d}\sum_{\eta\notin\bigcup_{i=1}^d A_i,\,\eta_j=1}\frac{p_\eta}{p_\eta^*})}\big).$$

To complete the proof of the convergence of (5.4), it remains to choose ε such that

$$0<\varepsilon<\min_{1\le j\le d-1}\bigg\{\frac{1}{2^d}\sum_{\eta\notin\bigcup_{i=1}^d A_i,\,\eta_j=1}\frac{p_\eta}{p_\eta^*}\bigg\}.$$

Step 3.2.2. It only remains to consider the case $j=1$ for (5.4). Denoting $B=\{\eta:\eta_1\ne1\}$ and $u=(\pi2^{-k_1},\pi2^{l_2},\dots,\pi2^{l_d})$, we can always choose $p_{\mathbf 0}^*>p_{\mathbf 0}$ and $p_\eta^*>p_\eta$, $\eta\in B$, such that

$$\sum_{\eta\in B}\frac{1}{p_\eta^*}+\sum_{\eta\notin B}\frac{1}{p_\eta}=2^{d-1}.$$

Then, applying Hölder's inequality and Lemmas 3.2 and 3.1, we obtain

$$\|\varDelta_u f\|_2=O\big(2^{-k_1[\frac{\varepsilon}{2^d p_{\mathbf 1}'}(1-\frac{p_{\mathbf 0}}{p_{\mathbf 0}^*})+\frac12]}2^{\sum_{j=2}^d\frac{l_j}{2^d}[\frac{\varepsilon}{p_{\mathbf 1}'}(1-\frac{p_{\mathbf 0}}{p_{\mathbf 0}^*})+2^{d-2}+\sum_{\eta\in B,\,\eta_j=1}\frac{p_\eta}{p_\eta^*}]}\big).$$

Choosing $\varepsilon>0$ such that

$$\frac{\varepsilon}{p_{\mathbf 1}'}\Big(1-\frac{p_{\mathbf 0}}{p_{\mathbf 0}^*}\Big)+\sum_{\eta\in B,\,\eta_j=1}\frac{p_\eta}{p_\eta^*}<2^{d-2}$$

yields the finiteness of (5.4).

This completes the proof of (a) of the theorem.

To prove (b), let us consider the function $m=m_{\alpha,\beta}$ from (1.1). Suppose that $p_\eta>d/(\beta-|\eta|(\alpha-1))$ for all η. Simple calculations show that $m\in L_{p_{\mathbf 0}}(\mathbb R^d)$ and $D^\eta m\in L_{p_\eta}(\mathbb R^d)$. If $\beta/\alpha\le d/2$, then $m\notin A(\mathbb R^d)$. The last inequality is equivalent to $2\beta/d-(\alpha-1)\le1$. Therefore,

$$\sum_{\mathbf 0\le\eta\le\mathbf 1}\frac{1}{p_\eta}<\frac{2^d\beta-d2^{d-1}(\alpha-1)}{d}<2^{d-1},$$

and so m provides the required counterexample. ∎

5.3. Proof of Theorem 2.4. The proof is very similar to those of the above theorems. We want to show the convergence of all series of the type

$$\sum_{k_1=0}^{\infty}\cdots\sum_{k_j=0}^{\infty}\sum_{l_{j+1}=1}^{\infty}\cdots\sum_{l_d=1}^{\infty}\frac{2^{\frac{1}{2}(k_1+\cdots+k_j)}}{2^{\frac{1}{2}(l_{j+1}+\cdots+l_d)}}\|\varDelta_u^{\mathbf{1},r}f\|_2, \tag{5.16}$$

where $u=(\pi 2^{-k_1},\dots,\pi 2^{-k_j},\pi 2^{l_{j+1}},\dots,\pi 2^{l_d})$ if $1\le j\le d-1$ and $u=(\pi 2^{l_1},\dots,\pi 2^{l_d})$ if $j=0$. In what follows, we suppose that $\sum_{i=n}^{m}=0$ and $\prod_{i=n}^{m}=0$ if $n>m$.

First, we choose $\varepsilon>0$, $p^*\in(p,p_0]$ and $p_i^*>p_i$, $j+1\le i\le d$, such that

$$\Big(1-\sum_{i=1}^{j}\frac{1+\varepsilon}{2r_i}-\sum_{i=j+1}^{d}\frac{1}{2r_i}\Big)\frac{1}{p^*}+\sum_{i=1}^{j}\frac{1+\varepsilon}{2r_ip_i}+\sum_{i=j+1}^{d}\frac{1}{2r_ip_i^*}=\frac{1}{2}$$

and

$$\sum_{i=1}^{j}\frac{1+\varepsilon}{2r_i}+\sum_{i=j+1}^{d}\frac{1}{2r_i}<1.$$

Applying Hölder's inequality, we obtain

$$\|\varDelta_u^{\mathbf{1},r}f\|_2\le S_1S_2S_3,$$

where

$$S_1=\|f\|_{p^*}^{1-\sum_{i=1}^{j}\frac{1+\varepsilon}{2r_i}-\sum_{i=j+1}^{d}\frac{1}{2r_i}},\qquad S_2=\prod_{i=1}^{j}\|\varDelta_{h(k_i)}^{e_i,r_i}f\|_{p_i}^{\frac{1+\varepsilon}{2r_i}},$$

$$S_3=\prod_{i=j+1}^{d}\|\varDelta_{h(-l_i)}^{e_i,r_i}f\|_{p_i^*}^{\frac{1}{2r_i}}.$$

Further, repeating the proof of estimates (5.7) and (5.8), it is easy to show that

$$\|\varDelta_u^{\mathbf{1},r}f\|_2=O\Big(2^{-\frac{1+\varepsilon}{2}\sum_{i=1}^{j}k_i+\frac{1}{2}\sum_{i=j+1}^{d}\frac{p_i}{p_i^*}l_i}\Big).$$

The last estimate readily yields the convergence of (5.16).

The proof of (b) is similar to the proof of (b) in Theorem 2.2. ∎

5.4. Proofs of Corollaries 2.6 and 2.7. We only prove Corollary 2.6 for condition (2.5). The proofs of Corollary 2.6 with (2.4) and of Corollary 2.7 go along the same lines.

Let us rewrite (2.5) as

$$\sum_{\mathbf{0}\le\eta\le\mathbf{1}}\gamma_\eta=d2^{d-1}+\epsilon.$$

For each η, let us choose p_η so that $\gamma_\eta p_\eta = d + \epsilon/2^d$. Then

$$\sum_{\mathbf{0}\le\eta\le\mathbf{1}} \frac{d+\epsilon/2^d}{p_\eta} = d2^{d-1} + \epsilon.$$

Since

$$\sum_{\mathbf{0}\le\eta\le\mathbf{1}} \frac{\epsilon/2^d}{p_\eta} < \epsilon,$$

we have

$$\sum_{\mathbf{0}\le\eta\le\mathbf{1}} \frac{d}{p_\eta} > d2^{d-1}.$$

This is equivalent to (2.2), and hence $f \in A(\mathbb{R}^d)$. ■

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