STUDIA MATHEMATICA 202 (3) (2011)

On the size of quotients of function spaces on a topological group

by

AHMED BOUZIAD (Rouen) and MAHMOUD FILALI (Oulu)

Abstract. For a non-precompact topological group G, we consider the space $C(G)$ of bounded, continuous, scalar-valued functions on G with the supremum norm, together with the subspace $LMC(G)$ of left multiplicatively continuous functions, the subspace $LUC(G)$ of left norm continuous functions, and the subspace $WAP(G)$ of weakly almost periodic functions.

We establish that the quotient space $LUC(G)/WAP(G)$ contains a linear isometric copy of ℓ_∞, and that the quotient space $C(G)/LMC(G)$ (and a fortiori $C(G)/LUC(G)$) contains a linear isometric copy of ℓ_∞ when G is a normal non-P-group. When G is not a P-group but not necessarily normal we prove that the quotient is non-separable. For non-discrete P-groups, the quotient may sometimes be trivial and sometimes non-separable. When G is locally compact, we show that the quotient space $LUC(G)/WAP(G)$ contains a linear isometric copy of $\ell_\infty(\kappa(G))$, where $\kappa(G)$ is the minimal number of compact sets needed to cover G. This leads to the extreme non-Arens regularity of the group algebra $L^1(G)$ when in addition either $\kappa(G)$ is greater than or equal to the smallest cardinality of an open base at the identity e of G, or G is metrizable. These results are improvements and generalizations of theorems proved by various authors along the last 35 years and until very recently.

1. Introduction. Throughout the paper, G is a Hausdorff topological group with identity e, and $C(G)$ is the space of bounded, continuous, scalar-valued functions on G with the supremum norm. For each $f \in C(G)$ and $s \in G$, f_s is the left translate of f by s, defined on G by $f_s(t) = f(st)$, and $f_G = \{f_s : s \in G\}$. We follow primarily the notation used in [1], from which we also recall the definitions of the following subspaces of $C(G)$. We start with the space of *weakly almost periodic* functions on G,

$$WAP(G) = \{f \in C(G) : f_G \text{ is relatively weakly compact in } C(G)\}.$$

2010 *Mathematics Subject Classification*: Primary 43A20; Secondary 22D15.
Key words and phrases: left multiplicatively continuous functions, left norm continuous functions, weakly almost periodic functions, quotient space, extreme non-Arens regularity.

DOI: 10.4064/sm202-3-3

Grothendieck's famous iterated limit criterion shows that a function f in $C(G)$ is in $WAP(G)$ if and only if, for any sequences (x_n) and (y_m) in G,

$$\lim_{n\to\infty}\lim_{m\to\infty} f(x_n y_m) = \lim_{m\to\infty}\lim_{n\to\infty} f(x_n y_m)$$

whenever these iterated limits exist; see for example [1, Appendix A].

Another space we will consider is

$$LUC(G) = \{f \in C(G) : s \mapsto f_s \text{ is norm continuous}\}.$$

This space is denoted by $LC(G)$ in [1] and is referred to as the space of *left norm continuous* functions on G. We note that these are in fact the bounded functions f on G which are uniformly continuous with respect to the right uniformity on G, i.e., for every $\epsilon > 0$, there is a neighbourhood O of the identity in G such that $|f(s) - f(t)| < \epsilon$ whenever $ts^{-1} \in O$. Thus, left norm continuous functions in the sense of [1] and right uniformly continuous functions in the sense of [25] are the same. It is well known that $WAP(G) \subseteq LUC(G)$ (see for example [1, Corollary 4.4.11]).

There will also be the space

$$\begin{aligned} LMC(G) &= \{f \in C(G) : s \mapsto f_s \text{ is } \sigma(C(G), \beta G)\text{-continuous}\} \\ &= \{f \in C(G) : s \mapsto x(f_s) \text{ is continuous for each } x \in \beta G\}; \end{aligned}$$

this is the space of *left multiplicatively continuous* functions on G, which can be easily identified with the functions $f \in C(G)$ such that

$$\lim_n \lim_m f(x_n y_m) = \lim_m \lim_n f(x_n y_m)$$

whenever (x_n) and (y_m) are nets in G with (x_n) converging in G and (y_m) converging in βG, the Stone–Čech compactification of G. It is clear that $LUC(G) \subseteq LMC(G)$, and when G is locally compact, $LUC(G) = LMC(G)$. But in general, the equality may fail as for example when G is the group of rationals with its usual topology (see [1, Example 4.5.8]).

When G is locally compact, we will also consider the Banach space $L^\infty(G)$ of scalar-valued measurable functions which are essentially bounded with respect to the Haar measure; two functions are identified if they differ only on a locally null set, and the norm is given by the essential supremum norm.

The paper is concerned with the Banach quotients $LUC(G)/WAP(G)$ and $C(G)/LMC(G)$ for a general topological group G, and the quotient $L^\infty(G)/WAP(G)$ when G is locally compact. In the following section, we prove that $LUC(G)/WAP(G)$ contains a linear isometric copy of ℓ_∞ for any non-precompact group G. This was established by different authors in some special cases and finally settled by Dzinotyiweyi in [9] for any locally compact non-compact group. In [31], Megrelishvili, Pestov and Uspenskij considered topological groups and proved that G is precompact if and only if $LUC(G) = WAP(G)$. A simplified proof of this theorem was provided later on by Bouziad and Troallic in [4].

In [9], Dzinotyiweyi also showed that the quotient $C(G)/LUC(G)$ is non-separable for any locally compact, non-discrete, non-compact group G. The third section of the present paper generalizes this result to a vast class of topological groups. We show that $C(G)/LMC(G)$ (and a fortiori $C(G)/LUC(G)$) contains in fact a linear isometric copy of ℓ_∞ whenever G is a non-precompact, normal, topological group which is not a P-group. Note that this improves Dzinotyiweyi's result also for locally compact groups. Without the hypothesis of normality, we prove that the quotient is non-separable when G is not a P-group. For P-groups, the quotient may contain a linear isometric copy of ℓ_∞, but may also be trivial even if G is non-discrete.

The last section specializes to locally compact groups and shows further that $LUC(G)/WAP(G)$ contains a linear isometric copy of $\ell_\infty(\kappa(G))$, where $\kappa(G)$ is the minimal number of compact sets required to cover G. This immediately yields the same property for the quotient $L^\infty(G)/WAP(G)$, and accordingly the extreme non-Arens regularity of the group algebra $L^1(G)$ whenever $\kappa(G)$ is greater than or equal to the smallest cardinality $b(G)$ of an open base at the identity e of G. The latter result includes [17, Theorem 4.3], and is precisely the dual of Hu's result in [26] which states that the Fourier algebra $A(G)$ is extremely non-Arens regular when $b(G) \geq \kappa(G)$.

We recall that a topological group G is *precompact* (or *totally bounded*) if for every neighbourhood U of the identity e there exists a finite subset F of G such that $G = UF$. A topological space X is said to be *pseudocompact* if every continuous real-valued function on X is bounded.

The following lemma presents the common argument finishing the proof of each of the theorems in the paper.

LEMMA 1.1. *Let $\mathcal{S}$, $\mathcal{T}$ be two families of nets on G with the property that if (s_i) is in $\mathcal{S}$ (respectively, in $\mathcal{T}$) then every subnet of (s_i) is in $\mathcal{S}$ (respectively, in $\mathcal{T}$). Let $h : G \to \mathbb{R}$ be a bounded function with the property that whenever $(s_i) \in \mathcal{S}$, $(t_j) \in \mathcal{T}$ and both iterated limits of $h(s_it_j)$ exist, then these limits are equal. Let $f : G \to \mathbb{R}$, and suppose there are nets $(s_i) \in \mathcal{S}$, $(t_j) \in \mathcal{T}$ with*

$$\lim_j \lim_i f(s_it_j) = a, \qquad \lim_i \lim_j f(s_it_j) = b.$$

Then $\|f + h\| \geq |a - b|/2$.

Proof. If $\lim_j \lim_i h(s_it_j) = \lim_i \lim_j h(s_it_j) = \ell$, then

$$\|f + h\| \geq \sup_{i,j} |f(s_it_j) + h(s_it_j)| \geq \max\{|a + \ell|, |b + \ell|\} \geq \frac{|a - b|}{2}.$$

Replacing $(s_i)_{i\in I}$ and $(t_j)_{j\in J}$ by subnets does not change the iterated limits of $f(s_it_j)$, so all we need to do is to show that for any nets $(s_i)_{i\in I}$, $(t_j)_{j\in J}$ we can find subnets for which the iterated limits of $(h(s_it_j))$ exist.

Take K with $\|h\| \le K$. For each $i \in I$, $(h(s_it_j))_{j\in J} \in [-K, K]^J$, and so by compactness of the product, for some subnet of (s_i), $(h(s_it_j))_{j\in J} \to (u_j)_{j\in J}$, say. We can then take a subnet of (u_j) for which the j-limit exists. Interchanging the roles of (s_i) and (t_j) we find further subnets for which the other iterated limit exists. ∎

2. On the quotient $LUC(G)/WAP(G)$**.** For locally compact non-compact groups, the quotient $LUC(G)/WAP(G)$ contains an isometric linear copy of ℓ_∞. This was proved by Granirer for amenable groups in [20], by Chou for E-groups in [6] and extended by Dzinotyiweyi to all locally compact groups in [9]. Accordingly, the same property holds if G is a non-precompact, locally precompact group. This is due to the known fact that the completion $\widetilde{G}$ of such a group is a locally compact non-compact group, and to the facts that $WAP(\widetilde{G})_{|G} = WAP(G)$ and $LUC(\widetilde{G})_{|G} = LUC(G)$ (see, for example, [1]).

In this section, we prove the result for any non-precompact topological group. Our method is inspired by that of Chou, but it is much simplified and covers all the locally compact case. Due to the highly technical proofs presented in both papers by Chou and Dzinotyiweyi, we shall present the proofs for any topological group including the locally compact case as well.

Recall that a topological group G is called a *SIN-group* if the left and the right uniform structures on G coincide. Equivalently, a topological group G is a SIN-group if every neighbourhood U of the identity e in G contains a neighbourhood of e which is invariant under all inner automorphisms of G.

LEMMA 2.1. *Let G be a non-precompact topological group. Suppose that G is either SIN* (*we can also take G as an E-group in the sense of* [6]) *or not locally precompact. Then G contains two infinite countable sets S and T and a symmetric neighbourhood V of e such that ST is right* (*left*) *uniformly discrete with respect to V^2; that is,*

$$(V^2 s_n t_m) \cap (ST) = \{s_n t_m\} \quad ((s_n t_m V^2) \cap (ST) = \{s_n t_m\})$$

for every pair $(n, m) \in \mathbb{N} \times \mathbb{N}$.

Proof. The case when G is not locally precompact is taken from [3] or [4]. Since G is not precompact, we may pick a symmetric neighbourhood U of e and an infinite subset $T = \{t_n : n \in \mathbb{N}\}$ of G which is right uniformly discrete with respect to U^2, that is, Ut_l and Ut_n are disjoint whenever $l, n \in \mathbb{N}$, $l < n$. Let W be a neighbourhood of e such that $W^2 \subseteq U$. Since W is in turn not precompact, we may pick again a symmetric neighbourhood V of e, $V \subseteq W$, and an infinite subset $S = \{s_m : m \in \mathbb{N}\}$ of W such that Vs_k and Vs_m

are disjoint whenever $k, m \in \mathbb{N}$, $k < m$. Consider now the set ST and let $(k,l) \neq (m,n)$ in $\mathbb{N} \times \mathbb{N}$. If $l \neq n$, then $Ut_l \cap Ut_n = \emptyset$, and so

$$(Vs_k t_l) \cap (Vs_m t_n) \subseteq (W^2 t_l) \cap (W^2 t_n) \subseteq (Ut_l) \cap (Ut_n) = \emptyset.$$

If $l = n$, then $k \neq m$, and so $(Vs_k) \cap (Vs_m) = \emptyset$, which clearly implies our claim.

Suppose now that G is a SIN-group, and pick a symmetric invariant neighbourhood U of e such that $G \neq UF$ (since G is not precompact) for any finite subset F of G. We start by putting $s_0 = t_0 = e$. We argue by induction, and suppose that s_0, $s_1, \ldots, s_n$ and $t_0, t_1, \ldots, t_n$ have been defined. First choose

$$s_{n+1} \notin U\{s_p t_q t_r^{-1} : p, q, r \leq n\},$$

then choose

$$t_{n+1} \notin U\{s_p^{-1} s_q t_r : p, q \leq n+1,\ r \leq n\}.$$

We claim that $s_k t_l \notin U s_m t_n$ whenever $(k,l) \neq (m,n)$.

Case 1: $k \geq l, m, n$. If $k = m$, then $s_k t_l \in U s_m t_n$ implies that $t_l \in U t_n$ (since U is invariant), and so $l = n$. Thus, $(k,l) = (m,n)$. If $k > m$ then $s_k \in U s_m t_n t_l^{-1}$ is not possible by the choice of s_k. Thus, we have again $s_k t_l \notin U s_m t_n$.

Case 2: $l \geq k, m, n$. If $l = n$, then $s_k t_l \in U s_m t_n$ implies that $s_k \in U s_m$, and so $k = m$. Thus, $(k,l) = (m,n)$. If $l > n$, then $t_l \in s_k^{-1} U s_m t_n$ is not possible by the choice of t_l (since U is invariant). Thus, we have again $s_k t_l \notin U s_m t_n$. Now any symmetric neighbourhood V of e such that $V^2 \subseteq U$ has the required property of the claim.

The other cases $m \geq k, l, n$ and $n \geq k, l, m$ are treated similarly. ∎

Theorem 2.2. *Let G be any non-precompact topological group. Then the quotient $LUC(G)/WAP(G)$ contains a linear isometric copy of ℓ_∞. In particular, the quotient space $LUC(G)/WAP(G)$ is non-separable.*

Proof. Suppose first that G is a non-SIN locally precompact group. Then the completion $\widetilde{G}$ of G is a non-SIN locally compact group. By [24], the quotient $LUC(\widetilde{G})/UC(\widetilde{G})$ contains an isometric copy of ℓ_∞ (see also [3]). It is then clear that the quotient $LUC(\widetilde{G})/WAP(\widetilde{G})$ has the same property, and so does $LUC(G)/WAP(G)$ as already noted at the beginning of this section.

Suppose now that our group G either is not locally precompact or is locally precompact and SIN. These are the groups considered in Lemma 2.1, so we may pick a symmetric neighbourhod V of e in G and two countably infinite subsets S and T of G such that ST is right uniformly discrete with respect to V^2, i.e., $(Vs_k t_l) \cap (Vs_m t_n) = \emptyset$ whenever $(k,l) \neq (m,n)$ in $\mathbb{N} \times \mathbb{N}$.

Partition $\mathbb{N}$ into infinitely many subsets I_n, $n \in \mathbb{N}$. Let $\phi \in LUC(G)$ be such that $0 \le \phi \le 1$, $\phi(e) = 1$ and $\phi(G \setminus V) = 0$ (see [25, Theorem 8.4], [34, Theorem 3.18.10], [27, p. 7] or [4, Lemma 2.4]). Fix $\xi \in \ell_\infty(\mathbb{N} \times \mathbb{N})$ with the property $\lim_i \xi(i,j) = 1$ for each $j \in \mathbb{N}$, and $\lim_j \xi(i,j) = -1$ for each $i \in \mathbb{N}$. Define, for each $k \in \mathbb{N}$, a function f_k on G by

$$f_k(s) = \sum_{i,j \in I_k} \xi(i,j)\phi(st_j^{-1}s_i^{-1}),$$

and for each bounded scalar sequence $c = (c_k)$, let

$$f_c = \sum_{k \in \mathbb{N}} c_k f_k.$$

Note that if $f_c(s) \neq 0$, then $f_k(s) \neq 0$ for some $k \in \mathbb{N}$, and so $\phi(st_j^{-1}s_i^{-1}) \neq 0$ for some $i, j \in I_k$. Accordingly, $st_j^{-1}s_i^{-1} \in V$, and so $s \in Vs_it_j$ for only one pair $(i,j) \in I_k \times I_k$. Thus, $f_c(s) = \xi(i,j)\phi(st_j^{-1}s_i^{-1})$. It follows that f_c is well-defined, and it is in $LUC(G)$ since ϕ is in $LUC(G)$, is supported in V and $(Vs_it_j) \cap (Vs_kt_l) = \emptyset$ whenever $(i,j) \neq (k,l)$ in $\mathbb{N} \times \mathbb{N}$ (see for example [1, Exercise 4.4.16] or [17, Lemma 1.3]).

We prove that the composition map

$$\ell_\infty \to LUC(G) \to LUC(G)/WAP(G) : c \mapsto f_c \mapsto f_c + WAP(G)$$

is a linear isometry, where each of the spaces is equipped with its usual norm. Since the map $\ell_\infty \to LUC(G) : c \mapsto f_c$ is clearly a linear isometry, we only need to check that the map $f_c \mapsto f_c + WAP(G)$ is an isometry. So let $f_c \in LUC(G)$ with c a function in ℓ_∞.

Note that $f_c(s_it_j) = c_k\xi(i,j)$ for each pair $(i,j) \in I_k \times I_k$. Therefore, when $(i,j) \in I_k \times I_k$,

$$\lim_j \lim_i f(s_it_j) = c_k \quad \text{and} \quad \lim_i \lim_j f(s_it_j) = -c_k.$$

By Lemma 1.1, we deduce that $\|f_c + h\| \ge |c_k|$ for every $h \in WAP(G)$ and for every $k \in \mathbb{N}$. Therefore,

$$\|f_c + h\| \ge \|c\| \quad \text{for every } h \in WAP(G).$$

If we let $\|\cdot\|_q$ denote the norm in $LUC(G)/WAP(G)$, then we have

$$\|f_c + WAP(G)\|_q = \inf\{\|f_c + h\| : h \in WAP(G)\} \ge \|c\|.$$

Of course, $\|f_c + WAP(G)\|_q \le \|f_c\| = \|c\|$ since the quotient map is bounded. Thus we obtain the required isometry. ■

COROLLARY 2.3. *Let G be a topological group. Then the following statements are equivalent:*

(i) *G is not precompact.*
(ii) *$LUC(G) \neq WAP(G)$.*

(iii) *$LUC(G)/WAP(G)$ is non-separable.*

Proof. (ii)⇒(i) is known, see for instance [1]. (iii)⇒(ii) is clear. (i)⇒(iii) by the theorem above. ∎

3. On the quotient $C(G)/LMC(G)$. In [9], Dzinotyiweyi proved that the Banach quotient $C(G)/LUC(G)$ is non-separable when G is a locally compact group which is neither compact nor discrete. To reach this conclusion, he showed in fact that $C(G)$ contains a linear isometric copy $\{f_c : c \in \ell_\infty\}$ of ℓ_∞ such that $\|f_c + LMC(G)\|_q \geq \frac{1}{2}\|c\|$ for every $c \in \ell_\infty$.

In this section, we improve and generalize Dzinotyiweyi's result. We show that for a wide class of topological groups the quotient $C(G)/LMC(G)$ contains in fact a linear isometric copy of ℓ_∞ (i.e., $\|f_c + LMC(G)\|_q = \|c\|$ for every $c \in \ell_\infty$).

Our methods are less technical and are inspired by [16], and our conclusion is valid for any non-precompact normal topological group which is not a P-group. Without normality, we still deduce that the quotient $C(G)/LMC(G)$ is non-separable. For a P-group, the conclusion may, or may not, hold. Indeed, we know from [16] that for Lindelöf P-groups, the quotient is trivial since $LUC(G) = C(G)$ for this class of groups. But we will also present in this section a class of P-groups for which the quotient contains a linear isometric copy of ℓ_∞.

Recall that a *P-space* is a topological space in which every countable intersection of open subsets remains open. For several interesting properties of P-spaces, see [18]. A *P-group* is a topological group which is also a P-space.

Theorem 3.1. *Let G be a non-precompact topological group which is not a P-group. Then*

(i) *The quotient $C(G)/LMC(G)$ is non-separable.*
(ii) *If in addition G is normal, then the quotient $C(G)/LMC(G)$ contains a linear isometric copy of ℓ_∞.*

Proof. Suppose that G is not precompact and pick a neighbourhood U of e in G and a sequence $(x_n)_{n\geq 1}$ such that $(Ux_n) \cap (Ux_m) = \emptyset$ whenever $n \neq m$. Note that the sequence $(x_n)_{n\geq 1}$ does not have any cluster point in G. Since G is not a P-group, there exists a countable decreasing family $\{U_n\}_{n\geq 1}$ of open symmetric neighbourhoods of e such that

$$\mathrm{Int}\Big(\bigcap_{n\geq 1} U_n\Big) = \emptyset.$$

We may assume that $U_1^2 \subseteq U$ and $\overline{U_{n+1}} \subsetneq U_n \subseteq U$ for every $n \geq 1$. Let $\{B_i\}_{i\in I}$ be a base at the identity e in G. Since the interior of $\bigcap_{n\geq 1} U_n$ is empty,

$$B_i \cap \Big(G \setminus \bigcap_{n\geq 1} U_n\Big) \neq \emptyset \quad \text{ for every } i \in I.$$

Hence, for every $i \in I$, we may choose the smallest index $n_i \geq 1$ such that

$$B_i \cap (U_{n_i} \setminus U_{n_i+1}) \neq \emptyset,$$

and so we may form a net $(s_i)_{i\in I}$ such that $s_i \in B_i \cap (U_{n_i} \setminus U_{n_i+1})$ for each $i \in I$. Note that $(s_i)_{i\in I}$ converges to e in G.

First suppose that G is only completely regular, and let $\{\phi_n\}_{n=1}^{\infty}$ be a family of continuous functions on G with values in $[0,1]$ satisfying

$$\phi_n(e) = 1 \quad \text{and} \quad \phi_n(G \setminus U_n) = 0 \quad \text{ for every } n \geq 1.$$

Partition $\mathbb{N}$ into infinitely many infinite subsets I_k, $k \in \mathbb{N}$, and for each $k \in \mathbb{N}$, define

$$f_k(s) = \sum_{n\in I_k} \phi_n(sx_n^{-1}).$$

Then, as before, since $\phi_n(sx_n^{-1}) \neq 0$ for at most one $n \in \mathbb{N}$, each function f_k is well-defined on G and, clearly, bounded and continuous.

Now define, for each bounded scalar sequence $c = (c_k)$, a function f_c on G by

$$f_c = \sum_{k\in\mathbb{N}} c_k f_k.$$

As in Theorem 2.2, the mapping

$$c \mapsto f_c : \ell_\infty \to C(G)$$

is a linear isometry. We claim further that $2\|f_c + h\| \geq \|c\|$ for every $h \in LMC(G)$. Note first that

$$\lim_i \lim_j f_c(s_i x_j) = 0 \quad \text{while} \quad \lim_j \lim_i f_c(s_i x_j) = c_k$$

for $j \in I_k$, and remember that $h \in LMC(G)$ implies

$$\lim_i \lim_j h(s_i y_j) = \lim_j \lim_i h(s_i y_j)$$

whenever the limits exist, since (s_i) is a net which converges in G. So Lemma 1.1 yields

$$\|f_c + h\| \geq \frac{1}{2}|c_k| \quad \text{ for every } k \in \mathbb{N} \text{ and } h \in LMC(G).$$

Therefore,

$$\|f_c + LMC(G)\|_q = \inf\{\|f_c + h\| : h \in LMC(G)\} \geq \frac{1}{2}\|c\|$$

for every $c \in \ell_\infty$, hence the quotient $C(G)/LMC(G)$ is non-separable.

If G is normal, then we may take a family $\{\phi_n\}_{n=1}^{\infty}$ of continuous functions on G with values in $[-1,1]$ satisfying

$$\phi_n(e)=1,\quad \phi_n(\overline{U_1}\setminus U_n)=-1 \quad\text{and}\quad \phi_n(G\setminus U)=0 \qquad\text{for every } n\ge 1.$$

As in the case above, we consider the functions $\{f_k\}_{k=1}^{\infty}$, and for each $c\in\ell_\infty$, the function f_c. Here, Lemma 1.1 implies that $\|f_c+h\|\ge\|c\|$ for every $h\in LMC(G)$. As in Theorem 2.2, this yields the required linear isometry of ℓ_∞ into $C(G)/LMC(G)$. ∎

Corollary 3.2. *Let G be topological group which is not a P-group, and consider the following statements:*

(i) *G is not precompact.*
(ii) *$C(G)/LMC(G)$ is non-separable.*
(iii) *$C(G)\ne LMC(G)$.*
(iv) *G is not pseudocompact.*

Then (i)⇒(ii)⇒(iii)⇔(iv).

Proof. (i)⇒(ii) follows by Theorem 3.1. (ii)⇒(iii) is clear. (iii)⇒(iv) is known (see for instance [1]). (iv)⇒(iii) is proved in [16]. ∎

For non-discrete P-groups the quotient $C(G)/LMC(G)$ is sometimes non-separable and sometimes trivial. In fact, for every Lindelöf P-group, the quotient is trivial since $LUC(G)=LMC(G)=C(G)$ (see [16, Theorem 5.1]. Next, we give a criterion which helps to construct examples of P-groups with a non-separable quotient $C(G)/LMC(G)$.

Theorem 3.3. *Let G be a non-discrete topological group with a base $\mathcal{B}$ at e consisting of open subgroups, and suppose that $|G/H|\ge|\mathcal{B}|$ for some $H\in\mathcal{B}$. Then the quotient $C(G)/LMC(G)$ contains a linear isometric copy of $\ell_\infty(|\mathcal{B}|)$, and so the quotient is non-separable.*

Proof. Let $\mathcal{B}=\{H_i : i\in I\}$. We may assume that $H_i\subseteq H$ for every $i\in I$. Since $|G/H|\ge|\mathcal{B}|$, we may pick, for each $i\in I$, an element $x_i\in G$ such that the family $\{Hx_i : i\in I\}$ is pairwise disjoint. Hence, the family $\{H_ix_i : i\in I\}$ is also pairwise disjoint. Partition I into $|\mathcal{B}|$ subsets I_k each with cardinality $|\mathcal{B}|$. Then we may define, for each $k<|\mathcal{B}|$, a function f_k by

$$f_k=\sum_{n\in I_k}(\chi_{H_nx_n}-\chi_{(H\setminus H_n)x_n}),$$

and for each $c\in\ell_\infty(|\mathcal{B}|)$, a function f_c by

$$f_c=\sum_{k<|\mathcal{B}|}c_kf_k=\sum_{k<|\mathcal{B}|}c_k\sum_{n\in I_k}(\chi_{H_nx_n}-\chi_{(H\setminus H_n)x_n}).$$

Since each H_n is open, f_c is continuous. The mapping

$$c\mapsto f_c:\ell_\infty(|\mathcal{B}|)\to C(G)$$

is clearly a linear isometry.

As in Theorem 3.1, pick s_i from each $H_i\setminus\{e\}$ and consider the net $(s_i)_{i\in I}$ which converges to e in G. Let x be any cluster point of $\{x_n : n \in I_k\}$ in βG (note that $x \in \beta G \setminus G$), and let (x_j) be a net in this set converging to x. Then, clearly,

$$\lim_j \lim_i f_k(s_i x_j) = \lim_j f_k(x_j) = \lim_j \chi_{H_j x_j}(x_j) = 1.$$

Since $s_i \notin H_j$ starting from some index j_0, we see

$$\lim_j f_k(s_i x_j) = \lim_j (\chi_{H_j x_j} - \chi_{(H\setminus H_j)x_j})(s_i x_j) = -1,$$

and so

$$\lim_i \lim_j f_k(s_i x_j) = \lim_i \lim_j (\chi_{H_j x_j} - \chi_{(H\setminus H_j)x_j})(s_i x_j) = -1.$$

Lemma 1.1 leads again to the required inequality, namely,

$$\|f_c + h\| \geq 1 \quad \text{for every } h \in LMC(G),$$

and so to the desired conclusion. ∎

There is a P-group with the quotient $C(G)/LMC(G)$ non-separable.

EXAMPLE 3.4. Let T be an uncountable set and let G be the permutation group of T (with composition of functions as operation). Let the sets

$$U_A = \{x \in G : x(t) = t \text{ for every } t \in A\},$$

where $A \subseteq T$ is countable, form a base at the identity e in G. Then G becomes a P-group. Fix $t \in T$. Then, for every $x \in G$,

$$xU_{\{t\}} = \{y \in G : y = xu \text{ where } u(t) = t\} = \{y \in G : y(t) = x(t)\}.$$

Therefore, $xU_{\{t\}} \cap x'U_{\{t\}} = \emptyset$ if and only if $x(t) \neq x'(t)$. Thus, $|G/U_{\{t\}}| \geq |T|$. Since $|\mathcal{B}| = |\{U_A : A \subseteq T \text{ is countable}\}| = |T|$, we see that $|G/U_{\{t\}}| \geq |\mathcal{B}|$, and so Theorem 3.3 implies that the quotient $C(G)/LMC(G)$ contains a linear isometric copy of $\ell_\infty(|\mathcal{B}|)$.

4. On extreme non-Arens regularity of the group algebra. This section is concerned with the extreme non-Arens regularity of the group algebra $L^1(G)$ with the convolution product, G being a locally compact group. Before stating our main theorem, we need to recall a few definitions about Banach algebras A. As is known ([7], [15] or [33]), two Arens multiplications may be introduced in the second Banach dual A^{**} of A; each extends that of A and makes A^{**} a Banach algebra. When these two products coincide, A is said to be *Arens regular*. Let $WAP(A^*)$ be the space of weakly almost periodic functionals on A. This is the space of all elements $a' \in A^*$ with the property that the set $\{a'a : a \in A,\ \|a\| \leq 1\}$ is relatively weakly compact in A^*, where $a'a \in A^*$ is defined on A by $\langle a'a, b\rangle = \langle a', ab\rangle$. In [35], Pym proved that A is Arens regular if and only if $WAP(A^*) = A^*$. Well-known examples of Arens regular Banach algebras are C^*-algebras. The group algebra $L^1(G)$ is Arens regular if and only if G is finite ([38]). For more details, see for example [15].

In [19], Granirer introduced the concept of extreme non-Arens regularity by defining A to be *extremely non-Arens regular* (ENAR for short) when the quotient $A^*/WAP(A^*)$ contains a closed linear subspace having A^* as a continuous linear image, and proved in particular that the Fourier algebras $A(\mathbb{R})$ and $A(\mathbb{T})$ are ENAR. (Thus the group algebras $L^1(\mathbb{R})$ and $\ell^1(\mathbb{Z})$ are ENAR.) In [26], Hu generalized Granirer's results and proved that the Fourier algebra $A(G)$ is extremely non-Arens regular whenever $b(G) \geq \kappa(G)$, where $\kappa(G)$ is the minimal number of compact sets required to cover G and $b(G)$ is the minimal cardinality of an open base at e.

In [17], Fong and Neufang proved that the group algebra $L^1(G)$ is ENAR when G is σ-compact or it contains an open σ-compact subgroup H which is either normal or such that $|H| < |G|$. In this section we prove the full dual of Hu's result, that is, $L^1(G)$ is ENAR whenever $\kappa(G) \geq b(G)$. Since by [37], $WAP(L^\infty(G)) = WAP(G)$, we are really concerned with the quotient $L^\infty(G)/WAP(G)$. To this end, we go back to Theorem 2.2, we use a different method and prove that in fact for any locally compact, non-compact group the quotient $LUC(G)/WAP(G)$ contains an isometric linear copy of $\ell_\infty(\kappa(G))$.

It may be worth reminding the reader that following [8], A is said to be *strongly Arens irregular* when the topological centre $Z(A^{**})$ of A^{**} is precisely A. (Recall that $Z(A^{**})$ is the set of $a'' \in A^{**}$ such that the mapping $A^{**} \ni b'' \mapsto a''b'' \in A^{**}$ is weak*-weak*-continuous.) By [28] (see also [8], [32] and [14]), the group algebra $L^1(G)$ is strongly Arens irregular. Extreme non-Arens regularity in the sense of Granirer does not imply strong Arens irregularity in the sense of Dales and Lau, since $A(\mathrm{SO}(3))$ is extremely non-Arens regular by [26] but it is not strongly Arens irregular as recently proved by Losert [30]. However, it is still not known whether strong Arens irregularity implies extreme non-Arens regularity.

Let G^{LUC} be the LUC-compactification of G. This is the largest semigroup compactification of G in the sense that every other semigroup compactification of G is a quotient of the LUC-compactification (see [1]). The LUC-compactification may be realized as the spectrum of the C^*-algebra $LUC(G)$, that is,

$$G^{LUC} = \{x \in LUC(G)^* \setminus \{0\} : x(fg) = x(f)x(g) \text{ for all } f, g \in LUC(G)\},$$

and may be regarded as a weak*-compact subsemigroup of $LUC(G)^*$. We shall identify G with its image in G^{LUC}.

For each $A \subseteq G$, let $\kappa(A)$ be the minimal number of compact sets required to cover A, and define the *height* of $x \in G^{LUC}$ by

$$\rho(x) = \min\{\kappa(A) : x \in \overline{A}\}.$$

Then the set

$$\mathcal{U}(G) = \{x \in G^{LUC} : \rho(x) = \kappa(G)\}$$

is a closed left ideal of G^{LUC}. When G is discrete, $G^{LUC} = \beta G$, $\kappa(G) = |G|$ and $\mathcal{U}(G)$ is known as the subspace of uniform ultrafilters in βG. For more details on the structure of G^{LUC}, see [5], [10]–[14], [29] and references therein.

The construction of the functions necessary for the proof of the main theorem in this section is included in the proof of the following lemma.

LEMMA 4.1. *Let G be a locally compact, non-compact group. Then G contains a left uniformly discrete subset T such that $|T| = \kappa(G)$ and any two distinct points x and y in $\overline{T}$ can be separated by some function f from $LUC(G) \setminus WAP(G)$. Moreover, if the points x and y are in $\mathcal{U}(T) = \mathcal{U}(G) \cap \overline{T}$, then two nets $(t_\alpha)_\alpha$ and $(t_\beta)_\beta$ may be chosen in T converging respectively to x and y in G^{LUC} such that*

$$f(y) = f(xy) = \lim_\alpha \lim_\beta f(t_\alpha t_\beta) = -1 \quad \text{and} \quad f(x) = \lim_\beta \lim_\alpha f(t_\alpha t_\beta) = 1.$$

Proof. Put $\kappa = \kappa(G)$. Let U be an open symmetric relatively compact neighbourhood of e in G. Let $(K_\alpha)_{\alpha<\kappa}$ be a covering family of symmetric compact subsets of G with $U \subseteq K_\alpha$ and such that any finite union of K_α's is contained in some K_γ. Now form an increasing family $(V_\alpha)_{\alpha<\kappa}$ of open sets inductively by writing

$$V_0 = U^2 K_0, \qquad V_\alpha = U^2\Big(\bigcup_{\gamma<\alpha} K_\gamma\Big),$$

and observe that $\kappa(V_\alpha) \le \alpha$ for $\alpha < \kappa$. Next choose inductively elements $t_\alpha \in G$ with

$$\overline{U}^2 V_\gamma t_\gamma V_\gamma \cap \overline{U}^2 V_\alpha t_\alpha V_\alpha = \emptyset \quad \text{when } \alpha \neq \gamma,$$

using

$$\kappa\Big(V_\alpha^{-1}\overline{U}^{-2}\overline{U}^2\Big(\bigcup_{\gamma<\alpha} V_\gamma t_\gamma V_\gamma\Big)V_\alpha^{-1}\Big) \le \alpha.$$

Put $T = \{t_\alpha : \alpha < \kappa\}$. Then T is left uniformly discrete, and so $\overline{T} \setminus T \subseteq G^{LUC} \setminus G$.

Take $x, y \in \mathcal{U}(T)$ with $x \neq y$. Let A, B be disjoint subsets of T with $x \in \overline{A}$ and $y \in \overline{B}$. Define w_{xy} to be 1 on $\bigcup\{UV_\alpha t_\alpha V_\alpha : t_\alpha \in A\}$, and -1 on $\bigcup\{UV_\alpha t_\alpha V_\alpha : t_\alpha \in B\}$, and 0 elsewhere. The function w_{xy} is in $L_\infty(G)$. Let $\varphi \in L^1(G)$ be such that

$$\varphi \ge 0, \qquad \operatorname{supp}\varphi \subseteq U, \qquad \int_G \varphi(u)\,du = 1.$$

Denote the modular function on G by Δ, let $\widetilde{\varphi}$ be the function defined on G by $\widetilde{\varphi}(s) = \Delta(s^{-1})\varphi(s^{-1})$, and consider the function $f = \widetilde{\varphi} * w_{xy}$. Then f is in $LUC(G)$. Since the sets $U^2V_\alpha t_\alpha V_\alpha$ are disjoint, f is 1 on $V_\alpha t_\alpha V_\alpha$ when $t_\alpha \in A$, and -1 on this set when $t_\alpha \in B$, and 0 off $\bigcup\{\overline{U}^2 V_\alpha t_\alpha V_\alpha : t_\alpha \in A \cup B\}$.

Now take any $t \in G$. Because $V_\alpha \to G$, eventually $t \in V_\alpha$. Therefore when $t_\beta \in B$ and β is sufficiently large, we find $tt_\beta \in V_\beta t_\beta V_\beta$ so that $f(tt_\beta) = -1$. Similarly, for $t_\alpha \in A$ and α sufficiently large, we find $t_\alpha t \in V_\alpha t_\alpha V_\alpha$ so that $f(t_\alpha t) = 1$. Thus,

$$f(xy) = \lim_{t_\alpha \in A} \lim_{t_\beta \in B} f(t_\alpha t_\beta) = -1 = f(y) \quad \text{and} \quad \lim_{t_\beta \in B} \lim_{t_\alpha \in A} f(t_\alpha t_\beta) = 1 = f(x).$$

Clearly, f separates x and y. ∎

THEOREM 4.2. *Let G be a locally compact non-compact group. Then the quotient $LUC(G)/WAP(G)$ contains a linear isometric copy of $\ell_\infty(\kappa(G))$.*

Proof. Put $\kappa = \kappa(G)$. Let T be a left uniformly discrete subset of G as in Lemma 4.1, and partition it into κ subsets T_λ in such a way that each T_λ also has cardinal κ. On each $\overline{T_\lambda}$, fix two distinct points x and y in $\mathcal{U}(G)$ together with two nets $(t_\alpha)_\alpha$ and $(t_\beta)_\beta$ picked in two disjoint subsets A_λ and B_λ of T_λ and a function $f_\lambda \in LUC(G) \setminus WAP(G)$ as constructed in the proof of Lemma 4.1 such that $-1 \le f_\lambda \le 1$,

$$\lim_{t_\beta \in B_\lambda} \lim_{t_\alpha \in A_\lambda} f_\lambda(t_\beta t_\alpha) = f_\lambda(x) = 1 \quad \text{and} \quad \lim_{t_\alpha \in A_\lambda} \lim_{t_\beta \in B_\lambda} f_\lambda(t_\alpha t_\beta) = f_\lambda(y) = -1.$$

Next we assign to each function c in $\ell_\infty(\kappa)$ the function

$$f_c = \sum_{\lambda < \kappa} c_\lambda f_\lambda.$$

Then f_c is well-defined since the supports of the functions f_λ are non-overlapping, and $f_c \in LUC(G)$. Again, it is clear that

$$c \to f_c : \ell_\infty(\kappa(G)) \to LUC(G)$$

is a linear isometry. So we only need to check that $f_c \mapsto f_c + WAP(G)$ is an isometry into the quotient $LUC(G)/WAP(G)$. But this follows again as in the previous theorems from Lemma 1.1. To see this, simply note that

$$\lim_{t_\alpha \in A_\lambda} \lim_{t_\beta \in B_\lambda} f_c(t_\alpha t_\beta) = -c_\lambda \quad \text{and} \quad \lim_{t_\beta \in B_\lambda} \lim_{t_\alpha \in A_\lambda} f_c(t_\alpha t_\beta) = c_\lambda,$$

so

$$\|f_c + h\| \ge |c_\lambda| \quad \text{for every } \lambda < \kappa \text{ and } h \in WAP(G).$$

Thus,

$$\|f_c + WAP(G)\|_q \ge \|c\|.$$

As in the previous theorems, this yields the required linear isometry of $\ell_\infty(\kappa(G))$ into $LUC(G)/WAP(G)$. ∎

REMARK 4.3. Originally, slowly oscillating functions were used to obtain the theorem above. These functions are the antipodes of the weakly almost periodic ones and have shown to be very effective in studying semigroup compactifications of a locally compact group and convolution algebras related to it (see [14]). But as the referee pointed out, slow oscillation is not necessary to obtain the theorem. This has greatly simplified and shortened the proof of Lemma 4.1.

Recall that $b(G)$ is the smallest cardinality of an open base at e, and let $d(G)$ be the smallest cardinality of a norm-dense subset of $L^1(G)$. As noted in [17], $d(G) = \max\{\kappa(G), b(G)\}$. The following theorem is the dual of [26, Corollary 4.2] and encompasses [17, Theorem 4.7].

THEOREM 4.4. *If an infinite locally compact group G satisfies $\kappa(G) \geq b(G)$, then $L^1(G)$ is extremely non-Arens regular.*

Proof. We follow the general argument given in [26, Lemma 2.1]. Suppose that $d(G) = \kappa(G)$ and let $\{\phi_i : i < \kappa(G)\}$ be norm-dense in $L^1(G)$. Define $\pi : L^\infty(G) \to \ell_\infty(\kappa(G))$ by

$$\pi(f)(i) = \langle f, \phi_i \rangle \quad \text{for every } f \in L^\infty(G),\ i < \kappa(G).$$

Then π is the required linear isometry. ∎

When $\kappa(G) \leq b(G)$, we have the following special case.

THEOREM 4.5. *If G is an infinite locally compact metrizable group, then $L^1(G)$ is extremely non-Arens regular.*

Proof. Here $b(G) = \omega$. If G is non-compact then $\kappa(G) = b(G) = \omega$, and so we are in the situation of the previous theorem. If G is compact, then $C(G) = LUC(G) = WAP(G)$ and $d(G) = \omega$. Since G is infinite, it is not discrete and so by [36] (see also [22]), G is not extremely disconnected. Now we follow the argument used by Gulick in [23, Lemma 5.2] showing that $L^\infty(G)/C(G)$ is non-separable. (As noted by Granirer in [22], Gulick's argument holds also for non-abelian groups). Let U and V be disjoint open subsets in G with $e \in \overline{U} \cap \overline{V}$. Let $\{W_k\}_{k\geq 1}$ be a family of neighbourhoods of e and $(x_k)_{k\geq 1}$ a sequence of elements in G such that $(W_k x_k) \cap (W_l x_l) = \emptyset$ whenever $k \neq l$. For every $k \in \mathbb{N}$, let $T_k = W_k \cap \overline{U}$ and $S_k = W_k \cap \overline{V}$. Then, for each bounded scalar sequence $c = (c_k)$, let f_c be the function on G given by

$$f_c = \sum_{k\in\mathbb{N}} c_k(\chi_{T_k x_k} - \chi_{S_k x_k}).$$

Then $f_c \in L^\infty(G)$. To conclude, we only need to check that the quotient map

$$f_c \mapsto f_c + C(G) : L^\infty(G) \to L^\infty(G)/C(G)$$

is an isometry, the rest being clear. We suppose otherwise that $\|f_c+f\|_\infty < 1$ for some $f \in C(G)$ and $c \in \ell_\infty$, $\|c\| = 1$, pick $\epsilon > 0$ such that $\|f_c + f\|_\infty < 1-\epsilon$, and let $k \in \mathbb{N}$ be such that $|c_k| > 1-\epsilon/2$. Then

$$-1+\epsilon-c_k < f(s) < 1-\epsilon-c_k \quad \text{for every } s \in (W_k \cap U)x_k,$$
$$-1+\epsilon+c_k < f(s) < 1-\epsilon+c_k \quad \text{for every } s \in (W_k \cap V)x_k.$$

If $c_k > 0$, then $f(s) < -\epsilon/2$ if $s \in (W_k \cap U)x_k$, while $f(s) > \epsilon/2$ if $s \in (W_k \cap V)x_k$. If $c_k < 0$, then $f(s) > \epsilon/2$ if $s \in (W_k \cap U)x_k$, while $f(s) < -\epsilon/2$ if $s \in (W_k \cap V)x_k$. In both situations, the function f cannot be continuous. Therefore, $\|f_c + f\|_\infty \geq 1$ for every $f \in C(G)$. This gives as in the previous theorem the linear isometry of ℓ^∞ into the quotient $L^\infty(G)/C(G)$.

As earlier, this shows that the quotient contains a linear isometric copy of $L^\infty(G)$, and so $L^1(G)$ is ENAR. ∎

REMARK 4.6. Theorem 3.1 is now proved without the hypothesis of normality (see [2]).

Acknowledgements. We would like to thank the referee for the constructive comments in the report, and in particular for suggesting that the argument now in Lemma 1.1 should be presented as a separate result.

This work started when the second author was visiting the University of Rouen in March–April 2007. He would like to express his warm thanks for the kind hospitality of Ahmed Bouziad and the LMRS (UMR CNRS 6085) of the University of Rouen.

References

[1] J. F. Berglund, H. D. Junghenn and P. Milnes, *Analysis on Semigroups: Function Spaces, Compactifications, Representations*, Wiley, New York, 1989.

[2] A. Bouziad and M. Filali, *The Stone–Čech compactification of a topological group as a semigroup and the SIN property*, Houston J. Math., to appear.

[3] A. Bouziad and J.-P. Troallic, *Nonseparability and uniformities in topological groups*, Topology Proc. 28 (2004), 343–359.

[4] —, —, *A precompactness test for topological groups in the manner of Grothendieck*, ibid. 31 (2007), 19–30.

[5] T. Budak and J. S. Pym, *Local topological structure in the LUC-compactification of a locally compact group and its relationship with Veech's theorem*, Semigroup Forum 73 (2006), 159–174.

[6] C. Chou, *Weakly almost periodic functions and almost convergent functions on a group*, Trans. Amer. Math. Soc. 206 (1975), 175–200.

[7] H. G. Dales, *Banach Algebras and Automatic Continuity*, Oxford Univ. Press, New York, 2000.

[8] H. G. Dales and A. T.-M. Lau, *The second duals of Beurling algebras*, Mem. Amer. Math. Soc. 177 (2005), no. 836.

[9] A. M. H. Dzinotyiweyi, *Nonseparability of quotient spaces of function algebras on topological semigroups*, Trans. Amer. Math. Soc. 272 (1982), 223–235.

[10] M. Filali, *The uniform compactification of a locally compact abelian group*, Math. Proc. Cambridge Philos. Soc. 108 (1990), 527–538.

[11] —, *On the actions of a locally compact group on some of its semigroup compactifications*, ibid. 143 (2007), 25–39.

[12] M. Filali and J. S. Pym, *Right cancellation in the LUC-compactification of a locally compact group*, Bull. London Math. Soc. 35 (2003), 128–134.

[13] M. Filali and P. Salmi, *One-sided ideals and right cancellation in the second dual of the group algebra and similar algebras*, J. London Math. Soc. (2) 75 (2007), 35–46.

[14] —, —, *Slowly oscillating functions in semigroup compactification and convolution algebras*, J. Funct. Anal. 250 (2007), 144–166.

[15] M. Filali and A. I. Singh, *Recent developments on Arens regularity and ideal structure of the second dual of a group algebra and some related topological algebras*, in: General Topological Algebras, Est. Math. Soc., Tartu, 2001, 95–124.

[16] M. Filali and T. Vedenjuoksu, *The Stone–Čech compactification of a topological group and the β-extension property*, Houston J. Math. 36 (2010), 477–488.

[17] C. K. Fong and M. Neufang, *On the quotient space $UC(G)/WAP(G)$ and extreme non-Arens regularity of $L_1(G)$*, preprint, 2006.

[18] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton, 1960.

[19] E. E. Granirer, *Day points for quotients of the Fourier algebra $A(G)$, extreme nonergodicity of their duals and extreme non-Arens regularity*, Illinois J. Math. 40 (1996), 402–419.

[20] —, *Exposed points of convex sets and weak sequential convergence*, Mem. Amer. Math. Soc. 123 (1972).

[21] —, *Weakly almost periodic and uniformly continuous functionals on the Fourier algebra of any locally compact group*, Trans. Amer. Math. Soc. 189 (1974), 371–382.

[22] —, *The radical of $L^\infty(G)^*$*, Proc. Amer. Math. Soc. 41 (1973), 321–324.

[23] S. L. Gulick, *Commutativity and ideals in the biduals of topological algebras*, Pacific J. Math. 18 (1966), 121–137.

[24] G. Hansel and J.-P. Troallic, *Nonseparability and uniform structures in locally compact groups*, Proc. Amer. Math. Soc. 123 (1995), 1613–1621.

[25] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis I*, Springer, Berlin, 1963.

[26] Z. Hu, *Extreme non-Arens regularity of quotients of the Fourier algebra $A(G)$*, Colloq. Math. 72 (1997), 237–249.

[27] J. R. Isbell, *Uniform Spaces*, Math. Surveys 12, Amer. Math. Soc., Providence, RI, 1964.

[28] A. T.-M. Lau and V. Losert, *On the second conjugate algebra of $L_1(G)$ of a locally compact group*, J. London Math. Soc. (2) 37 (1988), 464–470.

[29] A. T.-M. Lau, P. Milnes and J. S. Pym, *Locally compact groups, invariant means and the centres of compactifications*, ibid. 56 (1997), 77–90.

[30] V. Losert, talk at Abstract Harmonic Analysis conference, Istanbul, 2006.

[31] M. G. Megrelishvili, V. G. Pestov and V. V. Uspenskij, *A note on the precompactness of weakly almost periodic groups*, in: Nuclear Groups and Lie Groups (Madrid, 1999), Res. Exp. Math. 24, Heldermann, Lemgo, 2001, 209–216.

[32] M. Neufang, *A unified approach to the topological centre problem for certain Banach algebras arising in abstract harmonic analysis*, Arch. Math. (Basel) 82 (2004), 164–171.

[33] T. W. Palmer, *Banach Algebras and the General Theory of *-Algebras, Vol. I: Algebras and Banach Algebras*, Cambridge Univ. Press, Cambridge, 1994.

[34] L. S. Pontryagin, *Topological Groups*, Gordon and Breach, New York, 1966.

[35] J. S. Pym, *The convolution of functionals on spaces of bounded functions*, Proc. London Math. Soc. (3) 15 (1965), 84–104.

[36] M. Rajagopalan, *Fourier transform in locally compact groups*, Acta Sci. Math. (Szeged) 25 (1964), 86–89.

[37] A. Ülger, *Continuity of weakly almost periodic functionals on* $L^1(G)$, Quart. J. Math. Oxford Ser. (2) 37 (1986), 495–497.

[38] N. J. Young, *The irregularity of multiplication in group algebras*, ibid. 24 (1973), 59–62.

Ahmed Bouziad
Département de Mathématiques
Université de Rouen
UMR CNRS 6085
Avenue de l'Université, BP.12
F-76801 Saint-Étienne-du-Rouvray, France
E-mail: ahmed.bouziad@univ-rouen.fr

Mahmoud Filali
Department of Mathematical Sciences
University of Oulu
90014 Oulu, Finland
E-mail: mahmoud.filali@oulu.fi

Received July 15, 2010
Revised version December 6, 2010 (6946)