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Character contractibility of Banach algebras and homological properties of Banach modules

by

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Abstract. Let $\mathcal{A}$ be a Banach algebra and let ϕ be a nonzero character on $\mathcal{A}$. We give some necessary and sufficient conditions for the left ϕ-contractibility of $\mathcal{A}$ as well as several hereditary properties. We also study relations between homological properties of some Banach left $\mathcal{A}$-modules, the left ϕ-contractibility and the right ϕ-amenability of $\mathcal{A}$. Finally, we characterize the left character contractibility of various Banach algebras related to locally compact groups.

1. Introduction. A Banach algebra $\mathcal{A}$ is called *amenable* if the first cohomology group $H^1(\mathcal{A}, \mathcal{X}^*)$ vanishes for all Banach $\mathcal{A}$-bimodules $\mathcal{X}$. Johnson [J] showed that the amenability of the group algebra $L^1(G)$ for a locally compact group G is equivalent to the amenability of G; however, this equivalence does not remain true for the convolution semigroup algebra $\ell^1(S)$ of a discrete semigroup S; consider, for example the additive semigroup $\mathbb{N}$ of natural numbers.

Motivated by these considerations, Lau [L] introduced and investigated a large of class of Banach algebras which he called F-algebras; that is, Banach algebras $\mathcal{L}$ which are the preduals of W^*-algebras $\mathcal{M}$ such that the identity element u of $\mathcal{M}$ is a character on $\mathcal{L}$. Later, in [PI], F-algebras were termed *Lau algebras*. A Lau algebra $\mathcal{L}$ is said to be *left amenable* if $H^1(\mathcal{L}, \mathcal{X}^*) = \{0\}$ for all Banach $\mathcal{L}$-bimodules $\mathcal{X}$ with the left action defined by $l \cdot x = u(l)x$ for all $l \in \mathcal{L}$ and $x \in \mathcal{X}$; Lau [L] proved that $\ell^1(S)$ is left amenable if and only if S is left amenable (see also Lau and Wong [LW]).

Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A}) \cup \{0\}$, where $\Delta(\mathcal{A})$ is the spectrum of $\mathcal{A}$ consisting of all characters from $\mathcal{A}$ into the complex numbers. The Banach algebra $\mathcal{A}$ is called *left ϕ-amenable* if $H^1(\mathcal{A}, \mathcal{X}^*)$ vanishes for all Banach $\mathcal{A}$-bimodules $\mathcal{X}$ for which the left module action of $\mathcal{A}$ on $\mathcal{X}$ is

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defined by $a \cdot x = \phi(a)x$ ($a \in \mathcal{A}$, $x \in \mathcal{X}$). This concept of amenability has recently been introduced and studied by Kaniuth, Lau and Pym [KLP1], [KLP2] under the name of ϕ-amenability.

The Banach algebra $\mathcal{A}$ is called *left ϕ-contractible* if $H^1(\mathcal{A}, \mathcal{X})$ vanishes for any Banach $\mathcal{A}$-bimodule $\mathcal{X}$ for which the right module action of $\mathcal{A}$ on $\mathcal{X}$ is defined by $x \cdot a = \phi(a)x$ ($a \in \mathcal{A}$, $x \in \mathcal{X}$); moreover, $\mathcal{A}$ is called *left character contractible* if it is left ϕ-contractible for all $\phi \in \Delta(\mathcal{A}) \cup \{0\}$. These notions were recently introduced and studied by Hu, Monfared and Traynor [HMT] as right ϕ-contractibility and right character contractibility respectively; see also Monfared [M1]. Let us point out that left ϕ-contractibility is significantly stronger than left ϕ-amenability.

On the other hand, there is a considerable progress in the study of homological properties of Banach left $\mathcal{A}$-modules defined by Helemskii [H2]; in particular, homological properties of several Banach left $L^1(G)$-modules have recently been described by several authors: see for example [BNS], [DP], and [NS].

In this paper, we study the relation between the left character contractibility of the Banach algebra $\mathcal{A}$ and homological properties of Banach left $\mathcal{A}$-modules. In Section 2, we show that the left ϕ-contractibility of $\mathcal{A}$ is equivalent to the existence of a topological left invariant ϕ-mean in $\mathcal{A}$, that is, an element $m \in \mathcal{A}$ such that $\phi(m) = 1$ and $am = \phi(a)m$ for all $a \in \mathcal{A}$. We also characterize the left ϕ-contractibility of $\mathcal{A}$ in terms of derivations from $\mathcal{A}$ into certain Banach $\mathcal{A}$-bimodules.

In Section 3, we study hereditary properties of left ϕ-contractibility. We then investigate the left ϕ-contractibility and left character contractibility of unitizations, second duals and projective tensor products of Banach algebras.

In Section 4, we show that the projectivity of a Banach left $\mathcal{A}$-module $\mathcal{E}$ with $a \cdot \xi = \phi(a)\xi$ ($a \in \mathcal{A}$, $\xi \in \mathcal{E}$) is equivalent to the left ϕ-contractibility of $\mathcal{A}$.

In Section 5, we investigate the injectivity of a Banach left $\mathcal{A}$-module $\mathcal{E}$ with $a \cdot \xi = \phi(a)\xi$ ($a \in \mathcal{A}$, $\xi \in \mathcal{E}$) in terms of the right ϕ-amenability of $\mathcal{A}$. We also show that the injectivity of a dual Banach left $\mathcal{A}$-module $\mathcal{E}$ with $a \cdot \xi = \phi(a)\xi$ ($a \in \mathcal{A}$, $\xi \in \mathcal{E}$) is equivalent to the right ϕ-amenability of $\mathcal{A}$. We then show that the injectivity of such Banach left $\mathcal{A}$-modules is a sufficient, but not necessary condition for the right ϕ-amenability of $\mathcal{A}$.

Finally, in Section 6, we study the left character contractibility of several Banach algebras related to a locally compact group.

2. Characterization of left ϕ-contractibility. Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. A *derivation* is a linear map $D : \mathcal{A} \to \mathcal{X}$ such that $D(ab) = a \cdot D(b) + D(a) \cdot b$ for all $a, b \in \mathcal{A}$. For $x \in \mathcal{X}$, the linear map $\mathrm{ad}_x : \mathcal{A} \to \mathcal{X}$ defined by $\mathrm{ad}_x(a) = a \cdot x - x \cdot a$ for all $a \in \mathcal{A}$ is a derivation. A derivation $D : \mathcal{A} \to \mathcal{X}$ is *inner* if there is $x \in \mathcal{X}$ such that $D = \mathrm{ad}_x$.

Let $\phi \in \Delta(\mathcal{A})$ and recall from Hu, Monfared and Traynor [HMT] that $\mathcal{A}$ is left ϕ-contractible if $H^1(\mathcal{A}, \mathcal{X}) = \{0\}$ for all Banach $\mathcal{A}$-bimodules $\mathcal{X}$ with $x \cdot a = \phi(a)x$ for all $a \in \mathcal{A}$ and $x \in \mathcal{X}$, that is, every continuous derivation from $\mathcal{A}$ into $\mathcal{X}$ is inner.

Let us recall that the second dual $\mathcal{A}^{**}$ of $\mathcal{A}$ equipped with the first Arens product $\odot$ defined by the equations

$$(M \odot N)(f) = M(Nf), \quad (Nf)(a) = N(fa), \quad (fa)(b) = f(ab)$$

for all $M, N \in \mathcal{A}^{**}$, $f \in \mathcal{A}^*$ and $a, b \in \mathcal{A}$ is a Banach algebra. It is well-known from Kaniuth, Lau and Pym [KLP1, Theorem 1.1] that the left ϕ-amenability of $\mathcal{A}$ is equivalent to the existence of a *topological left invariant ϕ-mean* in $\mathcal{A}^{**}$, that is, a linear functional M in $\mathcal{A}^{**}$ satisfying $M(\phi) = 1$ and $a \odot M = \phi(a)M$ for all $a \in \mathcal{A}$. Also, they proved that the left ϕ-amenability of $\mathcal{A}$ is equivalent to the existence of a *bounded topological left invariant approximate ϕ-mean* in $\mathcal{A}$, that is, a bounded net $(a_\alpha) \subseteq \mathcal{A}$ such that $\phi(a_\alpha) \to 1$ and $\|aa_\alpha - \phi(a)a_\alpha\| \to 0$ for all $a \in \mathcal{A}$; see [KLP1, Theorem 1.4].

Our first result characterizes the left ϕ-contractibility of $\mathcal{A}$ in terms of topological left invariant ϕ-means in $\mathcal{A}$; we shall frequently use it without explicit reference.

THEOREM 2.1. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. Then the following assertions are equivalent:*

(i) *$\mathcal{A}$ is left ϕ-contractible.*
(ii) *There is a topological left invariant ϕ-mean in $\mathcal{A}$.*

Proof. (i)⇒(ii). Choose $b \in \mathcal{A}$ with $\phi(b) = 1$. Note that $\ker(\phi)$ is a Banach $\mathcal{A}$-bimodule by taking the right action to be $x \cdot a = \phi(a)x$ for all $x \in \ker(\phi)$ and $a \in \mathcal{A}$ and taking the left action to be the natural one. Then the formula $D(a) = ab - ba$ for all $a \in \mathcal{A}$ defines a derivation D from $\mathcal{A}$ into $\ker(\phi)$. Therefore there is $c \in \ker(\phi)$ such that $D = \mathrm{ad}_c$. Now, consider $m = b - c$ and note that $\phi(m) = 1$ and $am = \phi(a)m$ for all $a \in \mathcal{A}$.

(ii)⇒(i). We have to show that any continuous derivation $D : \mathcal{A} \to \mathcal{X}$ with $x \cdot a = \phi(a)x$ for all $x \in \mathcal{X}$ and $a \in \mathcal{A}$ is inner. To that end, choose a topological left invariant ϕ-mean m in $\mathcal{A}$, and note that

$$\begin{aligned} a \cdot D(m) - D(m) \cdot a &= D(a \cdot m) - D(a) \cdot m - D(m) \cdot a \\ &= \phi(a)D(m) - D(a)\phi(m) - \phi(a)D(m) = -D(a) \end{aligned}$$

for all $a \in \mathcal{A}$. This means that $D = \mathrm{ad}_{-D(m)}$ as required. ∎

As a consequence of Theorem 2.1, we have the following result in which we consider the Banach $\mathcal{A}$-bimodule $\ker(\phi)$ as in the proof of Theorem 2.1.

COROLLARY 2.2. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A}) \cup \{0\}$. Then the following statements are equivalent:*

(i) *$\mathcal{A}$ is left ϕ-contractible.*
(ii) *Any continuous derivation $D : \mathcal{A} \to \ker(\phi)$ is inner.*

Let $\Theta : \mathcal{A} \to \mathcal{B}$ be a Banach algebra homomorphism and $\phi \in \Delta(\mathcal{A}) \cup \{0\}$. Then $\mathcal{B}$ is a Banach $\mathcal{A}$-bimodule under the following module actions:

$$a \cdot b = \Theta(a)b, \quad b \cdot a = \phi(a)b \quad (a \in \mathcal{A},\, b \in \mathcal{B}).$$

We denote the above Banach $\mathcal{A}$-bimodule by $\mathcal{B}_\phi^\Theta$.

THEOREM 2.3. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A}) \cup \{0\}$. Then the following statements are equivalent:*

(i) *$\mathcal{A}$ is left ϕ-contractible.*
(ii) *For every Banach algebra $\mathcal{B}$ and every homomorphism $\Theta : \mathcal{A} \to \mathcal{B}$, any continuous derivation $D : \mathcal{A} \to \mathcal{B}_\phi^\Theta$ is inner.*
(iii) *For every Banach algebra $\mathcal{B}$ and every injective homomorphism $\Theta : \mathcal{A} \to \mathcal{B}$, any continuous derivation $D : \mathcal{A} \to \mathcal{B}_\phi^\Theta$ is inner.*

Proof. The implications (i)⇒(ii)⇒(iii) are trivial. We show that (iii) implies (i). Suppose that (iii) holds and let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule with the right action $x \cdot a = \phi(a)x$ for all $x \in \mathcal{X}$ and $a \in \mathcal{A}$. Also, let $D : \mathcal{A} \to \mathcal{X}$ be a continuous derivation. Consider the module extension Banach algebra $\mathcal{X} \oplus_1 \mathcal{A}$, that is, the space $\mathcal{X} \oplus \mathcal{A}$ endowed with the norm

$$\|(x, a)\| = \|x\| + \|a\|$$

and the product

$$(x, a) \cdot_1 (y, b) = (x \cdot b + a \cdot y, ab)$$

for all $x, y \in \mathcal{X}$ and $a, b \in \mathcal{A}$. Obviously the map $\Theta : \mathcal{A} \to \mathcal{X} \oplus_1 \mathcal{A}$ defined by $\Theta(a) = (0, a)$ for all $a \in \mathcal{A}$ is an injective Banach algebra homomorphism. Now, define $D_1 : \mathcal{A} \to \mathcal{X} \oplus_1 \mathcal{A}$ by $D_1(a) = (D(a), 0)$ for $a \in \mathcal{A}$. Then

$$\begin{aligned} D_1(ab) &= (D(ab), 0) = (\phi(b)D(a) + aD(b), 0) \\ &= \phi(b)D(a, 0) + (0, a) \cdot_1 (D(b), 0) \\ &= \phi(b)D_1(a) + \Theta(a) \cdot_1 D_1(b). \end{aligned}$$

Thus D_1 is a derivation from $\mathcal{A}$ into $(\mathcal{X} \oplus_1 \mathcal{A})_\phi^\Theta$, and so D_1 is inner by assumption. That is, there exist $a_0 \in \mathcal{A}$ and $x_0 \in \mathcal{X}$ such that $D_1 = \mathrm{ad}_{(x_0, a_0)}$. So, for each $a \in \mathcal{A}$ we have

$$\begin{aligned} (D(a), 0) &= \mathrm{ad}_{(x_0, a_0)}(a) = \Theta(a) \cdot_1 (x_0, a_0) - \phi(a)(x_0, a_0) \\ &= (0, a) \cdot_1 (x_0, a_0) - \phi(a)(x_0, a_0) = (ax_0 - \phi(a)x_0, aa_0 - \phi(a)a_0). \end{aligned}$$

Therefore $D = \mathrm{ad}_{x_0}$ and so for each Banach $\mathcal{A}$-bimodule $\mathcal{X}$ with the right action $x \cdot a = \phi(a)x$ for all $a \in \mathcal{A}$ and $x \in \mathcal{X}$, any continuous derivation $D : \mathcal{A} \to \mathcal{X}$ is inner, that is, $\mathcal{A}$ is left ϕ-contractible. ■

3. Hereditary properties of left ϕ-contractibility. Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. We denote by $\mathcal{A}^\sharp$ the unitization of $\mathcal{A}$ and by $\phi^\sharp \in \Delta(\mathcal{A}^\sharp)$ the unique extension of ϕ.

THEOREM 3.1. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. Then $\mathcal{A}$ is left ϕ-contractible if and only if $\mathcal{A}^\sharp$ is left $\phi^\sharp$-contractible.*

Proof. Suppose that $\mathcal{A}$ is left ϕ-contractible. Then there is a topological left invariant ϕ-mean $m \in \mathcal{A}$. Thus for each $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, we have $\phi^\sharp((m,0)) = \phi(m) = 1$ and

$$(a,\lambda)(m,0) = (am + \lambda m, 0) = \phi^\sharp(a,\lambda)(m,0).$$

That is, $(m,0)$ is a topological left invariant $\phi^\sharp$-mean in $\mathcal{A}^\sharp$.

Conversely, suppose that $\mathcal{A}^\sharp$ is left $\phi^\sharp$-contractible. Then there exists a topological left invariant $\phi^\sharp$-mean $(m,\alpha) \in \mathcal{A}^\sharp$, that is,

$$\phi^\sharp(m,\alpha) = 1 \quad \text{and} \quad (a,\lambda)(m,\alpha) = \phi^\sharp((a,\lambda))(m,\alpha)$$

for each $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. It follows that

$$\phi(a) + \alpha = 1, \qquad \alpha a = 0 \quad \text{and} \quad ma = \phi(a)m.$$

This implies that m is a topological left invariant ϕ-mean in $\mathcal{A}$. ■

It is well-known that $\Delta(\mathcal{A}^\sharp) = \{\phi^\infty\} \cup \{\phi^\sharp : \phi \in \Delta(\mathcal{A})\}$, where $\phi^\infty(a,\lambda) = \lambda$ for all $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$.

THEOREM 3.2. *Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}$ has a right identity if and only if $\mathcal{A}^\sharp$ is left ϕ^∞-contractible.*

Proof. Suppose that $\mathcal{A}$ has a right identity e. Then for each $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ we have $\phi^\infty(-e,1) = 1$ and

$$(a,\lambda)(-e,1) = (-ae - \lambda e + a, \lambda) = (-\lambda e, 1) = \phi^\infty(a,\lambda)(-e,1).$$

Thus $(-e,1)$ is a topological left invariant ϕ^∞-mean in $\mathcal{A}^\sharp$, and so $\mathcal{A}^\sharp$ is left ϕ^∞-contractible.

Conversely, suppose that $\mathcal{A}^\sharp$ is left ϕ^∞-contractible and choose a topological left invariant ϕ^∞-mean $(m,\alpha) \in \mathcal{A}^\sharp$. Thus for each $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$,

$$\phi^\infty((m,\alpha)) = 1 \quad \text{and} \quad (a,\lambda)(m,\alpha) = \lambda(m,\alpha).$$

It follows that $am + \alpha a = 0$ and $\alpha = 1$. So, $-m$ is a left identity for $\mathcal{A}$. ■

COROLLARY 3.3. *Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}$ is left character contractible if and only if $\mathcal{A}^\sharp$ is left character contractible.*

This result follows from Theorems 3.1 and 3.2, and the following result from [HMT, Corollary 2.5] and [J, Proposition 1.5].

Proposition 3.4. *Let $\mathcal{A}$ be a Banach algebra. Then:*

(i) *$\mathcal{A}$ is left 0-amenable if and only if it has a bounded right approximate identity.*
(ii) *$\mathcal{A}$ is left 0-contractible if and only if it has a right identity.*

Now, let us give another characterization of the left character amenability of $\mathcal{A}$ by the left character contractibility of its second dual.

Proposition 3.5. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. Then $\mathcal{A}^{**}$ is left ϕ-contractible if and only if $\mathcal{A}$ is left ϕ-amenable.*

Proof. The "only if" part is trivial. For the converse, suppose that $\mathcal{A}$ is left ϕ-amenable. Then there is $M \in \mathcal{A}^{**}$ such that $M(\phi) = 1$ and $a \odot M = \phi(a)M$ for all $a \in \mathcal{A}$. By the continuity of the first Arens product, we have

$$M(\phi) = 1 \quad \text{and} \quad N \odot M = N(\phi)M$$

for all $N \in \mathcal{A}^{**}$. Therefore M is a topological left invariant ϕ-mean in $\mathcal{A}^{**}$. Hence $\mathcal{A}^{**}$ is left ϕ-contractible. ∎

We can now introduce a large family of Banach algebras on which left ϕ-amenability coincides with left ϕ-contractibility.

Corollary 3.6. *Let $\mathcal{A}$ be a Banach algebra which is a left or right ideal in $\mathcal{A}^{**}$ and $\phi \in \Delta(\mathcal{A})$. Then $\mathcal{A}$ is left ϕ-contractible if and only if $\mathcal{A}$ is left ϕ-amenable.*

Proof. In view of Proposition 3.5, we only need to note that if M is a topological left invariant mean in $\mathcal{A}^{**}$ and a_0 is an element of $\mathcal{A}$ with $\phi(a_0) = 1$, then $a_0 \odot M$ and $M \odot a_0$ are topological left invariant means in $\mathcal{A}$. This means that $\mathcal{A}$ is left ϕ-contractible. ∎

We say that $\mathcal{A}$ is *left character amenable* if it is left ϕ-amenable for all $\phi \in \Delta(\mathcal{A}) \cup \{0\}$; this is the same as right character amenability in [HMT] (see also [M2]).

It is well-known that $\mathcal{A}$ has a bounded right approximate identity if and only if $\mathcal{A}^{**}$ has a right identity; see for example [BD, Proposition III.28.7]. So, as a consequence of Propositions 3.4 and 3.5, we have the following.

Corollary 3.7. *Let $\mathcal{A}$ be a Banach algebra. If $\mathcal{A}^{**}$ is left character contractible, then $\mathcal{A}$ is left character amenable.*

In Section 6, we will see that the converse of Corollary 3.7 is not valid. Our next result describes an interaction between the ϕ-contractibility of a Banach algebra and of its closed ideals.

PROPOSITION 3.8. *Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{I}$ be a closed two-sided ideal of $\mathcal{A}$ and $\phi \in \Delta(\mathcal{A})$ with $\mathcal{I} \nsubseteq \ker(\phi)$. Then the following statements are equivalent:*

(i) *$\mathcal{I}$ is left $\phi|_{\mathcal{I}}$-contractible.*
(ii) *$\mathcal{A}$ is left ϕ-contractible.*

Proof. First, suppose that (i) holds. Then there exists an element $m_0 \in \mathcal{I}$ such that $\phi|_{\mathcal{I}}(m_0) = 1$ and $bm_0 = \phi|_{\mathcal{I}}(b)m_0$ for all $b \in \mathcal{I}$. Fix $\iota_0 \in \mathcal{I}$ such that $\phi|_{\mathcal{I}}(\iota_0) = 1$ and set $m := \iota_0 m_0$. Now, $\phi(m) = \phi|_{\mathcal{I}}(\iota_0 m_0) = 1$ and for each $a \in \mathcal{A}$ we have

$$\begin{aligned} am - \phi(a)m &= a\iota_0 m_0 - \phi(a)\iota_0 m_0 = (a\iota_0)m_0 - \phi(a)\phi|_{\mathcal{I}}(\iota_0)m_0 \\ &= \phi(a)\phi|_{\mathcal{I}}(\iota_0)m_0 - \phi(a)\phi(\iota_0)m_0 = 0. \end{aligned}$$

Thus $\mathcal{A}$ is left ϕ-contractible.

Conversely, suppose that (ii) holds and let $m \in \mathcal{A}$ be such that

$$\phi(m) = 1 \quad \text{and} \quad am = \phi(a)m$$

for all $a \in \mathcal{A}$. Choose $\iota_0 \in \mathcal{I}$ such that $\phi(\iota_0) = 1$ and define $m_0 := \iota_0 m$. Thus $m_0 \in \mathcal{I}$, $\phi|_{\mathcal{I}}(m_0) = \phi(\iota_0 m) = \phi(m) = 1$ and for each $b \in \mathcal{I}$ we have

$$bm_0 - \phi|_{\mathcal{I}}(b)m_0 = b\iota_0 m - \phi|_{\mathcal{I}}(b)\iota_0 m = \phi|_{\mathcal{I}}(b)\phi(\iota_0)m - \phi|_{\mathcal{I}}(b)\phi(\iota_0)m = 0;$$

therefore $\mathcal{I}$ is left $\phi|_{\mathcal{I}}$-contractible. ■

As a consequence of Proposition 3.8, we have the following result.

COROLLARY 3.9. *Let $\mathcal{A}$ be a left character contractible Banach algebra and let $\mathcal{I}$ be a closed two-sided ideal of $\mathcal{A}$. Then $\mathcal{I}$ is left character contractible if and only if it has a right identity.*

This result was also obtained by Hu, Monfared and Traynor [HMT, Lemma 6.8], with a different proof. Before we give our next result, note that if $\mathcal{I}$ is a closed two-sided ideal of $\mathcal{A}$, then there is a unique $\psi \in \Delta(\mathcal{A}/\mathcal{I})$ with $\psi \circ q = \phi$ if and only if $\mathcal{I} \subseteq \ker(\phi)$, where $q : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ is the canonical epimorphism.

PROPOSITION 3.10. *Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{I}$ be a closed two-sided ideal of $\mathcal{A}$ and $\phi \in \Delta(\mathcal{A})$ with $\mathcal{I} \subseteq \ker(\phi)$. Suppose that $\mathcal{I}$ has a right identity and that $\mathcal{A}/\mathcal{I}$ is ψ-contractible, where $\psi \in \Delta(\mathcal{A}/\mathcal{I})$ with $\psi \circ q = \phi$. Then $\mathcal{A}$ is ϕ-contractible.*

Proof. Since $\mathcal{A}/\mathcal{I}$ is ψ-contractible, there exists $n \in \mathcal{A}$ such that $\psi(n+\mathcal{I}) = 1$ and $an + \mathcal{I} = \psi(a+\mathcal{I})n + \mathcal{I}$ for all $a \in \mathcal{A}$. Set $m := n - ne$, where e is a right identity for $\mathcal{I}$. Since $ne \in \mathcal{I}$ and $\phi(n) = 1$, we have $\phi(m) = 1$ and

$$\begin{aligned} am - \phi(a)m &= an - ane - \phi(a)n + \phi(a)ne \\ &= an - \phi(a)n - (an - \phi(a)n)e \end{aligned}$$

for all $a \in \mathcal{A}$. Since $an - \phi(a)n \in \mathcal{I}$, it follows that $am = \phi(a)m$. That is, m is a topological left invariant mean in $\mathcal{A}$, and so $\mathcal{A}$ is ϕ-contractible. ∎

Lemma 3.11. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $\psi \in \Delta(\mathcal{B})$. If there is a continuous epimorphism $\Theta : \mathcal{A} \to \mathcal{B}$ and $\mathcal{A}$ is $\psi \circ \Theta$-contractible, then $\mathcal{B}$ is ψ-contractible.*

Proof. By assumption, there is $m \in \mathcal{A}$ such that $\psi(\Theta(m)) = 1$ and $am = \psi(\Theta(a))m$ for all $a \in \mathcal{A}$. Define $n := \Theta(m)$ and note that $\psi(n) = 1$. Also, for each $b \in \mathcal{B}$, there exists $a \in \mathcal{A}$ such that $\Theta(a) = b$, and hence

$$bn - \psi(b)n = \Theta(a)\Theta(m) - \psi(\Theta(a))\Theta(m) = \Theta(am - \psi(\Theta(a))m) = 0.$$

So, n is a topological left invariant ψ-mean in $\mathcal{B}$; hence $\mathcal{B}$ is ψ-contractible. ∎

As a consequence of Lemma 3.11, we have the following result.

Proposition 3.12. *Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{I}$ be a closed two-sided ideal of $\mathcal{A}$ and $\phi \in \Delta(\mathcal{A})$ with $\mathcal{I} \subseteq \ker(\phi)$. Suppose that $\mathcal{A}$ is ϕ-contractible. Then $\mathcal{A}/\mathcal{I}$ is ψ-contractible, where $\psi \in \Delta(\mathcal{A}/\mathcal{I})$ with $\psi \circ q = \phi$.*

Before we present our next result, let us remark that for each collection $\{\mathcal{A}_\alpha : \alpha \in \Gamma\}$ of Banach algebras, we denote by $\prod_{\alpha\in\Gamma} \mathcal{A}_\alpha$ the product space of the collection, i.e., the space consisting of all mappings $a : \Gamma \to \bigcup_{\alpha\in\Gamma} \mathcal{A}_\alpha$ for which $a_\alpha \in \mathcal{A}_\alpha$, the linear operations being given coordinatewise. For $1 \le p < \infty$, we recall that the l^p direct sum of the collection is

$$\bigoplus_{\alpha\in\Gamma}^{p} \mathcal{A}_\alpha = \Big\{a \in \prod_{\alpha\in\Gamma} \mathcal{A}_\alpha : \|a\| = \Big(\sum_{\alpha} \|a_\alpha\|^p\Big)^{1/p} < \infty\Big\},$$

and the c_0 direct sum of the collection is

$$\bigoplus_{\alpha\in\Gamma}^{0} \mathcal{A}_\alpha = \Big\{a \in \prod_{\alpha\in\Gamma} \mathcal{A}_\alpha : \|a\|_\infty = \max_{\alpha} \|a_\alpha\| < \infty \text{ and } \lim_{\alpha} a_\alpha = 0\Big\}.$$

The sum $\bigoplus_{\alpha\in\Gamma}^{p} \mathcal{A}_\alpha$, $p \ge 1$ or $p = 0$, is a Banach algebra with multiplication being defined coordinatewise. If $\Lambda \subseteq \Gamma$, then $\bigoplus_{\alpha\in\Lambda}^{p} \mathcal{A}_\alpha$ can be identified with the complemented closed ideal of $\bigoplus_{\alpha\in\Gamma}^{p} \mathcal{A}_\alpha$, consisting of all a with $a_\alpha = 0$ for all $\alpha \notin \Lambda$. It is easy to see that

$$\Delta\Big(\bigoplus_{\alpha\in\Gamma}^{p} \mathcal{A}_\alpha\Big) = \{\phi_\beta^{\oplus} : \phi_\beta \in \Delta(\mathcal{A}_\beta),\ \beta \in \Gamma\},$$

where $\phi_\beta^{\oplus}$ is defined by $\phi_\beta^{\oplus}((a_\alpha)_{\alpha\in\Gamma}) = \phi_\beta(a_\beta)$ for all $(a_\alpha)_{\alpha\in\Gamma} \in \bigoplus_{\alpha\in\Gamma}^{p} \mathcal{A}_\alpha$.

Theorem 3.13. *Let $(\mathcal{A}_\alpha)_{\alpha\in\Gamma}$ be a family of Banach algebras, $\phi_\beta \in \Delta(\mathcal{A}_\beta)$ for some $\beta \in \Gamma$, and $p = 0$ or $p \ge 1$. Then $\mathfrak{A} = \bigoplus_{\alpha\in\Gamma}^{p} \mathcal{A}_\alpha$ is left $\phi_\beta^{\oplus}$-contractible if and only if $\mathcal{A}_\beta$ is left ϕ_β-contractible.*

Proof. Since for each $\beta \in \Gamma$, $\mathcal{A}_\beta$ is a closed two-sided ideal of $\mathfrak{A}$ and $\phi_\beta^{\oplus}$ is the unique extension of ϕ_β, the result follows from Proposition 3.8. ∎

There is no analogue of the above theorem for left character contractibility. For example, the Banach algebra $l^p = \bigoplus_{\imath\in\mathbb{N}}^p \mathbb{C}$, $p \geq 1$, does not have a right identity, and so it is not left character contractible whereas $\mathbb{C}$ is left character contractible.

Now, we study the relation between the left character contractibility of two Banach algebras and of their projective tensor product. As usual, we denote by $\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{B}$ the projective tensor product of the Banach algebras $\mathcal{A}$ and $\mathcal{B}$; we recall that for $f \in \mathcal{A}^*$ and $g \in \mathcal{B}^*$, $f \otimes g$ denotes the element of $(\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{B})^*$ satisfying $(f \otimes g)(a \otimes b) = f(a)g(b)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, and we note that $\Delta(\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{B}) = \{\phi \otimes \psi : \phi \in \Delta(\mathcal{A}),\ \psi \in \Delta(\mathcal{B})\}$.

THEOREM 3.14. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, $\phi \in \Delta(\mathcal{A})$ and $\psi \in \Delta(\mathcal{B})$. Then $\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{B}$ is left $\phi \otimes \psi$-contractible if and only if $\mathcal{A}$ is left ϕ-contractible and $\mathcal{B}$ is left ψ-contractible.*

Proof. Suppose that $\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{B}$ is left $\phi \otimes \psi$-contractible. Then there is an element $m \in \mathcal{A}$ with

$$\phi \otimes \psi(m) = 1 \quad \text{and} \quad (a \otimes b)m = \phi(a)\psi(b)m$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Now, we consider the linear map $\Upsilon : \mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{B} \to \mathcal{A}$ determined by $\Upsilon(a \otimes b) = a\psi(b)$. Then $\Upsilon(m) \in \mathcal{A}$. We show that $\Upsilon(m)$ is a topological left invariant mean. To that end, choose $a_0 \in \mathcal{A}$ and $b_0 \in \mathcal{B}$ such that $\phi(a_0) = 1$ and $\psi(b_0) = 1$. Thus

$$\begin{aligned} a\Upsilon(m) &= a\Upsilon((a_0 \otimes b_0)m)) = \Upsilon(aa_0 \otimes b_0 m) \\ &= \Upsilon(\phi(a)\phi(a_0)\psi(b_0)m) = \phi(a)\Upsilon(m). \end{aligned}$$

Moreover, since $\phi \circ \Upsilon = \phi \otimes \psi$, we have

$$\phi(\Upsilon(m)) = \phi \otimes \psi(m) = 1.$$

Similarly, there is a topological left invariant mean in $\mathcal{B}$.

Conversely, suppose that $\mathcal{A}$ is left ϕ-contractible and $\mathcal{B}$ is left ψ-contractible. Then, by assumption, there is an element $m_1 \in \mathcal{A}$ with

$$\phi(m_1) = 1 \quad \text{and} \quad m_1 = \phi(a)m_1$$

for all $a \in \mathcal{A}$. Moreover, there is an element $m_2 \in \mathcal{B}$ such that

$$\psi(m_2) = 1 \quad \text{and} \quad bm_2 = \psi(b)m_2$$

for all $b \in \mathcal{B}$. Now, set $m := m_1 \otimes m_2$ and note that

$$(\phi \otimes \psi)(m_1 \otimes m_2) = 1 \quad \text{and} \quad (a \otimes b)(m_1 \otimes m_2) = \phi(a)\psi(b)(m_1 \otimes m_2)$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$; it follows that $w(m_1 \otimes m_2) = \phi \otimes \psi(w)\ m_1 \otimes m_2$ for all $w \in \mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{B}$. Hence $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is left $\phi \otimes \psi$-contractible. ∎

COROLLARY 3.15. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. Then $\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{B}$ is left character contractible if and only if $\mathcal{A}$ and $\mathcal{B}$ are left character contractible.*

Proof. By Theorem 3.14, it is sufficient to show that $\mathcal{A} \widehat{\otimes} \mathcal{B}$ is left 0-contractible if and only if $\mathcal{A}$ and $\mathcal{B}$ are left 0-contractible. But this follows immediately form Proposition 3.4 together with the fact that $\mathcal{A} \widehat{\otimes} \mathcal{B}$ has a right identity if and only if $\mathcal{A}$ and $\mathcal{B}$ have right identities; see [LO, Theorem 1]. ∎

4. Left ϕ-contractibility and projectivity. Let $\mathcal{A}$ be a Banach algebra and $\mathcal{E}$ be a Banach left $\mathcal{A}$-module. In this case, the dual $\mathcal{E}^*$ of $\mathcal{E}$ is a Banach right $\mathcal{A}$-module with the action induced via $\langle \Lambda \cdot a, \xi \rangle = \langle \Lambda, a \cdot \xi \rangle$ for all $a \in \mathcal{A}$, $\xi \in \mathcal{E}$ and $\Lambda \in \mathcal{E}^*$. The Banach left $\mathcal{A}$-module $\mathcal{E}$ is called *faithful* if $\mathcal{A} \cdot \xi \neq \{0\}$ for all $\xi \in \mathcal{E} \setminus \{0\}$; it is called *essential* if the linear span of $\mathcal{A} \cdot \mathcal{E}$ is dense in $\mathcal{E}$.

PROPOSITION 4.1. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. Then any Banach left $\mathcal{A}$-module $\mathcal{E}$ with $a \cdot \xi = \phi(a)\xi$ for all $a \in \mathcal{A}$ and $\xi \in \mathcal{E}$ is faithful and essential.*

Proof. The result follows immediately from the assumption that $a \cdot \xi = \xi$ for all $a \in \mathcal{A}$ with $\phi(a) = 1$. ∎

Let $\mathcal{E}$ and $\mathcal{F}$ be two Banach spaces, and denote by $B(\mathcal{E}, \mathcal{F})$ the Banach space of all bounded operators from $\mathcal{E}$ into $\mathcal{F}$. An operator $T \in B(\mathcal{E}, \mathcal{F})$ is called *admissible* if $T \circ S \circ T = T$ for some $S \in B(\mathcal{F}, \mathcal{E})$. In the case where $\mathcal{E}$ and $\mathcal{F}$ are Banach left $\mathcal{A}$-modules, ${}_{\mathcal{A}}B(\mathcal{E}, \mathcal{F})$ denotes the closed linear subspace of $B(\mathcal{E}, \mathcal{F})$ consisting of all left $\mathcal{A}$-module morphisms. An operator $T \in {}_{\mathcal{A}}B(\mathcal{E}, \mathcal{F})$ is called a *retraction* if there exists $S \in {}_{\mathcal{A}}B(\mathcal{F}, \mathcal{E})$ with $T \circ S = I_{\mathcal{F}}$, and in this case $\mathcal{F}$ is called a *retract* of $\mathcal{E}$; T is a *coretraction* if there exists $S \in {}_{\mathcal{A}}B(\mathcal{F}, \mathcal{E})$ with $S \circ T = I_{\mathcal{E}}$.

A Banach left $\mathcal{A}$-module $\mathcal{P}$ is called *projective* if for any Banach left $\mathcal{A}$-modules $\mathcal{E}$ and $\mathcal{F}$, each admissible epimorphism $T \in {}_{\mathcal{A}}B(\mathcal{E}, \mathcal{F})$, and each $S \in {}_{\mathcal{A}}B(\mathcal{P}, \mathcal{F})$, there exists $R \in {}_{\mathcal{A}}B(\mathcal{P}, \mathcal{E})$ such that $T \circ R = S$.

Let $\mathcal{E}$ be a Banach space, and let $\mathcal{P} = \mathcal{A} \widehat{\otimes} \mathcal{E}$ be endowed with the module operation determined by

$$a \cdot (b \otimes \xi) = ab \otimes \xi \quad (a, b \in \mathcal{A},\ \xi \in \mathcal{E}).$$

So, we can define the canonical morphism $\pi \in {}_{\mathcal{A}}B(\mathcal{A} \widehat{\otimes} \mathcal{E}, \mathcal{E})$ by $\pi(a \otimes \xi) = a \cdot \xi$ for all $a \in \mathcal{A}$, $\xi \in \mathcal{E}$. The following characterization of projective Banach left $\mathcal{A}$-modules is given in [H2, Proposition IV.1.1].

PROPOSITION 4.2. *Let $\mathcal{A}$ be a Banach algebra and $\mathcal{E}$ be an essential Banach left $\mathcal{A}$-module. Then $\mathcal{E}$ is projective if and only if the canonical morphism $\pi \in {}_{\mathcal{A}}B(\mathcal{A} \widehat{\otimes} \mathcal{E}, \mathcal{E})$ is a retraction.*

Our first result characterizes the projectivity of certain Banach left $\mathcal{A}$-modules in terms of the left ϕ-contractibility of $\mathcal{A}$.

THEOREM 4.3. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. Then the following assertions are equivalent.*

(i) *Any Banach left $\mathcal{A}$-module $\mathcal{E}$ with $a \cdot \xi = \phi(a)\xi$ ($\xi \in \mathcal{E}$, $a \in \mathcal{A}$) is projective.*
(ii) *There is a projective Banach left $\mathcal{A}$-module $\mathcal{E}$ with $a \cdot \xi = \phi(a)\xi$ ($\xi \in \mathcal{E}$, $a \in \mathcal{A}$).*
(iii) *The Banach left $\mathcal{A}$-module $\mathbb{C}$ with $a \cdot z = \phi(a)z$ ($z \in \mathbb{C}$, $a \in \mathcal{A}$) is projective.*
(iv) *$\mathcal{A}$ is left ϕ-contractible.*

Proof. (ii)⇒(iii). Let $\mathcal{E}$ be a projective Banach left $\mathcal{A}$-module with $a\cdot\xi = \phi(a)\xi$ for all $\xi \in \mathcal{E}$ and $a \in \mathcal{A}$, and choose $\xi_0 \in \mathcal{E}$ and $\Lambda \in \mathcal{E}^*$ such that $\Lambda(\xi_0) = 1$. Define the left $\mathcal{A}$-module morphism $\gamma : \mathbb{C} \to \mathcal{E}$ by $\gamma(z) = z\xi_0$ for all $z \in \mathbb{C}$. So, $(\Lambda \circ \gamma)(z) = z$ for all $z \in \mathbb{C}$, and therefore $\mathbb{C}$ is a retract of $\mathcal{E}$ as a Banach left $\mathcal{A}$-module. Thus $\mathbb{C}$ is a projective Banach left $\mathcal{A}$-module by [H2, Proposition III.1.16].

(iii)⇒(iv). Suppose that $\mathbb{C}$ is a projective Banach left $\mathcal{A}$-module with $a \cdot z = \phi(a)z$ for all $z \in \mathbb{C}$ and $a \in \mathcal{A}$. Then there exists $\rho : \mathbb{C} \to \mathcal{A} \widehat{\otimes} \mathbb{C} \simeq \mathcal{A}$ such that $(\rho \circ \pi)(z) = z$ for all $z \in \mathbb{C}$, where $\pi : \mathcal{A} \widehat{\otimes} \mathbb{C} \to \mathbb{C}$ is canonical embedding. Define $m := \rho(1) \in \mathcal{A}$. Then $m(\phi) = \phi(m) = \pi(\rho(1)) = 1$ and $am = \rho(a \cdot 1) = \phi(a)m$. So, m is a topological left invariant ϕ-mean in $\mathcal{A}$, and thus $\mathcal{A}$ is a left ϕ-contractible.

(iv)⇒(i). Suppose that there is a topological left invariant mean m in $\mathcal{A}$. Then for each Banach left $\mathcal{A}$-module $\mathcal{E}$ with $a \cdot \xi = \phi(a)\xi$ for all $a \in \mathcal{A}$ and $\xi \in \mathcal{E}$, the map $\rho : \mathcal{E} \to \mathcal{A} \widehat{\otimes} \mathcal{E}$ with $\rho(\xi) = m \otimes \xi$ ($\xi \in \mathcal{E}$) has the following properties:

$$\rho(a \cdot \xi) = m \otimes a \cdot \xi = \phi(a)m \otimes \xi = am \otimes \xi = a \cdot \rho(\xi),$$

and

$$\pi \circ \rho(\xi) = \pi m \otimes \xi = \phi(m)\xi = \xi.$$

Therefore, ρ is a retraction of π, and so $\mathcal{E}$ is a projective Banach left $\mathcal{A}$-module by Proposition 4.2. ∎

COROLLARY 4.4. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. Then the Banach left $\mathcal{A}$-module $\mathcal{A}$ with $a \cdot b = \phi(a)b$ ($a, b \in \mathcal{A}$) is projective if and only if $\mathcal{A}$ is left ϕ-contractible.*

REMARK 4.5. In view of Theorem 4.3, hereditary properties of ϕ-contractibility of Banach algebras $\mathcal{A}$ in Section 3 can be reformulated in terms of projectivity of Banach left $\mathcal{A}$-modules. For example, Theorem 3.1 can be restated as follows.

Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. Then a Banach left $\mathcal{A}$-module $\mathcal{E}$ with $a \cdot \xi = \phi(a)\xi$ ($a \in \mathcal{A}$, $\xi \in \mathcal{E}$) is projective if and only if $\mathcal{E}$ is projective as a Banach left $\mathcal{A}^\sharp$-module such that

$$(a,\lambda)\cdot\xi = \phi^\sharp(a,\lambda)\xi \quad ((a,\lambda)\in\mathcal{A}^\sharp,\ \xi\in\mathcal{E}).$$

5. Right ϕ-amenability and injectivity. Let $\mathcal{A}$ be a Banach algebra and recall that a Banach left $\mathcal{A}$-module $\mathcal{J}$ is called *injective* if for any Banach left $\mathcal{A}$-modules $\mathcal{E}$ and $\mathcal{F}$, each admissible monomorphism $T\in {}_{\mathcal{A}}B(\mathcal{E},\mathcal{F})$, and each $S\in {}_{\mathcal{A}}B(\mathcal{E},\mathcal{J})$, there exists $R\in {}_{\mathcal{A}}B(\mathcal{F},\mathcal{J})$ such that $R\circ T=S$. Similar definitions apply for Banach right $\mathcal{A}$-modules. Note that each retraction of an injective Banach left $\mathcal{A}$-module is injective by [H2, Proposition III.1.16].

Let $\mathcal{E}$ be a Banach space. Then $B(\mathcal{A},\mathcal{E})$ is a Banach $\mathcal{A}$-bimodule for the following two module operations:

$$(a\cdot T)(b)=T(ba), \qquad (T\cdot a)(b)=T(ab)$$

for all $a,b\in\mathcal{A}$ and $T\in B(\mathcal{A},\mathcal{E})$; see [D, Example 2.6.2(viii)]. Now, we can consider the canonical embedding $\Pi:\mathcal{E}\to B(\mathcal{A},\mathcal{E})$ defined by the formula

$$\Pi(\xi)(a)=a\cdot\xi \quad (a\in\mathcal{A},\ \xi\in\mathcal{E}).$$

The following characterization of injective modules is given in [H2, Proposition III.1.31]:

PROPOSITION 5.1. *Let $\mathcal{A}$ be a Banach algebra, and let $\mathcal{E}$ be a faithful Banach left $\mathcal{A}$-module. Then $\mathcal{E}$ is injective if and only if the canonical embedding Π is a coretraction of $\mathcal{A}$-modules.*

Note that injectivity can be characterized in terms of the functor Ext whose definition can be found in [H2]. Indeed, a left Banach $\mathcal{A}$-module $\mathcal{J}$ is injective if and only if $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{E},\mathcal{J})=\{0\}$ for each Banach left $\mathcal{A}$-module $\mathcal{E}$; see [H2, Proposition III.4.5].

We are now ready to characterize the injectivity of some Banach $\mathcal{A}$-modules by the right ϕ-amenability of $\mathcal{A}$; that is, $H^1(\mathcal{A},\mathcal{X}^*)$ vanishes for every Banach $\mathcal{A}$-bimodule $\mathcal{X}$ with the right module action $x\cdot a=\phi(a)x$ ($a\in\mathcal{A}$, $x\in\mathcal{X}$). Note that this concept is equivalent to the existence of an element M in $\mathcal{A}^{**}$ such that $M(\phi)=1$ and $M\odot a=\phi(a)M$ for all $a\in\mathcal{A}$; such an element M is called a *topological right invariant ϕ-mean.*

THEOREM 5.2. *Let $\mathcal{A}$ be a Banach algebra and $\phi\in\Delta(\mathcal{A})$. Then the following assertions are equivalent:*

(i) *Any dual Banach left $\mathcal{A}$-module $\mathcal{E}$ with $a\cdot\xi=\phi(a)\xi$ ($\xi\in\mathcal{E}$, $a\in\mathcal{A}$) is injective.*
(ii) *The Banach left $\mathcal{A}$-module $\mathbb{C}$ with $a\cdot z=\phi(a)z$ ($z\in\mathbb{C}$, $a\in\mathcal{A}$) is injective.*
(iii) *There is an injective Banach left $\mathcal{A}$-module $\mathcal{E}$ with $a\cdot\xi=\phi(a)\xi$ ($\xi\in\mathcal{E}$, $a\in\mathcal{A}$).*
(iv) *$\mathcal{A}$ is right ϕ-amenable.*

Proof. The implications (i)⇒(ii)⇒(iii) are trivial.

(iii)⇒(iv). Suppose that (iii) holds, and let $\mathcal{E}$ be as in (iii). Then there is a left $\mathcal{A}$-module morphism $\rho : B(\mathcal{A},\mathcal{E}) \to \mathcal{E}$ such that $\rho \circ \Pi = I_{\mathcal{E}}$. For each $f \in \mathcal{A}^*$, let $T_f \in B(\mathcal{A},\mathcal{E})$ be defined by $T_f(a) = f(a)\xi_0$ for all $a \in \mathcal{A}$, where ξ_0 is a fixed nonzero element of $\mathcal{E}$. Now, define the functional $M \in \mathcal{A}^{**}$ by $M(f) = (\Lambda_0 \circ \rho)(T_f)$ for all $f \in \mathcal{A}^*$, where Λ_0 is an element of $\mathcal{E}^*$ with $\Lambda_0(\xi_0) = 1$. Then

$$M(\phi) = (\Lambda_0 \circ \rho)(T_\phi) = (\Lambda_0 \circ \rho)(\Pi(\xi_0)) = \Lambda_0(\xi_0) = 1.$$

Also, for each $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$, we have $T_{af} = a \cdot T_f$ and thus

$$\begin{aligned}(M \odot a)(f) = M(af) &= (\Lambda_0 \circ \rho)(a \cdot T_f) = \Lambda_0(a \cdot \rho(T_f)) \\ &= \phi(a)\Lambda_0(\rho(T_f)) = \phi(a)M(f),\end{aligned}$$

That is, M is a topological right invariant ϕ-mean in $\mathcal{A}^{**}$, and so (iv) holds.

(iv)⇒(i). Suppose that (iv) holds and let $\mathcal{E}$ be a dual Banach left $\mathcal{A}$-module with $a \cdot \xi = \phi(a)\xi$ for all $a \in \mathcal{A}$ and $\xi \in \mathcal{E}$. To prove that $\mathcal{E}$ is injective, it is sufficient to show $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{F},\mathcal{E}) = \{0\}$ for all Banach right $\mathcal{A}$-modules $\mathcal{F}$. To that end, choose a Banach right $\mathcal{A}$-module $\mathcal{E}_*$ with $z \cdot a = \phi(a)z$ for all $z \in \mathcal{E}_*$ and $a \in \mathcal{A}$ such that $(\mathcal{E}_*)^* = \mathcal{E}$. Then for each Banach right $\mathcal{A}$-module $\mathcal{F}$ we have

$$\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{F},\mathcal{E}) = H^1(\mathcal{A}, B(\mathcal{F},\mathcal{E})) = H^1(\mathcal{A}, (\mathcal{F} \mathbin{\widehat{\otimes}} \mathcal{E}_*)^*) = H^1(\mathcal{A}, (\mathcal{E}_* \mathbin{\widehat{\otimes}} \mathcal{F})^*).$$

Moreover, $\mathcal{E}_* \mathbin{\widehat{\otimes}} \mathcal{F}$ is a Banach $\mathcal{A}$-bimodule with

$$(z \otimes y) \cdot a = \phi(a)(z \otimes y)$$

for all $a \in \mathcal{A}$, $z \in \mathcal{E}_*$ and $y \in \mathcal{F}$. Since $\mathcal{A}$ is right ϕ-amenable, it follows that $H^1(\mathcal{A}, \mathcal{X}^*) = \{0\}$ for all Banach $\mathcal{A}$-bimodules $\mathcal{X}$ with $a \cdot x = \phi(a)x$ for all $a \in \mathcal{A}$ and $x \in \mathcal{X}$; in particular, $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{F},\mathcal{E}) = \{0\}$ as required. ∎

COROLLARY 5.3. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. Then the Banach left $\mathcal{A}$-module $\mathcal{A}^*$ with $a \cdot f = \phi(a)f$ ($f \in \mathcal{A}^*$, $a \in \mathcal{A}$) is injective if and only if $\mathcal{A}$ is right ϕ-amenable.*

THEOREM 5.4. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. Let $\mathcal{I}$ be a closed two-sided ideal of $\mathcal{A}$ with $\mathcal{I} \not\subseteq \ker(\phi)$. Then a Banach left $\mathcal{A}$-module $\mathcal{E}$ with $a \cdot \xi = \phi(a)\xi$ ($a \in \mathcal{A}$, $\xi \in \mathcal{E}$) is injective if and only if $\mathcal{E}$ is injective as a Banach left $\mathcal{I}$-module with $i \cdot \xi = \phi|_{\mathcal{I}}(i)\xi$ ($i \in \mathcal{I}$, $\xi \in \mathcal{E}$).*

Proof. Suppose that $\mathcal{E}$ is injective as a Banach left $\mathcal{A}$-module with $a \cdot \xi = \phi(a)\xi$ for all $a \in \mathcal{A}$ and $\xi \in \mathcal{E}$. Then by Proposition 5.1, there is a left $\mathcal{A}$-module morphism $\rho : B(\mathcal{A},\mathcal{E}) \to \mathcal{E}$ such that $\rho \circ \Pi = I_{\mathcal{E}}$. For each $T \in B(\mathcal{I},\mathcal{E})$, let $\tilde{T} \in B(\mathcal{A},\mathcal{E})$ be defined by $\tilde{T}(a) = T(\iota_0 a)$ for all $a \in \mathcal{A}$, where ι_0 is a fixed nonzero element of $\mathcal{I}$ such that $\phi(\iota_0) = 1$. Also, define the left $\mathcal{I}$-module morphism $\tilde{\rho} : B(\mathcal{I},\mathcal{E}) \to \mathcal{E}$ by the formula $\tilde{\rho}(T) = \rho(\tilde{T})$

for all $T \in B(\mathcal{I},\mathcal{E})$. Since $\tilde{\tilde{\Pi}}(\xi) = \Pi(\xi)$ and $(i \cdot T)^{\sim} = i \cdot \tilde{T}$, for all $\xi \in \mathcal{E}$, $i \in \mathcal{I}$ and $T \in B(\mathcal{I},\mathcal{E})$, we have

$$\tilde{\rho}(i \cdot T) = i \cdot \tilde{\rho}(T) \quad \text{and} \quad \tilde{\rho} \circ \tilde{\Pi} = I_{\mathcal{E}},$$

where $\tilde{\Pi} : \mathcal{E} \to B(\mathcal{I},\mathcal{E})$ is the canonical embedding. Thus $\mathcal{E}$ is injective as a Banach left $\mathcal{I}$-module such that $i \cdot \xi = \phi|_{\mathcal{I}}(i)\xi$ for all $i \in \mathcal{I}$ and $\xi \in \mathcal{E}$.

Conversely, suppose that $\mathcal{E}$ is injective as a Banach left $\mathcal{I}$-module such that $i \cdot \xi = \phi|_{\mathcal{I}}(i)\xi$ for all $i \in \mathcal{I}$ and $\xi \in \mathcal{E}$. Then by Proposition 5.1, there is a left $\mathcal{A}$-module morphism $\rho : B(\mathcal{I},\mathcal{E}) \to \mathcal{E}$ such that $\rho \circ \Pi = I_{\mathcal{E}}$. We define the left $\mathcal{A}$-module morphism $\tilde{\rho} : B(\mathcal{A},\mathcal{E}) \to \mathcal{E}$ by the formula $\tilde{\rho}(T) = \rho(T_{\mathcal{I}})$ for all $T \in B(\mathcal{A},\mathcal{E})$. Since $\tilde{\Pi}_{\mathcal{I}}(\xi) = \Pi(\xi)$ and $i \cdot T_{\mathcal{I}} = (i \cdot T)_{\mathcal{I}} = T(ia)$ for all $\xi \in \mathcal{E}$, $i \in \mathcal{I}$ and $T \in B(\mathcal{A},\mathcal{E})$, we have

$$\tilde{\rho}(a \cdot T) = a \cdot \tilde{\rho}(T) \quad \text{and} \quad \tilde{\rho} \circ \tilde{\Pi} = I_{\mathcal{E}},$$

where $\tilde{\Pi} : \mathcal{E} \to B(\mathcal{A},\mathcal{E})$ is the canonical embedding. Therefore by Proposition 5.1, $\mathcal{E}$ is injective as a Banach left $\mathcal{A}$-module such that $a \cdot \xi = \phi(a)\xi$ for all $a \in \mathcal{A}$ and $\xi \in \mathcal{E}$. ∎

As a consequence of Theorem 5.4, we have the following result.

COROLLARY 5.5. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. Then a Banach left $\mathcal{A}$-module $\mathcal{E}$ with $a \cdot \xi = \phi(a)\xi$ ($a \in \mathcal{A}$, $\xi \in \mathcal{E}$) is injective if and only if $\mathcal{E}$ is injective as a Banach left $\mathcal{A}^\sharp$-module with $(a,\lambda) \cdot \xi = \phi^\sharp(a,\lambda)\xi$ ($(a,\lambda) \in \mathcal{A}^\sharp$, $\xi \in \mathcal{E}$).*

Note that a Banach algebra $\mathcal{A}$ is right ϕ-amenable if any Banach left $\mathcal{A}$-module $\mathcal{E}$ with $a \cdot \xi = \phi(a)\xi$ for all $a \in \mathcal{A}$ and $\xi \in \mathcal{E}$ is injective; this follows from Theorem 5.2. We note that this is an "if and only if" statement for certain Banach algebras. First, we state another result on those Banach algebras $\mathcal{A}$ which are right ϕ-contractible; that is, $H^1(\mathcal{A},\mathcal{X}) = \{0\}$ for every Banach $\mathcal{A}$-bimodule $\mathcal{X}$ with the left module action $a \cdot x = \phi(a)x$ for all $a \in \mathcal{A}$ and $x \in \mathcal{X}$.

LEMMA 5.6. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. If $\mathcal{A}$ is right ϕ-contractible, then any Banach left $\mathcal{A}$-module $\mathcal{E}$ with $a \cdot \xi = \phi(a)\xi$ for all $a \in \mathcal{A}$ and $\xi \in \mathcal{E}$ is injective.*

Proof. Suppose that $\mathcal{A}$ is right ϕ-contractible. Then by a similar proof to that of Theorem 2.1, there exists $a_0 \in \mathcal{A}$ with $\phi(a_0) = 1$ and $a_0 a = \phi(a)a_0$ for all $a \in \mathcal{A}$. Define $\rho : B(\mathcal{A},\mathcal{E}) \to \mathcal{E}$ by $\rho(T) = T(a_0)$ for all $T \in B(\mathcal{A},\mathcal{E})$. Then for each $a \in \mathcal{A}$, we have

$$\rho(a \cdot T) = (a.T)(a_0) = T(a_0 a) = \phi(a)T(a_0) = a \cdot \rho(T)$$

and

$$(\rho \circ \Pi)(\xi) = \Pi(\xi)(a_0) = \phi(a_0)\xi = \xi.$$

Therefore by Proposition 5.1, $\mathcal{E}$ is injective. ∎

As a consequence of Corollary 3.6 and Lemma 5.6, we have the following result.

THEOREM 5.7. *Let $\mathcal{A}$ be a Banach algebra which is a left or right ideal in $\mathcal{A}^{**}$ and $\phi \in \Delta(\mathcal{A})$. Then $\mathcal{A}$ is right ϕ-amenable if and only if any Banach left $\mathcal{A}$-module such that $a \cdot \xi = \phi(a)\xi$ for all $a \in \mathcal{A}$ and $\xi \in \mathcal{E}$ is injective.*

The following example shows that Theorem 5.7 does not remain valid for all Banach algebras.

EXAMPLE 5.8. Let $\ell^1(\mathbb{N})$ denote the convolution semigroup algebra of the additive semigroup $\mathbb{N}$, and let $\phi_{\mathbb{N}} \in \Delta(\ell^1(\mathbb{N}))$ be defined by $\phi_{\mathbb{N}}((a_n)) = \sum_{n=1}^{\infty} a_n$ for all $(a_n) \in \ell^1(\mathbb{N})$. Then $\ell^1(\mathbb{N})$ is right $\phi_{\mathbb{N}}$-amenable. But we shall show that $c_0(\mathbb{N})$, the space of null sequences, is not injective as a Banach left $\ell^1(\mathbb{N})$-module with the action

$$(a_n) \cdot (g_n) = \Big(\sum_{m=1}^{\infty} a_m g_n\Big)$$

for all $(a_n) \in \ell^1(\mathbb{N})$ and $(g_n) \in c_0(\mathbb{N})$. On the contrary, suppose that $c_0(\mathbb{N})$ is an injective Banach left $\ell^1(\mathbb{N})$-module. Then by Proposition 5.1, there exists a left $\ell^1(\mathbb{N})$-module morphism $\rho : B(\ell^1(\mathbb{N}), c_0(\mathbb{N})) \to c_0(\mathbb{N})$ such that $\rho \circ \Pi = I_{c_0(\mathbb{N})}$, where $\Pi : c_0(\mathbb{N}) \to B(\ell^1(\mathbb{N}), c_0(\mathbb{N}))$ is the canonical embedding defined by $\Pi((g_n))((a_n)) = (a_n) \cdot (g_n)$ for all $(g_n) \in c_0(\mathbb{N})$ and $(a_n) \in \ell^1(\mathbb{N})$.

Now, let $P : \ell^\infty(\mathbb{N}) \to B(\ell^1(\mathbb{N}), c_0(\mathbb{N}))$ be the continuous map given by the formulae

$$P((f_n))((a_n)) = \Big(\Big(\sum_{m=n}^{\infty} a_m\Big) f_n\Big)_n,$$

for all $(f_n) \in \ell^\infty(\mathbb{N})$ and $(a_n) \in \ell^1(\mathbb{N})$, where $\ell^\infty(\mathbb{N})$ is the space of bounded sequences.

We show that the map $\rho \circ P$ is a projection of $\ell^\infty(\mathbb{N})$ onto $c_0(\mathbb{N})$, which contradicts [H1, Theorem 0.1.16]. To that end, it suffices to show that $\rho \circ P$ is the identity map on $c_{00}(\mathbb{N})$, the space of sequences with finite support. Take $(h_n) \in c_{00}(\mathbb{N})$ and define $T_0 = P((h_n)) - \Pi((h_n))$. If l is a natural number with $h_n = 0$ for all $n > l$, then we define $(b_n) \in \ell^1(\mathbb{N})$ by

$$b_n = \begin{cases} 1, & 1 \le n \le l, \\ -1, & l+1 \le n \le 2l, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $\phi_{\mathbb{N}}((b_n)) = 0$ and $T_0 = (b_n) \cdot S$, where $S : \ell^1(\mathbb{N}) \to c_0(\mathbb{N})$ is defined by $S((a_n)) = (a_n h_n)$ for all $(a_n) \in \ell^1(\mathbb{N})$. Therefore

$$\rho(T_0) = \rho((b_n) \cdot S) = \phi_{\mathbb{N}}((b_n))\rho(S) = 0.$$

That is, $\rho \circ P = \rho \circ \Pi$ as required.

6. Applications to group algebras. Let G be a locally compact group with left Haar measure λ_G and let $L^1(G)$ be the group algebra of G as defined in [HR] endowed with the norm $\|\cdot\|_1$ and the convolution product $*$. Let $L^\infty(G)$ be the usual Lebesgue space with the essential supremum norm $\|\cdot\|_\infty$, and let $M(G)$ be the measure algebra of G as defined in [HR]. Let X be a *left introverted* subspace of $L^\infty(G)$, i.e., $Ff, fh \in X$ for all $F \in X^*$, $f \in X$ and $h \in L^1(G)$, where

$$(Ff)(h) = F(fh) \quad \text{and} \quad (fh)(k) = f(h * k)$$

for all $k \in L^1(G)$. In this case, X^* is a Banach algebra with the first Arens multiplication $\odot$ defined by $(F \odot H)(f) = F(Hf)$ for all $F, H \in X^*$ and $f \in X$. Examples of closed left introverted subspace of $L^\infty(G)$ include the space $LUC(G)$ of all left uniformly continuous functions on G and the space $L_0^\infty(G)$ of all $f \in L^\infty(G)$ which vanish at infinity; this space was introduced and extensively studied by Lau and Pym in [LP].

Let $\widehat{G}$ denote the dual group of G consisting of all continuous homomorphisms from G into the circle group $\mathbb{T}$. For $\rho \in \widehat{G}$, define ϕ_ρ to be the character induced by ρ on $L^1(G)$, that is,

$$\phi_\rho(h) = \int_G \overline{\rho(s)} h(s)\, d\lambda_G(s) \qquad (h \in L^1(G)).$$

Note that there is always at least one character on $L^1(G)$, namely, the augmentation character ϕ_1. It is well-known that there is no other character on $L^1(G)$; that is,

$$\Delta(L^1(G)) = \{\phi_\rho : \rho \in \widehat{G}\};$$

see for example [HR, Theorem 23.7]. Let ϕ_ρ also denote the natural extensions of ϕ_ρ from $L^1(G)$ to a character on any one of the Banach algebras $M(G)$, $L_0^\infty(G)^*$, $LUC(G)^*$ and $L^1(G)^{**}$. Finally, let us recall that G is called *amenable* if $L^1(G)$ is ϕ_1-amenable; it is well-known that all locally compact abelian groups and compact groups are amenable, but the free group $\mathbb{F}_2$ on two generators is not amenable; see [R] for more details.

THEOREM 6.1. *Let G be a locally compact group and $\rho \in \widehat{G}$. Then the following assertions are equivalent:*

(i) *$L^1(G)$ is left ϕ_ρ-contractible.*
(ii) *$M(G)$ is left ϕ_ρ-contractible.*
(iii) *$L_0^\infty(G)^*$ is left ϕ_ρ-contractible.*
(iv) *G is compact.*

Proof. It is well-known that $L^1(G)$ is a closed two-sided ideal in $L_0^\infty(G)^*$ and $M(G)$; see [LP, Proposition 2.2] and [HR, Theorem 19.18]. So, the equivalences (i)⇔(ii)⇔(iii) are trivial by Proposition 3.8.

Now, suppose that (i) holds. Then there exists $h_0 \in L^1(G)$ such that $\phi_\rho(h_0) = 1$ and $h * h_0 = \phi_\rho(h)h_0$ for all $h \in L^1(G)$. So, $\bar{\rho}h_0 \in L^1(G)$, $\phi_1(\bar{\rho}h_0) = 1$ and $h * \bar{\rho}h_0 = \phi_1(h)\bar{\rho}h_0$ for all $h \in L^1(G)$. It follows that $\bar{\rho}h_0$ is a topological left invariant ϕ_1-mean in $L^1(G)$, and hence G is compact; see for example [R, Exercise 1.1.7]. Conversely, suppose that G is a compact group endowed with a normalized left Haar measure. Then $\rho \in L^1(G)$, $\phi_\rho(\rho) = 1$ and $h * \rho = \phi_\rho(h)\rho$ for all $h \in L^1(G)$. Therefore $L^1(G)$ is left ϕ_ρ-contractible. ∎

Corollary 6.2. *Let G be a locally compact group. Then the following assertions are equivalent:*

(i) *$L^1(G)$ is left character contractible.*
(ii) *$M(G)$ is left character contractible.*
(iii) *$L_0^\infty(G)^*$ is left character contractible.*
(iv) *G is finite.*

Proof. Suppose that (i) holds. By Proposition 3.4, $L^1(G)$ has a right identity. So, G is discrete; see [HR, Theorem 20.25]. Therefore $L_0^\infty(G)^* = M(G) = L^1(G)$ and thus (ii) and (iii) hold. So, the equivalences (i)⇔(ii)⇔(iii) follow from Corollaries 3.7 and 3.9.

The equivalence (i)⇔(iv) follows from Proposition 3.4 and Theorem 6.1. ∎

Corollary 6.3. *Let G be a locally compact group and $\rho \in \widehat{G}$. Then the following assertions are equivalent:*

(i) *$L^1(G)^{**}$ is left ϕ_ρ-contractible.*
(ii) *$LUC(G)^*$ is left ϕ_ρ-contractible.*
(iii) *G is amenable.*

Proof. Suppose that $L^1(G)^{**}$ is left ϕ_ρ-contractible and note that the restriction map $\Theta : L^1(G)^{**} \to LUC(G)^*$ is a continuous epimorphism. This together with Lemma 3.11 implies that $LUC(G)^*$ is ϕ_ρ-contractible. Now, suppose that (ii) holds. Then there is an $M \in LUC(G)^*$ such that $M(\phi_\rho)=1$ and $N \odot M = N(\phi_\rho)M$ for all $N \in LUC(G)^*$. Define $M_\rho(f) := M(f_\rho)$ for all $f \in LUC(G)$, where $f_\rho(s) = f(s)\overline{\rho(s)}$ for $s \in G$. Then $f_\rho \in LUC(G)$, and therefore M_ρ is well-defined. We also have

$$M_\rho(\phi_1) = M(\phi_\rho) = 1.$$

Indeed, $\phi_{1_\rho}(s) = \phi_1(s)\bar{\rho}(s) = \phi_\rho(s)$. On the other hand, for each $h \in L^1(G)$ and $f \in LUC(G)$ we have

$$\begin{aligned}(h \odot M_\rho)(f) &= M_\rho(fh) = M((fh)_\rho) = (\rho h \odot M)(f_\rho)\\ &= \phi_\rho(\rho h)M_\rho(f) = \phi_1(h)M_\rho(f).\end{aligned}$$

Therefore, M_ρ is a topological left invariant mean on $LUC(G)$ and so G is amenable by [R, Lemma 1.1.7 and Theorem 1.1.9].

Now, suppose that G is amenable. Then $L^1(G)$ is left ϕ_ρ-amenable by [M2, Corollary 2.4]. So, Proposition 3.5 shows that $L^1(G)^{**}$ is left ϕ_ρ-contractible. ∎

Remark 6.4. As in [LL, Corollary 4.4], it would be interesting to characterize the left ϕ-contractibility on $\mathcal{X}^*$, the dual Banach algebra of an introverted subspace $\mathcal{X}$ of the space of essentially bounded functions on G; see also [LL, Theorem 3.9].

Corollary 6.5. *Let G be a locally compact group and $\rho \in \widehat{G}$. Then the following assertions are equivalent:*

(i) *$L^1(G)^{**}$ is left character contractible.*
(ii) *$LUC(G)^*$ is left character contractible.*
(iii) *G is finite.*

Proof. Suppose that (i) holds and note that the restriction map from $L^1(G)^{**}$ onto $LUC(G)^*$ is a continuous epimorphism. This together with [HMT, Lemma 6.8] implies that $LUC(G)^*$ is character contractible. Now, suppose that (ii) holds. Then the restriction map from $LUC(G)^*$ onto $M(G)$ is a continuous epimorphism. Therefore, [HMT, Lemma 6.8] implies that $M(G)$ is character contractible, and so G is finite by Corollary 6.2.

If G is finite, it is trivial that $L^1(G)^{**}$ is left character contractible. ∎

In the following, let $A(G)$ denote the Fourier algebra of G as defined in [E], and recall the well-known fact that $\Delta(A(G)) = \{\phi_s : s \in G\}$, where $\phi_s(k) = k(s)$ for all $k \in A(G)$.

Proposition 6.6. *Let G be a locally compact group and $s \in G$. Then $A(G)$ is left ϕ_s-contractible if and only if G is discrete.*

Proof. First, suppose that $A(G)$ is left ϕ_s-contractible. Therefore there exists $m \in A(G)$ such that $m(\phi_s) = 1$ and $k(t)m(t) = k(s)m(t)$ for all $x \in G$. Thus $k = k(s)\chi_{\mathrm{coz}(m)}$ for all $k \in A(G)$. This shows that $m = \chi_{\{s\}}$. Since $m \in A(G)$, it follows that G is discrete. For the converse, suppose that G is discrete. Then $\chi_{\{s\}}$ is an element of $A(G)$ with $\phi_s(\chi_{\{s\}}) = 1$ and $k\chi_{\{s\}} = \phi_s(k)\chi_{\{s\}}$ for all $k \in A(G)$. This means $A(G)$ is ϕ_s-contractible. ∎

Corollary 6.7. *Let G be a locally compact group. Then $A(G)$ is left character contractible if and only if G is finite.*

Proof. We only need to recall from Propositions 3.4 and 6.6 that $A(G)$ is left character contractible if and only if G is discrete and $A(G)$ has a right identity; that is, G is finite. ∎

To prepare for the setting in our last results, set

$$\mathcal{L}A(G) := L^1(G) \cap A(G)$$

and define

$$|||h||| := \|h\|_1 + \|h\|_{A(G)} \quad (h \in \mathcal{L}A(G)).$$

Then $\mathcal{L}A(G)$ with the norm $|||\cdot|||$ is a Banach space; this space was introduced and extensively studied by Ghahramani and Lau [GL].

We recall that $\mathcal{L}A(G)$ with the convolution product is a Banach algebra and it is a dense left ideal of $L^1(G)$ such that $|||h * k||| \le \|h\|_1 |||k|||$ for all $k \in \mathcal{L}A(G)$ and $h \in L^1(G)$. Ghahramani and Lau [GL] called $\mathcal{L}A(G)$ endowed with convolution product the *Lebesgue–Fourier algebra* of G.

Proposition 6.8. *Let G be a locally compact group and $\rho \in \widehat{G}$. If the Lebesgue–Fourier algebra $\mathcal{L}A(G)$ is $\phi_\rho|_{\mathcal{L}A(G)}$-contractible, then G is compact.*

Proof. First note that ϕ_ρ is linear and multiplicative on $\mathcal{L}A(G)$ and we have

$$|\phi_\rho(h)| \le \|\phi_\rho\| \, \|h\|_1 \le \|\phi_\rho\| \, |||h|||$$

for all $h \in \mathcal{L}A(G)$. Since $\mathcal{L}A(G)$ is dense in $L^1(G)$, $\phi_\rho|_{\mathcal{L}A(G)}$ is a nonzero character on $\mathcal{L}A(G)$. Since $\mathcal{L}A(G)$ is a left ideal of $L^1(G)$ and $\|h\|_1 \le |||h|||$ for all $h \in \mathcal{L}A(G)$, an argument similar to the proof of Lemma 3.8 shows that $L^1(G)$ is ϕ_ρ-contractible, and so G is compact by Theorem 6.1. ∎

Corollary 6.9. *Let G be a locally compact group. Then the Lebesgue–Fourier algebra $\mathcal{L}A(G)$ is character contractible if and only if G is finite.*

Proof. The result follows immediately from Proposition 6.8 and the fact that $\mathcal{L}A(G)$ has an identity if and only if G is discrete. ∎

Let us recall from [GL] that $\mathcal{L}A(G)$ with the pointwise product is a Banach algebra which is a dense ideal of $A(G)$ and $|||k \cdot h||| \le \|k\|_{A(G)} |||h|||$ for all $h \in \mathcal{L}A(G)$ and $k \in A(G)$.

Proposition 6.10. *Let G be a locally compact group. Then $\mathcal{L}A(G)$ endowed with the pointwise product is character contractible if and only if G is finite.*

Proof. Suppose that $\mathcal{L}A(G)$ endowed with the pointwise product is character contractible. First note that $\phi \in \Delta(A(G))$ is linear and multiplicative on $\mathcal{L}A(G)$ and we have

$$|\phi(h)| \le \|\phi\| \, \|h\|_1 \le \|\phi\| \, |||h|||$$

for all $h \in \mathcal{L}A(G)$. Since $\mathcal{L}A(G)$ is dense in $A(G)$, $\phi|_{\mathcal{L}A(G)}$ is a nonzero character on $\mathcal{L}A(G)$. Now, since $\mathcal{L}A(G)$ is an ideal of $A(G)$ and $\|h\|_1 \le |||h|||$ for all $h \in \mathcal{L}A(G)$, the same argument in the proof of Lemma 3.8 shows that $A(G)$ is ϕ-contractible, and so G is discrete by Proposition 6.6. It is well-known that $\mathcal{L}A(G)$ endowed with the pointwise product has an identity if and only if G is compact; see [GL, Proposition 2.6]. Therefore the result follows from Theorem 3.8.

Conversely, if G is finite, then $A(G)$ is character contractible by Corollary 6.7, and hence $\mathcal{L}A(G) = A(G)$. Thus the norms $\|\!| \cdot |\!\|$ and $\| \cdot \|_{A(G)}$ are equivalent on $\mathcal{L}A(G)$ by the open mapping theorem, and so $\mathcal{L}A(G)$ is character contractible. ■

We conclude the paper by the following result on the Banach left $L^1(G)$-module $L^1(G)$ which does not remain valid for all Banach algebras; see Example 5.8.

COROLLARY 6.11. *Let G be a locally compact group. Then $L^1(G)$ is injective as a Banach left $L^1(G)$-module such that $h \cdot k = \phi_1(h)k$ for all $h, k \in L^1(G)$ if and only if G is amenable.*

Proof. By Proposition 5.2, the "only if" part is clear. Now suppose that G is amenable. Then $L^1(G)$ is right ϕ_1-amenable. So, $M(G)$ is injective as a Banach left $L^1(G)$-module with $h \cdot \mu = \phi_1(h)\mu$ for all $\mu \in M(G)$ and $h \in L^1(G)$, by Theorem 5.2. We also know $L^1(G)$ is a retract of $M(G)$; thus, $L^1(G)$ is injective as a Banach left $L^1(G)$-module. ■

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