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Triebel–Lizorkin spaces with non-doubling measures

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Abstract. Suppose that μ is a Radon measure on $\mathbb{R}^d$, which may be non-doubling. The only condition assumed on μ is a growth condition, namely, there is a constant $C_0 > 0$ such that for all $x \in \operatorname{supp}(\mu)$ and $r > 0$,

$$\mu(B(x,r)) \le C_0 r^n,$$

where $0 < n \le d$. The authors provide a theory of Triebel–Lizorkin spaces $\dot{F}^s_{pq}(\mu)$ for $1 < p < \infty$, $1 \le q \le \infty$ and $|s| < \theta$, where $\theta > 0$ is a real number which depends on the non-doubling measure μ, C_0, n and d. The method does not use the vector-valued maximal function inequality of Fefferman and Stein and is new even for the classical case. As applications, the lifting properties of these spaces by using the Riesz potential operators and the dual spaces are given.

1. Introduction. Suppose that μ is a Radon measure on $\mathbb{R}^d$, which may be non-doubling. The only condition we assume on μ is a growth condition, namely, there is a constant $C_0 > 0$ such that for all $x \in \operatorname{supp}(\mu)$ and $r > 0$,

$$\mu(B(x,r)) \le C_0 r^n, \tag{1.1}$$

where $0 < n \le d$.

Our goal in this paper is to develop a theory of Triebel–Lizorkin spaces associated to non-doubling measures. The theory of Besov spaces associated to non-doubling measures has already been established in [4].

It is well known that the doubling property of the underlying measure is a basic condition in the classical Calderón–Zygmund theory of harmonic analysis. Recently much attention has been payed to non-doubling measures. It has been shown that many results of this theory still hold without assuming the doubling property. See [18–21, 25–27, 31, 7, 8] for some results on

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Calderón–Zygmund operators, [17, 28–30] for some other results related to the spaces BMO(μ) and $H^1(\mu)$, and [9, 10, 22] for vector-valued inequalities for Calderón–Zygmund operators and weights.

However, there is still no counterpart of the Fefferman–Stein [5] vector-valued inequality for the non-centered maximal operator $M_{(\varrho)}f(x)$ defined by

$$M_{(\varrho)}f(x) = \sup_{x\in Q} \frac{1}{\mu(\varrho Q)} \int_Q |f(y)|\, d\mu(y),$$

where $\varrho > 1$. Such an inequality was a key tool to develop a theory of Triebel–Lizorkin spaces on $\mathbb{R}^d$ and spaces of homogeneous type. Thus, in the current circumstances, to develop a theory of Triebel–Lizorkin spaces with non-doubling measures, we need a new method without using the Fefferman–Stein inequality. We manage to overcome this difficulty. We remark that although García-Cuerva and Martell in [10] have already obtained some counterparts of Fefferman and Stein's result of [5] for some kind of vector-valued maximal operators, their inequalities are not suitable for our purposes.

Another key tool to study the Triebel–Lizorkin spaces (and some other function spaces) on $\mathbb{R}^d$ is the so-called Calderón reproducing formula which was first proved by Calderón in [1]. This formula says that given any suitable function ψ, there exists a function ϕ with similar properties such that

$$f = \sum_{k=-\infty}^{\infty} \phi_k * \psi_k * f, \tag{1.2}$$

where the series converges in both

$$\mathcal{S}_\infty(\mathbb{R}^d) = \Big\{ f \in \mathcal{S}(\mathbb{R}^d) : \int_{\mathbb{R}^d} x^\alpha f(x)\, dx = 0 \text{ for all } \alpha \in (\mathbb{N}\cup\{0\})^d \Big\}$$

and $\mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d)$, where $\mathcal{S}(\mathbb{R}^d)$ is the space of Schwartz test functions, and $\mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d)$ is the Schwartz distribution space modulo the space of all polynomials. It is well known that $\mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d)$ is naturally identified with the dual space of $\mathcal{S}_\infty(\mathbb{R}^d)$, $\mathcal{S}'_\infty(\mathbb{R}^d)$; see [6, 23, 33, 34] for more details.

Using Coifman's ideas, David, Journé and Semmes [3] developed the Littlewood–Paley theory on spaces of homogeneous type introduced by Coifman and Weiss [2]. More precisely, let $\{S_k\}_{k=-\infty}^{\infty}$ be an approximation to the identity whose kernels $\{S_k(x,y)\}_{k=1}^{\infty}$ satisfy certain size and regularity conditions. (See [3] for the construction of this approximation to the identity. It is worth pointing out that the doubling property plays an important role in this construction.) Set $D_k = S_k - S_{k-1}$. Based on Coifman's ideas (see [3] for the details), at least formally, the identity operator I can be written as

$$I=\sum_{k=-\infty}^{\infty}D_k=\Big(\sum_{k=-\infty}^{\infty}D_k\Big)\Big(\sum_{j=-\infty}^{\infty}D_j\Big)$$
$$=\sum_{|k-j|\le N}D_kD_j+\sum_{|k-j|>N}D_kD_j=T_N+R_N. \tag{1.3}$$

David, Journé and Semmes proved that if N is large enough, then R_N is bounded on $L^p(X), 1<p<\infty$, with operator norm less than 1. Thus, they obtained the following Calderón-type reproducing formulae:

$$f=\sum_{k=-\infty}^{\infty}T_N^{-1}D_k^ND_k(f)=\sum_{k=-\infty}^{\infty}D_kD_k^NT_N^{-1}(f), \tag{1.4}$$

where T_N^{-1} is the inverse of T_N and the series converge in $L^p(X),\ 1<p<\infty$.

Using these formulae, they were able to obtain the Littlewood–Paley theory for $L^p(X)$: There exists a constant $C>0$ such that for all $f\in L^p(X),\ 1<p<\infty$,

$$C^{-1}\|f\|_{L^p(X)}\le\Big\|\Big\{\sum_{k=-\infty}^{\infty}|D_k(f)|^2\Big\}^{1/2}\Big\|_{L^p(X)}\le C\|f\|_{L^p(X)}.$$

In [14], using the Littlewood–Paley theory, the Triebel–Lizorkin spaces were generalized to spaces of homogeneous type. More precisely, Sawyer and the first author [14] first introduced a test function space $\mathcal{M}(X)$, which is also called smooth molecular space in [11], and approximations to the identity $\{S_k\}_{k=-\infty}^{\infty}$ whose kernels satisfy all size and regularity conditions as in Coifman's construction, and furthermore, a second difference smoothness condition. They then proved that if N is large enough, R_N is bounded on $\mathcal{M}(X)$ with operator norm less than 1. Using this fact, Sawyer and the first author [14] obtained the Calderón reproducing formula. More precisely, let $\{S_k\}_{k=-\infty}^{\infty}$ be any approximation to the identity defined in [14] and $D_k=S_k-S_{k-1}$ for $k\in\mathbb{Z}$. Then there exist families of operators $\{\widetilde{D}_k\}_{k=-\infty}^{\infty}$ and $\{\overline{D}_k\}_{k=-\infty}^{\infty}$ such that

$$f=\sum_{k=-\infty}^{\infty}\widetilde{D}_kD_k(f)=\sum_{k=-\infty}^{\infty}D_k\overline{D}_k(f), \tag{1.5}$$

where the series converge in the $L^p(X)$ norm, $1<p<\infty$, in the norm of the test function space $\mathcal{M}(X)$, and in $(\mathcal{M}(X))^*$, the corresponding distribution space.

Notice that (1.5) is similar to (1.2) and the second difference smoothness condition plays a crucial role for the proof of (1.5). Thus, the theory of Triebel–Lizorkin spaces on spaces of homogeneous type can be developed as in the case of $\mathbb{R}^d$. See [12]–[16] for the details.

The main difficulty in developing a theory of Triebel–Lizorkin spaces with respect to a non-doubling measure μ which does not have any regularity property, apart from the growth condition (1.1), is the construction of an approximation to the identity. Recently, Tolsa constructed a "reasonable" approximation to the identity. More precisely, in [29] he constructed a sequence $\{S_k\}_{k=-\infty}^{\infty}$ of integral operators given by kernels $\{S_k(x,y)\}_{k=-\infty}^{\infty}$ defined on $\mathbb{R}^d \times \mathbb{R}^d$, satisfying some appropriate size and regularity conditions, and also

$$\int_{\mathbb{R}^d} S_k(x,y)\, d\mu(y) = 1$$

for all $x \in \operatorname{supp}(\mu)$ and $S_k(x,y) = S_k(y,x)$ for all $k \in \mathbb{Z}$. For each $k \in \mathbb{Z}$, set $D_k = S_k - S_{k-1}$. Then, again, based on Coifman's ideas mentioned above, and by use of the appropriate size and regularity conditions on $S_k(x,y)$, the Cotlar–Stein lemma (see [24]) and the Calderón–Zygmund theory associated to non-doubling measures, Tolsa proved that the Calderón-type reproducing formula in (1.4) still holds for non-doubling measures. Using this formula, he was able to produce a theory of Littlewood–Paley associated to non-doubling measures. However, the size and regularity conditions on $S_k(x,y)$ given by Tolsa are not enough to obtain a Calderón reproducing formula similar to (1.5). A crucial observation of this paper (see also [4]) is that if the norm $\|f\|_{\dot{F}^s_{pq}(\mu)}$ for all $L^2(\mu)$ functions f is defined by

$$\|f\|_{\dot{F}^s_{pq}(\mu)} = \Big\| \Big\{ \sum_{k=-\infty}^{\infty} 2^{skq} |D_k(f)|^q \Big\}^{1/q} \Big\|_{L^p(\mu)} < \infty,$$

where $\{D_k\}_{k=-\infty}^{\infty}$ are as in Tolsa's Calderón-type reproducing formula, then R_N in (1.3) is bounded with respect to this norm and its operator norm is less than 1 if N is large enough. Hence, T_N^{-1} is bounded with respect to this norm. This observation leads to introduce a new "test function space" defined by

$$\dot{\mathcal{F}}^s_{p,q}(\mu) = \{f \in L^2(\mu) : \|f\|_{\dot{F}^s_{pq}(\mu)} < \infty\}.$$

We will prove that the Calderón-type reproducing formulae (1.4) with Tolsa's approximations to the identity hold for the test function space $\dot{\mathcal{F}}^s_{p,q}(\mu)$.

To show that the formulae (1.4) still hold in the distribution space $(\dot{\mathcal{F}}^s_{p,q}(\mu))^*$ (as they do for spaces of homogeneous type), a second difference smoothness estimate of the approximation to the identity is needed. See [4] for similar formulae associated to Besov spaces $\dot{B}^s_{pq}(\mu)$.

The plan of this paper is the following. In the next section, we will show that the operator T_N^{-1} is bounded with respect to the norm $\|\cdot\|_{\dot{F}^s_{pq}(\mu)}$. To this end, we first prove that R_N in (1.3) is bounded with respect to this norm with small operator norm; see Theorem 1 below. The duality method and the technique of the proof of the Cotlar–Stein lemma (see [24]) are the key to the proof of Theorem 1. The main result of this section is the Calderón-type reproducing formulae in the distribution space $(\dot{\mathcal{F}}^s_{p,q}(\mu))^*$ (see Theorem 2). In Section 3, we introduce the Triebel–Lizorkin spaces $\dot{F}^s_{pq}(\mu)$ and give some of their applications. Specifically, we study the boundedness of Riesz potential operators on these spaces, and using them, we prove that these spaces have lifting properties. Finally, we consider their dual spaces. We point out that using the Littlewood–Paley theory of Tolsa [29], together with our result, it is easy to see that $\dot{F}^0_{p2}(\mu) = L^p(\mu)$ if $1 < p < \infty$. Thus, our Triebel–Lizorkin spaces $\dot{F}^s_{pq}(\mu)$ generalize $L^p(\mu)$ spaces.

Throughout the paper, the letter C is used for non-negative constants that may change from one occurrence to another. Constants with subscripts, such as C_0, do not change in different occurrences. The notation $A \sim B$ means that there is some constant $C > 0$ such that $C^{-1}A \le B \le CA$. For any index $q \in [1, \infty]$, we denote by q' the conjugate index, that is, $1/q + 1/q' = 1$. We also denote $\mathbb{N} \cup \{0\}$ by $\mathbb{Z}_+$.

2. Calderón-type reproducing formulae. Throughout this section, all definitions and notation are as in Tolsa [29]; see also [30]. To introduce an approximation to the identity for non-doubling measures, we need the following lemma.

LEMMA 1. *There exist a sequence $\{S_k\}_{k=-\infty}^{\infty}$ of operators with kernels $S_k(x, y)$ defined on $\mathbb{R}^d \times \mathbb{R}^d$ such that for each $k \in \mathbb{Z}$ the following properties hold*:

(a) $S_k(x, y) = S_k(y, x)$.

(b) $\int_{\mathbb{R}^d} S_k(x, y)\, d\mu(y) = 1$ *for* $x \in \operatorname{supp}(\mu)$.

(c) *If $Q_{x,k}$ is a transit cube* (*see Definition* 3.4 *in* [29, p. 67]), *then* $\operatorname{supp}(S_k(x, \cdot)) \subset Q_{x,k-1}$.

(d) *If $Q_{x,k}$ and $Q_{y,k}$ are transit cubes*, *then*

$$0 \le S_k(x, y) \le \frac{C}{(\ell(Q_{x,k}) + \ell(Q_{y,k}) + |x - y|)^n}.$$

(e) *If $Q_{x,k}$, $Q_{x',k}$, $Q_{y,k}$ are transit cubes*, *and $x, x' \in Q_{x_0,k}$ for some $x_0 \in \operatorname{supp}(\mu)$*, *then*

$$|S_k(x, y) - S_k(x', y)| \le C \frac{|x - x'|}{\ell(Q_{x_0,k})} \frac{1}{(\ell(Q_{x,k}) + \ell(Q_{y,k}) + |x - y|)^n}.$$

(f) *If $Q_{x,k}$, $Q_{x',k}$, $Q_{y,k}$ and $Q_{y',k}$ are transit cubes*, *$x, x' \in Q_{x_0,k}$ and $y, y' \in Q_{y_0,k}$ for some $x_0, y_0 \in \operatorname{supp}(\mu)$*, *then*

$$|[S_k(x,y)-S_k(x',y)]-[S_k(x,y')-S_k(x',y')]| \\ \leq C\,\frac{|x-x'|}{\ell(Q_{x_0,k})}\,\frac{|y-y'|}{\ell(Q_{y_0,k})}\,\frac{1}{(\ell(Q_{x,k})+\ell(Q_{y,k})+|x-y|)^n}.$$

This lemma basically belongs to Tolsa who constructed $\{S_k\}_{k=-\infty}^{\infty}$ and proved they satisfy (a)–(e) in [29]. The fact that they satisfy (f) was proved in [4].

Definition 1. A sequence of operators, $\{S_k\}_{k\in\mathbb{Z}}$, is said to be an *approximation to the identity associated to a non-doubling measure* μ if the kernels of $\{S_k\}_{k\in\mathbb{Z}}$, $\{S_k(x,y)\}_{k\in\mathbb{Z}}$, satisfy conditions (a)–(f) of Lemma 1.

Now, let $\{S_k\}_{k\in\mathbb{Z}}$ be an approximation to the identity as in Definition 1 and set $D_k=S_k-S_{k-1}$ for $k\in\mathbb{Z}$. Following [3] and [29], based on Coifman's idea, we can write

$$I=T_N+R_N, \tag{2.1}$$

where $T_N=\sum_{|k-j|\leq N}D_kD_j$ and $R_N=\sum_{|k-j|>N}D_kD_j$.

If we set $D_k^N=\sum_{|j|\leq N}D_k$ for $k\in\mathbb{Z}$, then we can also write

$$T_N=\sum_{k\in\mathbb{Z}}D_k^ND_k.$$

In what follows, unless explicitly stated otherwise, the following notations and assumptions will be used throughout the paper:

- $\{S_k\}_{k\in\mathbb{Z}}$, $\{A_k\}_{k\in\mathbb{Z}}$ and $\{P_k\}_{k\in\mathbb{Z}}$ are approximations to the identity as in Definition 1.
- $D_k=S_k-S_{k-1}$, $G_k=A_k-A_{k-1}$ and $E_k=P_k-P_{k-1}$ for $k\in\mathbb{Z}$.
- $1<p<\infty$, $1\leq q\leq\infty$.
- θ is half the maximum η such that Lemma 3.4 in [29] (see also Lemma 6.3 in [30]) holds. It is easy to see that θ depends on C_0, μ, n and d.
- $|s|<\theta$.
- T_N and R_N are as in (2.1).

As mentioned in the introduction, the following result is a crucial observation of this paper.

Theorem 1. *For all $f\in L^2(\mu)$ and $\nu\in(0,1/2)$ such that $|s|<2\nu\theta$,*

$$\Big\|\Big\{\sum_{j=-\infty}^{\infty}2^{jsq}|E_jR_Nf|^q\Big\}^{1/q}\Big\|_{L^p(\mu)} \\ \leq C_1(2^{-N(s+2\nu\theta)}+2^{-N(2\nu\theta-s)})\Big\|\Big\{\sum_{k=-\infty}^{\infty}2^{ksq}|G_kf|^q\Big\}^{1/q}\Big\|_{L^p(\mu)} \tag{2.2}$$

with C_1 independent of N, f, and ν; moreover, if we choose $N\in\mathbb{N}$ such *that*

$$C_1(2^{-N(s+2\nu\theta)} + 2^{-N(2\nu\theta-s)}) < 1, \tag{2.3}$$

then for all $f \in L^2(\mu)$,

$$\Big\|\Big\{\sum_{j=-\infty}^{\infty} 2^{jsq}|E_j T_N^{-1} f|^q\Big\}^{1/q}\Big\|_{L^p(\mu)} \le C\Big\|\Big\{\sum_{k=-\infty}^{\infty} 2^{ksq}|G_k f|^q\Big\}^{1/q}\Big\|_{L^p(\mu)}, \tag{2.4}$$

where C *is independent of* f.

To show Theorem 1, we recall that if $\varrho > 1$, then $M_{(\varrho)}$ is bounded on $L^p(\mu)$, $1 < p < \infty$, and of weak type $(1,1)$; see [28, pp. 126–127]. The following lemma states the basic properties of the composition of two approximations to the identity.

LEMMA 2. *The following assertions are true.*

(i) $\mathrm{supp}(E_j D_k)(x,\cdot) \subset Q_{x,\min(j,k)-3}$ *and* $\mathrm{supp}(E_j D_k)(\cdot,y) \subset Q_{y,\min(j,k)-3}$ *for all* $j,k \in \mathbb{Z}$ *and all* $x,y \in \mathrm{supp}(\mu)$.

(ii) *For all* $x,y \in \mathrm{supp}(\mu)$ *and all* $j,k \in \mathbb{Z}$,

$$|(E_j D_k)(x,y)| \le C2^{-2|j-k|\theta}\,\frac{1}{(\ell(Q_{x,\min(j,k)+1}) + \ell(Q_{y,\min(j,k)+1}) + |x-y|)^n}.$$

(iii) *For* $p \in [1,\infty]$, $j,k \in \mathbb{Z}$, *and all* $x \in X$,

$$\|E_j D_k\|_{L^p(\mu)\to L^p(\mu)} \le C_2 2^{-2|j-k|\theta}, \quad \|(E_j D_k)(x,\cdot)\|_{L^1(\mu)} \le C_2 2^{-2|j-k|\theta},$$

and

$$\|(E_j D_k)(\cdot,x)\|_{L^1(\mu)} \le C_2 2^{-2|j-k|\theta},$$

where $C_2 > 0$ *is a constant depending on* p, *but not on* j *and* k.

(iv) *For all* $f \in L^2_{\mathrm{c}}(\mu)$ *and all* $x \in \mathrm{supp}(\mu)$,

$$|(E_j D_k) f(x)| \le C_3 2^{-2|j-k|\theta} M_{(2)} f(x),$$

where $C_3 > 0$ *is independent of* j, k, f *and* x.

Proof. The proof is essentially contained in the proof of Lemma 6.1 in [29]; see also [4] for some details. For the reader's convenience, let us show (iv), whose proof is similar to that of Remark 8.1 in [29].

Let N_0 be the smallest integer such that $Q_{x,\min(j,k)-3} \subset 2^{N_0} Q_{x,\min(j,k)+1}$. Then Lemma 3.1 in [29] and the definition of $Q_{x,k}$ in [29] tell us that

$$\begin{aligned}
&\delta(Q_{x,\min(j,k)+1}, 2^{N_0+1} Q_{x,\min(j,k)+1})\\
&\qquad = \delta(Q_{x,\min(j,k)+1}, Q_{x,\min(j,k)-3}) + \delta(Q_{x,\min(j,k)-3}, 2^{N_0+1} Q_{x,\min(j,k)+1})\\
&\qquad = 4A \pm \varepsilon_1 + \delta(Q_{x,k-3}, 2^{N_0+1} Q_{x,k}) \le C.
\end{aligned}$$

This fact and (ii) imply that for all $f \in L^2_{\rm c}(\mu)$ and all $x \in \mathrm{supp}(\mu)$, if we write $Q_1 = Q_{x,\min(j,k)+1}$ for brevity, then

$$\begin{aligned}
|(E_jD_k)f(x)| &= \Big| \int\limits_{Q_{x,\min(j,k)-3}} (E_jD_k)(x,y)f(y)\,d\mu(y)\Big| \\
&\le C2^{-2|j-k|\theta}\bigg[\int\limits_{Q_1} \frac{1}{\ell(Q_1)^n}\,|f(y)|\,d\mu(y) \\
&\quad + \sum_{j=1}^{N_0} \int\limits_{2^jQ_1\setminus 2^{j-1}Q_1} \frac{1}{|x-y|^n}\,|f(y)|\,d\mu(y)\bigg] \\
&\le C2^{-2|j-k|\theta}\bigg[\frac{\mu(2Q_1)}{\ell(Q_1)^n}\,\frac{1}{\mu(2Q_1)}\int\limits_{Q_1}|f(y)|\,d\mu(y) \\
&\quad + \sum_{j=1}^{N_0} \frac{\mu(2^{j+1}Q_1)}{\ell(2^{j+1}Q_1)^n}\,\frac{1}{\mu(2^{j+1}Q_1)}\int\limits_{2^jQ_1}|f(y)|\,d\mu(y)\bigg] \\
&\le C2^{-2|j-k|\theta}[1+\delta(Q_1,2^{N_0+1}Q_1)]M_{(2)}f(x) \\
&\le C_3 2^{-2|j-k|\theta}M_{(2)}f(x),
\end{aligned}$$

where, in the third-to-last inequality, we used some equivalent definition of $\delta(Q,P)$; see [28]. This is the desired estimate.

Before we return to the proof of Theorem 1, we observe that by a result of Tolsa [29], if N is large enough, then for all $f \in L^2(\mu)$, we have

$$f = \sum_{k\in\mathbb{Z}} T_N^{-1}D_k^N D_k(f) = \sum_{k\in\mathbb{Z}} D_k^N D_k T_N^{-1}(f) \tag{2.5}$$

in the norm of $L^2(\mu)$. In fact, T_N^{-1} is bounded on $L^p(\mu)$ with $1 < p < \infty$. The formula (2.5) is called the *Calderón-type reproducing formula.* See [29] for more details.

We now write T_N^{-1} as

$$T_N^{-1} = \sum_{l=0}^{\infty}(R_N)^l \tag{2.6}$$

in the operator norm of $L^2(\mu)$, and for $l \in \mathbb{N}$,

$$(R_N)^l = \sum_{|k_1-j_1|>N} D_{k_1}D_{j_1} \sum_{|k_2-j_2|>N} D_{k_2}D_{j_2} \ldots \sum_{|k_l-j_l|>N} D_{k_l}D_{j_l} \tag{2.7}$$

also in the operator norm of $L^2(\mu)$.

Using Lemma 2, we can verify the following lemma.

LEMMA 3. *Let $\{f_k\}_{k\in\mathbb{Z}}$ be a sequence of measurable functions. For $j \in \mathbb{Z}$ and $N, N_1 \in \mathbb{N}$, let*

$$H_j(\{f_k\}_{k=-\infty}^{\infty})(x) = \sum_{l=0}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \sum_{|m|>N} \sum_{k_1=-\infty}^{\infty} \sum_{|m_1|>N_1} \cdots \sum_{k_l=-\infty}^{\infty} \times \sum_{|m_l|>N_1} E_j D_i D_{i+m} G_{k_1} G_{k_1+m_1} \ldots G_{k_l} G_{k_l+m_l} G_k^{N_1} f_k(x) \tag{2.8}$$

for $x \in \operatorname{supp}(\mu)$. Let $1 \le q < p < \infty$ and $\nu \in (0, 1/2)$ be such that $|s| < 2\nu\theta$. Then there is a constant $C_1 > 0$ such that for all $\{f_k\}_{k\in\mathbb{Z}}$, all $N \in \mathbb{N}$ and all $N_1 \in \mathbb{N}$ large enough (depending on C_2, C_3, s, ν and θ),

$$\Big\| \Big\{ \sum_{j=-\infty}^{\infty} 2^{jsq} |H_j(\{f_k\}_{k=-\infty}^{\infty})|^q \Big\}^{1/q} \Big\|_{L^p(\mu)} \le C_1(2^{-N(s+2\nu\theta)} + 2^{-N(2\nu\theta - s)}) \Big\| \Big\{ \sum_{k=-\infty}^{\infty} 2^{ksq} |f_k|^q \Big\}^{1/q} \Big\|_{L^p(\mu)}. \tag{2.9}$$

Proof. For $j \in \mathbb{Z}$ and $l \in \mathbb{Z}_+$, let

$$H_j^l(\{f_k\}_{k=-\infty}^{\infty})(x) = \sum_{k,i,m,\{k_t,m_t\}_1^l} E_j D_i D_{i+m} G_{k_1} G_{k_1+m_1} \ldots G_{k_l} G_{k_l+m_l} G_k^{N_1} f_k(x),$$

where

$$\sum_{k,i,m,\{k_t,m_t\}_1^l} = \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \sum_{|m|>N} \sum_{k_1=-\infty}^{\infty} \sum_{|m_1|>N_1} \cdots \sum_{k_l=-\infty}^{\infty} \sum_{|m_l|>N_1}$$

(we also use similar abbreviations for multiple sums below). Then the Minkowski inequality tells us that

$$\Big\| \Big\{ \sum_{j=-\infty}^{\infty} 2^{jsq} |H_j(\{f_k\}_{k=-\infty}^{\infty})|^q \Big\}^{1/q} \Big\|_{L^p(\mu)} \le \sum_{l=0}^{\infty} \Big\| \Big\{ \sum_{j=-\infty}^{\infty} 2^{jsq} |H_j^l(\{f_k\}_{k=-\infty}^{\infty})|^q \Big\}^{1/q} \Big\|_{L^p(\mu)}. \tag{2.10}$$

Let $r = p/q$. Then $r > 1$. For $g \in L^{r'}(\mu)$ with $g \ge 0$, the Hölder and Minkowski inequalities yield

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^d} \sum_{j=-\infty}^{\infty} 2^{jsq} |H_j^l(\{f_k\}_{k=-\infty}^{\infty})(x)|^q g(x)\, d\mu(x) \\
&\quad \leq \sum_{j=-\infty}^{\infty} 2^{jsq} \int\limits_{\mathbb{R}^d} \Big\{ \sum_{k,i,m,\{k_t,m_t\}_1^l} \\
&\qquad \times \Big[\int\limits_{\mathbb{R}^d} |E_j D_i D_{i+m} G_{k_1} G_{k_1+m_1} \dots G_{k_l} G_{k_l+m_l} G_k^{N_1}(x,y)|\, d\mu(y)\Big]^{1/q'} \\
&\qquad \times \Big[\int\limits_{\mathbb{R}^d} |E_j D_i D_{i+m} G_{k_1} G_{k_1+m_1} \dots G_{k_l} \\
&\qquad \times G_{k_l+m_l} G_k^{N_1}(x,y)|\, |f_k(y)|^q\, d\mu(y)\Big]^{1/q} \Big\}^q g(x)\, d\mu(x) \\
&\quad \leq \sum_{j=-\infty}^{\infty} 2^{jsq} \int\limits_{\mathbb{R}^d} \Big[\sum_{k,i,m,\{k_t,m_t\}_1^l} 2^{-ks} \int\limits_{\mathbb{R}^d} |E_j D_i D_{i+m} G_{k_1} G_{k_1+m_1} \dots G_{k_l} \\
&\qquad \times G_{k_l+m_l} G_k^{N_1}(x,y)|\, d\mu(y)\Big]^{q/q'} \\
&\qquad \times \Big[\sum_{k,i,m,\{k_t,m_t\}_1^l} 2^{-ks} \int\limits_{\mathbb{R}^d} |E_j D_i D_{i+m} G_{k_1} G_{k_1+m_1} \dots G_{k_l} G_{k_l+m_l} G_k^{N_1}(x,y)| \\
&\qquad \times 2^{ksq} |f_k(y)|^q\, d\mu(y)\Big] g(x)\, d\mu(x).
\end{aligned}
$$

By using a technique used in the proof of the Cotlar–Stein lemma (see [24]) and Lemma 2, we find that there is a constant $C_2 > 0$ such that

$$
\begin{aligned}
\text{(2.11)} \quad & \|E_j D_i D_{i+m} G_{k_1} G_{k_1+m_1} \dots G_{k_l} G_{k_l+m_l} G_k^{N_1}(x,\cdot)\|_{L^1(\mu)} \\
&= \|(E_j D_i)(D_{i+m} G_{k_1})(G_{k_1+m_1} G_{k_2}) \dots (G_{k_l+m_l} G_k^{N_1}(x,\cdot))\|_{L^1(\mu)} \\
&\leq C N_1 C_2^l 2^{-2\theta[|j-i|+|i+m-k_1|+\dots+|k_l+m_l-k|]}
\end{aligned}
$$

and

$$
\begin{aligned}
\text{(2.12)} \quad & \|E_j D_i D_{i+m} G_{k_1} G_{k_1+m_1} \dots G_{k_l} G_{k_l+m_l} G_k^{N_1}(x,\cdot)\|_{L^1(\mu)} \\
&= \|E_j (D_i D_{i+m})(G_{k_1} G_{k_1+m_1}) \dots (G_{k_l} G_{k_l+m_l}) G_k^{N_1}(x,\cdot)\|_{L^1(\mu)} \\
&\leq C N_1 C_2^l 2^{-2\theta[|m|+|m_1|+\dots+|m_l|]},
\end{aligned}
$$

where we also used the fact that $\|E_j(x,\cdot)\|_{L^1(\mu)} \leq C$ uniformly in j and

$$\|G_k^{N_1}(z,\cdot)\|_{L^1(\mu)} \leq C N_1$$

uniformly in k with C independent of N_1. The geometric mean of (2.11) and

(2.12) yields

$$\begin{aligned}(2.13)\qquad \|E_jD_iD_{i+m}G_{k_1}G_{k_1+m_1}\ldots G_{k_l}G_{k_l+m_l}G_k^{N_1}G_kf(x,\cdot)\|_{L^1(\mu)}&\\ \le CN_1C_2^l2^{-2\theta(1-\nu)[|j-i|+|i+m-k_1|+\ldots+|k_l+m_l-k|]}&\\ \times 2^{-2\theta\nu[|m|+|m_1|+\ldots+|m_l|]}.&\end{aligned}$$

From (2.13), it follows that

$$\begin{aligned}&\sum_{k,i,m,\{k_t,m_t\}_1^l} 2^{-ks}\int_{\mathbb{R}^d}|E_jD_iD_{i+m}G_{k_1}G_{k_1+m_1}\ldots G_{k_l}G_{k_l+m_l}G_k^{N_1}(x,y)|\,d\mu(y)\\ &\quad\le CN_1C_2^l\sum_{k,i,m,\{k_t,m_t\}_1^l}2^{-2\theta(1-\nu)[|j-i|+|i+m-k_1|+\ldots+|k_l+m_l-k|]}\\ &\quad\quad\times 2^{-2\theta\nu[|m|+|m_1|+\ldots+|m_l|]}.\end{aligned}$$

We now first sum over k and next over m_l; then we can estimate the last quantity in the above inequality by

$$\begin{aligned}&\le CN_1C_2^l\sum_{i,m,\{k_t,m_t\}_1^{l-1},k_l}2^{-k_ls}\\ &\quad\times\sum_{|m_l|>N_1}2^{-m_ls}\sum_{k=-\infty}^{\infty}2^{-2\theta(1-\nu)[|j-i|+|i+m-k_1|+\ldots+|k_l+m_l-k|]}2^{(m_l+k_l-k)s}\\ &\quad\times 2^{-2\theta\nu[|m|+|m_1|+\ldots+|m_l|]}\\ &\le CN_1C_2^l(2^{-N_1(s+2\nu\theta)}+2^{-N_1(2\nu\theta-s)})\\ &\quad\times\sum_{i,m,\{k_t,m_t\}_1^{l-1},k_l}2^{-2\theta(1-\nu)[|j-i|+|i+m-k_1|+\ldots+|k_{l-1}+m_{l-1}-k_l|]}\\ &\quad\times 2^{-2\theta\nu[|m|+|m_1|+\ldots+|m_{l-1}|]}.\end{aligned}$$

Repeating this process $l+1$ times, we finally obtain

$$\begin{aligned}(2.14)\qquad &\sum_{k,i,m,\{k_t,m_t\}_1^l}2^{-ks}\\ &\quad\times\int_{\mathbb{R}^d}|E_jD_iD_{i+m}G_{k_1}G_{k_1+m_1}\ldots G_{k_l}G_{k_l+m_l}G_k^{N_1}(x,y)|\,d\mu(y)\\ &\le CN_1(2^{-N(s+2\nu\theta)}+2^{-N(2\nu\theta-s)})\\ &\quad\times(CC_2)^l(2^{-N_1(s+2\nu\theta)}+2^{-N_1(2\nu\theta-s)})^l2^{-js}.\end{aligned}$$

From (2.14), it follows that

$$\begin{aligned}(2.15)\qquad &\int\limits_{\mathbb{R}^d}\sum_{j=-\infty}^{\infty}2^{jsq}|H_j^l(\{f_k\}_{k=-\infty}^{\infty})(x)|^q g(x)\,d\mu(x)\\ &\quad\le [CN_1(2^{-N(s+2\nu\theta)}+2^{-N(2\nu\theta-s)})\\ &\qquad\times (CC_2)^l(2^{-N_1(s+2\nu\theta)}+2^{-N_1(2\nu\theta-s)})^l]^{q/q'}\\ &\qquad\times \sum_{j,k,i,m,\{k_t,m_t\}_1^l} 2^{(j-k)s}\\ &\qquad\times\int\limits_{\mathbb{R}^d}\Big[\int\limits_{\mathbb{R}^d}|E_jD_iD_{i+m}G_{k_1}G_{k_1+m_1}\dots G_{k_l}G_{k_l+m_l}G_k^{N_1}(x,y)|\\ &\qquad\times g(x)\,d\mu(x)\Big]2^{ksq}|f_k(y)|^q\,d\mu(y).\end{aligned}$$

Lemma 2 and the trivial estimate

$$(2.16)\qquad |f(x)|\le CM_{(2)}f(x)$$

yield

$$\begin{aligned}(2.17)\qquad &\int\limits_{\mathbb{R}^d}|E_jD_iD_{i+m}G_{k_1}G_{k_1+m_1}\dots G_{k_l}G_{k_l+m_l}G_k^{N_1}(x,y)|g(x)\,d\mu(x)\\ &=\int\limits_{\mathbb{R}^d}|(E_jD_i)(D_{i+m}G_{k_1})(G_{k_1+m_1}G_{k_2})\dots(G_{k_l+m_l}G_k^{N_1})(x,y)|g(x)\,d\mu(x)\\ &\le CN_1C_3^l2^{-2\theta[|j-i|+|i+m-k_1|+|k_1+m_1-k_2|+\dots+|k_{l-1}+m_{l-1}-k_l|+|k_l+m_l-k|]}\\ &\quad\times M_{(2)}^{l+3}(g)(y)\end{aligned}$$

and

$$\begin{aligned}(2.18)\qquad &\int\limits_{\mathbb{R}^d}|E_jD_iD_{i+m}G_{k_1}G_{k_1+m_1}\dots G_{k_l}G_{k_l+m_l}G_k^{N_1}(x,y)|g(x)\,d\mu(x)\\ &=\int\limits_{\mathbb{R}^d}|E_j(D_iD_{i+m})(G_{k_1}G_{k_1+m_1})\dots(G_{k_l}G_{k_l+m_l})G_k^{N_1}(x,y)|g(x)\,d\mu(x)\\ &\le CN_1C_3^l2^{-2\theta[|m|+|m_1|+\dots+|m_{l-1}|+|m_l|]}M_{(2)}^{l+3}(g)(y),\end{aligned}$$

where $M_{(2)}^{l+3}=\underbrace{M_{(2)}\circ\dots\circ M_{(2)}}_{l+3\,\text{times}}$ for $l\in\mathbb{N}$, and we have also used the estimate

$$|E_jf(x)|\le CM_{(2)}f(x)$$

and

$$|[|G_k^{N_1}|]^*f(x)|\le CN_1M_{(2)}f(x),$$

which can be proved similarly to Lemma 2(iv); see also Remark 8.1 in [29].

Let ν be as in the theorem. The geometric mean of (2.17) and (2.18) yields

$$\begin{aligned}(2.19)\qquad \int\limits_{\mathbb{R}^d} |E_j D_i D_{i+m} G_{k_1} G_{k_1+m_1} \ldots G_{k_l} G_{k_l+m_l} G_k^{N_1}(x,y)| g(x)\, d\mu(x) \\ \le C N_1 C_3^l 2^{-2\theta(1-\nu)[|j-i|+|i+m-k_1|+|k_1+m_1-k_2|+\ldots+|k_{l-1}+m_{l-1}-k_l|]} \\ \times 2^{-2\theta(1-\nu)|k_l+m_l-k|} 2^{-2\theta\nu[|m|+|m_1|+\ldots+|m_{l-1}|+|m_l|]} M_{(2)}^{l+3}(g)(y).\end{aligned}$$

Inserting (2.19) into (2.15) leads to

$$\begin{aligned}\int\limits_{\mathbb{R}^d} \sum_{j=-\infty}^{\infty} 2^{jsq} |H_j^l(\{f_k\}_{k=-\infty}^{\infty})(x)|^q g(x)\, d\mu(x) \\ \le [C N_1 (2^{-N(s+2\nu\theta)} + 2^{-N(2\nu\theta-s)}) \\ \times (CC_2)^l (2^{-N_1(s+2\nu\theta)} + 2^{-N_1(2\nu\theta-s)})^l]^{q/q'} \\ \times C N_1 C_3^l \sum_{k,j,i,m,\{k_t,m_t\}_1^l} 2^{(j-k)s} \\ \times 2^{-2\theta(1-\nu)[|j-i|+|i+m-k_1|+|k_1+m_1-k_2|+\ldots+|k_{l-1}+m_{l-1}-k_l|+|k_l+m_l-k|]} \\ \times 2^{-2\theta\nu[|m|+|m_1|+\ldots+|m_{l-1}|+|m_l|]} \int\limits_{\mathbb{R}^d} M_{(2)}^{l+3}(g)(y) 2^{ksq} |f_k(y)|^q\, d\mu(y).\end{aligned}$$

If we sum first over j, then over i and finally over m, then the last term is dominated by

$$\begin{aligned}\le [C N_1 (2^{-N(s+2\nu\theta)} + 2^{-N(2\nu\theta-s)}) (CC_2)^l (2^{-N_1(s+2\nu\theta)} + 2^{-N_1(2\nu\theta-s)})^l]^{q/q'} \\ \times C N_1 C_3^l (2^{-N(s+2\nu\theta)} + 2^{-N(2\nu\theta-s)}) \sum_{k,\{k_t,m_t\}_1^l} 2^{(k_1-k)s} \\ \times 2^{-2\theta(1-\nu)[|k_1+m_1-k_2|+\ldots+|k_{l-1}+m_{l-1}-k_l|+|k_l+m_l-k|]} \\ \times 2^{-2\theta\nu[|m_1|+\ldots+|m_{l-1}|+|m_l|]} \int\limits_{\mathbb{R}^d} M_{(2)}^{l+3}(g)(y) 2^{ksq} |f_k(y)|^q\, d\mu(y).\end{aligned}$$

Repeating this procedure l times by the Hölder inequality we obtain

$$\begin{aligned}\int\limits_{\mathbb{R}^d} \sum_{j=-\infty}^{\infty} 2^{jsq} |H_j^l(\{f_k\}_{k=-\infty}^{\infty})(x)|^q g(x)\, d\mu(x) \\ \le [C N_1 (2^{-N(s+2\nu\theta)} + 2^{-N(2\nu\theta-s)}) \\ \times (CC_2)^l (2^{-N_1(s+2\nu\theta)} + 2^{-N_1(2\nu\theta-s)})^l]^{q/q'}\end{aligned}$$

$$\times CN_1(2^{-N(s+2\nu\theta)}+2^{-N(2\nu\theta-s)})(CC_3)^l(2^{-N_1(s+2\nu\theta)}+2^{-N_1(2\nu\theta-s)})^l$$
$$\times \int_{\mathbb{R}^d} M_{(2)}^{l+3}(g)(y)\Big\{\sum_{k=-\infty}^{\infty} 2^{ksq}|f_k(y)|^q\Big\}\,d\mu(y)$$
$$\le CN_1(2^{-N(s+2\nu\theta)}+2^{-N(2\nu\theta-s)})^q\overline{C}_1^{lq}(2^{-N_1(s+2\nu\theta)}+2^{-N_1(2\nu\theta-s)})^{lq}$$
$$\times \|M_{(2)}^{l+3}(g)\|_{L^{r'}(\mu)}\Big\|\Big\{\sum_{k=-\infty}^{\infty} 2^{ksq}|f_k|^q\Big\}^{1/q}\Big\|_{L^p(\mu)}^q$$
$$\le CN_1(2^{-N(s+2\nu\theta)}+2^{-N(2\nu\theta-s)})^q\overline{C}_1^{lq}(2^{-N_1(s+2\nu\theta)}+2^{-N_1(2\nu\theta-s)})^{lq}$$
$$\times \|g\|_{L^{r'}(\mu)}\Big\|\Big\{\sum_{k=-\infty}^{\infty} 2^{ksq}|f_k|^q\Big\}^{1/q}\Big\|_{L^p(\mu)}^q,$$

where, in the second-to-last inequality, we used the $L^{r'}(\mu)$-boundedness of $M_{(2)}$ and we let $\overline{C}_1 = C\max\{C_2, C_3\}$; see [28]. Taking the infimum over $g \in L^{r'}(\mu)$ with $\|g\|_{L^{r'}(\mu)} \le 1$ yields

$$\Big\|\Big\{\sum_{j=-\infty}^{\infty} 2^{jsq}|H_j^l(\{f_k\}_{k=-\infty}^{\infty})|^q\Big\}^{1/q}\Big\|_{L^p(\mu)}$$
$$\le CN_1(2^{-N(s+2\nu\theta)}+2^{-N(2\nu\theta-s)})\overline{C}_1^l(2^{-N_1(s+2\nu\theta)}+2^{-N_1(2\nu\theta-s)})^l$$
$$\times \Big\|\Big\{\sum_{k=-\infty}^{\infty} 2^{ksq}|f_k|^q\Big\}^{1/q}\Big\|_{L^p(\mu)}.$$

Combining this with (2.10), we finally obtain

$$\Big\|\Big\{\sum_{j=-\infty}^{\infty} 2^{jsq}|H_j(\{f_k\}_{k=-\infty}^{\infty})|^q\Big\}^{1/q}\Big\|_{L^p(\mu)}$$
$$\le CN_1(2^{-N(s+2\nu\theta)}+2^{-N(2\nu\theta-s)})\Big\{\sum_{l=0}^{\infty}\overline{C}_1^l(2^{-N_1(s+2\nu\theta)}+2^{-N_1(2\nu\theta-s)})^l\Big\}$$
$$\times \Big\|\Big\{\sum_{k=-\infty}^{\infty} 2^{ksq}|f_k|^q\Big\}^{1/q}\Big\|_{L^p(\mu)}$$
$$\le C_1(2^{-N(s+2\nu\theta)}+2^{-N(2\nu\theta-s)})\Big\|\Big\{\sum_{k=-\infty}^{\infty} 2^{ksq}|f_k|^q\Big\}^{1/q}\Big\|_{L^p(\mu)},$$

where C_1 is a constant independent of N and $\{f_k\}_{k\in\mathbb{Z}}$, and we chose $N_1 \in \mathbb{N}$ large enough such that

$$\overline{C}_1(2^{-N_1(s+2\nu\theta)}+2^{-N_1(2\nu\theta-s)}) < 1.$$

This completes the proof of Lemma 3.

Proof of Theorem 1. We first verify (2.2). If $1 \le p = q \le \infty$, then (2.2) was proved in [4]. If $1 \le q < p < \infty$, then Lemma 3 tells us that (2.2) in this case is also true.

We now suppose $1 < p < q \le \infty$. Recall that if $1 < p \le \infty$ and $0 < q \le \infty$, then

$$(L^{p'}(l^{q'})(\mu))^* = L^p(l^q)(\mu) \tag{2.20}$$

(see Proposition 2.11.1 in [33, p. 177]; the proof there is also valid for any non-doubling measure). Moreover, $L^p(l^q)(\mu)$ is the set of all sequences $\{f_k\}_{k=-\infty}^{\infty}$ of measurable functions such that

$$\|\{f_k\}_{k=-\infty}^{\infty}\|_{L^p(l^q)(\mu)} = \Big\|\Big\{\sum_{k=-\infty}^{\infty} |f_k|^q\Big\}^{1/q}\Big\|_{L^p(\mu)} < \infty.$$

Let $\{g_k\}_{k=-\infty}^{\infty} \in L^{p'}(l^{q'})(\mu)$ with $\|\{g_k\}_{k=-\infty}^{\infty}\|_{L^{p'}(l^{q'})(\mu)} \le 1$. Then

$$\Big\|\Big\{\sum_{j=-\infty}^{\infty} 2^{jsq}|E_jR_Nf|^q\Big\}^{1/q}\Big\|_{L^p(\mu)} = \sup\Big|\sum_{j=-\infty}^{\infty} 2^{js}\int_{\mathbb{R}^d}(E_jR_Nf)(x)g_j(x)\,d\mu(x)\Big|, \tag{2.21}$$

where the supremum is taken over all $\{g_k\}_{k=-\infty}^{\infty} \in L^{p'}(l^{q'})(\mu)$ as above.

The formulae (2.5)–(2.7) tell us that

$$(E_jR_Nf)(x) = \sum_{l,k,i,m,\{k_t,m_t\}_1^l} E_jD_iD_{i+m}G_{k_1}G_{k_1+m_1}\ldots G_{k_l}G_{k_l+m_l}G_k^{N_1}G_k(f)(x)$$

(the sum over l is from 0 to ∞). Thus,

$$\sum_{j=-\infty}^{\infty} 2^{js}\int_{\mathbb{R}^d}(E_jR_Nf)(x)g_j(x)\,d\mu(x) = \sum_{k,j,l,i,m,\{k_t,m_t\}_1^l}\int_{\mathbb{R}^d} G_k(f)(x) \times 2^{js}G_k^{N_1}G_{k_l+m_l}G_{k_l}\ldots G_{k_1+m_1}G_{k_1}D_{i+m}D_iE_j(g_j)(x)\,d\mu(x).$$

Noting that $1 < p < q \le \infty$ implies $1 \le q' < p' < \infty$, by Lemma 3 (and its proof), (2.20) and the Hölder inequality, we obtain

$$\begin{aligned}
&\Big|\sum_{j=-\infty}^{\infty} 2^{js}\int_{\mathbb{R}^d}(E_jR_Nf)(x)g_j(x)\,d\mu(x)\Big|\\
&\qquad\qquad\le \Big\|\Big\{\sum_{k=-\infty}^{\infty} 2^{ksq}\big|G_k(f)\big|^q\Big\}^{1/q}\Big\|_{L^p(\mu)}\\
&\qquad\qquad\quad\times\Big\|\Big\{\sum_{k=-\infty}^{\infty} 2^{-ksq'}\Big|\sum_{l,j,i,m,\{k_t,m_t\}_1^l} 2^{js}\\
&\qquad\qquad\quad\times G_k^{N_1}G_{k_l+m_l}G_{k_l}\ldots G_{k_1+m_1}G_{k_1}D_{i+m}D_iE_j(g_j)\Big|^{q'}\Big\}^{1/q'}\Big\|_{L^{p'}(\mu)}\\
&\qquad\qquad\le C_1(2^{-N(s+2\nu\theta)}+2^{-N(2\nu\theta-s)})\Big\|\Big\{\sum_{k=-\infty}^{\infty} 2^{ksq}|G_k(f)|^q\Big\}^{1/q}\Big\|_{L^p(\mu)}\\
&\qquad\qquad\quad\times\Big\|\Big\{\sum_{k=-\infty}^{\infty}|g_k|^{q'}\Big\}^{1/q'}\Big\|_{L^{p'}(\mu)}\\
&\qquad\qquad\le C_1(2^{-N(s+2\nu\theta)}+2^{-N(2\nu\theta-s)})\Big\|\Big\{\sum_{k=-\infty}^{\infty} 2^{ksq}|G_k(f)|^q\Big\}^{1/q}\Big\|_{L^p(\mu)}.
\end{aligned}$$

Combining this with (2.21) finally yields (2.2) in the case $1 < p < q \le \infty$, and so we have completed the proof of (2.2).

To verify (2.4) under the assumption (2.3), in fact, we only need to note that in this case, we have (2.6). Thus, using (2.2) and the Minkowski inequality, we further obtain

$$\begin{aligned}
&\Big\|\Big\{\sum_{j=-\infty}^{\infty} 2^{jsq}|E_jT_N^{-1}f|^q\Big\}^{1/q}\Big\|_{L^p(\mu)}\\
&\qquad\le \sum_{l=0}^{\infty}\Big\|\Big\{\sum_{j=-\infty}^{\infty} 2^{jsq}|E_j(R_N)^lf|^q\Big\}^{1/q}\Big\|_{L^p(\mu)}\\
&\qquad\le \sum_{l=0}^{\infty} C_1^l(2^{-N(s+2\nu\theta)}+2^{-N(2\nu\theta-s)})^l\Big\|\Big\{\sum_{k=-\infty}^{\infty} 2^{ksq}|G_k(f)|^q\Big\}^{1/q}\Big\|_{L^p(\mu)}\\
&\qquad\le C\Big\|\Big\{\sum_{k=-\infty}^{\infty} 2^{ksq}|G_k(f)|^q\Big\}^{1/q}\Big\|_{L^p(\mu)},
\end{aligned}$$

where C is independent of f. This proves (2.4) and finishes the proof of Theorem 1.

We now use the approximation to the identity in Definition 1 to introduce the "test function space".

DEFINITION 2. For all $f \in L^2(\mu)$, we define

$$\|f\|_{\dot F^s_{pq}(\mu)} = \Big\|\Big\{\sum_{k=-\infty}^{\infty} 2^{ksq}|D_k f|^q\Big\}^{1/q}\Big\|_{L^p(\mu)},$$
$$\dot{\mathcal F}^s_{pq}(\mu) = \{f \in L^2(\mu) : \|f\|_{\dot F^s_{pq}(\mu)} < \infty\}.$$

To show that Definition 2 is independent of the chosen approximations to the identity, we first establish the following lemma.

LEMMA 4. *For all $f \in L^2(\mu)$ and N so large that* (2.3) *holds*,

$$\Big\|\Big\{\sum_{j=-\infty}^{\infty} 2^{jsq}\Big|\sum_{k=-\infty}^{\infty} E_j D_k^N D_k(f)\Big|^q\Big\}^{1/q}\Big\|_{L^p(\mu)} \le C\Big\|\Big\{\sum_{k=-\infty}^{\infty} 2^{ksq}|D_k(f)|^q\Big\}^{1/q}\Big\|_{L^p(\mu)},$$

where C is independent of f.

Proof. The essence of the proof is the same as in the proof of (2.2). We sketch it for the reader's convenience.

If $1 \le p = q \le \infty$, then Lemma 2 and the Hölder inequality tell us that

$$\begin{aligned}
&\Big\|\Big\{\sum_{j=-\infty}^{\infty} 2^{jsq}\Big|\sum_{k=-\infty}^{\infty} E_j D_k^N D_k(f)\Big|^q\Big\}^{1/q}\Big\|_{L^p(\mu)} \\
&\quad = \Big\{\sum_{j=-\infty}^{\infty} 2^{jsq}\Big\|\sum_{k=-\infty}^{\infty} E_j D_k^N D_k(f)\Big\|_{L^p(\mu)}^q\Big\}^{1/q} \\
&\quad \le C\Big\{\sum_{j=-\infty}^{\infty} 2^{jsq}\Big[\sum_{k=-\infty}^{\infty} 2^{-2|j-k|\theta}\|D_k(f)\Big\|_{L^p(\mu)}\Big]^q\Big\}^{1/q} \\
&\quad \le C\Big\{\sum_{j=-\infty}^{\infty}\Big[\sum_{k=-\infty}^{\infty} 2^{(j-k)s-2|j-k|\theta}2^{ksq}\|D_k(f)\|_{L^p(\mu)}^q\Big]\Big\}^{1/q} \\
&\quad \le C\Big\{\sum_{k=-\infty}^{\infty} 2^{ksq}\|D_k(f)\|_{L^p(\mu)}^q\Big\}^{1/q} \\
&\quad = C\Big\|\Big\{\sum_{k=-\infty}^{\infty} 2^{ksq}|D_k(f)|^q\Big\}^{1/q}\Big\|_{L^p(\mu)}.
\end{aligned}$$

If $1 \le q < p < \infty$, let $r = p/q$. Then $r > 1$. For $g \in L^{r'}(\mu)$ with $g \ge 0$ and $\|g\|_{L^{r'}(\mu)} \le 1$, the Hölder inequality, the Minkowski inequality, Lemma 2 and the $L^p(\mu)$-boundedness of $M_{(2)}$ yield

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty} 2^{jsq} \int_{\mathbb{R}^d} \Big| \sum_{k=-\infty}^{\infty} E_j D_k^N D_k(f)(x)\Big|^q g(x)\, d\mu(x) \\
&\quad \le \sum_{j=-\infty}^{\infty} 2^{jsq} \int_{\mathbb{R}^d} \Big[\sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^d} |(E_j D_k^N)(x,y)|\, d\mu(y)\Big]^{q/q'} \\
&\qquad \times \Big[\sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^d} |(E_j D_k^N)(x,y)||D_k(f)(y)|^q\, d\mu(y)\Big] g(x)\, d\mu(x) \\
&\quad \le C \sum_{j=-\infty}^{\infty} \Big[\sum_{k=-\infty}^{\infty} 2^{(j-k)s-2|j-k|\theta}\Big]^{q/q'} \Big\{ \sum_{k=-\infty}^{\infty} 2^{(j-k)s} \\
&\qquad \times \int_{\mathbb{R}^d} \Big[\int_{\mathbb{R}^d} |(E_j D_k^N)(x,y)| g(x)\, d\mu(x)\Big] 2^{ksq} |D_k(f)(y)|^q\, d\mu(y)\Big\} \\
&\quad \le C \sum_{j=-\infty}^{\infty} \Big\{ \sum_{k=-\infty}^{\infty} 2^{(j-k)s-2|j-k|\theta} \int_{\mathbb{R}^d} M_{(2)} g(y) 2^{ksq} |D_k(f)(y)|^q\, d\mu(y)\Big\} \\
&\quad \le C \|M_{(2)} g\|_{L^{r'}(\mu)} \Big\| \Big\{ \sum_{k=-\infty}^{\infty} 2^{ksq} |D_k(f)|^q \Big\}^{1/q} \Big\|_{L^p(\mu)} \\
&\quad \le C \Big\| \Big\{ \sum_{k=-\infty}^{\infty} 2^{ksq} |D_k(f)|^q \Big\}^{1/q} \Big\|_{L^p(\mu)}.
\end{aligned}$$

We obtain the desired inequality by taking the supremum over the above g.

Finally, using (2.20) and the case $1 \le q < p < \infty$, we can also verify the assertion for $1 < p < q \le \infty$; this finishes the proof of Lemma 4.

Applying Theorem 1 and Lemma 4, we can now verify that the test function space $\dot{\mathcal{F}}^s_{pq}(\mu)$ in Definition 2 is independent of the chosen approximations to the identity.

PROPOSITION 1. *For all* $f \in L^2(\mu)$,

$$\Big\| \Big\{ \sum_{k=-\infty}^{\infty} 2^{ksq} |D_k f|^q \Big\}^{1/q} \Big\|_{L^p(\mu)} \sim \Big\| \Big\{ \sum_{k=-\infty}^{\infty} 2^{ksq} |E_k f|^q \Big\}^{1/q} \Big\|_{L^p(\mu)}.$$

Proof. For given $|s| < \theta$, we choose $\nu \in (0, 1/2)$ such that $|s| < 2\nu\theta$. By (2.5), for any $j \in \mathbb{Z}$, we can write

$$E_j f(x) = \sum_{k=-\infty}^{\infty} E_j D_k^N D_k T_N^{-1}(f)(x),$$

where $N \in \mathbb{N}$ is large enough such that (2.3) holds. Then Lemma 4 and Theorem 1 yield

$$\begin{aligned}
&\Big\|\Big\{\sum_{j=-\infty}^{\infty} 2^{jsq}|E_j f|^q\Big\}^{1/q}\Big\|_{L^p(\mu)} \\
&\qquad = \Big\|\Big\{\sum_{j=-\infty}^{\infty} 2^{jsq}\Big|\sum_{k=-\infty}^{\infty} E_j D_k^N D_k T_N^{-1}(f)\Big|^q\Big\}^{1/q}\Big\|_{L^p(\mu)} \\
&\qquad \le C\Big\|\Big\{\sum_{k=-\infty}^{\infty} 2^{ksq}|D_k T_N^{-1}(f)|^q\Big\}^{1/q}\Big\|_{L^p(\mu)} \\
&\qquad \le C\Big\|\Big\{\sum_{k=-\infty}^{\infty} 2^{ksq}|D_k(f)|^q\Big\}^{1/q}\Big\|_{L^p(\mu)}.
\end{aligned}$$

By symmetry, the proof of Proposition 1 is finished.

The following theorem is one of the main results of this paper.

Theorem 2. *If $1 < p < \infty$ and $1 \le q < \infty$, then for all $f \in \dot{\mathcal{F}}^s_{pq}(\mu)$,*

$$f = \sum_{k\in\mathbb{Z}} D_k^N D_k T_N^{-1}(f) = \sum_{k\in\mathbb{Z}} T_N^{-1} D_k^N D_k(f) \tag{2.22}$$

in both the norm $\|\cdot\|_{\dot{F}^s_{pq}(\mu)}$ and the norm $\|\cdot\|_{\dot{F}^s_{p\infty}(\mu)}$. Moreover, for all $g \in \dot{\mathcal{F}}^s_{pq}(\mu)$ with $1 < p < \infty$ and $1 \le q < \infty$,

$$\begin{aligned}
\langle f, g\rangle &= \sum_{k\in\mathbb{Z}} \langle D_k D_k^N T_N^{-1}(f), g\rangle \\
&= \sum_{k\in\mathbb{Z}} \langle T_N^{-1} D_k D_k^N(f), g\rangle
\end{aligned} \tag{2.23}$$

for all $f \in (\dot{\mathcal{F}}^s_{pq}(\mu))^*$ with $1 < p < \infty$ and $1 \le q \le \infty$.

Proof. We only show the first equality in (2.22). The proof for the second equality in (2.22) is similar. The proof that (2.22) holds in the norm $\|\cdot\|_{\dot{F}^s_{p\infty}(\mu)}$ is easy by noting that $\dot{\mathcal{F}}^s_{pq}(\mu) \subset \dot{\mathcal{F}}^s_{p\infty}(\mu)$ for $1 \le q < \infty$, which is a simple consequence of the monotonicity of l^q; see the proof of Proposition 2.3.2/2 in [33, p. 47].

Let $f \in \dot{\mathcal{F}}^s_{pq}(\mu)$, $1 < p < \infty$ and $1 \le q < \infty$. It suffices to show that

$$\lim_{L\to\infty} \Big\|\sum_{|k|>L} D_k^N D_k T_N^{-1}(f)\Big\|_{\dot{F}^s_{pq}(\mu)} = 0. \tag{2.24}$$

Lemma 4 and Theorem 1 lead to

$$\begin{aligned}\Big\|\sum_{|k|>L} D_k^N D_k T_N^{-1}(f)\Big\|_{\dot F^s_{pq}(\mu)} &= \Big\|\Big\{\sum_{j=-\infty}^{\infty} 2^{jsq}\Big|D_j\Big(\sum_{|k|>L} D_k^N D_k T_N^{-1}(f)\Big)\Big|^q\Big\}^{1/q}\Big\|_{L^p(\mu)}\\ &\le C\Big\|\Big\{\sum_{|k|>L} 2^{ksq}|D_k T_N^{-1}(f)|^q\Big\}^{1/q}\Big\|_{L^p(\mu)} \to 0 \quad \text{as } L\to\infty,\end{aligned}$$

since $T_N^{-1}(f)\in \dot{\mathcal F}^s_{pq}(\mu)$. Thus, (2.24) holds, and therefore the first equality in (2.22) holds.

From (2.22) we can deduce the second equality in (2.23). In fact, for all $g\in\dot{\mathcal F}^s_{pq}(\mu)$ with $1<p<\infty$ and $1\le q<\infty$, we have

$$\langle f,g\rangle = \Big\langle f, \sum_{k\in\mathbb Z} D_k^N D_k T_N^{-1}(g)\Big\rangle = \sum_{k\in\mathbb Z}\langle f, D_k^N D_k T_N^{-1}(g)\rangle,$$

where $f\in(\dot{\mathcal F}^s_{pq}(\mu))^*$.

To finish the proof, we only need to verify that for any $k\in\mathbb Z$,

$$\langle f, D_k^N D_k T_N^{-1}(g)\rangle = \langle D_k D_k^N T_N^{-1}(f), g\rangle. \tag{2.25}$$

To this end, for any $M>0$, let $Q_{0,M}$ be the cube centered at the origin with side length $2M$. Define

$$g_{k,M}(x) = \int_{Q_{0,M}} D_k^N(x,y)(D_kT_N^{-1})(g)(y)\,d\mu(y).$$

We claim that

$$\lim_{M\to\infty}\|D_k^N D_k T_N^{-1}(g) - g_{k,M}\|_{\dot F^s_{pq}(\mu)} = 0. \tag{2.26}$$

In fact, Theorem 1 tells us that $T_N^{-1}g\in \dot F^s_{pq}(\mu)$, and Lemma 2 and the boundedness of $M_{(2)}$ in $L^p(\mu)$ further yield

$$\begin{aligned}&\|D_k^N D_k T_N^{-1}(g) - g_{k,M}\|_{\dot F^s_{pq}(\mu)}\\ &\quad= \Big\|\Big\{\sum_{l=-\infty}^{\infty} 2^{lsq}\Big|D_l\Big[\int_{\mathbb R^d\setminus Q_{0,M}} D_k^N(\cdot,y)(D_kT_N^{-1})(g)(y)\,d\mu(y)\Big]\Big|^q\Big\}^{1/q}\Big\|_{L^p(\mu)}\\ &\quad\le CN\Big\|\Big\{\sum_{l=-\infty}^{\infty} 2^{(l-k)sq-2|l-k|\theta q}\Big\}^{1/q} 2^{ks} M_{(2)}[\chi_{\mathbb R^d\setminus Q_{0,M}} D_kT_N^{-1}(g)]\Big\|_{L^p(\mu)}\\ &\quad\le CN2^{ks}\Big[\int_{\mathbb R^d\setminus Q_{0,M}} |(D_kT_N^{-1})(g)(y)|^p\,d\mu(y)\Big]^{1/p}\to 0\end{aligned}$$

as $M\to\infty$, where we used the facts that $|s|<\theta$ and $1<p<\infty$. Thus, (2.26) holds. Therefore,

$$\langle f, D_k^N D_k T_N^{-1}(g)\rangle = \lim_{M\to\infty}\langle f, g_{k,M}\rangle. \tag{2.27}$$

Let $S = Q_{0,M}\cap \operatorname{supp}(\mu)$. For any $z\in S$, there is a cube $Q_{z,k+N}$ centered at z. Thus, $\{Q_{z,k+N}\}_{z\in S}$ is a covering of S. By the compactness of S, we can find a finite number of cubes, $\{Q_{z_i,k+N}\}_{i=1}^{\nu}\subset\{Q_{z,k+N}\}_{z\in S}$, such that $\bigcup_{i=1}^{\nu} Q_{z_i,k+N}\supset S$. We now decompose S into the union of a finite number of cubes with disjoint interiors, $\{Q_j\}_{j=1}^{N_0}$, such that each Q_j for $j\in\{1,\dots,N_0\}$ is contained in some $Q_{z_i,k+N}$ for some $i\in\{1,\dots,\nu\}$. We then divide each Q_j into a union of cubes, $\{Q_j^i\}_{i=1}^{N_j}$, such that $\ell(Q_j^i)\sim 2^{-J}$, where $N_j\sim 2^J\ell(Q_j)$ for $j=1,\dots,N_0$. Now we write

$$\begin{aligned}
g_{k,M}(x) &= \sum_{j=1}^{N_0}\int_{Q_j} D_k^N(x,y)(D_kT_N^{-1})(g)(y)\,d\mu(y)\\
&= \sum_{j=1}^{N_0}\sum_{i=1}^{N_j}\int_{Q_j^i}[D_k^N(x,y)-D_k^N(x,y_{Q_j^i})](D_kT_N^{-1})(g)(y)\,d\mu(y)\\
&\quad+\sum_{j=1}^{N_0}\sum_{i=1}^{N_j} D_k^N(x,y_{Q_j^i})\int_{Q_j^i}(D_kT_N^{-1})(g)(y)\,d\mu(y)\\
&= g_{k,M}^1(x)+g_{k,M}^2(x),
\end{aligned}$$

where $y_{Q_j^i}$ is any point in the cube Q_j^i. We now claim that for any fixed k and M,

$$\lim_{J\to\infty}\|g_{k,M}^1\|_{\dot F_{pq}^s(\mu)} = 0. \tag{2.28}$$

To prove this claim, let

$$F_{k,i,j}(z,y) = [D_k^N(z,y)-D_k^N(z,y_{Q_j^i})]\chi_{Q_j^i}(y).$$

Lemmas 2.4 and 2.5 in [29] tell us that

$$\operatorname{supp} F_{k,i,j}(\cdot,y)\subset Q_{y,k-N-3},\quad \operatorname{supp} F_{k,i,j}(z,\cdot)\subset Q_{z,k-N-3}; \tag{2.29}$$

$$\int_{\mathbb{R}^d} F_{k,i,j}(z,y)\,d\mu(z) = 0; \tag{2.30}$$

$$\begin{aligned}
|F_{k,i,j}(z,y)| &\le C_4 2^{-J}\ell(Q_{z_{i_0},k+N})^{-1}\\
&\qquad\times\frac{1}{(\ell(Q_{z,k+N})+\ell(Q_{y,k+N})+|z-y|)^n}
\end{aligned} \tag{2.31}$$

if $Q_j^i\subset Q_{z_{i_0},k+N}$ for some $i_0\in\{1,\dots,\nu\}$; and

$$(2.32)\quad |F_{k,i,j}(z,y) - F_{k,i,j}(z',y)| \le C_4 2^{-J} \ell(Q_{z_{i_0},k+N})^{-1} \frac{|z-z'|}{\ell(Q_{x_0,k+N})} \frac{1}{(\ell(Q_{z,k+N}) + \ell(Q_{y,k+N}) + |z-y|)^n}$$

if $z, z' \in Q_{x_0,k+N}$ for some $x_0 \in \operatorname{supp}(\mu)$ and $Q_j^i \subset Q_{z_{i_0},k+N}$ for some $i_0 \in \{1,\dots,\nu\}$. Here C_4 depends on N. From (2.29)–(2.32), Lemma 2 and its proof, it follows that for all $l, k \in \mathbb{Z}$ and $x, y \in \operatorname{supp}(\mu)$,

$$(2.33)\quad \operatorname{supp}(D_l F_{k,i,j})(\cdot, y) \subset Q_{y,\min(l,k-N-1)-3},$$

$$(2.34)\quad \operatorname{supp}(D_l F_{k,i,j})(x, \cdot) \subset Q_{x,\min(l,k-N-1)-3},$$

and for all $x \in \operatorname{supp}(\mu)$ and $y \in Q_j^i \subset Q_{z_{i_0},k+N}$ for some $i_0 \in \{1,\dots,\nu\}$,

$$(2.35)\quad |(D_l F_{k,i,j})(x,y)| \le C_4 2^{-J} 2^{-2|l-k|\theta} \ell(Q_{z_{i_0},k+N})^{-1} \times \frac{1}{(\ell(Q_{x,\min(l,k+N)+1}) + \ell(Q_{y,\min(l,k+N)+1}) + |x-y|)^n}.$$

Let

$$C_5 = \max\left\{C_4, \frac{1}{\ell(Q_{z_i,k+N})} : i = 1,\dots,\nu\right\}.$$

Then C_5 depends on N, k, but not on J and l. Set

$$K(x,y) = \sum_{j=1}^{N_0} \sum_{i=1}^{N_j} (D_l F_{k,i,j})(x,y).$$

Then, by (2.34) and (2.35), we have

$$\begin{aligned}
(2.36)\quad & \Big| \int_{\mathbb{R}^d} K(x,y)(D_k T_N^{-1})(g)(y)\, d\mu(y) \Big| \\
&\le C C_5 2^{-J} 2^{-2|l-k|\theta} \\
&\quad \times \sum_{j=1}^{N_0} \sum_{i=1}^{N_j} \int_{Q_j^i \cap Q_{x,\min(l,k-N-1)-3}} \frac{|(D_k T_N^{-1})(g)(y)|}{(\ell(Q_{x,\min(l,k+N)+1}) + |x-y|)^n}\, d\mu(y) \\
&= C C_5 2^{-J} 2^{-2|l-k|\theta} \\
&\quad \times \sum_{j=1}^{N_0} \int_{Q_j \cap Q_{x,\min(l,k-N-1)-3}} \frac{|(D_k T_N^{-1})(g)(y)|}{(\ell(Q_{x,\min(l,k+N)+1}) + |x-y|)^n}\, d\mu(y) \\
&\le C C_5 N_0 2^{-J} 2^{-2|l-k|\theta} [1 + \delta(Q_{x,\min(l,k-N-1)-3}, Q_{x,\min(l,k+N)+1})] \\
&\quad \times M_{(2)}[(D_k T_N^{-1})(g)](x) \\
&\le C_6 2^{-J} 2^{-2|l-k|\theta} M_{(2)}[(D_k T_N^{-1})(g)](x),
\end{aligned}$$

where C_6 is independent of J and l, but it may depend on M, N and k. Therefore, from (2.36) and the $L^p(\mu)$-boundedness of $M_{(2)}$, it follows that

$$\text{(2.37)}\quad \|g^1_{k,M}\|_{\dot F^s_{pq}(\mu)} = \Big\|\Big\{\sum_{l=-\infty}^{\infty} 2^{lsq}\Big|D_l\Big(\sum_{j=1}^{N_0}\sum_{i=1}^{N_j}\int_{Q^i_j}[D^N_k(\cdot,y)-D^N_k(\cdot,y_{Q^i_j})] \times (D_kT_N^{-1})(g)(y)\,d\mu(y)\Big)\Big|^q\Big\}^{1/q}\Big\|_{L^p(\mu)}$$
$$\le C_6 2^{-J}\Big\{\sum_{l=-\infty}^{\infty} 2^{lsq}2^{-2|l-k|\theta q}\Big\}^{1/q}\|(D_kT_N^{-1})(g)\|_{L^p(\mu)}$$
$$\le CC_6 2^{-J}2^{ks}\|(D_kT_N^{-1})(g)\|_{L^p(\mu)} \to 0$$

as $J\to\infty$. Obviously, (2.37) implies (2.28). By (2.27) and (2.28), we have

$$\text{(2.38)}\quad \langle f, D^N_k D_k T_N^{-1}(g)\rangle = \lim_{M\to\infty}\langle f, g_{k,M}\rangle = \lim_{M\to\infty}\lim_{J\to\infty}\langle f, g^2_{k,M}\rangle$$
$$= \lim_{M\to\infty}\lim_{J\to\infty}\sum_{j=1}^{N_0}\sum_{i=1}^{N_j} D^N_k(f)(y_{Q^i_j})\int_{Q^i_j}(D_kT_N^{-1})(g)(y)\,d\mu(y).$$

We now write

$$\sum_{i=1}^{N_j} D^N_k(f)(y_{Q^i_j})\int_{Q^i_j}(D_kT_N^{-1})(g)(y)\,d\mu(y)$$
$$= \sum_{i=1}^{N_j}\int_{Q^i_j} D^N_k(f)(y)(D_kT_N^{-1})(g)(y)\,d\mu(y)$$
$$+ \int_{\mathbb{R}^d}\Big\{\sum_{i=1}^{N_j}[D^N_k(f)(y_{Q^i_j}) - D^N_k(f)(y)]\chi_{Q^i_j}(y)\Big\}(D_kT_N^{-1})(g)(y)\,d\mu(y).$$

Using the second difference property of the approximation to the identity in Lemma 1(f), by a proof similar to that for (2.37), we can show that

$$\Big\|\sum_{i=1}^{N_j}[D^N_k(y_{Q^i_j},\cdot) - D^N_k(y,\cdot)]\chi_{Q^i_j}(\cdot)\Big\|_{\dot F^s_{pq}(\mu)} \le C_7 2^{-J},$$

where C_7 is independent of J. It follows that

$$\Big|\sum_{i=1}^{N_j}[D^N_k(f)(y_{Q^i_j}) - D^N_k(f)(y)]\chi_{Q^i_j}(y)\Big| \le C_7 2^{-J}\|f\|_{(\dot{\mathcal F}^s_{pq}(\mu))^*}$$

for all $y\in\operatorname{supp}(\mu)$. Noting that $(D_kT_N^{-1})(g)\in L^q(\mu)$ by Theorem 1 and the construction of $\{Q^i_j\}$ for $j\in\{1,\dots,N_0\}$ and $i\in\{1,\dots,N_j\}$, by the

Lebesgue dominated convergence theorem we have

$$\lim_{J\to\infty}\int\limits_{\mathbb{R}^d}\Big\{\sum_{i=1}^{N_j}[D_k^N(f)(y_{Q_j^i})-D_k^N(f)(y)]\chi_{Q_j^i}(y)\Big\}(D_kT_N^{-1})(g)(y)\,d\mu(y)=0.$$

Thus, together with (2.38), we further have

$$\begin{aligned}\langle f, D_k^N D_k T_N^{-1}(g)\rangle &= \lim_{M\to\infty}\lim_{J\to\infty}\sum_{j=1}^{N_0}\sum_{i=1}^{N_j}\int\limits_{Q_j^i} D_k^N(f)(y)(D_kT_N^{-1})(g)(y)\,d\mu(y)\\ &= \int\limits_{\mathbb{R}^d} D_k^N(f)(y)(D_kT_N^{-1})(g)(y)\,d\mu(y)\\ &= \langle T_N^{-1}D_kD_k^N(f), g\rangle.\end{aligned}$$

That is, (2.25) holds and we have completed the proof of Theorem 2.

3. Triebel–Lizorkin spaces. It is easy to see that $D_k(x,\cdot)\in L^2(\mu)$ with compact support for all $x\in\operatorname{supp}(\mu)$ and all $k\in\mathbb{Z}$. We will show that $D_k(x,\cdot)\in\dot{\mathcal{F}}^s_{pq}(\mu)$ for all $x\in\operatorname{supp}(\mu)$. We first recall the definition of the space $\dot{\mathcal{B}}^s_{pq}(\mu)$ in [4].

DEFINITION 3. For all $1\le p,q\le\infty$ and $f\in L^2(\mu)$, we define

$$\|f\|_{\dot{B}^s_{pq}(\mu)}=\Big\{\sum_{k=-\infty}^{\infty}2^{ksq}\|D_kf\|^q_{L^p(\mu)}\Big\}^{1/q},$$

$$\dot{\mathcal{B}}^s_{pq}(\mu)=\{f\in L^2(\mu):\|f\|_{\dot{B}^s_{pq}(\mu)}<\infty\}.$$

LEMMA 5. *The following assertions are true.*

(i) $\dot{\mathcal{B}}^s_{p,\min(p,q)}(\mu)\subset\dot{\mathcal{F}}^s_{pq}(\mu)\subset\dot{\mathcal{F}}^s_{p,\max(p,q)}(\mu)$;

(ii) *Let* $\{D_k\}_{k=-\infty}^{\infty}$ *be as in Theorem* 1. *Then* $D_k(x,\cdot)$ *and* $D_k(\cdot,x)$ *are in* $\dot{\mathcal{F}}^s_{pq}(\mu)$ *for all* $x\in\operatorname{supp}(\mu)$ *and all* $k\in\mathbb{Z}$.

Proof. (i) is obvious by the Minkowski inequality and the monotonicity of l^q for $q\in(0,\infty]$; see the proof of Proposition 2.3.2/2 in [33, p. 47].

It was proved in [4] that for all $|s|<\theta$, $1\le p,q\le\infty$, all $x\in\operatorname{supp}(\mu)$ and all $k\in\mathbb{Z}$, $D_k(x,\cdot)$ and $D_k(\cdot,x)$ are in $\dot{\mathcal{B}}^s_{pq}(\mu)$. From this and (i), it is easy to deduce (ii). This proves the lemma. ∎

We can now introduce the Triebel–Lizorkin spaces $\dot{F}^s_{pq}(\mu)$.

DEFINITION 4. Let p' and q' be the conjugate indices of p and q, respectively. We define

$$\dot{F}^s_{pq}(\mu)=\{f\in(\dot{\mathcal{F}}^{-s}_{p',q'}(\mu))^*:\|f\|_{\dot{F}^s_{pq}(\mu)}<\infty\},$$

where

$$\|f\|_{\dot{F}^s_{pq}(\mu)} = \Big\|\Big\{\sum_{k=-\infty}^{\infty} 2^{ksq}|D_k f|^q\Big\}^{1/q}\Big\|_{L^p(\mu)}.$$

Based on Lemma 5 and Theorem 2, for all $f \in (\dot{\mathcal{F}}^{-s}_{p',q'}(\mu))^*$, we have

$$E_j f(x) = \sum_{k=-\infty}^{\infty} E_j D_k^N D_k T_N^{-1}(f)(x),$$

where $N \in \mathbb{N}$ is large enough such that (2.22) holds. The above equality and the same proof of Proposition 1 show that the spaces $\dot{F}^s_{pq}(\mu)$ are independent of the choice of approximations to the identity in Definition 1. We leave these details to the reader.

It is well known that the Schwartz test function space is dense in Triebel–Lizorkin spaces on $\mathbb{R}^d$. The following result shows that our test function space $\dot{\mathcal{F}}^s_{p,q}(\mu)$ is also dense in the Triebel–Lizorkin space $\dot{F}^s_{pq}(\mu)$. More precisely, we have

PROPOSITION 2. *Let $\overline{\dot{\mathcal{F}}^s_{pq}(\mu)}$ be the closure of $\dot{\mathcal{F}}^s_{pq}(\mu)$ with respect to the norm $\|f\|_{\dot{F}^s_{pq}(\mu)}$. Then*

$$\overline{\dot{\mathcal{F}}^s_{pq}(\mu)} = \dot{F}^s_{pq}(\mu). \tag{3.1}$$

Proof. We first claim that if $f \in \dot{\mathcal{F}}^s_{pq}(\mu)$, then $f \in (\dot{\mathcal{F}}^{-s}_{p',q'}(\mu))^*$ and

$$\|f\|_{(\dot{\mathcal{F}}^{-s}_{p',q'}(\mu))^*} \le C\|f\|_{\dot{F}^s_{pq}(\mu)}. \tag{3.2}$$

To show this claim, let $f \in \dot{\mathcal{F}}^s_{pq}(\mu)$ and $g \in \dot{\mathcal{F}}^{-s}_{p',q'}(\mu)$. Let $\{D_k\}_{k\in\mathbb{Z}}$ be as before. It is easy to see that D_k^N has the same properties as D_k with a constant depending on N, namely CN, if C is the constant appearing in the properties satisfied by D_k for $k \in \mathbb{Z}$.

Noting that $(D_k^N)^* = D_k^N$, by (2.5), the Hölder inequality, Theorem 1 and Proposition 1, we obtain

$$\begin{aligned}
|f(g)| &= |\langle f, g\rangle| \quad (\text{in the sense of } (L^2(\mu))^* = L^2(\mu)) \\
&= \Big|\int_{\mathbb{R}^d} \sum_{k=-\infty}^{\infty} D_k^N D_k T_N^{-1}(f) g\, d\mu\Big| \\
&= \Big|\sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^d} D_k T_N^{-1}(f) D_k^N(g)\, d\mu\Big| \\
&\le \int_{\mathbb{R}^d} \Big\{\sum_{k=-\infty}^{\infty} 2^{ksq}|D_k T_N^{-1}(f)|^q\Big\}^{1/q}\Big\{\sum_{k=-\infty}^{\infty} 2^{-ksq'}|D_k^N(g)|^{q'}\Big\}^{1/q'} d\mu
\end{aligned}$$

$$\leq \Big\| \Big\{ \sum_{k=-\infty}^{\infty} 2^{ksq} |D_k T_N^{-1}(f)|^q \Big\}^{1/q} \Big\|_{L^p(\mu)}$$
$$\times \Big\| \Big\{ \sum_{k=-\infty}^{\infty} 2^{-ksq'} |D_k^N(g)|^{q'} \Big\}^{1/q'} \Big\|_{L^{p'}(\mu)}$$
$$\leq C \|T_N^{-1}(f)\|_{\dot{F}^s_{pq}(\mu)} \|g\|_{\dot{F}^{-s}_{p',q'}(\mu)} \leq C \|f\|_{\dot{F}^s_{pq}(\mu)} \|g\|_{\dot{F}^{-s}_{p',q'}(\mu)}.$$

Thus, $f \in (\dot{\mathcal{F}}^{-s}_{p',q'}(\mu))^*$ and

$$\|f\|_{(\dot{\mathcal{F}}^{-s}_{p',q'}(\mu))^*} \leq C \|f\|_{\dot{F}^s_{pq}(\mu)}.$$

That is, (3.2) holds.

Now to show that $\overline{\dot{\mathcal{F}}^s_{pq}(\mu)} \subset \dot{F}^s_{pq}(\mu)$, let $\{f_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence in $\dot{\mathcal{F}}^s_{pq}(\mu)$ in the norm $\|\cdot\|_{\dot{F}^s_{pq}(\mu)}$. Then, by (3.2), it is also a Cauchy sequence in the norm $\|\cdot\|_{(\dot{\mathcal{F}}^{-s}_{p',q'}(\mu))^*}$. Since $(\dot{\mathcal{F}}^{-s}_{p',q'}(\mu))^*$ is a Banach space (see [35]), there is an $f \in (\dot{\mathcal{F}}^{-s}_{p',q'}(\mu))^*$ such that $f_k \to f$ in $(\dot{\mathcal{F}}^{-s}_{p',q'}(\mu))^*$ as $k \to \infty$. We still need to verify that $\|f\|_{\dot{F}^s_{pq}(\mu)} < \infty$. From Lemma 5 and

$$|D_k(f_n - f)(x)| \leq \|D_k(x,\cdot)\|_{\dot{F}^s_{pq}(\mu)} \|f_n - f\|_{(\dot{\mathcal{F}}^{-s}_{p',q'}(\mu))^*},$$

it follows that for all $x \in \operatorname{supp}(\mu)$ and all $k \in \mathbb{Z}$,

$$\lim_{n\to\infty} D_k f_n(x) = D_k f(x). \tag{3.3}$$

Thus, the fact that $\|f_n\|_{\dot{F}^s_{pq}(\mu)} \leq C$ with C independent of n, Definition 4, the Fatou lemma and (3.3) tell us that

$$\|f\|_{\dot{F}^s_{pq}(\mu)} \leq C,$$

which shows $f \in \dot{F}^s_{pq}(\mu)$ and $f_k \to f$ in $\dot{F}^s_{pq}(\mu)$ as $k \to \infty$.

We now prove the other direction: $\dot{F}^s_{pq}(\mu) \subset \overline{\dot{\mathcal{F}}^s_{pq}(\mu)}$. This comes from Theorem 2 and its proof. More precisely, if $f \in \dot{F}^s_{pq}(\mu)$, then by Theorem 2 and its proof, we can write (2.23) as

$$f = \sum_{k\in\mathbb{Z}} D_k D_k^N T_N^{-1}(f),$$

where the series converges in the norm of $\dot{F}^s_{pq}(\mu)$. As in the proof of Theorem 2, if we define $g_{k,M}(x)$ by

$$g_{k,M}(x) = \int_{Q_{0,M}} D_k^N(x,y)(D_k T_N^{-1})(f)(y)\, d\mu(y),$$

it is easy to check that $g_{k,M}(x)$ belongs to $\dot{\mathcal{F}}^s_{pq}(\mu)$ and f can be approximated by a finite sum of $g_{k,M}(x)$. We leave the details to the reader. This shows that $\dot{F}^s_{pq}(\mu) \subset \overline{\dot{\mathcal{F}}^s_{pq}(\mu)}$ and completes the proof of Proposition 2.

We remark that, in particular, Proposition 2 shows that $\dot{F}^s_{pq}(\mu)$ is a Banach space.

We now establish the boundedness of Riesz operators defined via the approximation to the identity in the spaces $\dot{F}^s_{pq}(\mu)$; then we show that the spaces $\dot{F}^s_{pq}(\mu)$ have the lifting property by using these operators.

DEFINITION 5. For $\alpha \in \mathbb{R}$, $f \in L^2(\mu)$ and all $x \in \operatorname{supp}(\mu)$, we define the *Riesz potential operator* I_α by

$$I_\alpha f(x) = \sum_{k=-\infty}^{\infty} 2^{-k\alpha} D_k f(x).$$

THEOREM 3. *Let $|s| < \theta$ and $|s+\alpha| < \theta$. Then I_α is bounded from $\dot{F}^s_{pq}(\mu)$ to $\dot{F}^{s+\alpha}_{pq}(\mu)$, that is, there is a constant $C > 0$ such that for all $f \in \dot{F}^s_{pq}(\mu)$,*

$$\|I_\alpha f\|_{\dot{F}^{s+\alpha}_{pq}(\mu)} \le C\|f\|_{\dot{F}^s_{pq}(\mu)}.$$

Proof. If $p = q$, then $\dot{B}^s_{pq}(\mu) = \dot{F}^s_{pq}(\mu)$ and the conclusion of the theorem was proved in [4].

If $1 \le q < p < \infty$, let $r = p/q$ and $g \in L^{r'}(\mu)$ with $g \ge 0$ and $\|g\|_{L^{r'}(\mu)} \le 1$. By Theorem 2 and the Hölder inequality, we then have

$$\begin{aligned}
(3.4)\qquad & \sum_{j=-\infty}^{\infty} 2^{j(s+\alpha)q} \int_{\mathbb{R}^d} |D_j I_\alpha f(x)|^q g(x)\, d\mu(x) \\
&= \sum_{j=-\infty}^{\infty} \int_{\mathbb{R}^d} 2^{j(s+\alpha)q} \Big| \sum_{k=-\infty}^{\infty} D_j I_\alpha D_k^N D_k T_N^{-1} f(x)\Big|^q g(x)\, d\mu(x) \\
&= \sum_{j=-\infty}^{\infty} 2^{j(s+\alpha)q} \int_{\mathbb{R}^d} \Big| \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} 2^{-i\alpha} D_j D_i D_k^N D_k T_N^{-1} f(x)\Big|^q g(x)\, d\mu(x) \\
&= \sum_{j=-\infty}^{\infty} 2^{j(s+\alpha)q} \int_{\mathbb{R}^d} \Big| \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} 2^{-i\alpha} \int_{\mathbb{R}^d} (D_j D_i D_k^N)(x,y) \\
&\quad \times (D_k T_N^{-1} f)(y)\, d\mu(y)\Big|^q g(x)\, d\mu(x) \\
&\le \sum_{j=-\infty}^{\infty} 2^{j(s+\alpha)q} \int_{\mathbb{R}^d} \Big\{ \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} 2^{-i\alpha-ks} \int_{\mathbb{R}^d} |(D_j D_i D_k^N)(x,y)|\, d\mu(y)\Big\}^{q/q'} \\
&\quad \times \Big\{ \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} 2^{-i\alpha-ks} \int_{\mathbb{R}^d} |(D_j D_i D_k^N)(x,y)| \\
&\quad \times 2^{ksq} |(D_k T_N^{-1} f)(y)|^q\, d\mu(y)\Big\} g(x)\, d\mu(x),
\end{aligned}$$

where we assume that N satisfies (2.3).

Since $|s| < \theta$ and $|s+\alpha| < \theta$, we can choose $\nu \in (0, 1/2)$ such that $|s+\alpha| < 2\nu\theta$, $|s| < 2\nu\theta$ and $|s| < 2(1-\nu)\theta$. Similarly to (2.11) and (2.12), by Lemma 2, we have

$$\|D_j D_i D_k^N(x,\cdot)\|_{L^1(\mu)} \le C2^{-2\theta|j-i|}, \tag{3.5}$$

$$\|D_j D_i D_k^N(x,\cdot)\|_{L^1(\mu)} \le C2^{-2\theta|i-k|}. \tag{3.6}$$

The geometric mean of (3.5) and (3.6) tells us that

$$\|D_j D_i D_k^N(x,\cdot)\|_{L^1(\mu)} \le C2^{-2\theta\nu|j-i|}2^{-2\theta(1-\nu)|i-k|}. \tag{3.7}$$

Inserting (3.7) into (3.4) leads to

$$\begin{aligned}\sum_{j=-\infty}^{\infty} 2^{j(s+\alpha)q} \int_{\mathbb{R}^d} |D_j I_\alpha f(x)|^q g(x)\, d\mu(x) & \\ \le C \sum_{j=-\infty}^{\infty} 2^{j(s+\alpha)q} & \\ \times \int_{\mathbb{R}^d} \Big\{ \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} 2^{-i\alpha-ks} 2^{-2\theta\nu|j-i|} 2^{-2\theta(1-\nu)|i-k|} \Big\}^{q/q'} & \\ \times \Big\{ \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} 2^{-i\alpha-ks} & \\ \times \int_{\mathbb{R}^d} |(D_j D_i D_k^N)(x,y)| 2^{ksq} |(D_k T_N^{-1} f)(y)|^q\, d\mu(y) \Big\} g(x)\, d\mu(x) & \\ \le C \sum_{j=-\infty}^{\infty} 2^{j(s+\alpha)} \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} 2^{-i\alpha-ks} & \\ \times \int_{\mathbb{R}^d} \Big\{ \int_{\mathbb{R}^d} |(D_j D_i D_k^N)(x,y)| g(x)\, d\mu(x) \Big\} 2^{ksq} |(D_k T_N^{-1} f)(y)|^q\, d\mu(y). &\end{aligned} \tag{3.8}$$

Some arguments similar to those for (2.17) and (2.18) tell us that

$$\int_{\mathbb{R}^d} |(D_j D_i D_k^N)(x,y)| g(x)\, d\mu(x) \le C2^{-2\theta|j-i|} M_{(2)}^2 g(y), \tag{3.9}$$

$$\int_{\mathbb{R}^d} |(D_j D_i D_k^N)(x,y)| g(x)\, d\mu(x) \le C2^{-2\theta|i-k|} M_{(2)}^2 g(y). \tag{3.10}$$

The geometric mean of (3.9) and (3.10) yields

$$\int_{\mathbb{R}^d} |(D_j D_i D_k^N)(x,y)| g(x)\, d\mu(x) \le C2^{-2\theta\nu|j-i|} 2^{-2\theta(1-\nu)|i-k|} M_{(2)}^2 g(y). \tag{3.11}$$

By inserting (3.11) into (3.8) and applying the $L^p(\mu)$-boundedness of $M_{(2)}$, we obtain

$$(3.12)\qquad \sum_{j=-\infty}^{\infty} 2^{j(s+\alpha)q} \int_{\mathbb{R}^d} |D_j I_\alpha f(x)|^q g(x)\,d\mu(x)$$
$$\le C \sum_{j=-\infty}^{\infty} 2^{j(s+\alpha)} \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} 2^{-i\alpha-ks} 2^{-2\theta\nu|j-i|} 2^{-2\theta(1-\nu)|i-k|}$$
$$\times \int_{\mathbb{R}^d} M_{(2)}^2 g(y) 2^{ksq} |(D_k T_N^{-1} f)(y)|^q\,d\mu(y)$$
$$\le C \int_{\mathbb{R}^d} M_{(2)}^2 g(y) \sum_{k=-\infty}^{\infty} 2^{ksq} |(D_k T_N^{-1} f)(y)|^q\,d\mu(y)$$
$$\le C \|M_{(2)}^2 g\|_{L^r(\mu)} \Big\| \Big\{ \sum_{k=-\infty}^{\infty} 2^{ksq} |(D_k T_N^{-1} f)(y)|^q \Big\}^{1/q} \Big\|_{L^p(\mu)}^q$$
$$\le C \|g\|_{L^r(\mu)} \Big\| \Big\{ \sum_{k=-\infty}^{\infty} 2^{ksq} |(D_k T_N^{-1} f)(y)|^q \Big\}^{1/q} \Big\|_{L^p(\mu)}^q$$
$$\le C \Big\| \Big\{ \sum_{k=-\infty}^{\infty} 2^{ksq} |(D_k T_N^{-1} f)(y)|^q \Big\}^{1/q} \Big\|_{L^p(\mu)}^q.$$

Taking the supremum in (3.12) over g leads to

$$(3.13)\qquad \|I_\alpha f\|_{F_{pq}^{s+\alpha}(\mu)} \le C \|f\|_{\dot{F}_{pq}^{s}(\mu)}$$

if $1 \le q < p < \infty$.

Let now $1 < p < q \le \infty$. Note that then $1 \le q' < p' < \infty$. If $\{g_i\}_{i=-\infty}^{\infty} \in L^{p'}(l^{q'})(\mu)$ and

$$\|\{g_i\}_{i=-\infty}^{\infty}\|_{L^{p'}(l^{q'})(\mu)} \le 1,$$

then an argument similar to that for (3.13) can be used to show

$$(3.14)\qquad \Big\| \Big\{ \sum_{k=-\infty}^{\infty} 2^{-ksq'} \Big| \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} 2^{j(s+\alpha)} 2^{-i\alpha} D_k^N D_i D_j g_j \Big|^{q'} \Big\}^{1/q'} \Big\|_{L^{p'}(\mu)}$$
$$\le C \|\{g_i\}_{i=-\infty}^{\infty}\|_{L^{p'}(l^{q'})(\mu)} \le C.$$

The Hölder inequality, Theorem 1 and the estimate (3.14) then yield

$$\|I_\alpha f\|_{F_{pq}^{s+\alpha}(\mu)} = \Big\| \Big\{ \sum_{j=-\infty}^{\infty} 2^{j(s+\alpha)q} |D_j I_\alpha f|^q \Big\}^{1/q} \Big\|_{L^p(\mu)}$$

$$
\begin{aligned}
&= \sup_{\|\{g_i\}_{i=-\infty}^{\infty}\|_{L^{p'}(l^{q'})(\mu)}\le 1} \Big| \sum_{j=-\infty}^{\infty} 2^{j(s+\alpha)} \int_{\mathbb{R}^d} (D_j I_\alpha f)(x) g_j(x)\, d\mu(x)\Big| \\
&= \sup_{\|\{g_i\}_{i=-\infty}^{\infty}\|_{L^{p'}(l^{q'})(\mu)}\le 1} \Big| \sum_{j=-\infty}^{\infty} 2^{j(s+\alpha)} \sum_{i=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} 2^{-i\alpha} \\
&\quad \times \int_{\mathbb{R}^d} D_j D_i D_k^N D_k T_N^{-1}(f)(x) g_j(x)\, d\mu(x)\Big| \\
&= \sup_{\|\{g_i\}_{i=-\infty}^{\infty}\|_{L^{p'}(l^{q'})(\mu)}\le 1} \Big| \sum_{j=-\infty}^{\infty} 2^{j(s+\alpha)} \\
&\quad \times \sum_{i=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} 2^{-i\alpha} \int_{\mathbb{R}^d} D_k T_N^{-1}(f)(x) D_k^N D_i D_j g_j(x)\, d\mu(x)\Big| \\
&\le \sup_{\|\{g_i\}_{i=-\infty}^{\infty}\|_{L^{p'}(l^{q'})(\mu)}\le 1} \int_{\mathbb{R}^d} \Big\{ \sum_{k=-\infty}^{\infty} 2^{ksq} |D_k T_N^{-1}(f)(x)|^q \Big\}^{1/q} \\
&\quad \times \Big\{ \sum_{k=-\infty}^{\infty} 2^{-ksq'} \Big| \sum_{j=-\infty}^{\infty} 2^{j(s+\alpha)} \sum_{i=-\infty}^{\infty} 2^{-i\alpha} D_k^N D_i D_j g_j(x)\Big|^{q'} \Big\}^{1/q'} d\mu(x) \\
&\le \|T_N^{-1}(f)\|_{\dot F^s_{pq}(\mu)} \sup_{\|\{g_i\}_{i=-\infty}^{\infty}\|_{L^{p'}(l^{q'})(\mu)}\le 1} \\
&\quad \times \Big\| \Big\{ \sum_{k=-\infty}^{\infty} 2^{-ksq'} \Big| \sum_{j=-\infty}^{\infty} 2^{j(s+\alpha)} \sum_{i=-\infty}^{\infty} 2^{-i\alpha} D_k^N D_i D_j g_j \Big|^{q'} \Big\}^{1/q'} \Big\|_{L^{p'}(\mu)} \\
&\le C\|f\|_{\dot F^s_{pq}(\mu)}.
\end{aligned}
$$

This proves that I_α is bounded from $\dot F^s_{pq}(\mu)$ to $\dot F^{s+\alpha}_{pq}(\mu)$ and completes the proof of Theorem 3.

We now establish the converse of Theorem 3. To this end, we first show that when α is very small, the composition $I_\alpha I_{-\alpha}$ is invertible in the spaces $\dot F^s_{pq}(\mu)$. To do so, for any given $N_1 \in \mathbb{N}$, we decompose $I - I_\alpha I_{-\alpha}$ into

$$
\begin{aligned}
I - I_\alpha I_{-\alpha} &= \sum_{i=-\infty}^{\infty} \sum_{|m|\le N_1} (1-2^{m\alpha}) D_i D_{i+m} + \sum_{i=-\infty}^{\infty} \sum_{|m|> N_1} (1-2^{m\alpha}) D_i D_{i+m} \\
&= L^1_{N_1} + L^2_{N_1}.
\end{aligned}
$$

We will show that if N_1 is large enough and if α is small enough, then the operator norms of $L^i_{N_1}$ in $\dot F^s_{pq}(\mu)$ will be very small for $i = 1, 2$. Thus, $I_\alpha I_{-\alpha}$ is invertible in $\dot F^s_{pq}(\mu)$.

The same procedure as in the proof of Theorem 3 can be used to verify the following theorem. We leave the details to the reader.

THEOREM 4. *Let $|s| < \theta$ and $|s-\alpha| < \theta$. Then for any $\nu \in (0,1/2)$ such that $|s| < 2\nu\theta$ and $|s-\alpha| < 2\nu\theta$,*

$$\|L^1_{N_1}\|_{\dot{F}^s_{pq}(\mu)\to\dot{F}^s_{pq}(\mu)} \le C_8 \sum_{|m|\le N_1} |1-2^{m\alpha}|2^{-2\theta\nu|m|-ms},$$

$$\|L^2_{N_1}\|_{\dot{F}^s_{pq}(\mu)\to\dot{F}^s_{pq}(\mu)} \le C_8 \sum_{|m|> N_1} |1-2^{m\alpha}|2^{-2\theta\nu|m|-ms},$$

where C_8 is independent of N_1 and α.

From Theorem 4, it is easy to deduce the following result.

COROLLARY 1. *Let $|s| < \theta$ and $|s-\alpha| < \theta$. Then there is $\alpha_0(s) > 0$ such that if $|\alpha| < \alpha_0(s)$, $\nu \in (0,1/2)$, $|s| < 2\nu\theta$ and $|s-\alpha| < 2\nu\theta$, then*

$$C_8\Big\{\sum_{|m|\le N_1} |1-2^{m\alpha}|2^{-2\theta\nu|m|-ms} + \sum_{|m|>N_1} |1-2^{m\alpha}|2^{-2\theta\nu|m|-ms}\Big\} < 1.$$

Thus, if $|\alpha| < \alpha_0(s)$, then $(I_\alpha I_{-\alpha})^{-1}$ exists in $\dot{F}^s_{pq}(\mu)$ and

$$\|(I_\alpha I_{-\alpha})^{-1}\|_{\dot{F}^s_{pq}(\mu)\to\dot{F}^s_{pq}(\mu)} \le C.$$

If we change the order of I_α and $I_{-\alpha}$, we have a similar result which is a simple corollary of the above Corollary 1.

COROLLARY 2. *Let $|s| < \theta$ and $|s+\alpha| < \theta$. Then there is $\alpha_0(s) > 0$ such that if $|\alpha| < \alpha_0(s)$, $\nu \in (0,1/2)$, $|s| < 2\nu\theta$ and $|s+\alpha| < 2\nu\theta$, then*

$$C_8\Big\{\sum_{|m|\le N_1} |1-2^{-m\alpha}|2^{-2\theta\nu|m|-ms} + \sum_{|m|>N_1} |1-2^{-m\alpha}|2^{-2\theta\nu|m|-ms}\Big\} < 1.$$

Thus, if $|\alpha| < \alpha_0(s)$, then $(I_{-\alpha} I_\alpha)^{-1}$ exists in $\dot{F}^s_{pq}(\mu)$ and

$$\|(I_{-\alpha} I_\alpha)^{-1}\|_{\dot{F}^s_{pq}(\mu)\to\dot{F}^s_{pq}(\mu)} \le C.$$

Theorem 3 and Corollary 2 imply the following lifting theorem for the spaces $\dot{F}^s_{pq}(\mu)$.

THEOREM 5. *Let $|s| < \theta$ and $|s+\alpha| < \theta$. Let $\alpha_0(s)$ be as in Corollary 2 and $|\alpha| < \alpha_0(s)$. Then there is a constant $C > 0$ such that for all $f \in \dot{F}^s_{pq}(\mu)$,*

$$C^{-1}\|f\|_{\dot{F}^s_{pq}(\mu)} \le \|I_\alpha f\|_{\dot{F}^{s+\alpha}_{pq}(\mu)} \le C\|f\|_{\dot{F}^s_{pq}(\mu)}.$$

Proof. We only need to verify the left-hand inequality. In fact, by Corollary 2, we have

$$\|f\|_{\dot{F}^s_{pq}(\mu)} = \|(I_{-\alpha} I_\alpha)^{-1} I_{-\alpha} I_\alpha\|_{\dot{F}^s_{pq}(\mu)} \le C\|I_{-\alpha} I_\alpha\|_{\dot{F}^s_{pq}(\mu)} \le C\|I_\alpha f\|_{\dot{F}^{s+\alpha}_{pq}(\mu)}.$$

This completes the proof of Theorem 5.

Finally, we study the dual spaces of the spaces $\dot{F}^s_{pq}(\mu)$. To begin with, we establish the following lemma.

LEMMA 6. *Suppose that* $\{g_k\}_{k\in\mathbb{Z}}$ *is a sequence of functions on* $\mathbb{R}^d$*. If* $1 < p < \infty$, $1 \le q < \infty$ *and*

$$\Big\|\Big\{\sum_{k=-\infty}^{\infty} 2^{ksq}|g_k|^q\Big\}^{1/q}\Big\|_{L^p(\mu)} < \infty,$$

then $g(x) = \sum_{k\in\mathbb{Z}} D_k g_k(x) \in \dot{F}^s_{pq}(\mu)$ *and*

$$\|g\|_{\dot{F}^s_{pq}(\mu)} \le C\Big\|\Big\{\sum_{k=-\infty}^{\infty} 2^{ksq}|g_k|^q\Big\}^{1/q}\Big\|_{L^p(\mu)},$$

where $C > 0$ *is a constant.*

Proof. For $L_1, L_2 \in \mathbb{Z}$ and $L_1 < L_2$, we define

$$g^{L_2}_{L_1}(x) = \sum_{k=L_1}^{L_2} D_k g_k(x).$$

Then for $f \in \dot{\mathcal{F}}^{-s}_{p',q'}(\mu)$, noting that $D_k(x,y) = D_k(y,x)$ and by the Hölder inequality, we have

$$\begin{aligned}|\langle g^{L_2}_{L_1}, f\rangle| &= \Big|\sum_{k=L_1}^{L_2} \langle D_k g_k, f\rangle\Big| \le \sum_{k=L_1}^{L_2} |\langle g_k, D_k f\rangle| \\ &\le \Big\|\Big\{\sum_{k=L_1}^{L_2} 2^{ksq}|g_k|^q\Big\}^{1/q}\Big\|_{L^p(\mu)}\Big\|\Big\{\sum_{k=L_1}^{L_2} 2^{-ksq'}|D_k f|^{q'}\Big\}^{1/q'}\Big\|_{L^{p'}(\mu)} \\ &\le \Big\|\Big\{\sum_{k=L_1}^{L_2} 2^{ksq}|g_k|^q\Big\}^{1/q}\Big\|_{L^p(\mu)}\|f\|_{\dot{\mathcal{F}}^{-s}_{p',q'}(\mu)}.\end{aligned}$$

Thus, $g^{L_2}_{L_1} \in (\dot{\mathcal{F}}^{-s}_{p',q'}(\mu))^*$ and

$$\|g^{L_2}_{L_1}\|_{(\dot{\mathcal{F}}^{-s}_{p',q'}(\mu))^*} \le \Big\|\Big\{\sum_{k=L_1}^{L_2} 2^{ksq}|g_k|^q\Big\}^{1/q}\Big\|_{L^p(\mu)}.$$

It follows that $g \in (\dot{\mathcal{F}}^{-s}_{p',q'}(\mu))^*$; and Lemma 4 tells us that

$$\|g\|_{\dot{F}^s_{pq}(\mu)} \le C\Big\|\Big\{\sum_{k=-\infty}^{\infty} 2^{ksq}|g_k|^q\Big\}^{1/q}\Big\|_{L^p(\mu)}.$$

That is, $f \in \dot{F}^s_{pq}(\mu)$, which finishes the proof of Lemma 6.

We can now establish the dual theorem for the spaces $\dot{F}^s_{pq}(\mu)$.

THEOREM 6. *If* $1 \le p, q \le \infty$ *and* $g \in \dot{F}^s_{pq}(\mu)$, *then*

$$\mathcal{L}_g(f) = \langle g, f\rangle$$

defines a linear functional on $\dot{\mathcal{F}}^{-s}_{p',q'}(\mu)$ *and*

$$\|\mathcal{L}_g\|_{(\dot{\mathcal{F}}^{-s}_{p',q'}(\mu))^*} \le C\|g\|_{\dot{F}^s_{pq}(\mu)}. \tag{3.15}$$

Conversely, if $1 < p, q < \infty$ *and* $\mathcal{L}$ *is a linear functional on* $\dot{F}^s_{pq}(\mu)$, *then there exists a unique* $g \in \dot{F}^{-s}_{p',q'}(\mu)$ *such that*

$$\mathcal{L}(f) = \langle g, f\rangle$$

on $\dot{\mathcal{F}}^s_{pq}(\mu)$ *and*

$$\|g\|_{\dot{F}^{-s}_{p',q'}(\mu)} \le C\|\mathcal{L}\|_{(\dot{F}^s_{pq}(\mu))^*}. \tag{3.16}$$

Proof. (3.15) is just (3.1) in Proposition 2.

Conversely, suppose that $\mathcal{L}$ is a linear functional on $\dot{F}^s_{pq}(\mu)$. By Proposition 2, it is easy to see that $\mathcal{L}$ is also a linear functional on $\dot{\mathcal{F}}^s_{pq}(\mu)$, and therefore, for all $f \in \dot{\mathcal{F}}^s_{pq}(\mu)$,

$$|\mathcal{L}(f)| \le \|\mathcal{L}\|_{(\dot{F}^s_{pq}(\mu))^*}\|f\|_{\dot{F}^s_{pq}(\mu)}.$$

Let $\{D_k\}_{k\in\mathbb{Z}}$ be as before. If $f \in \dot{\mathcal{F}}^s_{pq}(\mu)$, then the sequence $\{D_kf\}_{k\in\mathbb{Z}}$ is in the sequence space

$$L^p(l^q_s)(\mu) = \Big\{\{f_k\}_{k\in\mathbb{Z}} : \\ \|\{f_k\}_{k\in\mathbb{Z}}\|_{L^p(l^q_s)(\mu)} = \Big\|\Big\{\sum_{k=-\infty}^{\infty} 2^{ksq}|f_k|^q\Big\}^{1/q}\Big\|_{L^p(\mu)} < \infty\Big\}.$$

Define $\widetilde{\mathcal{L}}$ on this subset of $L^p(l^q_s)(\mu)$ by

$$\widetilde{\mathcal{L}}[\{D_kf\}_{k\in\mathbb{Z}}] = \mathcal{L}(f).$$

Then, if $f \in \dot{\mathcal{F}}^s_{pq}(\mu)$, we have

$$\begin{aligned}|\widetilde{\mathcal{L}}[\{D_kf\}_{k\in\mathbb{Z}}]| &= |\mathcal{L}(f)| \le \|\mathcal{L}\|_{(\dot{F}^s_{pq}(\mu))^*}\|f\|_{\dot{F}^s_{pq}(\mu)} \\ &= \|\mathcal{L}\|_{(\dot{F}^s_{pq}(\mu))^*}\|\{D_kf\}_{k\in\mathbb{Z}}\|_{L^p(l^q_s)(\mu)}.\end{aligned}$$

Thus, $\widetilde{\mathcal{L}}$ is bounded on this subset. The Hahn–Banach theorem tells us that $\widetilde{\mathcal{L}}$ can be extended to a functional on $L^p(l^q_s)(\mu)$. Since it is well known that $L^p(l^q_s)(\mu)^* = L^{p'}(l^{q'}_{-s})(\mu)$ for $1 < p < \infty$ and $1 \le q < \infty$ (see [32]), there exists a unique sequence $\{g_k\}_{k\in\mathbb{Z}} \in L^{p'}(l^{q'}_{-s})(\mu)$ such that

$$\|\{g_k\}_{k\in\mathbb{Z}}\|_{L^{p'}(l^{q'}_{-s})(\mu)} \le C\|\widetilde{\mathcal{L}}\|_{(L^p(l^q_s)(\mu))^*} \le C\|\mathcal{L}\|_{(\dot{F}^s_{pq}(\mu))^*}$$

and

$$\widetilde{\mathcal{L}}[\{f_k\}_{k\in\mathbb{Z}}] = \sum_{k=-\infty}^{\infty} \langle g_k, f_k\rangle$$

for all $\{f_k\}_{k\in\mathbb{Z}} \in L^p(l_s^q)(\mu)$. Thus, if $f \in \dot{\mathcal{F}}^s_{pq}(\mu)$, then Lemma 6 yields

$$\begin{aligned}\mathcal{L}(f) = \widetilde{\mathcal{L}}(\{D_k f\}_{k\in\mathbb{Z}}) &= \sum_{k=-\infty}^{\infty} \langle g_k, D_k(f)\rangle \\ &= \sum_{k=-\infty}^{\infty} \langle D_k(g_k), f\rangle = \Big\langle \sum_{k=-\infty}^{\infty} D_k(g_k), f\Big\rangle,\end{aligned}$$

since $D_k^* = D_k$. Let

$$g = \sum_{k=-\infty}^{\infty} D_k(g_k).$$

Then Lemma 6 tells us that $g \in \dot{F}^{-s}_{p',q'}(\mu)$ and

$$\|g\|_{\dot{F}^{-s}_{p',q'}(\mu)} \le C\|\{g_k\}_{k\in\mathbb{Z}}\|_{L^{p'}(l^{q'}_{-s})(\mu)} \le C\|\mathcal{L}\|_{(\dot{F}^s_{pq}(\mu))^*}.$$

Thus, (3.16) holds.

This finishes the proof of Theorem 6.

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