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On a theorem of Vesentini

by

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Abstract. Let $\mathcal{A}$ be a Banach algebra over $\mathbb{C}$ with unit $\mathbf{1}$ and $f:\mathbb{C}\to\mathbb{C}$ an entire function. Let $\mathbf{f}:\mathcal{A}\to\mathcal{A}$ be defined by

$$\mathbf{f}(a)=f(a)\quad (a\in\mathcal{A}),$$

where $f(a)$ is given by the usual analytic calculus. The connections between the periods of f and the periods of $\mathbf{f}$ are settled by a theorem of E. Vesentini. We give a new proof of this theorem and investigate further properties of periods of $\mathbf{f}$, for example in C^*-algebras.

Throughout this paper $\mathcal{A}$ denotes a complex unital Banach algebra with unit $\mathbf{1}$. For $a\in\mathcal{A}$ we write

$$\sigma(a)=\{\lambda\in\mathbb{C}: a-\lambda\mathbf{1}\text{ is not invertible in }\mathcal{A}\}$$

for the *spectrum* of a. The center of $\mathcal{A}$ is the subset $\mathcal{A}^{\mathrm{c}}$ of $\mathcal{A}$ given by

$$\mathcal{A}^{\mathrm{c}}=\{x\in\mathcal{A}: xa=ax\text{ for all }a\in\mathcal{A}\}.$$

By $H(\mathbb{C})$ we denote the collection of all entire functions $f:\mathbb{C}\to\mathbb{C}$. If $f\in H(\mathbb{C})$ and $a\in\mathcal{A}$, then $f(a)$ is defined by the well known analytic calculus (see [3]). If

$$f(z)=\sum_{n=0}^{\infty}a_nz^n\quad (z\in\mathbb{C})$$

is the power series representation of f, then by [3],

$$f(a)=\sum_{n=0}^{\infty}a_na^n=a_0\mathbf{1}+a_1a+a_2a^2+\dots$$

for $a\in\mathcal{A}$. Therefore, given $f\in H(\mathbb{C})$, we define the mapping $\mathbf{f}:\mathcal{A}\to\mathcal{A}$ by

$$\mathbf{f}(a)=f(a).$$

Hence $\mathbf{f}':\mathcal{A}\to\mathcal{A}$ is given by

$$\mathbf{f}'(a)=f'(a)=\sum_{n=1}^{\infty}na_na^{n-1}$$

(thus $\mathbf{f}'$ does not denote the derivative of the mapping $\mathbf{f}$).

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For $f \in H(\mathbb{C})$ put

$$P(f) = \{\omega \in \mathbb{C} : f(z+\omega) = f(z) \text{ for all } z \in \mathbb{C}\},$$
$$P(\mathbf{f}) = \{p \in \mathcal{A} : \mathbf{f}(a+p) = \mathbf{f}(a) \text{ for all } a \in \mathcal{A}\}.$$

Observe that $0 \in P(f)$ and $0 \in P(\mathbf{f})$.

Throughout this paper f will denote an element of $H(\mathbb{C})$ with power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (a_0, a_1, \ldots \in \mathbb{C}).$$

Proposition 1. *Let $\omega \in \mathbb{C}$, $q \in \mathcal{A}$ and $q^2 = q$.*

(1) $\mathbf{f}(\omega q) = a_0\mathbf{1} + (f(\omega) - a_0)q$.
(2) *If $\omega \in P(f)$, then $\mathbf{f}(\omega q) = a_0\mathbf{1}$.*

Proof. (1) We have

$$\mathbf{f}(\omega q) = \sum_{n=0}^{\infty} a_n\omega^n q^n = a_0\mathbf{1} + \Big(\sum_{n=1}^{\infty} a_n\omega^n\Big)q = a_0\mathbf{1} + (f(\omega) - a_0)q.$$

(2) Since $f(\omega) = f(0) = a_0$, it follows from (1) that $\mathbf{f}(\omega q) = a_0\mathbf{1}$. ■

Proposition 2. *Suppose that $a, b \in \mathcal{A}$, $ab = ba$ and that $\phi : \mathbb{C} \to \mathcal{A}$ is defined by $\phi(z) = \mathbf{f}(za+b)$ $(z \in \mathbb{C})$. Then ϕ is an $\mathcal{A}$-valued analytic function and*

$$\phi'(z) = \mathbf{f}'(za+b)a \quad \textit{for all } z \in \mathbb{C}.$$

Proof. We have

$$\phi(z) = \sum_{n=0}^{\infty} a_n(za+b)^n \quad (z \in \mathbb{C}).$$

It follows from [3, §59, §97] that ϕ is analytic and

$$\phi'(z) = \sum_{n=0}^{\infty} a_n \frac{d}{dz}(za+b)^n \quad (z \in \mathbb{C}).$$

Since $ab = ba$,

$$\frac{d}{dz}(za+b)^n = n(za+b)^{n-1}a \quad \text{for } n \geq 1,$$

thus

$$\phi'(z) = \Big(\sum_{n=1}^{\infty} na_n(za+b)^{n-1}\Big)a = \mathbf{f}'(za+b)a$$

for $z \in \mathbb{C}$. ■

Theorem 1. *Let $\omega \in P(f)$, $q \in \mathcal{A}^{\mathrm{c}}$ and $q^2 = q$. Then $\omega q \in P(\mathbf{f})$.*

Proof. Fix $a \in \mathcal{A}$ and define $\phi, \psi : \mathbb{C} \to \mathcal{A}$ by

$$\phi(z) = \mathbf{f}(za + \omega q), \qquad \psi(z) = \mathbf{f}(za).$$

Proposition 2 gives

$$\phi^{(k)}(z) = \mathbf{f}^{(k)}(za + \omega q)a^k, \qquad \psi^{(k)}(z) = \mathbf{f}^{(k)}(za)a^k$$

for $z \in \mathbb{C}$ and $k = 0, 1, \ldots$ Hence

$$\phi^{(k)}(0) = \mathbf{f}^{(k)}(\omega q)a^k, \qquad \psi^{(k)}(0) = \mathbf{f}^{(k)}(0)a^k$$

for $k \geq 0$. Since $\omega \in P(f^{(k)})$ for $k \geq 0$, Proposition 1 shows that

$$\phi^{(k)}(0) = f^{(k)}(0)a^k = \mathbf{f}^{(k)}(0)a^k = \psi^{(k)}(0)$$

for $k = 0, 1, \ldots$ Therefore $\phi = \psi$ on $\mathbb{C}$. Hence

$$\mathbf{f}(a + \omega q) = \phi(1) = \psi(1) = \mathbf{f}(a).$$

Since $a \in \mathcal{A}$ was arbitrary, $\omega q \in P(\mathbf{f})$. ∎

Corollary 1. $\{\omega \mathbf{1} : \omega \in P(f)\} \subseteq P(\mathbf{f})$.

Theorem 2. *Let $p \in P(\mathbf{f})$ and suppose that f is non-constant. Then*:

(1) $\sigma(p) \subseteq P(f)$.
(2) $p \in \mathcal{A}^{\mathrm{c}}$.
(3) $p \in P(\mathbf{f}')$.

Proof. (1) We have $f(p) = \mathbf{f}(p) = \mathbf{f}(0) = a_0\mathbf{1}$. Put $g(z) = f(z) - a_0$. Then $g(p) = 0$. The spectral mapping theorem ([3, Satz 99.2]) gives

$$g(\sigma(p)) = \sigma(g(p)) = \{0\}.$$

Since $\sigma(p)$ is compact and g is non-constant, $\sigma(p)$ is finite, say $\sigma(p) = \{\omega_1, \ldots, \omega_n\}$, and $g(\omega_j) = 0$ $(j = 1, \ldots, n)$. Therefore $f(\omega_j) = f(0)$. Fix $z_0 \in \mathbb{C}$ and define $h \in H(\mathbb{C})$ by $h(z) = g(z + z_0)$. Then

$$\begin{aligned}\mathbf{h}(a + p) &= \mathbf{g}(a + p + z_0\mathbf{1}) = \mathbf{f}((a + z_0\mathbf{1}) + p) - a_0\mathbf{1}\\ &= \mathbf{f}(a + z_0\mathbf{1}) - a_0\mathbf{1} = \mathbf{g}(a + z_0\mathbf{1}) = \mathbf{h}(a)\end{aligned}$$

for all $a \in \mathcal{A}$. This shows that $p \in P(\mathbf{h})$. As above, $h(\omega_j) = h(0)$ $(j = 1, \ldots, n)$. Thus

$$f(z_0) - a_0 = g(z_0) = h(0) = h(\omega_j) = g(\omega_j + z_0) = f(\omega_j + z_0) - a_0$$

for $j = 1, \ldots, n$. Consequently, $\omega_j \in P(f)$ $(j = 1, \ldots, n)$.

(2) Since f is non-constant, there is some $z_0 \in \mathbb{C}$ such that $f'(z_0) \neq 0$. Without loss of generality we can assume that $z_0 = 0$, so $a_1 \neq 0$. Now take $a \in \mathcal{A}$. Then

$$(a + p)\mathbf{f}(a) = (a + p)\mathbf{f}(a + p) = \mathbf{f}(a + p)(a + p) = \mathbf{f}(a)(a + p).$$

So $p\mathbf{f}(a) = \mathbf{f}(a)p$ for all $a \in \mathcal{A}$. Therefore

$$p\mathbf{f}(za) = \mathbf{f}(za)p \quad \text{for } a \in \mathcal{A} \text{ and } z \in \mathbb{C}.$$

This gives

$$\sum_{n=0}^{\infty} a_n z^n p a^n = \sum_{n=0}^{\infty} a_n z^n a^n p$$

for $a \in \mathcal{A}$ and $z \in \mathbb{C}$. Comparing coefficients yields

$$a_n p a^n = a_n a^n p \quad \text{for } a \in \mathcal{A} \text{ and } n \geq 0.$$

For $n = 1$ we get $a_1 pa = a_1 ap$ ($a \in \mathcal{A}$). Since $a_1 \neq 0$, $p \in \mathcal{A}^{\mathrm{c}}$.

(3) We have $\mathbf{f}(za + p) = \mathbf{f}(za)$ for $z \in \mathbb{C}$ and $a \in \mathcal{A}$. According to Proposition 2,

$$\mathbf{f}'(za + p)a = \mathbf{f}'(za)a \quad (z \in \mathbb{C},\, a \in \mathcal{A}).$$

Thus for $z = 1$,

$$(*) \qquad \mathbf{f}'(a + p)a = \mathbf{f}'(a)a \quad \text{for each } a \in \mathcal{A}.$$

Now fix $a \in \mathcal{A}$ and define $\phi : \mathbb{C} \to \mathcal{A}$ by

$$\phi(z) = \mathbf{f}'(a - z\mathbf{1} + p) - \mathbf{f}'(a - z\mathbf{1}).$$

By $(*)$, $\phi(z)(a - z\mathbf{1}) = 0$ for every $z \in \mathbb{C}$. If $|z| > \|a\|$, then $z \notin \sigma(a)$, thus $a - z\mathbf{1}$ is invertible in $\mathcal{A}$. Therefore $\phi(z) = 0$ for $z \in \mathbb{C}$ with $|z| > \|a\|$. Since ϕ is analytic on $\mathbb{C}$, we get $\phi(z) = 0$ for each $z \in \mathbb{C}$. Consequently,

$$\mathbf{f}'(a - z\mathbf{1} + p) = \mathbf{f}'(a - z\mathbf{1}) \quad (z \in \mathbb{C}).$$

Thus, for $z = 0$, $\mathbf{f}'(a + p) = \mathbf{f}'(a)$. Since $a \in \mathcal{A}$ was arbitrary, $p \in P(\mathbf{f}')$. ∎

PROPOSITION 3. *Suppose that f is non-constant. Then there exists $z_0 \in \mathbb{C}$ such that the function $h \in H(\mathbb{C})$ given by $h(z) = f(z + z_0) - f(z_0)$ has only simple zeros.*

Proof. First we show that there is some $c \in f(\mathbb{C})$ such that $f - c$ has only simple zeros. To this end assume to the contrary that for each $c \in f(\mathbb{C})$ the function $f - c$ has a zero of order ≥ 2. Therefore for each $c \in f(\mathbb{C})$ there is $z_c \in \mathbb{C}$ with

$$f(z_c) = c, \qquad f'(z_c) = 0.$$

It follows that $z_{c_1} \neq z_{c_2}$ if $c_1 \neq c_2$. Since f is non-constant, $f(\mathbb{C})$ is a region in $\mathbb{C}$, hence $f(\mathbb{C})$ is uncountable. This shows that the set $\{z_c : c \in f(\mathbb{C})\}$ is uncountable. Hence the set of zeros of f' is uncountable, a contradiction. Thus we have shown that there is some $z_0 \in \mathbb{C}$ such that $f - f(z_0)$ has only simple zeros. If $h \in H(\mathbb{C})$ is defined by $h(z) = f(z + z_0) - f(z_0)$, then h has the desired property. ∎

The following theorem contains a characterization of the periods of $\mathbf{f}$, and is due to E. Vesentini [5]. Vesentini's proof makes extensive use of the Dunford functional calculus and is essentially different from the proof given here.

THEOREM 3 (Vesentini). *Suppose that f is non-constant. Then the following assertions are equivalent*:

(1) $p \in P(\mathbf{f})$.

(2) *There are* $\omega_1, \ldots, \omega_n \in P(f)$ *and* $q_1, \ldots, q_n \in \mathcal{A}^{\mathrm{c}}$ *such that*

$$\mathbf{1} = q_1 + \ldots + q_n, \quad 0 \neq q_j^2 = q_j \quad (j = 1, \ldots, n), \quad q_j q_k = 0 \quad (j \neq k)$$

and

$$p = \omega_1 q_1 + \ldots + \omega_n q_n.$$

Proof. (1)⇒(2). By Proposition 3 there is $z_0 \in \mathbb{C}$ such that the entire function $h(z) = f(z + z_0) - f(z_0)$ has only simple zeros. It is clear that $P(h) = P(f)$. As in the proof of Theorem 2, $p \in P(\mathbf{h})$. By Theorem 2(1) we derive $\sigma(p) = \{\omega_1, \ldots, \omega_n\} \subseteq P(h) = P(f)$. Since $h(p) = \mathbf{h}(p) = \mathbf{h}(0) = h(0)\mathbf{1} = 0$ and h has only simple zeros, Proposition 8.11 in [2] shows that there are idempotents $q_1, \ldots, q_n \in \mathcal{A} \backslash \{0\}$ with

$$q_j q_k = 0 \quad (j \neq k), \quad q_1 + \ldots + q_n = \mathbf{1}, \quad p q_j = \omega_j q_j \quad (j = 1, \ldots, n).$$

Thus $p = p(q_1 + \ldots + q_n) = \omega_1 q_1 + \ldots \omega_n q_n$. From [2, Remark (2), p. 37] it follows that

$$q_j b = b q_j \quad \text{for each } b \in \mathcal{A} \text{ with } pb = bp$$

$(j = 1, \ldots, n)$. By Theorem 2(2) we derive $q_j \in \mathcal{A}^{\mathrm{c}}$ $(j = 1, \ldots, n)$.

(2)⇒(1). Use Theorem 1 to get $\omega_j q_j \in P(\mathbf{f})$ for $j = 1, \ldots, n$. Thus $p \in P(\mathbf{f})$. ■

EXAMPLES. (1) If

$$f(z) = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

then $p \in P(\mathbf{exp})$ if and only if there are $k_1, \ldots, k_n \in \mathbb{Z}$ and $q_1, \ldots, q_n \in \mathcal{A}^{\mathrm{c}}$ with $q_j^2 = q_j$ $(j = 1, \ldots, n)$ and $p = 2k_1 \pi i q_1 + \ldots + 2k_n \pi i q_n$.

(2) If

$$f(z) = \cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!},$$

then $p \in P(\mathbf{cos})$ if and only if there are $k_1, \ldots, k_n \in \mathbb{Z}$ and $q_1, \ldots, q_n \in \mathcal{A}^{\mathrm{c}}$ with $q_j^2 = q_j$ $(j = 1, \ldots, n)$ and $p = 2k_1 \pi q_1 + \ldots + 2k_n \pi q_n$.

(3) Let $w \in \mathbb{C}^m$ denote the vector $(1, \ldots, 1)$, and consider the Banach algebra

$$\mathcal{A} = \{A \in \mathbb{C}^{m \times m} : \exists \lambda \in \mathbb{C} : A^{\mathrm{T}} w = A w = \lambda w\}.$$

Put

$$Q = \frac{1}{m} \begin{pmatrix} 1 & \ldots & 1 \\ \vdots & \vdots & \vdots \\ 1 & \ldots & 1 \end{pmatrix}.$$

Then $\mathbf{1} \neq Q = Q^2$ and $Q \in \mathcal{A}^{\mathrm{c}}$. Therefore $2\pi i Q \in P(\mathbf{exp})$ and $2\pi Q \in P(\mathbf{cos})$. ■

(4) Let X be a complex Banach space and let $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators on X. Assume that $P_0 \in \mathcal{B}(X)$ and $x \in X$, $x \neq 0$ are such that $\exp(P_0)x = x$. We consider the following P_0-invariant closed subspace of X:

$$Y = \overline{[P_0^k x : k \in \mathbb{N}_0]}.$$

Let $P : Y \to Y$ be the restriction of P_0 to Y, and consider the commutative subalgebra of $\mathcal{B}(Y)$ defined by

$$\mathcal{A} = \overline{[P^k : k \in \mathbb{N}_0]}.$$

Obviously $\exp(A + P) = \exp(A)$ for all $A \in \mathcal{A}$, that is, $P \in P(\mathbf{exp})$. Hence there exist $k_1, \ldots, k_n \in \mathbb{Z}$ and $Q_1, \ldots, Q_n \in \mathcal{A}^{\mathrm{c}}$ with $Q_j^2 = Q_j$ $(j = 1, \ldots, n)$ and

$$P = 2k_1\pi i Q_1 + \ldots + 2k_n\pi i Q_n.$$

Moreover $v_j := Q_j x$ satisfies $Pv_j = 2k_j\pi i v_j$ $(j = 1, \ldots, n)$, and

$$x = v_1 + \ldots + v_n.$$

Therefore, the eigenvector x of $\exp(P_0)$ can be written as a finite sum of eigenvectors of P_0.

In this context, let X be a normable complete topological subspace of the Fréchet space $H(\mathbb{C})$ with $f' \in X$ for each $f \in X$. Let $D : X \to X$ denote the differential operator $Df = f'$, and let $g \in X$ with $\omega \in P(g)$, $\omega \neq 0$. Then

$$(\exp(\omega D)g)(z) = g(z + \omega) = g(z) \quad (z \in \mathbb{C}).$$

Thus $g = f_1 + \ldots + f_n$ with $f_1, \ldots, f_n \in X$ satisfying $\omega D f_j = 2k_j\pi i f_j$. Therefore g has the form

$$g(z) = \sum_{j=1}^{n} \gamma_j \exp\left(\frac{2k_j\pi i}{\omega} z\right)$$

with $k_1, \ldots, k_n \in \mathbb{Z}$ and $\gamma_1, \ldots, \gamma_n \in \mathbb{C}$.

In particular, there is no normable complete topological subspace X of $H(\mathbb{C})$ such that $f' \in X$ for all $f \in X$, containing the function

$$\sum_{k=0}^{\infty} \frac{1}{k!} \exp(2k\pi i z),$$

for example.

The next result contains further characterizations of periods of $\mathbf{f}$.

THEOREM 4. *If f is non-constant and $p \in \mathcal{A}$ then the following assertions are equivalent*:

(1) $p \in P(\mathbf{f})$.

(2) $p \in \mathcal{A}^{\mathrm{c}}$, $\sigma(p) \subseteq P(f)$ *and each* $\omega \in \sigma(p)$ *is a simple pole of the resolvent* $r(\lambda, p) = (\lambda\mathbf{1} - p)^{-1}$.

(3) $p \in \mathcal{A}^{\mathrm{c}}$ *and there are* $\omega_1, \ldots, \omega_n \in P(f)$ *such that* $\omega_j \neq \omega_k$ $(j \neq k)$ *and*

$$(p - \omega_1\mathbf{1}) \ldots (p - \omega_n\mathbf{1}) = 0.$$

Proof. (1)⇒(2). By Theorems 2 and 3, $p \in \mathcal{A}^{\mathrm{c}}$, $\sigma(p) = \{\omega_1, \ldots, \omega_n\} \subseteq P(f)$ and there are $q_1, \ldots, q_n \in \mathcal{A}^{\mathrm{c}}$ such that

$$\mathbf{1} = q_1 + \ldots + q_n, \quad 0 \neq q_j^2 = q_j, \quad q_j q_k = 0 \quad (j \neq k)$$

and

$$p = \omega_1 q_1 + \ldots + \omega_n q_n.$$

We can assume that $\omega_j \neq \omega_k$ for $j \neq k$. Define the analytic function $\phi : \mathbb{C} \setminus \sigma(p) \to \mathcal{A}$ by

$$\phi(\lambda) = \sum_{j=1}^{n} \frac{q_j}{\lambda - \omega_j}.$$

Since $p \in \mathcal{A}^{\mathrm{c}}$ and $pq_j = \omega_j q_j$ $(j = 1, \ldots, n)$,

$$(\lambda\mathbf{1} - p)\phi(\lambda) = \phi(\lambda)(\lambda\mathbf{1} - p) = \sum_{j=1}^{n} \frac{\lambda q_j - pq_j}{\lambda - \omega_j} = \sum_{j=1}^{n} q_j = \mathbf{1}.$$

This shows that $\phi(\lambda) = r(\lambda, p)$ $(\lambda \in \mathbb{C} \setminus \sigma(p))$. Since $q_j \neq 0$, it follows that each ω_j is a simple pole of $r(\lambda, p)$.

(2)⇒(1). We have $\sigma(p) = \{\omega_1, \ldots, \omega_n\} \subseteq P(f)$ with $\omega_j \neq \omega_k$ for $j \neq k$. By [2, Proposition 7.9] there exist $q_1, \ldots, q_n \in \mathcal{A}$ such that

$$\mathbf{1} = q_1 + \ldots + q_n, \quad q_j q_k = 0 \quad (j \neq k), \quad 0 \neq q_j^2 = q_j \quad (j = 1, \ldots, n),$$

and

$$\sigma(pq_j) = \{0, \omega_j\} \quad (j = 1, \ldots, n) \qquad \text{if } \; n > 1.$$

Furthermore (see [2, Remark (2), p. 37]), $q_j a = a q_j$ for each $a \in \mathcal{A}$ with $pa = ap$. Since $p \in \mathcal{A}^{\mathrm{c}}$, we derive $q_j \in \mathcal{A}^{\mathrm{c}}$ $(j = 1, \ldots, n)$. Next we show that $pq_1 = \omega_1 q_1$. Let $r > 0$ be so small that $\omega_2, \ldots, \omega_n \notin U = \{\lambda \in \mathbb{C} : |\lambda - \omega_1| < r\}$. Put $\gamma(t) = \omega_1 + re^{it}$ $(t \in [0, 2\pi])$. Then (see [2, Remark (1), p. 37])

$$q_1 = \frac{1}{2\pi i} \int_{\gamma} r(z, p)\, dz.$$

Since ω_1 is a simple pole of $r(\lambda, p)$, the Laurent expansion of $r(\lambda, p)$ on $U \setminus \{\omega_1\}$ has the form

$$r(\lambda, p) = \frac{q_1}{\lambda - \omega_1} + g(\lambda),$$

where $g : U \to \mathcal{A}$ is analytic (see [3, Satz 97.4]). For $\lambda \in U \setminus \{\omega_1\}$ it follows that

$$\mathbf{1} = (\lambda\mathbf{1} - p)r(\lambda, p) = \frac{(\lambda\mathbf{1} - p)q_1}{\lambda - \omega_1} + (\lambda\mathbf{1} - p)g(\lambda),$$

thus

$$(\lambda - \omega_1)\mathbf{1} = (\lambda\mathbf{1} - p)q_1 + (\lambda - \omega_1)(\lambda\mathbf{1} - p)g(\lambda).$$

If $\lambda \to \omega_1$ it follows that $pq_1 = \omega_1 q_1$. A similar proof shows that $pq_j = \omega_j q_j$ for $j = 2, \ldots, n$. Then we have

$$p = p(q_1 + \ldots + q_n) = \omega_1 q_1 + \ldots + \omega_n q_n.$$

Theorem 3 shows now that $p \in P(\mathbf{f})$.

(1)⇒(3). Let $h \in H(\mathbb{C})$ be as in the proof of Theorem 3. Then $P(h) = P(f) = \sigma(p) = \{\omega_1, \ldots, \omega_n\}$ and $h(p) = 0$. Since h has only simple zeros, Proposition 8.11 in [2] shows that

$$(p - \omega_1\mathbf{1}) \ldots (p - \omega_n\mathbf{1}) = 0.$$

From $p \in P(\mathbf{f})$, we get $p \in \mathcal{A}^{\mathrm{c}}$ (Theorem 2).

(3)⇒(1). Let $\varphi(z) = (z - \omega_1) \ldots (z - \omega_n)$ $(z \in \mathbb{C})$. Then $\varphi \in H(\mathbb{C})$, φ has only simple zeros and $\varphi(p) = 0$. Again by [2, Proposition 8.11], there exist non-zero idempotents $q_1, \ldots, q_n \in \mathcal{A}$ such that

$$\mathbf{1} = q_1 + \ldots + q_n, \quad q_j q_k = 0 \quad (j \neq k), \quad pq_j = \omega_j q_j \quad (j = 1, \ldots, n).$$

It follows from [2, Remark (2), p. 37] that $q_j a = a q_j$ for each $a \in \mathcal{A}$ with $ap = pa$. Since $p \in \mathcal{A}^{\mathrm{c}}$, also $q_j \in \mathcal{A}^{\mathrm{c}}$ $(j = 1, \ldots, n)$. From $p = p(q_1 + \ldots + q_n) = \omega_1 q_1 + \ldots + \omega_n q_n$ we see now that $p \in P(\mathbf{f})$ (Theorem 3). ■

Now we consider special types of Banach algebras.

A *representation* of $\mathcal{A}$ on a normed linear space X is a homomorphism of $\mathcal{A}$ into the algebra $\mathcal{B}(X)$ of all bounded linear operators on X. A representation T is said to be *strictly irreducible* if $T \neq 0$ and if $\{0\}$ and X are the only invariant subspaces of X for T (i.e. Y with $T(a)Y \subseteq Y$ for all $a \in \mathcal{A}$). We call $\mathcal{A}$ *primitive* if there is an injective strictly irreducible representation of $\mathcal{A}$ on a Banach space.

Example. If X is a complex Banach space, then $\mathcal{B}(X)$ is a primitive Banach algebra (see [1, F.2.2]).

Proposition 4. *If $\mathcal{A}$ is primitive, then $\mathcal{A}^{\mathrm{c}} = \{\alpha\mathbf{1} : \alpha \in \mathbb{C}\}$.*

Proof. [4, Corollary 2.4.5]. ■

Theorem 5. *Let $\mathcal{A}$ be a primitive Banach algebra and suppose that $f \in H(\mathbb{C})$ is non-constant. Then*

$$P(\mathbf{f}) = \{\omega\mathbf{1} : \omega \in P(f)\}.$$

Proof. That $\{\omega\mathbf{1} : \omega \in P(f)\} \subseteq P(\mathbf{f})$ follows from Corollary 1. Now take $p \in P(\mathbf{f})$. By Theorem 2(2) and Proposition 4, $p = \omega\mathbf{1}$ for some $\omega \in \mathbb{C}$. Theorem 2(1) gives

$$\{\omega\} = \sigma(p) \subseteq P(f).$$

Thus $\omega \in P(f)$. ∎

REMARK. There is an elementary proof of Theorem 5 if $\mathcal{A}$ is the Banach algebra $\mathcal{B}(X)$ (X a complex Banach space): Because of Theorem 3 it suffices to show that if $0 \neq Q^2 = Q \in \mathcal{B}(X)^{\mathrm{c}}$, then $Q = I$ (where I denotes the identity on X). Therefore let $0 \neq Q^2 = Q \in \mathcal{B}(X)^{\mathrm{c}}$. Then

$$X = Q(X) \oplus N(Q),$$

where $Q(X) = \{Qx : x \in X\} = \{x \in X : Qx = x\}$ and $N(Q) = \{x \in X : Qx = 0\}$. We have to show that $N(Q) = \{0\}$. Assume to the contrary that there is $z_0 \in N(Q)$ with $z_0 \neq 0$. Since $Q \neq 0$ there exists $y_0 \in Q(X)$ such that $y_0 \neq 0$. Now put $x_0 = y_0 + z_0$. Since $z_0 \neq 0$, $x_0 \notin Q(X)$. Furthermore, $Q(X)$ is a closed subspace of X, thus, by the Hahn–Banach Theorem, there is a bounded linear functional φ on X with

$$\varphi(x_0) \neq 0, \qquad \varphi(Qx) = 0 \quad \text{for all } x \in X.$$

Now define the operator $A \in \mathcal{B}(X)$ by

$$Ax = \varphi(x)x_0 \quad (x \in X).$$

Then $AQx_0 = \varphi(Qx_0)\, x_0 = 0$ and $QAx_0 = \varphi(x_0)\, Qx_0$. Since $Q \in \mathcal{B}(X)^{\mathrm{c}}$ and $\varphi(x_0) \neq 0$, we get $Qx_0 = 0$. From $x_0 = y_0 + z_0$ and $z_0 \in N(Q)$ it follows that $Qy_0 = 0$, thus $y_0 = Qy_0 = 0$, a contradiction. ∎

PROPOSITION 5. *Let $\mathcal{A}$ be a C^*-algebra and let $q \in \mathcal{A}^{\mathrm{c}}$ and $q^2 = q$. Then $q^* = q$.*

Proof. By [1, BA.4.3] there exists $e = e^2 = e^* \in \mathcal{A}$ such that $qe = e$ and $eq = q$. Since $q \in \mathcal{A}^{\mathrm{c}}$, we have $qe = eq$, thus $q = e$ and therefore $q^* = q$. ∎

For the next result observe that by Corollary 1 and Theorem 2, we have $P(f) = \{0\} \Leftrightarrow P(\mathbf{f}) = \{0\}$.

COROLLARY 2. *Let $\mathcal{A}$ be a C^*-algebra and suppose that f is non-constant and that $P(f) \neq \{0\}$. Then*:

(1) *Each $p \in P(\mathbf{f})$ is normal.*
(2) *$P(f) \subseteq \mathbb{R} \Leftrightarrow p = p^*$ for each $p \in P(\mathbf{f})$.*
(3) *$P(f) \subseteq i\mathbb{R} \Leftrightarrow p = -p^*$ for each $p \in P(\mathbf{f})$.*

Proof. For (1), notice that since $p \in \mathcal{A}^{\mathrm{c}}$ (Theorem 2), $pp^* = p^*p$. For (2) and (3) let $\omega_0 \in P(f) \setminus \{0\}$ with $|\omega_0|$ minimal. If $p \in P(\mathbf{f})$ then, by

Theorem 3, there are $k_1,\dots,k_n \in \mathbb{Z}$ and $q_1,\dots,q_n \in \mathcal{A}^{\mathrm{c}}$ with $q_j^2 = q_j$ ($j = 1,\dots,n$) and

$$p = \omega_0(k_1q_1 + \ldots + k_nq_n).$$

Proposition 5 gives $p^* = \overline{\omega}_0(k_1q_1 + \ldots + k_nq_n)$, thus

$$p - p^* = (\omega_0 - \overline{\omega}_0)(k_1q_1 + \ldots + k_nq_n),$$
$$p + p^* = (\omega_0 + \overline{\omega}_0)(k_1q_1 + \ldots + k_nq_n).$$

This shows that (2) and (3) hold. ∎

Corollary 3. *Let $\mathcal{A}$ and f be as in Corollary 2. Then the following assertions are equivalent*:

(1) *$P(\mathbf{f})$ is a $*$-subset (i.e., $p \in P(\mathbf{f})$ implies $p^* \in P(\mathbf{f})$).*
(2) *$P(f) \subseteq \mathbb{R}$ or $P(f) \subseteq i\mathbb{R}$.*

Proof. (1)⇒(2). Take $\omega_0 \in P(f) \setminus \{0\}$. By Corollary 1, $\omega_0\mathbf{1} \in P(\mathbf{f})$, hence $\overline{\omega}_0\mathbf{1} \in P(\mathbf{f})$. Theorem 2(1) gives

$$\sigma(\overline{\omega}_0\mathbf{1}) \subseteq P(f);$$

thus $\overline{\omega}_0 \in P(f)$. It follows that $\overline{\omega}_0 = \omega_0$ or $\overline{\omega}_0 = -\omega_0$.

(2)⇒(1). Use Corollary 2. ∎

Corollary 4. *Assume that $\mathcal{A}$ and f are as in Corollary 2. If the coefficients $a_0, a_1, \ldots$ of f are real, then $P(\mathbf{f})$ is a $*$-subset.*

Proof. For $a \in \mathcal{A}$ we have $\mathbf{f}(a^*) = \sum_{n=0}^{\infty} a_n(a^*)^n$, thus $\mathbf{f}(a^*)^* = \mathbf{f}(a)$. Now take $p \in P(\mathbf{f})$. Then, for each $a \in \mathcal{A}$,

$$\mathbf{f}(a+p^*) = \mathbf{f}((a^*)^* + p^*) = \mathbf{f}((a^*+p)^*) = \mathbf{f}(a^*+p)^* = \mathbf{f}(a^*)^* = \mathbf{f}(a);$$

thus $p^* \in P(\mathbf{f})$. ∎

In C^*-algebras each $p \in P(\mathbf{f})$ is normal. The following corollary shows that in a general Banach algebra, elements in $P(\mathbf{f})$ share some properties of normal operators (on complex Hilbert spaces) with closed range.

Corollary 5. *For $p \in P(\mathbf{f})$ we have*:

(1) *There is $q \in \mathcal{A}$ with $pqp = p$ and $qpq = q$ (hence p has a pseudo-inverse).*
(2) *$p\mathcal{A} = \{pa : a \in \mathcal{A}\}$ is closed.*
(3) *$\mathcal{A} = p\mathcal{A} \oplus \{a \in \mathcal{A} : pa = 0\}$.*
(4) *If $a \in \mathcal{A}$ and $p^2a = 0$, then $pa = 0$ (hence the ascent of p is ≤ 1).*
(5) *$p^2\mathcal{A} = p\mathcal{A}$ (hence the descent of p is ≤ 1).*

Proof. By Theorems 2 and 3, $p \in \mathcal{A}^{\mathrm{c}}$, $\sigma(p) = \{\omega_1,\dots,\omega_n\} \subseteq P(f)$ ($\omega_j \ne \omega_k$ for $j \ne k$) and there are $q_1,\dots,q_n \in \mathcal{A}^{\mathrm{c}}$ with

$$\mathbf{1} = q_1 + \ldots + q_n, \quad 0 \ne q_j = q_j^2 \quad (j = 1,\dots,n),$$
$$q_jq_k = 0 \quad (j \ne k), \quad p = \omega_1q_1 + \ldots + \omega_nq_n.$$

If $0 \notin \sigma(p)$, then we are done. Hence let $0 \in \sigma(p)$. We can assume that $\omega_1 = 0$. Thus $p = \omega_2 q_2 + \ldots + \omega_n q_n$.

(1) Put $q = \omega_2^{-1} q_2 + \ldots + \omega_n^{-1} q_n$. Then $pq = q_2 + \ldots + q_n = \mathbf{1} - q_1$. Thus $pqp = (\mathbf{1} - q_1)p = p - pq_1 = p - \omega_1 q_1 = p$ and $qpq = q(\mathbf{1} - q_1) = q - qq_1 = q$.

(2) Put $r = pq$. Then $r^2 = pqpq = pq = r$, thus $r\mathcal{A}$ is closed. But

$$r\mathcal{A} = pq\mathcal{A} \subseteq p\mathcal{A} = pqp\mathcal{A} \subseteq pq\mathcal{A} = r\mathcal{A},$$

hence $p\mathcal{A} = r\mathcal{A}$.

(3) Since $r^2 = r$, we have $\mathcal{A} = r\mathcal{A} \oplus (\mathbf{1} - r)\mathcal{A} = p\mathcal{A} \oplus (\mathbf{1} - r)\mathcal{A}$. It is easy to see that $(\mathbf{1} - r)\mathcal{A} = \{a \in \mathcal{A} : pa = 0\}$.

(4) Let $a \in \mathcal{A}$ and $p^2 a = 0$. Then $pa \in p\mathcal{A} \cap (\mathbf{1} - r)\mathcal{A} = \{0\}$.

(5) It is clear that $p^2\mathcal{A} \subseteq p\mathcal{A}$. Since $pqp = p$ and $p \in \mathcal{A}^{\mathrm{c}}$, it follows that $p\mathcal{A} = p^2 q\mathcal{A} \subseteq p^2\mathcal{A}$. ∎

References

[1] B. A. Barnes, G. J. Murphy, M. R. F. Smyth and T. T. West, *Riesz and Fredholm Theory in Banach Algebras*, Pitman, 1982.

[2] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer, 1973.

[3] H. Heuser, *Funktionalanalysis*, Teubner, 1986.

[4] C. E. Rickart, *General Theory of Banach Algebras*, Van Nostrand, 1960.

[5] E. Vesentini, *Periodic entire functions and reducible Banach algebras*, in: Several Complex Variables I (College Park, MD, 1970), Springer, 1970, 203–211.

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