STUDIA MATHEMATICA 214 (1) (2013)

Universal Jamison spaces and Jamison sequences for C_0-semigroups

by

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Abstract. An increasing sequence $(n_k)_{k\geq 0}$ of positive integers is said to be a Jamison sequence if for every separable complex Banach space X and every $T \in \mathcal{B}(X)$ which is partially power-bounded with respect to $(n_k)_{k\geq 0}$, the set $\sigma_p(T) \cap \mathbb{T}$ is at most countable. We prove that for every separable infinite-dimensional complex Banach space X which admits an unconditional Schauder decomposition, and for any sequence $(n_k)_{k\geq 0}$ which is not a Jamison sequence, there exists $T \in \mathcal{B}(X)$ which is partially power-bounded with respect to $(n_k)_{k\geq 0}$ and has the set $\sigma_p(T) \cap \mathbb{T}$ uncountable. We also investigate the notion of Jamison sequences for C_0-semigroups and we give an arithmetic characterization of such sequences.

1. Introduction. Let X be a separable infinite-dimensional complex Banach space and $T \in \mathcal{B}(X)$ a bounded linear operator on X. In the whole paper, $\mathbb{T} = \{\lambda \in \mathbb{C};\ |\lambda| = 1\}$ stands for the unit circle of the complex plane and $\sigma_p(T) = \{\lambda \in \mathbb{C};\ \mathrm{Ker}(T-\lambda) \neq \{0\}\}$ is the point spectrum of T. The set $\sigma_p(T) \cap \mathbb{T}$ will be called the *unimodular point spectrum* of T.

It is now well-known that the behaviour of the sequence $(\|T^n\|)_{n\geq 0}$ of the norms of the iterates of T is closely related to the size of the unimodular point spectrum $\sigma_p(T) \cap \mathbb{T}$: Jamison [10] proved that if T is power-bounded, that is, $\sup_{n\geq 0} \|T^n\| < \infty$, then the unimodular point spectrum of T is at most countable. The influence of (partial) power-boundedness on the size of $\sigma_p(T) \cap \mathbb{T}$ has been studied by Ransford [14], Ransford and Roginskaya [15], Badea and Grivaux [1, 2] and more recently by Grivaux and Eisner [5].

If $(n_k)_{k\geq 0}$ is an increasing sequence of positive integers, we say that the operator $T \in \mathcal{B}(X)$ is *partially power-bounded with respect to* $(n_k)_{k\geq 0}$ if $\sup_{k\geq 0} \|T^{n_k}\| < \infty$. We say that $(n_k)_{k\geq 0}$ is a *Jamison sequence* if for every separable complex Banach space X and every bounded linear operator T

2010 *Mathematics Subject Classification*: 47A10, 47B37, 47D06.
Key words and phrases: partially power-bounded operators, unimodular point spectrum, Jamison sequences.

DOI: 10.4064/sm214-1-5

on X which is partially power-bounded with respect to $(n_k)_{k\geq 0}$, the set $\sigma_p(T)\cap\mathbb{T}$ is at most countable.

The notion of Jamison sequence has been intensively studied by the previously mentioned authors, and C. Badea and S. Grivaux found an arithmetic characterization of Jamison sequences in [2]. Under the assumption that $n_0 = 1$, one can define a distance $d_{(n_k)}$ on $\mathbb{T}$ by setting

$$\forall(\lambda,\mu)\in\mathbb{T}^2,\qquad d_{(n_k)}(\lambda,\mu)=\sup_{k\geq 0}|\lambda^{n_k}-\mu^{n_k}|.$$

The characterization of [2] is as follows.

THEOREM 1.1 ([2, Theorem 2.1]). *Let $(n_k)_{k\geq 0}$ be an increasing sequence of positive integers such that $n_0 = 1$. The following assertions are equivalent:*

(1) *$(n_k)_{k\geq 0}$ is a Jamison sequence;*
(2) *there exists $\varepsilon > 0$ such that any two distinct points of $\mathbb{T}$ are ε-separated for the distance $d_{(n_k)}$:*

$$\forall(\lambda,\mu)\in\mathbb{T}^2,\qquad \lambda\neq\mu\ \Rightarrow\ \sup_{k\geq 0}|\lambda^{n_k}-\mu^{n_k}|\geq\varepsilon.$$

The hard part of the proof of Theorem 1.1 is the following: when condition (2) is not satisfied, C. Badea and S. Grivaux had to construct a separable Banach space X and a bounded linear operator T on X which is partially power-bounded with respect to $(n_k)_{k\geq 0}$, but $\sigma_p(T)\cap\mathbb{T}$ is uncountable. In this paper, we are interested in this problem of construction. We can state the problem as follows.

QUESTION 1.2. If $(n_k)_{k\geq 0}$ is not a Jamison sequence, on which separable complex Banach spaces X can we construct a partially power-bounded operator T with respect to $(n_k)_{k\geq 0}$ with uncountable unimodular point spectrum?

A separable complex Banach space for which the answer to the above question is affirmative will be called a *universal Jamison space* (Definition 2.1). Eisner and Grivaux [5] proved that a separable infinite-dimensional complex Hilbert space is a universal Jamison space. Recall that if T is a bounded linear operator on a separable complex Banach space X, one says that T has a *perfectly spanning set of eigenvectors associated to unimodular eigenvalues* (see [3]) if there exists a continuous probability measure σ on $\mathbb{T}$ such that for any Borel subset B of $\mathbb{T}$ with $\sigma(B) = 1$,

$$\overline{\operatorname{span}}\,[\operatorname{Ker}(T-\lambda);\ \lambda\in B]=X.$$

Grivaux [9, Theorem 4.1] proved that T has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues if and only if for any countable subset D of $\mathbb{T}$,

$$\overline{\operatorname{span}}\,[\operatorname{Ker}(T-\lambda);\ \lambda\in\mathbb{T}\setminus D]=X.$$

Theorem 1.3 ([5, Theorem 2.1]). *Let $(n_k)_{k\geq 0}$ be an increasing sequence of positive integers (with $n_0 = 1$) such that for any $\varepsilon > 0$ there exists $\lambda \in \mathbb{T} \setminus \{1\}$ such that*

$$\sup_{k\geq 0} |\lambda^{n_k} - 1| \leq \varepsilon.$$

Let δ be any positive real number. There exists a bounded linear operator T on the complex Hilbert space $\ell_2(\mathbb{N})$ such that T has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues and

$$\sup_{k\geq 0} \|T^{n_k}\| \leq 1 + \delta.$$

In particular the unimodular point spectrum of T is uncountable.

In Section 2 of this paper, we generalize Theorem 1.3 by providing a large class of separable complex Banach spaces (the class of Banach spaces which admit unconditional Schauder decompositions) which are universal Jamison spaces (Theorem 2.8). We also give several concrete examples of Banach spaces which are universal Jamison spaces and we prove that, in contrast, hereditarily indecomposable spaces are never universal Jamison spaces.

In Section 3, we investigate *Jamison sequences for C_0-semigroups* (Definition 3.2) which are the analog of Jamison sequences in the context of operator semigroups. We give an arithmetic characterization of these sequences (Theorem 3.3) by using the characterization of Jamison sequences (Theorem 1.1). We also consider *universal Jamison spaces for C_0-semigroups* (Definition 3.11) and prove that every separable complex Banach space which admits an unconditional Schauder decomposition is a universal Jamison space for C_0-semigroups (Theorem 3.12).

At the end of Section 3, we study the Hausdorff dimension of the unimodular point spectrum in the context of operator semigroups. We prove that if $(t_k)_{k\geq 0}$ is an increasing sequence of positive real numbers such that $t_0 = 1$ and $t_{k+1}/t_k \to \infty$, then there exists a separable complex Banach space X and a C_0-semigroup $(T_t)_{t\geq 0}$ of bounded linear operators on X (with infinitesimal generator A) with $\sup_{k\geq 0} \|T_{t_k}\| < \infty$ and such that the set $\sigma_p(A) \cap i\mathbb{R}$ has Hausdorff dimension 1 (Theorem 3.15).

2. Universal Jamison spaces.

In this section, we investigate the notion of universal Jamison space which is the kind of Banach space mentioned in Question 1.2.

Definition 2.1. Let X be a separable infinite-dimensional complex Banach space. We say that X is a *universal Jamison space* if for any increasing sequence $(n_k)_{k\geq 0}$ of positive integers which is not a Jamison sequence, there exists $T \in \mathcal{B}(X)$ which is partially power-bounded with respect to $(n_k)_{k\geq 0}$ and has uncountable unimodular point spectrum.

Example 2.2. According to Theorem 1.3, $\ell_2(\mathbb{N})$ is a universal Jamison space. More generally, if we replace 2 by $p \in [1, \infty[$ in the proof of Theorem 1.3, we easily see that $\ell_p(\mathbb{N})$ is also a universal Jamison space.

Our aim is to generalize Theorem 1.3 to a broader class of Banach spaces X, namely to the class of separable complex Banach spaces which admit an unconditional Schauder decomposition.

2.1. Unconditional Schauder decompositions. Let us now briefly recall a few known facts about unconditional Schauder decompositions.

Definition 2.3. Let X be a separable infinite-dimensional complex space. We say that X admits an *unconditional Schauder decomposition* if there exists a sequence $(X_\ell)_{\ell \geq 1}$ of closed subspaces of X (different from $\{0\}$) such that any vector x of X can be written in a unique way as an unconditionally convergent series $\sum_{\ell \geq 1} x_\ell$, where $x_\ell \in X_\ell$ for all ℓ.

There are many examples of such spaces. For instance, it is clear that if X has an unconditional Schauder basis, then X admits an unconditional Schauder decomposition. Recall that the space $C([0,1])$ of continuous functions on $[0,1]$ is *universal* in the sense that it contains an isometric copy of any separable Banach space. We denote by $c_0(\mathbb{N})$ the space of all complex sequences which converge to zero.

Example 2.4. A separable complex Banach space X admits an unconditional Schauder decomposition whenever it contains a complemented copy of a Banach space which admits an unconditional Schauder decomposition. In particular, if X contains a copy of $c_0(\mathbb{N})$ then it admits an unconditional Schauder decomposition. For instance, $C([0,1])$ admits an unconditional Schauder decomposition.

Proof. In the first case, the fact that X admits an unconditional Schauder decomposition follows directly from Definition 2.3. If X contains a copy of $c_0(\mathbb{N})$, then a result of Sobczyk says that this copy is complemented in X. The last assertion comes from the fact that $C([0,1])$ is a universal space and so contains a copy of $c_0(\mathbb{N})$. ∎

Remark 2.5. If $(X_\ell)_{\ell \geq 1}$ is an unconditional Schauder decomposition of X and $(I_k)_{k \geq 1}$ is any partition of $\mathbb{N}$ into finite or infinite subsets, let Y_k denote the closed linear span of the spaces X_ℓ, where $\ell \in I_k$. Then $(Y_k)_{k \geq 1}$ is also an unconditional Schauder decomposition of X. Hence, we will always suppose that if $(X_\ell)_{\ell \geq 1}$ is an unconditional Schauder decomposition of X then all the subspaces X_ℓ are infinite-dimensional.

This assumption will allow us to define a weighted backward shift on a space which admits an unconditional Schauder decomposition. To do this, recall the notion of *biorthogonal system.* We denote by $\delta_{i,j}$ the Kronecker symbol.

Definition 2.6 ([11, Definition 1.f.1]). Let Z be a separable infinite-dimensional complex Banach space. For any positive integer i, let z_i and z_i^* be elements of Z and Z^* respectively. The sequence $((z_i)_{i\geq 1}, (z_i^*)_{i\geq 1})$ is called a *biorthogonal system* in Z if $\langle z_i^*, z_j\rangle = \delta_{i,j}$ for any positive integers i, j.

The following result will be fundamental.

Theorem 2.7 ([11, Theorem 1.f.4]). *Let Z be a separable infinite-dimensional complex Banach space. Then there exists a biorthogonal sequence $((z_i)_{i\geq 1}, (z_i^*)_{i\geq 1})$ in Z (where $z_i \in Z$ and $z_i^* \in Z^*$) such that:*

(1) $\sup_{i\geq 1} \|z_i\| \, \|z_i^*\| < \infty$;
(2) *the linear span of the vectors z_i, $i \geq 1$, is a dense subspace of Z;*
(3) *if $z \in Z$ and $\langle z_i^*, z\rangle = 0$ for any positive integer i, then $z = 0$.*

For more information on biorthogonal systems, we refer the reader to [11].

2.2. The result. We are now ready to prove our result about Jamison universal spaces. The proof is very close to that of [5, Theorem 2.1].

Theorem 2.8. *Let X be a separable complex Banach space which admits an unconditional Schauder decomposition. Then X is a universal Jamison space.*

Proof. Let us fix an increasing sequence $(n_k)_{k\geq 0}$ of positive integers (with $n_0 = 1$) which is not a Jamison sequence. Our task is to construct a bounded linear operator T on X which is partially power-bounded with respect to $(n_k)_{k\geq 0}$ and $\sigma_p(T) \cap \mathbb{T}$ is uncountable.

By definition, there exists a sequence $(X_\ell)_{\ell\geq 1}$ of infinite-dimensional closed subspaces of X such that any vector x of X can be written in a unique way as an unconditionally convergent series $\sum_{\ell\geq 1} x_\ell$, where $x_\ell \in X_\ell$ for all ℓ. In particular, $X = \bigoplus_{\ell\geq 1} X_\ell$. According to Theorem 2.7, there exists a biorthogonal sequence $((x_{i,\ell})_{i\geq 1}, (x_{i,\ell}^*)_{i\geq 1})$ in X_ℓ such that

$$\|x_{i,\ell}\| = 1 \quad \text{for any } i \geq 1 \quad \text{and} \quad M_\ell := \sup_{i\geq 1} \|x_{i,\ell}^*\| < \infty. \tag{2.1}$$

Definition of T. As in the proof of [5, Theorem 2.1], we define T as the sum of a diagonal operator and a weighted backward shift. The construction depends on two sequences to be chosen in the proof: a sequence $(\lambda_n)_{n\geq 1}$ of distinct unimodular complex numbers and a sequence $(w_n)_{n\geq 1}$ of positive weights.

Let us first define the operator $D : X \to X$ associated to the sequence $(\lambda_n)_{n\geq 1}$ by setting

$$D\Big(\sum_{\ell\geq 1} x_\ell\Big) = \sum_{\ell\geq 1} \lambda_\ell x_\ell$$

for any vector $x = \sum_{\ell \geq 1} x_\ell$ of X. Since the decomposition $X = \bigoplus_{\ell \geq 1} X_\ell$ is unconditional, D defines a bounded linear operator on X which is partially power-bounded with respect to $(n_k)_{k \geq 0}$. Let us then define a weighted backward shift on X by using the biorthogonal systems $((x_{i,\ell})_{i \geq 1}, (x^*_{i,\ell})_{i \geq 1})$. Since

$$X = X_\ell \oplus \overline{\operatorname{span}}\Big(\bigcup_{p \neq \ell} X_p\Big),$$

we first observe that we can extend the functionals $x^*_{i,\ell}$ to X by setting $x^*_{i,\ell} = 0$ on $\overline{\operatorname{span}}(\bigcup_{p \neq \ell} X_p)$. If we denote by $C > 0$ the unconditional constant of the decomposition $X = \bigoplus_{\ell \geq 1} X_\ell$ then $\|x^*_{i,\ell}\| \leq CM_\ell$ for any i and ℓ. Since $\langle x^*_{i,\ell}, x_{i,\ell} \rangle = 1$ and $\|x_{i,\ell}\| = 1$, we have $CM_\ell \geq 1$ for any ℓ.

Let $((w_{i,\ell})_{i \geq 1})_{\ell \geq 1}$ be a double sequence of positive real numbers to be defined later. Let finally $j : \mathbb{N} \setminus \{1\} \to \mathbb{N}$ satisfy the following two conditions:

(i) for any $n \geq 2$, $j(n) < n$;
(ii) for any $k \geq 1$, the set $\{n \geq 2;\ j(n) = k\}$ is infinite.

Our weighted backward shift will be defined as

$$Bx := \sum_{\ell \geq 2} \sum_{i \geq 1} \langle x^*_{i,\ell}, x \rangle \alpha_{i,\ell-1} x_{i,\ell-1} \qquad (x \in X) \tag{2.2}$$

where

$$\alpha_{i,1} := w_{i,1} |\lambda_2 - \lambda_{j(2)}| \qquad \text{for all } i \geq 1, \tag{2.3}$$

$$\alpha_{i,\ell} := w_{i,\ell} \frac{|\lambda_{\ell+1} - \lambda_{j(\ell+1)}|}{|\lambda_\ell - \lambda_{j(\ell)}|} \qquad \text{for all } i \geq 1 \text{ and } \ell \geq 2, \tag{2.4}$$

where $w_{i,\ell} > 0$. We now choose the coefficients $\alpha_{i,\ell}$ in such a way that (2.2) defines a bounded linear operator on X. For any positive integers i and ℓ, let us define

$$w_{i,\ell} := \frac{w_\ell}{2^i CM_{\ell+1}}, \tag{2.5}$$

where w_ℓ is an arbitrary positive number. Under this condition, the series

$$\mathcal{N} := \sum_{\ell \geq 2} \sum_{i \geq 1} \alpha_{i,\ell-1} \|x_{i,\ell-1}\| \, \|x^*_{i,\ell}\|$$

is convergent. Indeed, conditions (2.1) and (2.5) and definitions (2.3) and (2.4) yield

$$\mathcal{N} = \sum_{\ell \geq 1} \sum_{i \geq 1} \alpha_{i,\ell} \, \|x^*_{i,\ell+1}\| \leq w_1 |\lambda_2 - \lambda_{j(2)}| + \sum_{\ell=2}^{\infty} w_\ell \frac{|\lambda_{\ell+1} - \lambda_{j(\ell+1)}|}{|\lambda_\ell - \lambda_{j(\ell)}|}.$$

For arbitrary weights $w_\ell > 0$, we can take λ_ℓ sufficiently close to $\lambda_{j(\ell)}$ for any positive integer ℓ so that the series

$$\sum_{\ell\geq 2} w_\ell \frac{|\lambda_{\ell+1}-\lambda_{j(\ell+1)}|}{|\lambda_\ell-\lambda_{j(\ell)}|}$$

is convergent. More precisely, we first choose λ_3 close to $\lambda_{j(3)}$ (recall that $j(3)\in\{1,2\}$ by definition of j) such that

$$|\lambda_3-\lambda_{j(3)}|\leq \frac{|\lambda_2-\lambda_1|}{2^3 w_2}$$

and $\lambda_3\notin\{\lambda_1,\lambda_2\}$. Then at step ℓ we choose $\lambda_\ell\in\mathbb{T}$ such that

$$|\lambda_\ell-\lambda_{j(\ell)}|\leq \frac{|\lambda_{\ell-1}-\lambda_{j(\ell-1)}|}{2^\ell w_{\ell-1}}$$

and $\lambda_\ell\notin\{\lambda_1,\dots,\lambda_{\ell-1}\}$. Under these conditions the series $\mathcal{N}$ is convergent, so that B is a nuclear operator. It follows that $T=D+B$ is a bounded linear operator on X.

Unimodular eigenvectors of T. Consider the closed subspace

$$\hat{X}_1 := \overline{\operatorname{span}}[x_{1,\ell};\ \ell\geq 1]$$

of X. Since $Bx_{1,1}=0$ and $Bx_{1,\ell}=\alpha_{1,\ell-1}x_{1,\ell-1}$ for any positive integer ℓ, $\hat{X}_1$ is T-invariant. Hence one can consider the operator $T_1:\hat{X}_1\to\hat{X}_1$ induced by T. Since the decomposition $X=\bigoplus_{\ell\geq 1}X_\ell$ is unconditional, the sequence $(x_{1,\ell})_{\ell\geq 1}$ is an unconditional basis of $\hat{X}_1$. Let us now describe the unimodular eigenvectors of T_1: if $x=\sum_{\ell\geq 1}c_\ell x_{1,\ell}\in\hat{X}_1$, then the algebraic equation $T_1x=\lambda x$ is satisfied if and only if

$$c_\ell=\frac{(\lambda-\lambda_{\ell-1})\dots(\lambda-\lambda_1)}{\alpha_{1,\ell-1}\dots\alpha_{1,1}}c_1 \quad \text{for every } \ell\geq 2.$$

It follows that for any positive integer n, the eigenspace $\operatorname{Ker}(T_1-\lambda_n)$ is 1-dimensional and spanned by

$$u_1^{(n)} := x_{1,1}+\sum_{\ell=2}^{n}\frac{(\lambda_n-\lambda_{\ell-1})\dots(\lambda_n-\lambda_1)}{\alpha_{1,\ell-1}\dots\alpha_{1,1}}x_{1,\ell}.$$

We now need the following.

FACT 2.9 ([5, Lemma 2.4]). *By choosing $w_{1,n}$ and λ_n suitably, it is possible to ensure that for any integer $n\geq 2$,*

$$\|u_1^{(n)}-u_1^{(j(n))}\|\leq 2^{-n}.$$

In order to prove this, let us write

$$
\begin{aligned}
&u_1^{(n)} - u_1^{(j(n))} \\
&\quad = \sum_{\ell=2}^{j(n)} \bigg(\frac{(\lambda_n - \lambda_{\ell-1}) \ldots (\lambda_n - \lambda_1)}{\alpha_{1,\ell-1} \ldots \alpha_{1,1}} - \frac{(\lambda_{j(n)} - \lambda_{\ell-1}) \ldots (\lambda_{j(n)} - \lambda_1)}{\alpha_{1,\ell-1} \ldots \alpha_{1,1}} \bigg) x_{1,\ell} \\
&\qquad + \sum_{\ell=j(n)+1}^{n} \frac{(\lambda_n - \lambda_{\ell-1}) \ldots (\lambda_n - \lambda_1)}{\alpha_{1,\ell-1} \ldots \alpha_{1,1}} x_{1,\ell} =: a_1^{(n)} + b_1^{(n)}.
\end{aligned}
$$

Since the quantities $\alpha_{1,\ell-1} \ldots \alpha_{1,1}$ for $\ell \le j(n)$ do not depend on λ_n, we can ensure that $\|a_1^{(n)}\| \le 2^{-(n+1)}$ by taking $|\lambda_n - \lambda_{j(n)}|$ sufficiently small.

Let us now estimate $b_1^{(n)}$. We can assume that $|\lambda_p - \lambda_q| \le 1$ for any positive integers p and q. By (2.3) and (2.4), we have

$$
\begin{aligned}
\|b_1^{(n)}\| &\le \sum_{\ell=j(n)+1}^{n} \left| \frac{(\lambda_n - \lambda_{\ell-1}) \ldots (\lambda_n - \lambda_1)}{\alpha_{1,\ell-1} \ldots \alpha_{1,1}} \right| \\
&\le \sum_{\ell=j(n)+1}^{n} \frac{1}{w_{1,\ell-1} \ldots w_{1,1}} \cdot \left| \frac{\lambda_n - \lambda_{j(n)}}{\lambda_\ell - \lambda_{j(\ell)}} \right|.
\end{aligned}
$$

We now choose the coefficients λ_n and $w_{1,\ell}$ as follows. At stage n of the construction we take $w_{1,n-1}$ so large with respect to $w_{1,1}, \ldots, w_{1,n-2}$ that $(w_{1,n-1} w_{1,n-2} \ldots w_{1,1})^{-1}$ is very small (this means that we take w_{n-1} very large, by (2.5)). Next we take λ_n extremely close to $\lambda_{j(n)}$ so that the quantities

$$
\frac{1}{w_{1,\ell-1} \ldots w_{1,1}} \cdot \left| \frac{\lambda_n - \lambda_{j(n)}}{\lambda_\ell - \lambda_{j(\ell)}} \right| \qquad (\ell \in \{j(n)+1, \ldots, n-1\})
$$

are very small. We can thus ensure that $\|b_1^{(n)}\|$ is less than $2^{-(n+1)}$ and the result is proved.

Fact 2.9 is useful when applying the following theorem.

Theorem 2.10 ([9, Theorem 4.2]). *Let Y be a separable infinite-dimensional complex Banach space and let S be a bounded linear operator on Y. Suppose that there exists a sequence $(u_i)_{i \ge 1}$ in Y with the following properties:*

- (i) *for each $i \ge 1$, u_i is an eigenvector of S associated to an eigenvalue μ_i with $|\mu_i| = 1$ and the complex numbers μ_i are all distinct;*
- (ii) $\operatorname{span}[u_i;\, i \ge 1]$ *is dense in Y;*
- (iii) *for any $i \ge 1$ and any $\varepsilon > 0$, there exists an $n \ne i$ such that $\|u_n - u_i\| < \varepsilon$.*

Then S has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues. In particular, $\sigma_p(S)\cap\mathbb{T}$ is uncountable.

Applying Theorem 2.10 with $Y := \hat{X}_1$, $S := T_1$ and $u_i := u_1^{(i)}$, we deduce that T_1 has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues (the only condition we really need to check is (iii), and it is true by Fact 2.9). Hence the unimodular point spectrum of T_1 is uncountable. Since $\sigma_p(T_1)\cap\mathbb{T}\subset\sigma_p(T)\cap\mathbb{T}$, the unimodular point spectrum of T is also uncountable. For this part of the proof, we also refer the reader to the proof of [5, Proposition 2.5].

Estimate of the norms $\|T^{n_k}\|$. As in the *hard part* of the proof of [5, Theorem 2.1], we now need to prove that if the coefficients w_n and λ_n are suitably chosen, then $\|T^{n_k}-D^{n_k}\|\le 1$ for all positive integers k. For all positive integers k,ℓ,i,n, we put

$$t_{k,\ell}^{(i,n)}=\langle x_{i,k}^*, T^n x_{i,\ell}\rangle.$$

It is then clear that $t_{k,\ell}^{(i,n)}=0$ when $k>\ell$ or $\ell-k>n$. Moreover $t_{k,k}^{(i,n)}=\lambda_k^n$. The lemma below gives the expression of $t_{k,\ell}^{(i,n)}$ for $1\le\ell-k\le n$.

Lemma 2.11. *For any positive integers $k,\ell,i,n\ge 1$ with $1\le\ell-k\le n$,*

$$t_{k,\ell}^{(i,n)}=\alpha_{i,\ell-1}\alpha_{i,\ell-2}\dots\alpha_{i,k}\sum_{j_k+\dots+j_\ell=n-(\ell-k)}\lambda_k^{j_k}\dots\lambda_\ell^{j_\ell}.$$

Proof. For any fixed positive integer i, define the operator B_i on the space $\hat{X}_i := \overline{\operatorname{span}}\,[x_{i,\ell};\ \ell\ge 1]$ by setting $B_i x_{i,\ell} := Bx_{i,\ell}=\alpha_{i,\ell-1}x_{i,\ell-1}$ if $\ell\ge 2$ and $B_i x_{i,1}=0$. Then B_i is a weighted backward shift as in the proof of [5, Theorem 2.1]. Moreover, since the decomposition $X=\bigoplus_{\ell\ge1}X_\ell$ is unconditional, $(x_{i,\ell})_{\ell\ge1}$ is an unconditional Schauder basis of $\hat{X}_i$ and any vector x of $\hat{X}_i$ can be written in a unique way as $x=\sum_{\ell\ge1}\langle x_{i,\ell}^*,x\rangle x_{i,\ell}$. It follows that

$$B_i x=\sum_{\ell\ge2}\langle x_{i,\ell}^*,x\rangle\alpha_{i,\ell-1}x_{i,\ell-1}.$$

Then [5, Lemma 2.6] gives the desired expression of $t_{k,\ell}^{(i,n)}$. ∎

We now want to estimate $\|T^{n_p}-D^{n_p}\|$. In every subspace X_ℓ of X, we consider the linear subspace

$$\tilde{X}_\ell := \operatorname{span}\,[x_{i,\ell};\ i\ge1].$$

Since $\tilde{X}_\ell$ contains all the vectors $x_{i,\ell}$ $(i\ge1)$, it is a dense subspace of X_ℓ. It follows that $\tilde{X} := \bigoplus_{\ell\ge1}\tilde{X}_\ell$ is dense in X. To prove our result it suffices to establish that $\|(T^{n_p}-D^{n_p})x\|\le\|x\|$ for any $x\in\tilde{X}$. Indeed, $x=\sum_{\ell\ge1}x_\ell$

where

$$x_\ell = \sum_{i=1}^{i_\ell} \langle x_{i,\ell}^*, y\rangle x_{i,\ell} \in X_\ell$$

for some positive integer i_ℓ. Then, for any $n \geq 2$,

$$\begin{aligned} T^n x = \sum_{\ell=1}^{\infty}\sum_{i=1}^{i_\ell} \langle x_{i,\ell}^*, x\rangle T^n x_{i,\ell} &= \sum_{\ell=1}^{\infty}\sum_{i=1}^{i_\ell} \langle x_{i,\ell}^*, x\rangle \sum_{k=\max(1,\ell-n)}^{\ell} \langle x_{i,k}^*, T^n x_{i,\ell}\rangle x_{i,k} \\ &= \sum_{\ell=1}^{\infty}\sum_{i=1}^{i_\ell} \langle x_{i,\ell}^*, x\rangle \sum_{k=\max(1,\ell-n)}^{\ell} t_{k,\ell}^{(i,n)} x_{i,k} \end{aligned}$$

and

$$D^n x = \sum_{\ell=1}^{\infty} \lambda_\ell^n \sum_{i=1}^{i_\ell} \langle x_{i,\ell}^*, x\rangle x_{i,\ell} = \sum_{\ell=1}^{\infty}\sum_{i=1}^{i_\ell} \langle x_{i,\ell}^*, x\rangle t_{\ell,\ell}^{(i,n)} x_{i,\ell}.$$

We deduce that

$$(T^n - D^n)x = \sum_{\ell=2}^{\infty}\sum_{i=1}^{i_\ell} \langle x_{i,\ell}^*, x\rangle \sum_{k=\max(1,\ell-n)}^{\ell-1} t_{k,\ell}^{(i,n)} x_{i,k}$$

and according to (2.1),

$$\begin{aligned} \|(T^n - D^n)x\| &\leq \|x\| \sum_{\ell=2}^{\infty}\sum_{i=1}^{i_\ell} \|x_{i,\ell}^*\| \Big(\sum_{k=\max(1,\ell-n)}^{\ell-1} |t_{k,\ell}^{(i,n)}| \Big) \\ &\leq C\|x\| \sum_{\ell=2}^{\infty} M_\ell \Big(\sum_{i=1}^{i_\ell} \sum_{k=\max(1,\ell-n)}^{\ell-1} |t_{k,\ell}^{(i,n)}| \Big). \end{aligned}$$

We know from Lemma 2.11 that

$$t_{k,\ell}^{(i,n)} = \alpha_{i,\ell-1}\alpha_{i,\ell-2}\dots\alpha_{i,k} s_{k,\ell}^{(n)}, \quad \text{where} \quad s_{k,\ell}^{(n)} := \sum_{j_k+\dots+j_\ell = n-(\ell-k)} \lambda_k^{j_k}\dots\lambda_\ell^{j_\ell}.$$

By (2.4) and (2.5), this yields

$$t_{k,\ell}^{(i,n)} = \frac{w_{\ell-1}w_{\ell-2}\dots w_k}{2^{(\ell-k)i} CM_\ell CM_{\ell-1}\dots CM_{k+1}} \cdot \frac{|\lambda_\ell - \lambda_{j(\ell)}|}{|\lambda_k - \lambda_{j(k)}|} \cdot s_{k,\ell}^{(n)}.$$

As $CM_p \geq 1$ for any positive integer p, this implies in particular that

$$|t_{k,\ell}^{(i,n)}| \leq \frac{w_{\ell-1}w_{\ell-2}\dots w_k}{2^{(\ell-k)i} CM_\ell} \cdot \frac{|\lambda_\ell - \lambda_{j(\ell)}|}{|\lambda_k - \lambda_{j(k)}|} \cdot |s_{k,\ell}^{(n)}|$$

and so

$$\|(T^{n_p} - D^{n_p})x\| \le \|x\| \sum_{\ell=2}^{\infty} \sum_{k=\max(1,\ell-n_p)}^{\ell-1} w_{\ell-1}\dots w_k \cdot \frac{|\lambda_\ell - \lambda_{j(\ell)}|}{|\lambda_k - \lambda_{j(k)}|} \cdot |s_{k,\ell}^{(n_p)}|. \tag{2.6}$$

It remains to estimate the quantity

$$\sum_{k=\max(1,\ell-n_p)}^{\ell-1} w_{\ell-1}\dots w_k \cdot \frac{|\lambda_\ell - \lambda_{j(\ell)}|}{|\lambda_k - \lambda_{j(k)}|} \cdot |s_{k,\ell}^{(n_p)}|, \tag{2.7}$$

which is essentially the difficult part of the proof of [5, Theorem 2.1]. It is in this part of the proof that we use the fact that $(n_k)_{k\ge1}$ is not a Jamison sequence: according to Theorem 1.1 this assumption means that for every $\varepsilon > 0$ and every $\lambda \in \mathbb{T}$, there exists $\lambda' \in \mathbb{T} \setminus \{\lambda\}$ such that $|\lambda - \lambda'| \le d_{(n_k)}(\lambda, \lambda') \le \varepsilon$. In their proof, T. Eisner and S. Grivaux prove that if we take λ_ℓ sufficiently close to $\lambda_{j(\ell)}$ (for the distance $d_{(n_k)}$, hence for the Euclidean norm $|\cdot|$), then the sum

$$\sum_{k=\max(1,\ell-n_p)}^{\ell-1} w_{\ell-1}^2\dots w_k^2 \cdot \frac{|\lambda_\ell - \lambda_{j(\ell)}|^2}{|\lambda_k - \lambda_{j(k)}|^2} \cdot |s_{k,\ell}^{(n_p)}|^2$$

can be made arbitrarily small. By rewriting that proof, one can see that if we take λ_ℓ close to $\lambda_{j(\ell)}$, the sum (2.7) is less than $2^{1-\ell}$. By density, we deduce from (2.6) that $\|T^{n_p} - D^{n_p}\| \le 1$ for any positive integer p, which concludes the proof. ∎

Example 2.4 allows us to give new examples of universal Jamison spaces.

2.3. Examples. Recall that the *James space* $\mathcal{J}$ is the set of all complex sequences $x = (x_n)_{n\ge1}$ belonging to $c_0(\mathbb{N})$ such that

$$\|x\|_{\mathcal{J}} := \sup\{|x_{p_1} - x_{p_2}|^2 + \dots + |x_{p_{k-1}} - x_{p_k}|^2\} < \infty,$$

where the supremum is taken over all k-tuples $(p_1, \dots, p_k)$ of positive integers such that $p_1 < \dots < p_k$. We refer the reader to [7] for more information on the James space. In particular, it is not difficult to see that the subspace

$$\mathcal{J}_2 = \{x \in \mathcal{J};\ x_{2k} = 0 \text{ for all } k \ge 1\}$$

of $\mathcal{J}$ is linearly isomorphic to $\ell_2(\mathbb{N})$ and complemented in $\mathcal{J}$. Then Example 2.4 gives us the following result.

Example 2.12. The James space $\mathcal{J}$ is a universal Jamison space.

Other examples of universal Jamison spaces are given by spaces which contain a copy of $c_0(\mathbb{N})$. For instance, $C([0, 1])$ is a universal Jamison space. On the other hand, it is not difficult to exhibit a class of Banach spaces which are not universal Jamison spaces, namely the class of *hereditarily indecomposable Banach spaces.*

DEFINITION 2.13. An infinite-dimensional Banach space X is said to be *decomposable* if there exist infinite-dimensional closed subspaces Y and Z of X such that $X = Y \oplus Z$. We say that X is *hereditarily indecomposable* if no infinite-dimensional closed subspace of X is decomposable.

The famous Gowers dichotomy highlights the fact that the notion of unconditional Schauder decomposition is in a sense opposite to the notion of hereditarily indecomposable space. More precisely, Gowers [8] proves that if X is an arbitrary Banach space then it either contains an unconditional basic sequence, or contains a hereditarily indecomposable subspace. In fact, any hereditarily indecomposable complex Banach space X fails to be a universal Jamison space. Indeed, each bounded linear operator T on X is of the form $\lambda I + S$ where $\lambda \in \mathbb{C}$ and $S \in \mathcal{B}(X)$ is a *strictly singular operator*, that is, S fails to be an isomorphism when restricted to any infinite-dimensional closed subspace of X. In particular, the unimodular point spectrum of T is at most countable (see [12] for more details). We have thus proved:

PROPOSITION 2.14. *If X is a hereditarily indecomposable Banach space, then X is not a universal Jamison space.*

3. Jamison sequences for C_0-semigroups. In this section, we study Jamison sequences in the context of strongly continuous semigroups acting on separable complex Banach spaces. Starting from the work of C. Badea and S. Grivaux [1, 2], we give a characterization of Jamison sequences for C_0-semigroups (Definition 3.2 below).

We begin by recalling some definitions and facts about C_0-semigroups. Let X be a complex Banach space. A family $(T_t)_{t\geq 0}$ of bounded linear operators on X is called a C_0-*semigroup* if

- $T_0 = \mathrm{Id}_X$;
- for any $s, t \geq 0$, $T_{s+t} = T_s T_t$;
- for any $x \in X$, $\lim_{t\to 0^+} \|T_t x - x\| = 0$.

The *infinitesimal generator* of the C_0-semigroup $(T_t)_{t\geq 0}$ is the map $A : D(A) \to X$ defined by

$$D(A) := \left\{ x \in X;\ \lim_{t\to 0^+} \frac{T_t x - x}{t} \text{ exists} \right\},$$

$$Ax := \lim_{t\to 0^+} \frac{T_t x - x}{t} \qquad (x \in D(A)).$$

We recall that $\sigma_p(T_t) \setminus \{0\} = \exp(t\sigma_p(A))$ for any $t \geq 0$ (see for instance

[6, Chapter 4]) and

$$\sigma_p(T_1) \cap \mathbb{T} = \exp(\sigma_p(A) \cap i\mathbb{R}).$$

We now introduce the notion of a semigroup partially bounded with respect to a sequence of positive real numbers.

DEFINITION 3.1. Let X be a separable complex Banach space. Let $(t_k)_{k\geq 0}$ be an increasing sequence of positive real numbers and $(T_t)_{t\geq 0}$ a semigroup of bounded linear operators on X. We say that $(T_t)_{t\geq 0}$ is *partially bounded with respect to* $(t_k)_{k\geq 0}$ if $\sup_{k\geq 0} \|T_{t_k}\| < \infty$.

If $(t_k)_{k\geq 0}$ is bounded, then any C_0-semigroup is bounded with respect to $(t_k)_{k\geq 0}$. So we will restrict ourselves to increasing sequences $(t_k)_{k\geq 0}$ which tend to infinity. Moreover, it is clear that we can assume that $t_0 = 1$.

DEFINITION 3.2. Let $(t_k)_{k\geq 0}$ be an increasing sequence of positive real numbers which tends to infinity. We say that $(t_k)_{k\geq 0}$ is a *Jamison sequence for C_0-semigroups* if for every separable complex Banach space X and for every C_0-semigroup $(T_t)_{t\geq 0}$ on X (with infinitesimal generator A) which is partially bounded with respect to $(t_k)_{k\geq 0}$, the set $\sigma_p(A) \cap i\mathbb{R}$ is at most countable.

Our aim in this section is to give a characterization of Jamison sequences for C_0-semigroups. For this, we will need the characterization of Jamison sequences obtained in [2, Theorem 2.1]. For $\theta \in \mathbb{R}$, define

$$\|\theta\| := \operatorname{dist}(\theta, \mathbb{Z}) = \inf\{|\theta - n|;\, n \in \mathbb{Z}\}.$$

Recall that there exist constants $C_1, C_2 > 0$ such that

$$C_2\|\theta\| \leq |e^{2i\pi\theta} - 1| \leq C_1\|\theta\| \quad \text{for any real } \theta.$$

We are going to prove the following result.

THEOREM 3.3. *Let $(t_k)_{k\geq 0}$ be an increasing sequence of positive real numbers such that $t_0 = 1$ and $t_k \to \infty$. The following assertions are equivalent:*

(1) *$(t_k)_{k\geq 0}$ is a Jamison sequence for C_0-semigroups;*
(2) *there exists $\varepsilon > 0$ such that for any $\theta \in \,]0, 1/2]$,*

$$\sup_{k\geq 0} \|t_k\theta\| \geq \varepsilon.$$

In order to be able to use the characterization of Jamison sequences of [2, Theorem 2.1], we first only consider *sequences of positive integers* which are Jamison sequences for C_0-semigroups. We show that an integer sequence is a Jamison sequence if and only if it is a Jamison sequence for C_0-semigroups (Theorem 3.4). From this, we will deduce Theorem 3.3.

3.1. Integer Jamison sequences. Let $(n_k)_{k\geq 0}$ be an increasing sequence of positive integers such that $n_0 = 1$. As in the proof of [2, Theorem 2.8], we associate to it a distance $d_{(n_k)}$ on the unit circle $\mathbb{T}$ by setting

$$d_{(n_k)}(\lambda,\mu) = \sup_{k\geq 0} |\lambda^{n_k} - \mu^{n_k}| \quad \text{for any } \lambda,\mu \in \mathbb{T}.$$

We now prove

THEOREM 3.4. *Let $(n_k)_{k\geq 0}$ be an increasing sequence of positive integers such that $n_0 = 1$. The following assertions are equivalent:*

(1) *$(n_k)_{k\geq 0}$ is a Jamison sequence for C_0-semigroups;*
(2) *$(n_k)_{k\geq 0}$ is a Jamison sequence;*
(3) *there exists $\varepsilon > 0$ such that for any distinct $\lambda,\mu \in \mathbb{T}$,*

$$d_{(n_k)}(\lambda,\mu) \geq \varepsilon.$$

Proof. (2)⇒(3). See [2, Theorem 2.1].

(3)⇒(1). Let X be a separable complex Banach space and $(T_t)_{t\geq 0}$ a C_0-semigroup on X such that $M := \sup_{k\geq 0} \|T_{n_k}\| < \infty$. We denote by A the infinitesimal generator of $(T_t)_{t\geq 0}$. Let $i\eta$ and $i\xi$ be two eigenvalues of A ($\eta,\xi \in \mathbb{R}$) and let x_η and x_ξ be eigenvectors such that $\|x_\eta\| = \|x_\xi\| = 1$ and $Ax_\phi = i\phi x_\phi$ for $\phi \in \{\eta,\xi\}$. We know that for any k and for $\phi \in \{\eta,\xi\}$, we have $T_{n_k}x_\phi = e^{i\phi n_k}x_\phi$. By the triangle inequality,

$$|e^{i\eta n_k} - e^{i\xi n_k}| - \|x_\eta - x_\xi\| \leq \|T_{n_k}(x_\eta - x_\xi)\| \leq M\|x_\eta - x_\xi\|. \tag{3.1}$$

Setting $\lambda_\theta = e^{i\theta}$ for $\theta \in \mathbb{R}$, we deduce from (3.1) that

$$\|x_\eta - x_\xi\| \geq \frac{\sup_{k\geq 0} |\lambda_{\eta-\xi}^{n_k} - 1|}{M+1}.$$

Assume that $\eta,\xi \in [2\ell\pi, 2(\ell+1)\pi[$ for some integer ℓ and $\eta \neq \xi$. Then $\lambda_{\eta-\xi} \in \mathbb{T}\setminus\{1\}$ and $\sup_{k\geq 0} |\lambda_{\eta-\xi}^{n_k} - 1| \geq \varepsilon$. It follows that

$$\|x_\eta - x_\xi\| \geq \frac{\varepsilon}{M+1}.$$

Since X is separable, we see that $\sigma_p(A) \cap [2i\ell\pi, 2i(\ell+1)\pi[$ is at most countable. Hence so is $\sigma_p(A)\cap i\mathbb{R}$, which proves that $(t_k)_{k\geq 0}$ is a Jamison sequence for C_0-semigroups.

(1)⇒(3). Assume that (3) is not satisfied. We know from [2, Theorem 2.8] that there exists an uncountable subset K of $\mathbb{T}$ such that the metric space $(K, d_{(n_k)})$ is separable; we want to prove that $(n_k)_{k\geq 0}$ is not a Jamison sequence for C_0-semigroups. To do this, we construct a separable Banach space X and a C_0-semigroup $(S_t)_{t\geq 0}$ on X (with infinitesimal generator A) which is bounded with respect to $(n_k)_{k\geq 0}$, but $\sigma_p(A) \cap i\mathbb{R}$ is uncountable.

Let

$$X = \left\{ f : [0,\infty[\to \mathbb{R} \text{ measurable}; \ \|f\| := \left(\int_0^\infty \frac{|f(t)|^2}{1+t^2}\, dt \right)^{1/2} < \infty \right\}$$

and let $(S_t)_{t\geq 0}$ be the translation semigroup on X defined by

$$S_t f(x) = f(x+t) \quad (f \in X,\ t, x \geq 0).$$

We set $X_* := \{f \in X;\ \|f\|_* < \infty\}$ where

$$\|f\|_* := \max\Big(\|f\|,\ \sup_{j\geq 0} 4^{-(j+1)} \sup_{k_0,\dots,k_j \geq 0} \Big\| \prod_{\ell=0}^{j} (S_{n_{k_\ell}} - I) f \Big\| \Big).$$

It is rather easy to check that S_t is a bounded linear operator on X_*. In a first step, we prove that the semigroup $(S_t)_{t\geq 0}$ is bounded with respect to $(n_k)_{k\geq 0}$. Then we construct a separable subspace of X_* on which $(S_t)_{t\geq 0}$ is strongly continuous and such that $\sigma_p(A) \cap i\mathbb{R}$ is uncountable, where A denotes the infinitesimal generator of $(S_t)_{t\geq 0}$.

Boundedness of $(S_t)_{t\geq 0}$ *with respect to* $(n_k)_{k\geq 0}$. Let $f \in X_*$ and $k \in \mathbb{Z}_+$. Then $\|S_{n_k} f\|_*$ is the maximum of

$$\|S_{n_k} f\| \quad \text{and} \quad \sup_{j\geq 0} 4^{-(j+1)} \sup_{k_0,\dots,k_j \geq 0} \Big\| \prod_{\ell=0}^{j} (S_{n_{k_\ell}} - I) S_{n_k} f \Big\|.$$

On the one hand,

$$\begin{aligned} \|S_{n_k} f\| = \|f + (S_{n_k} - I) f\| &\leq \|f\| + \|(S_{n_k} - I) f\| \\ &\leq \|f\|_* + 4 \cdot \frac{1}{4} \|(S_{n_k} - I) f\| \leq 5\|f\|_*; \end{aligned}$$

on the other hand, for any $j \in \mathbb{Z}_+$ and any $(j+1)$-tuple $(k_0, \dots, k_j)$ of nonnegative integers,

$$\begin{aligned} 4^{-(j+1)} \Big\| \prod_{\ell=0}^{j} (S_{n_{k_\ell}} - I) S_{n_k} f \Big\| &\leq 4 \cdot 4^{-(j+2)} \Big\| \prod_{\ell=0}^{j} (S_{n_{k_\ell}} - I)(S_{n_k} - I) f \Big\| \\ &\quad + 4^{-(j+1)} \Big\| \prod_{\ell=0}^{j} (S_{n_{k_\ell}} - I) f \Big\|. \end{aligned}$$

This proves that

$$\|S_{n_k} f\|_* \leq 4\|f\|_* + \|f\|_* = 5\|f\|_*$$

and so $\sup_{k\geq 0} \|S_{n_k}\|_* \leq 5$.

Eigenvectors of A. For any $\eta \in [0, 2\pi[$, we put $e_\eta(x) = e^{i\eta x}$ ($x \in \mathbb{R}$). It is clear that e_η is an eigenvector of A associated to the eigenvalue $i\eta$ and that $e_\eta \in X_*$. Indeed, $\|e_\eta\| = \sqrt{\pi/2}$ and for any nonnegative integer j and any $(j+1)$-tuple $(k_0, \ldots, k_j)$ of nonnegative integers, we have

$$\Big\| \prod_{\ell=0}^{j} (S_{n_{k_\ell}} - I) e_\eta \Big\| = \Big(\prod_{\ell=0}^{j} |e^{i\eta n_{k_\ell}} - 1| \Big) \sqrt{\pi/2} \leq 2^{j+1} \sqrt{\pi/2},$$

which proves that $\|e_\eta\|_* = \sqrt{\pi/2}$.

Making X_* separable. At this stage of the proof, we use our assumption: there exists an uncountable subset K of $\mathbb{T}$ such that the metric space $(K, d_{(n_k)})$ is separable. We define

$$I_K := \{\eta \in [0, 2\pi[;\ e^{i\eta} \in K\}.$$

The subspace of X_* we are going to consider is

$$X_*^K := \overline{\operatorname{span}}^{\|\cdot\|_*}[e_\eta;\ \eta \in I_K],$$

equipped with the norm $\|\cdot\|_*$. Since $\sigma_p(A) \cap i\mathbb{R}$ contains I_K, it is uncountable. Furthermore, the semigroup $(S_t)_{t\geq 0}$ is still bounded with respect to $(n_k)_{k\geq 0}$ and it is easy to prove that it is also strongly continuous, by using a density argument. The only thing we really need to prove is that X_*^K is separable. This is a consequence of the following lemma.

Lemma 3.5. *The eigenvector field $E : I_K \to X_*$ defined by $E(\eta) = e_\eta$ is continuous on I_K.*

Proof. Let $\eta, \mu \in I_K$. We need to estimate the quantity

$$\|e_\eta - e_\xi\|_* = \max\Big(\|e_\eta - e_\xi\|, \sup_{j\geq 0} 4^{-(j+1)} \sup_{k_0,\ldots,k_j \geq 0} \Big\| \prod_{\ell=0}^{j} (e^{i\eta n_{k_\ell}} - 1) e_\eta - \prod_{\ell=0}^{j} (e^{i\xi n_{k_\ell}} - 1) e_\xi \Big\| \Big).$$

For any nonnegative integer j and any $(k_0, \ldots, k_j) \in (\mathbb{Z}_+)^{j+1}$, we have

$$\Big\| \prod_{\ell=0}^{j} (e^{i\eta n_{k_\ell}} - 1) e_\eta - \prod_{\ell=0}^{j} (e^{i\xi n_{k_\ell}} - 1) e_\xi \Big\| \leq \|e_\eta - e_\xi\| \prod_{\ell=0}^{j} |e^{i\eta n_{k_\ell}} - 1| + \|e_\xi\| \Big| \prod_{\ell=0}^{j} (e^{i\eta n_{k_\ell}} - 1) - \prod_{\ell=0}^{j} (e^{i\xi n_{k_\ell}} - 1) \Big|$$

and so

$$(3.2)\qquad \Big\|\prod_{\ell=0}^{j}(e^{i\eta n_{k_\ell}}-1)e_\eta-\prod_{\ell=0}^{j}(e^{i\xi n_{k_\ell}}-1)e_\xi\Big\|\le 2^{j+1}\|e_\eta-e_\xi\| + \sqrt{\pi/2}\Big|\prod_{\ell=0}^{j}(e^{i\eta n_{k_\ell}}-1)-\prod_{\ell=0}^{j}(e^{i\xi n_{k_\ell}}-1)\Big|.$$

We need to estimate the quantity

$$d_j(e^{i\eta},e^{i\xi}) := \sup_{k_0,\dots,k_j\ge 0}\Big|\prod_{\ell=0}^{j}(e^{i\eta n_{k_\ell}}-1)-\prod_{\ell=0}^{j}(e^{i\xi n_{k_\ell}}-1)\Big|.$$

According to the identity

$$\prod_{\ell=0}^{j}(e^{i\eta n_{k_\ell}}-1)-\prod_{\ell=0}^{j}(e^{i\xi n_{k_\ell}}-1)=(e^{i\eta n_{k_0}}-e^{i\xi t_{k_0}})\prod_{\ell=1}^{j}(e^{i\eta n_{k_\ell}}-1) + (e^{i\xi n_{k_0}}-1)\Big(\prod_{\ell=1}^{j}(e^{i\eta n_{k_\ell}}-1)-\prod_{\ell=1}^{j}(e^{i\xi n_{k_\ell}}-1)\Big),$$

we get the estimate

$$d_j(e^{i\eta},e^{i\xi})\le 2^j d_{(n_k)}(e^{i\eta},e^{i\xi})+2d_{j-1}(e^{i\eta},e^{i\xi}).$$

Then it follows from an easy induction argument that

$$d_j(e^{i\eta},e^{i\xi})\le (j+1)2^j d_{(n_k)}(e^{i\eta},e^{i\xi})$$

for any j. The above estimates show that

$$\|e_\eta-e_\xi\|_* \le \max\big(\|e_\eta-e_\xi\|,\sup_{j\ge 0}\big(2^{-(j+1)}\|e_\eta-e_\xi\|+\sqrt{\pi/2}(j+1)2^{-(j+2)}d_{(n_k)}(e^{i\eta},e^{i\xi})\big)\big).$$

Then there exists a constant $C>0$ such that for any $\eta,\mu\in I_K$,

$$\|e_\eta-e_\xi\|_*\le C(\|e_\eta-e_\xi\|+d_{(n_k)}(e^{i\eta},e^{i\xi})).$$

We now fix $\varepsilon>0$. There exists $A_\varepsilon>0$ such that

$$\int_{A_\epsilon}^{\infty}\frac{|e_\eta(t)-e_\xi(t)|^2}{1+t^2}\,dt\le\int_{A_\epsilon}^{\infty}\frac{4}{1+t^2}\,dt\le\varepsilon^2$$

and so

$$\|e_\eta-e_\xi\|_*\le C\bigg(\bigg(\int_0^{A_\epsilon}\frac{|e^{i\eta t}-e^{i\xi t}|^2}{1+t^2}\,dt\bigg)^{1/2}+d_{(n_k)}(e^{i\eta},e^{i\xi})+\varepsilon\bigg).$$

It follows that the eigenvector field $E:\eta\mapsto e_\eta$ is continuous on I_K. ■

As a consequence, X_*^K is separable, which proves (1)⇒(3). ■

With the help of Theorem 3.4, we will be able to prove our result on the characterization of Jamison sequences for C_0-semigroups (Theorem 3.3).

3.2. Relationship with real Jamison sequences for C_0-semigroups. Let $(t_k)_{k\geq 0}$ be an increasing sequence of positive real numbers such that $t_0 = 1$ and $t_k \to \infty$. For any k, let n_k be the integer part of t_k. In particular, $n_0 = 1$. To begin with, we have the following easy fact.

FACT 3.6. *If $(T_t)_{t\geq 0}$ is a C_0-semigroup on a Banach space X such that $\sup_{k\geq 0} \|T_{n_k}\| < \infty$, then $\sup_{k\geq 0} \|T_{t_k}\| < \infty$.*

Proof. Since the family $\{T_s;\ s \in [0,1]\}$ is bounded, setting $\varepsilon_k = t_k - n_k \in [0,1[$ we have

$$\|T_{t_k}\| = \|T_{n_k}T_{\varepsilon_k}\| \leq \|T_{n_k}\|\,\|T_{\varepsilon_k}\| \leq \Big(\sup_{0\leq s\leq 1} \|T_s\|\Big)\|T_{n_k}\|,$$

and the conclusion follows from the assumption. ∎

The characterization of Jamison sequences for C_0-semigroups is a consequence of the following two lemmas.

LEMMA 3.7.

(1) *If $(t_k)_{k\geq 0}$ is a Jamison sequence for C_0-semigroups, then so is $(n_k)_{k\geq 0}$.*
(2) *If $(1, (n_k + 1)_{k\geq 0})$ is a Jamison sequence for C_0-semigroups, then so is $(t_k)_{k\geq 0}$.*

Proof. Assume that $(t_k)_{k\geq 0}$ is a Jamison sequence for C_0-semigroups. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup (with infinitesimal generator A) on a separable complex Banach space X such that $\sup_{k\geq 0} \|T_{n_k}\| < \infty$. According to Fact 3.6, $\sup_{k\geq 0} \|T_{t_k}\| < \infty$ and it follows from the assumption that $\sigma_p(A) \cap i\mathbb{R}$ is at most countable. Then $(n_k)_{k\geq 0}$ is a Jamison sequence for C_0-semigroups and (1) is proved.

We prove (2) with the same method: assume that $(1, (n_k + 1)_{k\geq 0})$ is a Jamison sequence for C_0-semigroups and let $(T_t)_{t\geq 0}$ be a C_0-semigroup (with infinitesimal generator A) on a separable complex Banach space X such that $\sup_{k\geq 0} \|T_{t_k}\| < \infty$. By the definition of n_k, we have $\varepsilon_k = n_k + 1 - t_k \in]0, 1]$ and the same proof as that of Fact 3.6 shows that $\sup_{k\geq 0} \|T_{n_k+1}\| < \infty$. Since $(1, (n_k+1)_{k\geq 0})$ is a Jamison sequence for C_0-semigroups, the set $\sigma_p(A)\cap i\mathbb{R}$ is at most countable and so $(t_k)_{k\geq 0}$ is a Jamison sequence for C_0-semigroups. ∎

In the second lemma, we establish a relationship between the integer sequences $(n_k)_{k\geq 0}$ and $(1, (n_k + 1)_{k\geq 0})$ from the point of view of Jamison sequences.

LEMMA 3.8. *The sequence $(n_k)_{k\geq 0}$ is a Jamison sequence if and only if $(1, (n_k + 1)_{k\geq 0})$ is.*

Proof. Assume that $(n_k)_{k\geq 0}$ is a Jamison sequence. By [2, Theorem 2.1], there exists $\varepsilon > 0$ such that $\sup_{k\geq 0} |\lambda^{n_k} - 1| \geq \varepsilon$ for any $\lambda \in \mathbb{T} \setminus \{1\}$. Let $\lambda \in \mathbb{T} \setminus \{1\}$ be such that $|\lambda - 1| \leq \varepsilon/2$. For any k, we get

$$|\lambda^{n_k+1} - 1| = |\lambda(\lambda^{n_k} - 1) + \lambda - 1| \geq |\lambda^{n_k} - 1| - \varepsilon/2$$

and so

$$\sup_{k\geq 0} |\lambda^{n_k+1} - 1| \geq \varepsilon/2\cdot$$

It follows that

$$\max\Big(|\lambda - 1|, \sup_{k\geq 0} |\lambda^{n_k+1} - 1|\Big) \geq \varepsilon/2\cdot$$

By [2, Theorem 2.1], $(1, (n_k + 1)_{k\geq 0})$ is a Jamison sequence.

Conversely, if $(1, (n_k + 1)_{k\geq 0})$ is a Jamison sequence then it is straightforward to check that an operator which is partially power-bounded with respect to $(n_k)_{k\geq 0}$ is also partially power-bounded with respect to $(1, (n_k + 1)_{k\geq 0})$. This proves that $(n_k)_{k\geq 0}$ is also a Jamison sequence. ■

The above lemmas and Theorem 3.4 allow us to prove the following result.

THEOREM 3.9. *Let $(t_k)_{k\geq 0}$ be an increasing sequence of positive real numbers such that $t_0 = 1$ and $t_k \to \infty$. For any k, let n_k be the integer part of t_k. The following assertions are equivalent:*

(1) *$(t_k)_{k\geq 0}$ is a Jamison sequence for C_0-semigroups;*
(2) *$(n_k)_{k\geq 0}$ is a Jamison sequence;*
(3) *there exists $\varepsilon > 0$ such that $\sup_{k\geq 0} |\lambda^{n_k} - 1| \geq \varepsilon$ for any $\lambda \in \mathbb{T} \setminus \{1\}$.*

Proof. (2)⇒(3). See [2, Theorem 2.1].

(1)⇒(2). Apply Lemma 3.7 and Theorem 3.4.

(2)⇒(1). If $(n_k)_{k\geq 0}$ is a Jamison sequence, then so is $(1, (n_k + 1)_{k\geq 0})$ by Lemma 3.8. According to Theorem 3.4, $(1, (n_k+1)_{k\geq 0})$ is a Jamison sequence for C_0-semigroups. Now Lemma 3.7 shows that $(t_k)_{k\geq 0}$ is a Jamison sequence for C_0-semigroups. ■

Theorem 3.3 is a consequence of Theorem 3.9 and the next proposition.

PROPOSITION 3.10. *Let $(t_k)_{k\geq 0}$ be an increasing sequence of positive real numbers such that $t_0 = 1$ and $t_k \to \infty$. For any k, let n_k be the integer part of t_k. The following assertions are equivalent:*

(i) *there exists $\varepsilon > 0$ such that $\sup_{k\geq 0} |\lambda^{n_k} - 1| \geq \varepsilon$ for any $\lambda \in \mathbb{T} \setminus \{1\}$;*
(ii) *there exists $\varepsilon' > 0$ such that $\sup_{k\geq 0} \|t_k\theta\| \geq \varepsilon'$ for any $\theta \in]0, 1/2]$.*

Proof. Setting $\lambda_\theta = e^{2i\pi\theta}$ for $\theta \in]0, 1/2]$, we know that $C_2\|\theta\| \leq |\lambda_\theta - 1| \leq C_1\|\theta\|$ where C_1 and C_2 are positive constants which do not depend on θ. By using these inequalities, it is easy to check that (i) and (ii) are equivalent. We leave the details to the reader. ■

3.3. Universal Jamison spaces for C_0-semigroups. We can prove an analog of Theorem 2.8 in the context of C_0-semigroups. To begin with, we define the notion of *universal Jamison space for C_0-semigroups*.

DEFINITION 3.11. Let X be a separable infinite-dimensional complex Banach space. We say that X is a *universal Jamison space for C_0-semigroups* if the following property holds true: for any increasing sequence $(t_k)_{k\geq 0}$ of positive real numbers which is not a Jamison sequence for C_0-semigroups, there exists a C_0-semigroup $(T_t)_{t\geq 0}$ on X (with infinitesimal generator A) which is bounded with respect to $(t_k)_{k\geq 0}$ and such that $\sigma_p(A)\cap i\mathbb{R}$ is uncountable.

The analog of Theorem 2.8 in the context of C_0-semigroups is the following.

THEOREM 3.12. *Let X be a separable complex Banach space which admits an unconditional Schauder decomposition. Then X is a universal Jamison space for C_0-semigroups.*

Proof. Let $(t_k)_{k\geq 0}$ be an increasing sequence of positive real numbers such that $t_0 = 1$ and $t_k \to \infty$. We assume that $(t_k)_{k\geq 0}$ is not a Jamison sequence for C_0-semigroups; we need to construct a C_0-semigroup (with infinitesimal generator A) on X which is bounded with respect to $(t_k)_{k\geq 0}$ and $\sigma_p(A)\cap i\mathbb{R}$ is uncountable. For any k, let n_k be the integer part of t_k. Since $(t_k)_{k\geq 0}$ is not a Jamison sequence for C_0-semigroups, Theorem 3.9 shows that $(n_k)_{k\geq 0}$ is not a Jamison sequence. We can then consider the operator $T = D + B$ constructed in the proof of Theorem 2.8: it is partially power-bounded with respect to $(n_k)_{k\geq 0}$, and $\sigma_p(T)\cap\mathbb{T}$ is uncountable.

In the proof of Theorem 2.8, we can take λ_1 with $|\lambda_1| = 1$ and $|\lambda_1 - 1| = 1/3$. Then we choose λ_n on the arc between 1 and λ_1 for any n. We have $\sigma(D) = \overline{\{\lambda_n;\, n\geq 1\}}$ and if we take λ_n sufficiently close to $\lambda_{j(n)}$, we get $\|B\| < 1/3$, which implies that $\sigma(T)\subset\sigma(D)_{1/3}$. Here if K is a compact set of the complex plane, we set

$$K_\varepsilon := \{z\in\mathbb{C};\, \mathrm{dist}(z,K) < \varepsilon\}.$$

In particular, $\sigma(T)\subset\mathcal{P}_{1/2} := \{z\in\mathbb{C};\, \mathfrak{Re}\, z > 1/2\}$. Since the complex logarithm is an analytic function in this domain, the operator $\operatorname{Log} T$ is well-defined (by the functional calculus) and bounded on X. Then the C_0-semigroup $(T_t)_{t\geq 0}$ with infinitesimal generator $\operatorname{Log} T$ is such that for any $t\geq 0$, $T_t = e^{t\operatorname{Log} T}$. Since $e^{n\operatorname{Log} z} = z^n$ for any $z\in\mathcal{P}_{1/2}$, we conclude that $T_{n_k} = T^{n_k}$ for any nonnegative integer k. Furthermore, Fact 3.6 shows that $(T_t)_{t\geq 0}$ is bounded with respect to $(t_k)_{k\geq 0}$ and since

$$\sigma_p(T)\cap\mathbb{T} = \sigma_p(T_1)\cap\mathbb{T} = \exp(\sigma_p(\operatorname{Log} T)\cap i\mathbb{R}),$$

the set $\sigma_p(\operatorname{Log} T)\cap i\mathbb{R}$ is uncountable. ■

We finish this paper by proving a result concerning the Hausdorff dimension of $\sigma_p(A) \cap i\mathbb{R}$ which fits into the framework of [2].

3.4. Hausdorff dimension of $\sigma_p(A) \cap i\mathbb{R}$. Ransford and Roginskaya [15, Theorem 4.1] proved that if a C_0-semigroup (with infinitesimal generator A) on a separable complex Banach space is bounded with respect to an increasing sequence of positive real numbers which tends to infinity, then $\sigma_p(A) \cap i\mathbb{R}$ has Lebesgue measure zero. It is then natural to study the Hausdorff dimension of this set. The same authors proved that the Hausdorff dimension $\dim_{\mathrm{H}}(\sigma_p(A) \cap i\mathbb{R})$ can be controlled by the growth of the sequence $(t_k)_{k\geq 0}$ ([15, Theorem 4.1]). The results below are the same as in [2, Section 3] but in the context of C_0-semigroups. To begin with, we prove the following theorem.

THEOREM 3.13. *Let $(n_k)_{k\geq 0}$ be an increasing sequence of positive integers such that $n_0 = 1$. Let $\mathcal{S}$ be any class of subsets of the unit circle $\mathbb{T}$ such that every subset of an element of $\mathcal{S}$ is an element of $\mathcal{S}$ itself. The following assertions are equivalent:*

(1) *for every separable complex Banach space X and every C_0-semigroup $(T_t)_{t\geq 0}$ on X which is bounded with respect to $(n_k)_{k\geq 0}$, the set $\sigma_p(T_1) \cap \mathbb{T}$ belongs to $\mathcal{S}$;*
(2) *for every subset K of $\mathbb{T}$ not belonging to $\mathcal{S}$, the metric space $(K, d_{(n_k)})$ is non-separable;*
(3) *for every subset K of $\mathbb{T}$ not belonging to $\mathcal{S}$, there exists $\varepsilon > 0$ such that K contains an uncountable ε-separated family for $d_{(n_k)}$.*

Proof. (2)⇔(3) is clear.

(3)⇒(1). Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X such that $M := \sup_{k\geq 0} \|T_{n_k}\| < \infty$ and assume that $\sigma_p(T_1) \cap \mathbb{T} \notin \mathcal{S}$. According to (3), there exists $\varepsilon > 0$ such that $\sigma_p(T_1) \cap \mathbb{T}$ contains an uncountable ε-separated family for the distance $d_{(n_k)}$. Let λ and μ be unimodular eigenvalues of T_1 with associated eigenvectors (of norm 1) e_λ and e_μ. As in the proof of Theorem 3.4, we obtain

$$\|e_\lambda - e_\mu\| \geq \frac{d_{(n_k)}(\lambda, \mu)}{M+1} \geq \frac{\varepsilon}{M+1},$$

which contradicts the separability of X.

(1)⇒(2). The proof is the same as in Theorem 3.4. ∎

Let $\mathcal{S}$ stand for the Borel subsets of $\mathbb{T}$ with Hausdorff dimension strictly less than 1. We then get the following corollary.

COROLLARY 3.14. *Let $(n_k)_{k\geq 0}$ be an increasing sequence of positive integers such that $n_0 = 1$. The following assertions are equivalent:*

(1) *there exists a separable complex Banach space X and a C_0-semigroup $(T_t)_{t\geq 0}$ on X which is bounded with respect to $(n_k)_{k\geq 0}$ and such that $\sigma_p(T_1)\cap\mathbb{T}$ is of Hausdorff dimension equal to 1;*

(2) *there exists a subset K of $\mathbb{T}$ of Hausdorff dimension 1 such that the metric space $(K, d_{(n_k)})$ is separable.*

When $n_{k+1}/n_k \to \infty$, C. Badea and S. Grivaux showed that there exists a subset K of $\mathbb{T}$ of Hausdorff dimension 1 such that the space $(K, d_{(n_k)})$ is separable (see [2, Theorem 3.4]). Then we have the following result.

THEOREM 3.15. *Let $(t_k)_{k\geq 0}$ be an increasing sequence of positive real numbers such that $t_0 = 1$ and $t_{k+1}/t_k \to \infty$. Then there exists a separable complex Banach space X and a C_0-semigroup $(T_t)_{t\geq 0}$ (with infinitesimal generator A) on X which is bounded with respect to $(t_k)_{k\geq 0}$ and such that* $\dim_{\mathrm{H}}(\sigma_p(A)\cap i\mathbb{R}) = 1$.

Proof. For any k, let n_k be the integer part of t_k. Since $t_{k+1}/t_k \to \infty$, we have $n_{k+1}/n_k \to \infty$ as well. But we know from the proof of [2, Theorem 3.4] that there exists a subset K of $\mathbb{T}$ of Hausdorff dimension 1 such that $(K, d_{(n_k)})$ is separable. According to Corollary 3.14, there exists a separable complex Banach space X and a C_0-semigroup $(T_t)_{t\geq 0}$ (with infinitesimal generator A) on X which is bounded with respect to $(n_k)_{k\geq 0}$ and such that $\dim_{\mathrm{H}}(\sigma_p(T_1)\cap\mathbb{T}) = 1$. According to Fact 3.6, $(T_t)_{t\geq 0}$ is also bounded with respect to $(t_k)_{k\geq 0}$. Furthermore, it is well-known that a Lipschitz function decreases the Hausdorff dimension. If $f : i\mathbb{R}\to\mathbb{C}$ is the exponential function, then f is a Lipschitz function and we have $\sigma_p(T_1)\cap\mathbb{T} = f(\sigma_p(A)\cap i\mathbb{R})$. Hence

$$1 = \dim_{\mathrm{H}}(\sigma_p(T_1)\cap\mathbb{T}) \leq \dim_{\mathrm{H}}(\sigma_p(A)\cap i\mathbb{R}) \leq 1. \quad \blacksquare$$

Acknowledgements. I am grateful to the referee for valuable suggestions on the presentation of the paper. I am also grateful to my advisors, Catalin Badea and Sophie Grivaux, for helpful discussions on Jamison sequences.

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Received July 5, 2012
Revised version March 11, 2013 (7558)