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Frequently hypercyclic semigroups

by

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Abstract. We study frequent hypercyclicity in the context of strongly continuous semigroups of operators. More precisely, we give a criterion (sufficient condition) for a semigroup to be frequently hypercyclic, whose formulation depends on the Pettis integral. This criterion can be verified in certain cases in terms of the infinitesimal generator of the semigroup. Applications are given for semigroups generated by Ornstein–Uhlenbeck operators, and especially for translation semigroups on weighted spaces of p-integrable functions, or continuous functions that, multiplied by the weight, vanish at infinity.

1. Introduction. The hypercyclic behaviour of strongly continuous one-parameter semigroups was studied in a systematic way for the first time in the paper by Desch, Schappacher, and Webb [21]. They gave a sufficient condition for hypercyclicity of a semigroup based on the analysis of the point spectrum of the generator of the semigroup. Moreover they characterized hypercyclic translation semigroups defined on weighted spaces of continuous or integrable functions on the real line. However, during the last years it was shown that hypercyclicity appears in C_0-semigroups associated to "birth and death" equations for cell populations, transport equations, first order partial differential equations and diffusion operators as Ornstein–Uhlenbeck operators (see [13] for a survey on the subject; further references on hypercyclic semigroups and related properties are, e.g., [2, 3, 7–9, 16, 18, 20, 24, 28–31, 32, 34, 36, 37]).

We recall that, if X is a separable infinite-dimensional Banach space, a C_0-semigroup $(T_t)_{t\geq 0}$ of linear and continuous operators on X is said to be *hypercyclic* if there exists $x \in X$ (called a *hypercyclic vector* for the semigroup) such that the set $\{T_t x : t \geq 0\}$ is dense in X. An element $x \in X$ is said to be a *periodic point* for the semigroup if there exist $t > 0$ such that $T_t x = x$. A semigroup $(T_t)_{t\geq 0}$ is called *chaotic* if it is hypercyclic and the set of periodic points is dense in X.

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In [15], the second author, in collaboration with A. Conejero and V. Müller, proved that if $x \in X$ is a hypercyclic vector for $(T_t)_{t\geq 0}$ then for every $t > 0$ the set $\{T_{nt}x : n \in \mathbb{N}\}$ is dense in X, i.e. x is a hypercyclic vector for each single operator T_t, $t > 0$. In particular, hypercyclicity is inherited by discrete subsemigroups. However, this is not the case in general if we change the index set [17], or if we consider the chaos property [4].

Motivated by Birkhoff's ergodic theorem, Bayart and Grivaux introduced the notion of frequently hypercyclic operators [5] (see [6] and the references quoted therein, see also [10, 11, 27]), trying to quantify the frequency with which an orbit meets the open sets. This concept was extended to C_0-semigroups in [1]. We recall that the *lower density* of a measurable set $M \subset \mathbb{R}_+$ is defined by

$$\underline{\text{Dens}}(M) := \liminf_{N\to\infty} \mu(M \cap [0, N])/N,$$

where μ is the Lebesgue measure on $\mathbb{R}_+$. A C_0-semigroup $(T_t)_{t\geq 0}$ is said to be *frequently hypercyclic* if there exists $x \in X$ such that $\underline{\text{Dens}}(\{t \in \mathbb{R}_+ : T_t x \in U\}) > 0$ for any non-empty open set $U \subset X$.

If the lower density of a set $A \subset \mathbb{N}$ is defined by

$$\underline{\text{dens}}(A) := \liminf_{N\to\infty} \#\{n \leq N : n \in A\}/N,$$

an operator $T \in L(X)$ is said to be *frequently hypercyclic* if there exists $x \in X$ (called a *frequently hypercyclic vector*) such that, for every non-empty open subset $U \subset X$, the set $\{n \in \mathbb{N} : T^n x \in U\}$ has positive lower density. In [15] it was proved that if $x \in X$ is a frequently hypercyclic vector for $(T_t)_{t\geq 0}$, then for every $t > 0$ the x is a frequently hypercyclic vector for the operator T_t.

In [10, 11], Bonilla and Grosse-Erdmann improve a result of Bayart and Grivaux and give the following Frequent Hypercyclicity Criterion for operators (see also [25] for a probabilistic criterion).

PROPOSITION 1.1. *Let T be a continuous operator on a separable Banach space X. Assume that there exist a dense subset $X_0 \subseteq X$ and a map $S : X_0 \to X_0$ satisfying:*

(i) *$TSx = x$ for all $x \in X_0$;*
(ii) *$\sum_{n=1}^{\infty} T^n x$ is unconditionally convergent for all $x \in X_0$;*
(iii) *$\sum_{n=1}^{\infty} S^n x$ is unconditionally convergent for all $x \in X_0$.*

Then T is frequently hypercyclic.

The aim of the present paper is to give a continuous version of the Frequent Hypercyclicity Criterion. The unconditional convergence of the series in Proposition 1.1 will be replaced by the Pettis integrability of orbits under the semigroup. Thanks to this criterion we will show that, e.g., the wellknown Desch–Schappacher–Webb criterion for chaotic semigroups (see [21]) is actually a condition for frequent hypercyclicity. Moreover we prove that chaotic translation semigroups on weighted spaces of integrable functions defined on $[0,\infty[$ are frequently hypercyclic. We give a necessary condition on the weight for frequent hypercyclicity. Since several properties of the Pettis integral are used in the proofs, for the convenience of the reader we recall in an appendix the main definitions and basic results.

2. Frequent Hypercyclicity Criterion for semigroups

PROPOSITION 2.1. *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable Banach space X. Then the following conditions are equivalent:*

(i) *$(T_t)_{t\geq 0}$ is frequently hypercyclic.*
(ii) *For every $t>0$ the operator T_t is frequently hypercyclic.*
(iii) *There exists $t>0$ such that T_t is frequently hypercyclic.*

Proof. The implication (i)⇒(ii) was proved in [15]. It remains to prove that (iii) implies (i). We can assume that $t=1$; let x be a frequently hypercyclic vector for T_1. Let $y\in X$, and let U,V be 0-neighbourhoods such that $V+V\subseteq U$. By the strong continuity of $(T_t)_{t\geq 0}$, there exists $0<\delta<1$ such that $T_s y - y\in V$ for every $s\in[0,\delta]$. Moreover, by the local equicontinuity of $(T_t)_{t\geq 0}$, there exists a 0-neighbourhood V' such that $T_s(V')\subseteq V$ for every $s\in[0,\delta]$. By assumption,

$$\underline{\mathrm{dens}}\{n\in\mathbb{N}: T^n x\in y+V'\}>0.$$

If $T^n x\in y+V'$, then for every $t\in[n,n+\delta]$,

$$T_t x - y = T_{t-n}(T_n x - y) + T_{t-n}y - y \in T_{t-n}(V') + V\subseteq V+V\subseteq U.$$

Thus, for every $N\in\mathbb{N}$,

$$\frac{\mu\{t\leq N: T_t x\in y+U\}}{N}\geq \delta\frac{\#\{n\leq N: T_n x\in y+V'\}}{N},$$

hence

$$\liminf_{N\to\infty}\frac{\mu\{t\leq N: T_t x\in y+U\}}{N}\geq \delta\liminf_{N\to\infty}\frac{\#\{n\leq N: T_n x\in y+V'\}}{N}>0. \ ■$$

THEOREM 2.2. *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable Banach space X. Assume that there exist a dense subset $X_0\subseteq X$ and maps $S_t: X_0\to X$, $t>0$, satisfying:*

(i) *$T_tS_t x = x$, $T_tS_r x = S_{r-t}x$ for all $x\in X_0$, $t>0$, $r>t>0$;*
(ii) *$t\mapsto T_t x$ is Pettis integrable on $[0,\infty[$ for all $x\in X_0$;*
(iii) *$t\mapsto S_t x$ is Pettis integrable on $[0,\infty[$ for all $x\in X_0$.*

Then $(T_t)_{t\geq 0}$ is frequently hypercyclic.

Proof. We will show that T_1 is a frequently hypercyclic operator. The assertion will follow from the previous result. First observe that for any $x \in X_0$, the map $t \mapsto S_t x$ is continuous; indeed, if we fix $r > t$, $S_t x = T_{r-t}(S_r x)$.

To verify that T_1 is frequently hypercyclic, we will follow the proof of Theorem 2.4 in [10], by considering suitable unconditionally convergent series of integrals.

We can assume that $X_0 = \{y_1, y_2, \dots\}$ is a countable set. Conditions (ii), (iii) and Corollary 4.4 imply that there is an increasing sequence $\{N_l\}_{l\in\mathbb{N}}$ in $\mathbb{N}$ such that, for all $\lambda \le l$ and all compact sets $K \subset [N_l, \infty[$, we have

$$\Big\|\int_K T_t y_\lambda \,dt\Big\| < \frac{1}{l2^l}, \qquad \Big\|\int_K S_t y_\lambda \,dt\Big\| < \frac{1}{l2^l}. \tag{2.1}$$

For every $l, \nu \in \mathbb{N}$, set $\rho(l,\nu) = \nu$ and apply Lemma 2.5 of [11] to find pairwise disjoint sets $A(l,\nu) \subseteq \mathbb{N}$, $l, \nu \in \mathbb{N}$, of positive lower density such that, for all $n \in A(l,\nu)$, $m \in A(k,\mu)$ with $n \ne m$ and

$$n \ge \nu, \qquad |n-m| \ge \nu + \mu. \tag{2.2}$$

Define now

$$z_n = \begin{cases} y_l, & n \in A(l, N_l),\\ 0, & \text{otherwise}, \end{cases} \tag{2.3}$$

and set

$$x := \sum_{n\ge 1} \int_n^{n+1} S_t z_n \,dt. \tag{2.4}$$

To see that this series is convergent, observe that, for each $l \in \mathbb{N}$,

$$\sum_{n\in A(l,N_l)} \int_n^{n+1} S_t z_n \,dt = \sum_{n\in A(l,N_l)} \int_n^{n+1} S_t y_l \,dt \tag{2.5}$$

converges unconditionally by (2.1). On the other hand, for every finite subset $F \subset A(l, N_l)$, by (2.2) we get $\bigcup_{n\in F}[n, n+1] \subset [N_l, \infty[$, hence, by (2.1),

$$\Big\|\sum_{n\in F} \int_n^{n+1} S_t y_l \,dt\Big\| \le \frac{1}{l2^l}. \tag{2.6}$$

Therefore we easily see that the series in (2.4) is convergent. Fix $l \in \mathbb{N}$ and $n \in A(l, N_l)$. Then

$$(2.7)\quad T_{n+1}x = \sum_{j\neq n} T_{n+1}\Big(\int_j^{j+1} S_t z_j\,dt\Big) + T_{n+1}\Big(\int_n^{n+1} S_t z_n\,dt\Big)$$
$$= \sum_{j<n} \int_j^{j+1} T_{n+1-t} z_j\,dt + \int_n^{n+1} T_{n+1-t} y_l\,dt + \sum_{j>n}\int_j^{j+1} S_{t-n-1} z_j\,dt$$
$$= \sum_{m=1}^{n} \int_m^{m+1} T_s z_{n-m}\,ds + u_l + \sum_{m=1}^{\infty}\int_{m-1}^{m} S_r z_{n+m}\,dr,$$

where $u_l = \int_0^1 T_t y_l\,dt$. We analyze the first summand in (2.7):

$$\sum_{m=1}^{n}\int_m^{m+1} T_s z_{n-m}\,ds$$
$$= \sum_{\lambda=1}^{l}\Big(\sum_{n-m\in A(\lambda,N_\lambda)} \int_m^{m+1} T_s y_\lambda\,ds\Big) + \sum_{\lambda>l}\Big(\sum_{n-m\in A(\lambda,N_\lambda)} \int_m^{m+1} T_s y_\lambda\,ds\Big).$$

By (2.2), since $n \in A(l,N_l)$, $n-m \in A(\lambda,N_\lambda)$, we necessarily have $m = n-(n-m) \geq N_l + N_\lambda$. Thus

$$(2.8)\qquad \Big\|\sum_{m=1}^{n}\int_m^{m+1} T_s z_{n-m}\,ds\Big\| \leq \sum_{\lambda=1}^{l}\frac{1}{l2^l} + \sum_{\lambda>l}\frac{1}{\lambda 2^\lambda} < \frac{2}{2^l}.$$

Analogously, by (2.1), we obtain

$$(2.9)\qquad \Big\|\sum_{m=1}^{\infty}\int_{m-1}^{m} S_r z_{n+m}\,dr\Big\| < \frac{2}{2^l},$$

which gives, for every $n \in A(l,N_l)$,

$$(2.10)\qquad \|T_{n+1}x - u_l\| < \frac{4}{2^l}.$$

Since $A(l,N_l)$ has positive lower density for each $l\in\mathbb{N}$, we are done if we show that $(u_l)_l$ is a dense sequence in X. Indeed, $u_l = Ry_l$, $l\in\mathbb{N}$, where R is the continuous operator defined by

$$Rx := \int_0^1 T_t x\,dt.$$

We need to prove that R has dense range. First observe that $I - T_1$ has dense range. Indeed, otherwise there would exist $\phi\in X'$, $\phi\neq 0$, such that $\langle\phi, x - T_1 x\rangle = 0$ for all $x\in X$. This implies that, for every $n\in\mathbb{N}$ and $x\in X$,

$\langle\phi, x\rangle = \langle\phi, T_n x\rangle = 0$ for all $x \in X$. In particular, if $s > 0$, then

$$\int_n^{n+s} \langle\phi, T_t y_l\rangle\, dt = \int_0^s \langle\phi, T_{u+n} y_l\rangle\, du = \int_0^s \langle\phi, T_u y_l\rangle\, du.$$

The left term tends to 0, by (2.1), as $n \to \infty$. Since the right term is fixed and $s > 0, l \in \mathbb{N}$ were arbitrary, we have $\langle\phi, x\rangle = 0$ for all $x \in X$, which is a contradiction. Finally observe that if $(A, D(A))$ is the generator of $(T_t)_{t\geq 0}$, then for every $x \in D(A)$,

$$(I - T_1)x = \int_0^1 T_t Ax\, dt = R(Ax),$$

thus $(I - T_1)(D(A)) \subseteq R(X)$. By the density of $D(A)$ in X, we get

$$X = \overline{(I - T_1)(X)} = \overline{(I - T_1)(D(A))} \subseteq \overline{R(X)}. \ \blacksquare$$

COROLLARY 2.3. *Let X be a separable complex Banach space, and $(T_t)_{t\geq 0}$ a C_0-semigroup with generator A. Assume that there exists a family $(f_j)_{j\in\Gamma}$ of locally bounded measurable maps $f_j : I_j \to X$ such that I_j is an interval in $\mathbb{R}$, $Af_j(t) = itf_j(t)$ for every $t \in I_j$, $j \in \Gamma$ and* $\operatorname{span}\{f_j(t) : j \in \Gamma, t \in I_j\}$ *is dense in X. If either*

(a) *$f_j \in C^2(I_j, X)$, $j \in \Gamma$, or*
(b) *X does not contain c_0 and $\langle\varphi, f_j\rangle \in C^1(I_j)$, $\varphi \in X'$, $j \in \Gamma$,*

then $(T_t)_{t\geq 0}$ is frequently hypercyclic.

First we prove the following:

(a)′ If (a) holds then there exists a family $(g_\lambda)_{\lambda\in\Lambda}$ of functions $g_\lambda \in C^2(\mathbb{R}, X)$ with compact support such that $Ag_\lambda(t) = itg_\lambda(t)$ for every $t \in \mathbb{R}$ and $\lambda \in \Lambda$, and $\operatorname{span}\{g_\lambda(t) : \lambda \in \Lambda,\ t \in \mathbb{R}\}$ is dense in X.
(b)′ If (b) holds then there exists a family $(g_\lambda)_{\lambda\in\Lambda}$ of bounded measurable functions $g_\lambda : \mathbb{R} \to X$ with compact support such that $\langle\varphi, g_\lambda\rangle \in C^1(\mathbb{R})$ for every $\varphi \in X'$, $Ag_\lambda(t) = itg_\lambda(t)$ for every $t \in \mathbb{R}$ and $\lambda \in \Lambda$, and $\operatorname{span}\{g_\lambda(t) : \lambda \in \Lambda, t \in \mathbb{R}\}$ is dense in X.

If $I_j =]x_j - r_j, x_j + r_j[$ is a bounded interval, consider a sequence $(\phi_n^j)_n \subset C^\infty(\mathbb{R})$ such that $\phi_n^j(s) = 1$ if $|s - x_j| \leq r_j - 1/n$ and $\phi_n^j(s) = 0$ if $|s - x_j| > r_j - 1/(2n)$. If we extend f_j outside I_j setting $f_j = 0$ in $\mathbb{R} \setminus I_j$, then $\phi_n^j f_j \in C^2(\mathbb{R}, X)$ for every $n \in \mathbb{N}$ if (a) holds, and $\langle\varphi, \phi_n^j f_j\rangle \in C^1(\mathbb{R})$ for every $\varphi \in X'$ if (b) holds. Moreover $(\phi_n^j f_j)_n$ converges pointwise to f_j and

$$A(\phi_n^j(t) f_j(t)) = \phi_n^j(t) Af_j(t) = it\phi_n^j(t) f_j(t)$$

for every $t \in \mathbb{R}$ and $j \in \Gamma$. If the interval I_j is unbounded, for example $I_j =]a_j, \infty[$, the argument runs analogously, by considering functions $\phi_n^j \in$

$C^\infty(\mathbb{R})$ with support in $]a_j+1/n, n[$. It remains to show that

$$\overline{\operatorname{span}\{\phi_n^j f_j(t) : j\in\Gamma,\ t\in I_j,\ n\in\mathbb{N}\}} = X. \tag{2.11}$$

If $\varphi\in X'$ and $\langle\varphi,\phi_n^j(t)f_j(t)\rangle=0$ for every $j\in\Gamma$, $t\in I_j$, $n\in\mathbb{N}$, then, by taking the limit as $n\to\infty$, we get $\langle\varphi,f_j(t)\rangle=0$ for every $t\in I_j$ and $j\in\Gamma$. Then, by the assumption on the ranges of the f_j, it follows that $\varphi=0$.

Proof of Corollary 2.3. From now on, let $\Lambda=\{(j,n): j\in\Gamma,\ n\in\mathbb{N}\}$ and, for every $\lambda=(j,n)\in\Lambda$, set $g_\lambda=\phi_n^j f_j$. Then the family $(g_\lambda)_{\lambda\in\Lambda}$ satisfies the assertion in (a)′ (resp. (b)′) if $(f_j)_{j\in J}$ satisfies (a) (resp. (b)). We will show that $(T_t)_{t\ge 0}$ is frequently hypercyclic. For every $r\in\mathbb{R}$ and $\lambda\in\Lambda$, set

$$\psi_{r,\lambda} := \int_{\mathbb{R}} e^{-irs} g_\lambda(s)\,ds = \mathcal{F}(g_\lambda)(r),$$

where $\mathcal{F}$ denotes the X-valued Fourier transform. The set $\{\psi_{r,\lambda} : r\in\mathbb{R}, \lambda\in\Lambda\}$ is dense in X. Indeed, let $\varphi\in X'$ be such that for all $r\in\mathbb{R}$ and $\lambda\in\Lambda$,

$$\langle\varphi,\psi_{r,\lambda}\rangle = \int_{\mathbb{R}} e^{-irs}\langle\varphi,g_\lambda(s)\rangle\,ds = 0.$$

This means that the Fourier transform of the (scalar) function $s\mapsto\langle\varphi,g_\lambda(s)\rangle$ vanishes on $\mathbb{R}$, hence, taking into account that $\langle\varphi,g_\lambda\rangle$ is continuous, we get $\langle\varphi,g_\lambda\rangle=0$ on $\mathbb{R}$, therefore $\varphi=0$. For every $t>0$ set

$$S_t\psi_{r,\lambda} := \int_{\mathbb{R}} e^{-i(t+r)s} g_\lambda(s)\,ds = \mathcal{F}(g_\lambda)(r+t) = \psi_{r+t,\lambda}.$$

We have

$$T_t\psi_{r,\lambda} = \int_{\mathbb{R}} e^{-i(r-t)s} g_\lambda(s)\,ds = \mathcal{F}(g_\lambda)(r-t) = \psi_{r-t,\lambda},$$

and $T_tS_t\psi_{r,\lambda}=\psi_{r,\lambda}$, $T_tS_s\psi_{r,\lambda}=S_{s-t}\psi_{r,\lambda}$ for all $\lambda\in\Lambda$, $r\in\mathbb{R}$, $s>t>0$.

It remains to show that the maps $t\mapsto S_t\psi_{r,\lambda}$ and $t\mapsto T_t\psi_{r,\lambda}$ are Pettis integrable on $[0,\infty[$ for every $r\in\mathbb{R}$ and $\lambda\in\Lambda$.

In case (a)′, $\mathcal{F}(g_\lambda)$ is Bochner integrable. Indeed, $g_\lambda\in C^2(\mathbb{R},X)$ and has compact support. Hence g_λ'' is Fourier integrable and

$$\mathcal{F}(g_\lambda'')(r) = -r^2\mathcal{F}(g_\lambda).$$

Therefore $\mathcal{F}(g_\lambda)$ is Bochner integrable on $\mathbb{R}$. It follows that $t\mapsto T_t(\psi_{r,\lambda})$ and $t\mapsto S_t(\psi_{r,\lambda})$ are Bochner integrable on $[0,\infty[$.

In case (b)′, we prove that $\mathcal{F}(g_\lambda)$ is Pettis integrable on $[0,\infty[$. It will follow that $t\mapsto T_t(\psi_{r,\lambda})$ and $t\mapsto S_t(\psi_{r,\lambda})$ are Pettis integrable on $[0,\infty[$. First observe that $\mathcal{F}(g_\lambda)$ is continuous, hence measurable. Let $\varphi\in X'$ and consider $g(s) = \langle \varphi, g_\lambda(s)\rangle \in C_c^1(\mathbb{R})$. We have

$$\langle \varphi, \mathcal{F}(g_\lambda)(r)\rangle = \int_{\mathbb{R}} e^{-irs}\langle \varphi, g_\lambda(s)\rangle\, ds = \mathcal{F}(g)(r).$$

Moreover, $g' \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\mathcal{F}(g')(r) = ir\mathcal{F}(g)(r) \in L^2(\mathbb{R})$. Hence, for $a > 0$,

$$\int_{|r|>a} |\mathcal{F}(g)(r)|\, dr \le \Big(\int_{|r|\ge a} \frac{1}{r^2}\, dr\Big)^{1/2} \Big(\int_{|r|>a} r^2 |\mathcal{F}(g)|^2\, dr\Big)^{1/2} < \infty.$$

Therefore $\mathcal{F}(g) \in L^1(\mathbb{R})$. By Theorem 4.5, this implies that $\mathcal{F}(g_\lambda)$ is Pettis integrable on $[0,\infty[$. ∎

Remarks 2.4. (1) With the same argument as in [23, Remark 2.2], one can show that the Desch–Schappacher–Webb criterion for chaos of C_0-semigroups (see [21]) implies frequent hypercyclicity.

(2) There is a connection between Corollary 2.3 and the recent results of S. Grivaux in [26]. Indeed assume that one of the conditions (a) or (b) (or equivalently (a)′ or (b)′) holds for a countable family of locally bounded functions $\{f_j\}_{j\in\mathbb{Z}}$. For every $j,k \in \mathbb{Z}$ and $\theta \in [0,2\pi[$ define

$$E_{j,k}(e^{i\theta}) = f_j(\theta + 2k\pi).$$

The family $\{E_{j,k} : \mathbb{T} \to X : j,k \in \mathbb{Z}\}$ is a countable family of bounded eigenvector fields for the operator T_1, where $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, such that $\operatorname{span}\{E_{j,k}(\lambda) : \lambda \in \mathbb{T},\ j,k \in \mathbb{Z}\}$ is dense in X. Actually $\operatorname{span}\{E_{j,k}(\lambda) : \lambda \in \mathbb{T}\setminus D,\ j,k \in \mathbb{Z}\}$ is dense in X for every countable subset D of $\mathbb{T}$. Indeed, if $D = \{e^{i\theta_n} : n \in \mathbb{N}\}$ with $\theta_n \in [0,2\pi[$, then

$$\begin{aligned}&\operatorname{span}\{E_{j,k}(\lambda) : \lambda \in \mathbb{T}\setminus D,\ j,k \in \mathbb{Z}\}\\ &\qquad = \operatorname{span}\{f_j(s) : s \in \mathbb{R}\setminus\{\theta_n + 2k\pi : n \in \mathbb{N},\ k \in \mathbb{Z}\},\ j \in \mathbb{Z}\},\end{aligned}$$

which is dense in X by the weak continuity of each f_j. Thus, by [26, Proposition 4.1], T_1 has perfectly spanning unimodular eigenvectors, i.e. there exists a probability measure σ on the unit circle $\mathbb{T}$ such that for every σ-measurable subset of A with $\sigma(A) = 1$, $\operatorname{span}\{\ker(T-\lambda) : \lambda \in A\}$ is dense in X.

Example 2.5. Consider the linear perturbation of the one-dimensional Ornstein–Uhlenbeck operator

$$\mathcal{A}_\alpha u = u'' + bxu' + \alpha u,$$

where $\alpha \in \mathbb{R}$, with domain

$$D(\mathcal{A}_\alpha) = \{u \in L_2(\mathbb{R}) \cap W^{2,2}_{\mathrm{loc}}(\mathbb{R}) : \mathcal{A}_\alpha u \in L_2(\mathbb{R})\}.$$

In [14], it was proved that if $\alpha > b/2 > 0$, then the semigroup generated by $\mathcal{A}_\alpha$ in $L_2(\mathbb{R})$ is chaotic. Actually the semigroup is frequently hypercyclic.

Indeed, for every $\mu \in \mathbb{C}$ with $\Re\mu < -b/2 + \alpha$ the functions u_μ^1 and u_μ^2 whose Fourier transforms are

$$\widehat{u_\mu^1}(\xi) = e^{-\xi^2/2b}\xi|\xi|^{-(2+(\mu-\alpha)/b)}, \quad \widehat{u_\mu^2}(\xi) = e^{-\xi^2/2b}|\xi|^{-(1+(\mu-\alpha)/b)},$$

are eigenfunctions of $\mathcal{A}_\alpha$ (see [14, 33]). For each $s \in \mathbb{R}$, consider the functions $f_1(s) = u_{is}^1$ and $f_2(s) = u_{is}^2$. For every $\phi \in X' = L_2(\mathbb{R})$ and $j = 1, 2$, by the Parseval equality, we have

$$\langle \phi, f_j(s)\rangle = \int_{\mathbb{R}} \phi(x) u_{is}^j(x)\, dx = \int_{\mathbb{R}} \widehat{\phi}(x)\widehat{u}_{is}^j(x)\, dx, \qquad s \in \mathbb{R}.$$

It is immediate to verify that $\langle \phi, f_j\rangle \in C^1(\mathbb{R})$, by Lebesgue's theorem. The argument of [14] shows that $\operatorname{span}\{f_i(s) : i = 1, 2,\ s \in \mathbb{R}\}$ is dense in $L^2(\mathbb{R})$. Therefore the semigroup is frequently hypercyclic by Corollary 2.3.

We will see that the Frequent Hypercyclicity Criterion for semigroups implies chaos for each single operator of the semigroup. It is interesting to observe that this is in general stronger than the chaoticity of the semigroup since, by recent results of Bayart and Bermúdez [4], there are chaotic C_0-semigroups $(T_t)_{t\geq 0}$ such that no single operator T_t is chaotic, and chaotic C_0-semigroups $(T_t)_{t\geq 0}$ containing non-chaotic operators T_{t_0}, $t_0 > 0$, and at the same time chaotic T_{t_1} for some $t_1 > 0$.

Proposition 2.6. *Let X be a separable Banach space and let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X that satisfies the Frequent Hypercyclicity Criterion of Theorem* 2.2. *Then the operator T_{t_0} is chaotic for every $t_0 > 0$.*

Proof. Given $t_0 > 0$, we know that T_{t_0} is frequently hypercyclic, thus hypercyclic [15]. Given $x \in X$ and $\varepsilon > 0$ we want to find a T_{t_0}-periodic point $z \in X$ such that $\|x - z\| < \varepsilon$. Indeed, let $y \in X_0$ be such that $\|x - y\| < \varepsilon$. By continuity, we fix $\delta > 0$ such that

$$\Big\| x - \delta^{-1}\int_0^\delta T_s y\, ds\Big\| < \varepsilon.$$

Now, let $n \in \mathbb{N}$ be large enough so that, by Corollary 4.4 and for $t := nt_0$, the element

$$z := \delta^{-1}\Big[\sum_{k\geq 1}\int_0^\delta S_{kt-s} y\, ds + \int_0^\delta T_s y\, ds + \sum_{k\geq 1}\int_0^\delta T_{kt+s} y\, ds\Big]$$

satisfies $\|x - z\| < \varepsilon$. Finally, observe that the hypothesis of the Frequent Hypercyclicity Criterion and continuity give $T_{t_0}^n z = T_t z = z$. ∎

We point out the connection between the Frequent Hypercyclicity Criterion for semigroups and the Frequent Hypercyclicity Criterion for operators.

PROPOSITION 2.7. *Let X be a separable Banach space and let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. Assume that there exist a dense subset $X_0 \subseteq X$ with $T_t(X_0) \subseteq X_0$ for every $t > 0$, and maps $S_t : X_0 \to X_0$, $t > 0$, satisfying*

(i) *$T_t S_t x = x$, $S_r T_t x = T_t S_r x = S_{r-t} x$ for all $x \in X_0$, $r > t > 0$;*
(ii) *$t \mapsto T_t x$ is Pettis integrable on $[0, \infty[$ for all $x \in X_0$;*
(iii) *$t \mapsto S_t x$ is Pettis integrable on $[0, \infty[$ for all $x \in X_0$.*

Then the operator T_t satisfies the Frequent Hypercyclicity Criterion for every $t > 0$.

Proof. For the sake of simplicity, let $t = 1$. First observe that $S_n x = S_n T_1 S_1 x = S_{n-1} S_1 x = S_{n-2} S_1^2 = \cdots = S_1^n x$ for every $x \in X_0$. For every $x \in X_0$, set $y = \int_0^1 T_t x\,dt$. Then the series

$$\sum_{n=1}^{\infty} T_n y = \sum_{n=1}^{\infty} \int_n^{n+1} T_t x\,dt$$

is unconditionally convergent by Proposition 4.3. Analogously, since

$$\int_{n-1}^{n} S_s x\,ds = \int_0^1 S_{n-1+s} x\,ds = \int_0^1 S_{n-u} x\,ds = S_n \int_0^1 T_s x\,ds = S_n y,$$

the series $\sum_{n=1}^{\infty} S_n y$ is unconditionally convergent. Finally we observe that, by the same argument used in the proof of Theorem 2.2, the set $\{\int_0^1 T_t x\,dt : x \in X_0\}$ is dense. ∎

Proposition 2.7 establishes a link between the continuous and discrete criteria for frequent hypercyclicity which, together with some analogous connections in the case of hypercyclicity [16] and mixing [7], motivate the following natural questions.

PROBLEM 2.8. *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup and $t_0 > 0$ such that T_{t_0} satisfies the Frequent Hypercyclicity Criterion of Proposition* 1.1. *Does $(T_t)_{t\geq 0}$ satisfy the Frequent Hypercyclicity Criterion of Theorem* 2.2*? Does it follow at least that every single operator T_t, $t > 0$, satisfies the Frequent Hypercyclicity Criterion for operators?*

We thank the referee for the second question in the above problem.

3. Translation semigroups. An *admissible weight* function on $[0, \infty[$ is a measurable function $\rho : [0, \infty[\to \mathbb{R}$ satisfying the following conditions:

(i) $\rho(t) > 0$ for all $t \in [0, \infty[$;
(ii) there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\rho(\tau) \leq M e^{\omega t} \rho(\tau + t)$ for all $\tau \in [0, \infty[$ and all $t > 0$.

We recall the following useful result.

Lemma 3.1 ([21]). *If ρ is an admissible weight, then for every $l > 0$ there exist $A, B > 0$ such that for every $\sigma \in [0,\infty[$ and every $t \in [\sigma, \sigma + l]$,*

$$A\rho(\sigma) \le \rho(t) \le B\rho(\sigma + l).$$

We also need the following lemma, which in particular unifies different characterizations of chaos for translation semigroups (see [19] and [32]).

Lemma 3.2. *Let $\rho : [0,\infty[\to \mathbb{R}^+$ be an admissible weight.*

(1) *The following conditions are equivalent:*

- (i) *For all $b \ge 0$ the series $\sum_{k=1}^{\infty} \rho(b + k)$ is convergent.*
- (ii) *For all $b \ge 0$ there exists $P > 0$ such that $\sum_{k=1}^{\infty} \rho(b + kP)$ is convergent.*
- (iii) *There exists $D \subseteq \mathbb{N}$ with bounded gaps (i.e. there exists $M > 0$ such that $D \cap [n, n + M] \ne \emptyset$ for every $n \in \mathbb{N}$) such that $\sum_{k\in D} \rho(k)$ is convergent.*
- (iv) *The series $\sum_{k=1}^{\infty} \rho(k)$ is convergent.*
- (v) *$\int_0^{\infty} \rho(s)\, ds < \infty$.*

(2) *ρ is bounded if and only if there exists $D \subseteq \mathbb{N}$ with bounded gaps such that ρ is bounded on D.*

Proof. We prove (2). The proof of (1) can be obtained by similar considerations and usual comparisons between integrals and series. Assume that $\rho(h) \le K$ for every $h \in D$, where $D \subseteq \mathbb{N}$ with bounded gaps. Hence, there is $M \in \mathbb{N}$ such that $[Mn, Mn + M] \cap D \ne \emptyset$ for every $n \in \mathbb{N}$. Choose $h_n \in [Mn, Mn+M] \cap D$. By Lemma 3.1, there exist $A_M, B_M > 0$ such that

$$A_M \rho(Mn) \le \rho(h_n) \le B_M \rho(Mn + M), \qquad n \in \mathbb{N},$$

hence $\rho(Mn) \le A_M^{-1} K$ for every $n \in \mathbb{N}$. On the other hand, for every $s \ge 0$ there exists $k \in \mathbb{N} \cup \{0\}$ such that $x \in [Mk, Mk + M]$, and therefore

$$\rho(s) \le B_M \rho(Mk + M) \le K A_M^{-1} B_M. \ \blacksquare$$

We consider the following function spaces:

$$L_p^{\rho}([0,\infty[) = \{u : [0,\infty[\to \mathbb{R} : u \text{ is measurable and } \|u\|_p < \infty\},$$

where $\|u\|_p = (\int_0^{\infty} |u(t)|^p \rho(t)\, dt)^{1/p}$, and

$$C_0^{\rho}([0,\infty[) = \{u : [0,\infty[\to \mathbb{R} : u \text{ is continuous and } \lim_{x\to\infty} u(x)\rho(x) = 0\},$$

with $\|u\|_{\infty} = \sup_{t\in[0,\infty[} |u(t)|\rho(t)$. If X is any of the spaces above, the translation semigroup $(T_t)_{t\ge0}$ is defined as usual by $T_t f(x) = f(x+t)$, $t \ge 0$, $x \in I$, and it is a C_0-semigroup (see e.g. [21]).

Hypercyclic and chaotic translation semigroups have been characterized in [21, 19, 32]. If X is one of the spaces $L_p^{\rho}([0,\infty[)$ or $C_0^{\rho}([0,\infty[)$ with an admissible weight function ρ, the translation semigroup $(T_t)_{t\ge0}$ on X is hypercyclic if and only if $\liminf_{t\to\infty} \rho(t) = 0$. If $X = C_0^\rho([0,\infty[)$, then the translation semigroup $(T_t)_{t\geq 0}$ on X is chaotic if and only if $\lim_{x\to\infty} \rho(x) = 0$. For $X = L_p^\rho([0,\infty[)$, $(T_t)_{t\geq 0}$ is chaotic if and only if any of the conditions of Lemma 3.2(1) are satisfied.

PROPOSITION 3.3. *Let ρ be an admissible weight on $[0,\infty[$, $X = L_p^\rho([0,\infty[)$, $1 \leq p < \infty$ and $(T_t)_{t\geq 0}$ the translation semigroup on X. Then $(T_t)_{t\geq 0}$ is chaotic if and only if it satisfies the Frequent Hypercyclicity Criterion for semigroups.*

Proof. If $(T_t)_{t\geq 0}$ is chaotic, then $\int_0^\infty \rho(s)\,ds$ is finite. Let X_0 be the space generated by the characteristic functions of bounded subintervals of $[0,\infty[$, which is dense in $L_p^\rho([0,\infty[)$. For every $t > 0$ and $f \in X_0$ we set

$$S_t f(s) = \begin{cases} f(s-t), & s \geq t, \\ 0, & s \in [0,t[. \end{cases}$$

Observe that $T_t S_t f = f$ and $T_t S_r f = S_{r-t} f$ for all $f \in X_0$, $t > 0$, $r > t > 0$. Moreover $\int_{\mathbb{R}+} \|T_t f\|\,dt$ converges for all $f \in X_0$, because of the compact support of f, hence $\int_{\mathbb{R}+} T_t f\,dt$ is Pettis integrable. On the other hand, consider $f = \chi_{[a,b]}$, with $0 \leq a < b$. If $p = 1$, we have

$$\|S_t f\| = \int_t^{t+b} \rho(s)\,ds = \int_0^b \rho(s+t)\,ds \leq bB\rho(t+b)$$

where B is a positive constant such that $\rho(s+t) \leq B\rho(t+b)$ for all $s \in [0,b]$ and $t \geq 0$. Since $\int_0^\infty \rho(t+b)\,dt$ is finite, we see that $t \mapsto S_t f$ is Pettis integrable.

Let $p > 1$ and let $\phi \in L_{p'}^\rho([0,\infty[)$, where $1/p + 1/p' = 1$. To prove that $t \mapsto S_t f$ is Pettis integrable, by Theorem 4.5, we have to show that $t \mapsto \langle \phi, S_t f\rangle \in L_1([0,\infty[)$. We have

$$\langle \phi, S_t f\rangle = \int_t^\infty f(s-t)\rho(s)\,ds = \int_0^\infty f(u)\rho(t+u)\,du.$$

A straightforward application of the Tonelli and Fubini theorems (as for the proof of the integrability of convolution) gives the assertion. ∎

With similar techniques we can prove the following result for translation semigroups on weighted spaces of continuous functions.

PROPOSITION 3.4. *Let ρ be an admissible weight on $[0,\infty[$ and $(T_t)_{t\geq 0}$ the translation semigroup on $C_0^\rho([0,\infty[)$. If $\int_0^\infty \rho(s)\,ds < \infty$, then $(T_t)_{t\geq 0}$ satisfies the Frequent Hypercyclicity Criterion for semigroups.*

REMARK 3.5. It should be observed that, for the translation semigroup $(T_t)_{t\geq 0}$ on $L_p^\rho([0,\infty[)$, $(T_t)_{t\geq 0}$ is chaotic if and only if every operator T_t satisfies the Frequent Hypercyclicity Criterion for operators by Proposition 2.7.

In [27] the authors obtain a necessary condition for frequent hypercyclicity of unilateral weighted shifts on ℓ^p. Inspired by their condition, we can obtain an analogous one for translation semigroups. The proof partially follows the one in [27], and standard arguments using Lemmas 3.1 and 3.2(2).

PROPOSITION 3.6. *Let ρ be an admissible weight on $[0,\infty[$, and $(T_t)_{t\geq 0}$ the translation semigroup in $L_p^\rho([0,\infty[)$. If $(T_t)_{t\geq 0}$ is frequently hypercyclic, then for every $\varepsilon > 0$ there exists a sequence $(n_k)_k$ in $\mathbb{N}$ with positive lower density such that $\sum_{k>i}\rho(n_k - n_i) < \varepsilon$ for all $i \in \mathbb{N}$. Moreover, ρ is bounded.*

If $(T_t)_{t\geq 0}$ is a frequently hypercyclic translation semigroup on $C_0^\rho([0,\infty[)$, then for every $\varepsilon > 0$ there exists a sequence $(n_k)_k$ in $\mathbb{N}$ with positive lower density such that $\rho(n_k - n_i) < \varepsilon$ for all $i \in \mathbb{N}$ and $k > i$.

EXAMPLE 3.7. Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a C^1 function with derivative bounded by $\omega > 0$ and such that $\limsup_{s\to\infty}\phi(s) = \infty$, and $\liminf_{s\to\infty}\phi(s) = -\infty$. (For example, consider a C^1 function such that $\phi(s) = s\sin(\log s)$ if $s \geq 1$.) Set $\rho = e^{-\phi}$. Clearly $\rho > 0$ and, if $t, \tau > 0$, we have

$$\frac{\rho(\tau)}{\rho(t+\tau)} = e^{-\int_\tau^{t+\tau}\phi'(s)\,ds} \leq e^{\omega\tau}.$$

Hence ρ is an admissible weight. The translation semigroup on $L_p^\rho([0,\infty[)$ is hypercyclic, since $\liminf_{s\to\infty}\rho(s) = 0$, but it is not frequently hypercyclic, since ρ is unbounded.

4. Appendix. We recall in this Appendix the main definitions and results about Pettis integrability. Let X be a Banach space and (Ω,μ) a σ-finite measure space. A function $f : \Omega \to X$ is said to be *weakly μ-measurable* if the scalar function $\varphi \circ f$ is μ-measurable for every $\varphi \in X'$, where X' denotes the topological dual of X; f is said to be *μ-measurable* if there exists a sequence $(f_n)_n$ of simple functions such that $\lim_{n\to\infty}|f_n - f| = 0$ μ-a.e.

LEMMA 4.1 (Dunford). *If f is weakly μ-measurable and $\varphi\circ f \in L_1(\Omega,\mu)$ for every $\varphi \in X'$, then for every measurable $E \subseteq \Omega$ there exists $x_E \in X''$ such that*

$$x_E(\varphi) = \int_E \varphi\circ f\,d\mu \quad \text{for every } \varphi \in X'.$$

DEFINITION 4.2. If $f : \Omega \to X$ is weakly μ-measurable and $\varphi \circ f \in L_1(\Omega,\mu)$ for every $\varphi \in X'$, then f is called *Dunford integrable*. The *Dunford integral* of f over a measurable $E \subseteq \Omega$ is defined to be the element $x_E \in X''$ such that

$$x_E(\varphi) = \int_E \varphi\circ f d\mu \quad \text{for every } \varphi \in X'.$$

In the case that $x_E \in X$ for every measurable E, then f is called *Pettis integrable* and x_E is called the *Pettis integral* of f over E and will be denoted by $(P)\text{-}\int_E f\,d\mu$.

Clearly the Dunford and Pettis integrals coincide if X is a reflexive space. Moreover, if $\|f\|$ is integrable on Ω (i.e. f is Bochner integrable on Ω), then f is Pettis integrable on X.

THEOREM 4.3 (Pettis). *If f is Pettis integrable, then for every sequence $(E_n)_n$ of disjoint measurable sets in Ω*

$$\int\limits_{\bigcup_{n\in\mathbb{N}} E_n} f\,d\mu = \sum_{n\in\mathbb{N}} \int\limits_{E_n} f\,d\mu,$$

where the series converges unconditionally.

COROLLARY 4.4. *If $f : [0,\infty[\, \to X$ is Pettis integrable on $[0,\infty[$, then for every $\varepsilon > 0$ there exists $N > 0$ such that for every compact set $K \subset [N,\infty[$,*

$$\Big\|\int\limits_K f(t)\,dt\Big\| < \varepsilon.$$

Proof. Assume that there exists $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ there exists a compact set $K_n \subseteq [n,\infty[$ such that $\|\int_{K_n} f(s)\,ds\| > \varepsilon$. It is easy to find a sequence $(k_n)_n$ of natural numbers such that the sets K_{k_n} are mutually disjoint. Then

$$\int\limits_{\bigcup_n K_{k_n}} f(s)\,ds = \sum_{n=1}^{\infty} \int\limits_{K_n} f(s)\,ds,$$

hence $\lim_{n\to\infty} \int_{K_n} f(s)\,ds = 0$, a contradiction. ∎

THEOREM 4.5. *If the Banach space X does not contain c_0 and (Ω,μ) is σ-finite measure space, then a measurable Dunford integrable function $f : \Omega \to X$ is Pettis integrable.*

The proofs of all these results can be found in [22] for the case of a finite measure space, but they easily extend to σ-finite measure spaces. In particular, the proof of the deep Theorem 4.5 follows analogously to the finite measure space case ([22, Theorem 7, p. 54]) taking into account the following decomposition theorem due to J. K. Brooks (see [12, Theorem 1]).

THEOREM 4.6. *Let (Ω,μ) be a σ-finite measure space. If $f : \Omega \to X$ is a measurable weakly integrable function, then f can be represented in the form $f = g + h$ μ-a.e. where g is a bounded Bochner integrable function and h assumes at most a countable number of values in X.*

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References

[1] C. Badea and S. Grivaux, *Unimodular eigenvalues, uniformly distributed sequences and linear dynamics*, Adv. Math. 211 (2007), 766–793.

[2] J. Banasiak, *Birth-and-death type systems with parameter and chaotic dynamics of some linear kinetic models*, Z. Anal. Anwendungen 24 (2005), 675–690.

[3] J. Banasiak and M. Moszyński, *A generalization of Desch–Schappacher–Webb criteria for chaos*, Discrete Contin. Dynam. Systems 12 (2005), 959–972.

[4] F. Bayart and T. Bermúdez, *Semigroups of chaotic operators*, Bull. London Math. Soc. 41 (2009), 823–830.

[5] F. Bayart and S. Grivaux, *Frequently hypercyclic operators*, Trans. Amer. Math. Soc. 358 (2006), 5083–5117.

[6] F. Bayart and E. Matheron, *Dynamics of Linear Operators*, Cambridge Tracts in Math. 179, Cambridge Univ. Press, Cambridge, 2009.

[7] T. Bermúdez, A. Bonilla, J. A. Conejero and A. Peris, *Hypercyclic, topologically mixing and chaotic semigroups on Banach spaces*, Studia Math. 170 (2005), 57–75.

[8] T. Bermúdez, A. Bonilla, and H. Emamirad, *Chaotic tensor product semigroups*, Semigroup Forum 71 (2005), 252–264.

[9] L. Bernal-González and K.-G. Grosse-Erdmann, *Existence and nonexistence of hypercyclic semigroups*, Proc. Amer. Math. Soc. 135 (2007), 755–766.

[10] A. Bonilla and K.-G. Grosse-Erdmann, *Frequently hypercyclic operators and vectors*, Ergodic Theory Dynam. Systems 27 (2007), 383–404.

[11] —, —, *Frequently hypercyclic operators and vectors, Erratum*, ibid. 29 (2009), 1993–1994.

[12] J. K. Brooks, *Representations of weak and strong integrals in Banach spaces*, Proc. Nat. Acad. Sci. U.S.A. 63 (1969), 266–270.

[13] J. A. Conejero and E. M. Mangino, *Spectral conditions for hypercyclic C_0-semigroups*, in: Function Theory on Infinite Dimensional Spaces X (Madrid, 2007), J. A. Jaramillo et al. (ed.), Univ. Complutense, Madrid, 2008, 17–26.

[14] —, —, *Hypercyclic semigroups generated by Ornstein–Uhlenbeck operators*, Mediterr. J. Math. 7 (2010), 101–109.

[15] J. A. Conejero, V. Müller, and A. Peris, *Hypercyclic behaviour of operators in a hypercyclic C_0-semigroup*, J. Funct. Anal. 244 (2007), 342–348.

[16] J. A. Conejero and A. Peris, *Linear transitivity criteria*, Topology Appl. 153 (2005), 767–773.

[17] —, —, *Hypercyclic translation C_0-semigroups on complex sectors*, Discrete Contin. Dynam. Systems 25 (2009), 1195–1208.

[18] G. Costakis and A. Peris, *Hypercyclic semigroups and somewhere dense orbits*, C. R. Math. Acad. Sci. Paris 335 (2002), 895–898.

[19] R. deLaubenfels and H. Emamirad, *Chaos for functions of discrete and continuous weighted shift operators*, Ergodic Theory Dynam. Systems 21 (2001), 1411–1427.

[20] W. Desch and W. Schappacher, *On products of hypercyclic semigroups*, Semigroup Forum 7 (2005), 301–311.

[21] W. Desch, W. Schappacher, and G. F. Webb, *Hypercyclic and chaotic semigroups of linear operators*, Ergodic Theory Dynam. Systems 17 (1997), 793–819.

[22] J. Diestel and J. J. Uhl, *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., Providence, RI, 1977.

[23] S. El Mourchid, *The imaginary point spectrum and hypercyclicity*, Semigroup Forum 73 (2006), 313–316.

[24] S. El Mourchid, G. Metafune, A. Rhandi, and J. Voigt, *On the chaotic behaviour of size structured cell populations*, J. Math. Anal. Appl. 339 (2008), 918–924.

[25] S. Grivaux, *A probabilistic version of the Frequent Hypercyclicity Criterion*, Studia Math. 176 (2006), 279–290.

[26] —, *A new class of frequently hypercyclic operators with applications*, Indiana Univ. Math. J., to appear.

[27] K.-G. Grosse-Erdmann and A. Peris, *Frequently dense orbits*, C. R. Math. Acad. Sci. Paris 341 (2005), 123–128.

[28] L. Ji and A. Weber, *Dynamics of the heat semigroup on symmetric spaces*, Ergodic Theory Dynam. Systems 30 (2010), 457–468.

[29] T. Kalmes, *On chaotic C_0-semigroups and infinitely regular hypercyclic vectors*, Proc. Amer. Math. Soc. 134 (2006), 2997–3002.

[30] —, *Hypercyclic, mixing, and chaotic C_0-semigroups induced by semiflows*, Ergodic Theory Dynam. Systems 27 (2007), 1599–1631.

[31] —, *Hypercyclic C_0-semigroups and evolution families generated by first order differential operators*, Proc. Amer. Math. Soc. 137 (2009), 3833–3848.

[32] M. Matsui, M. Yamada, and F. Takeo, *Erratum to: "Supercyclic and chaotic translation semigroups" [Proc. Amer. Math. Soc. 131(11): 3535–3546, 2003]*, Proc. Amer. Math. Soc. 132 (2003), 3751–3752.

[33] G. Metafune, *L^p-spectrum of Ornstein–Uhlenbeck operators*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 30 (2001), 97–124.

[34] R. Rudnicki, *Chaos for some infinite-dimensional dynamical systems*, Math. Methods Appl. Sci. 27 (2004), 723–738.

[35] C.-L. Stewart and R. Tijdeman, *On infinite-difference sets*, Canad. J. Math. 31 (1979), 897–910.

[36] F. Takeo, *Chaos and hypercyclicity for solution semigroups to chaos and hypercyclicity for solution semigroups to some partial differential equations*, Nonlinear Anal. 63 (2005), e1943–e1953.

[37] A. Weber, *Tensor products of recurrent hypercyclic semigroups*, J. Math. Anal. Appl. 351 (2009), 603–606.

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