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On local aspects of topological weak mixing in dimension one and beyond

by

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Abstract. We introduce the concept of weakly mixing sets of order n and show that, in contrast to weak mixing of maps, a weakly mixing set of order n does not have to be weakly mixing of order $n+1$. Strictly speaking, we construct a minimal invertible dynamical system which contains a non-trivial weakly mixing set of order 2, whereas it does not contain any non-trivial weakly mixing set of order 3.

In dimension one this difference is not that much visible, since we prove that every continuous map f from a topological graph into itself has positive topological entropy if and only if it contains a non-trivial weakly mixing set of order 2 if and only if it contains a non-trivial weakly mixing set of all orders.

1. Introduction. The following property of a continuous mapping f acting on a compact metric space X is well known in topological dynamics. We say that f is *weakly mixing* if for any non-empty open sets U_1, U_2, V_1, V_2 there is $k > 0$ such that $f^k(U_i) \cap V_i \neq \emptyset$ for $i = 1, 2$. It was proved more than forty years ago by Furstenberg that if in the definition of weak mixing we allow n pairs of sets $U_1, \dots, U_n$, $V_1, \dots, V_n$ instead of only two of them then the property does not change (i.e. we obtain a definition equivalent to weak mixing).

Blanchard and Huang introduced in [BH] a local version of weak mixing. In [BH], a closed set A with at least two elements is said to be *weakly mixing* if for any open sets $U_1, \dots, U_n$, $V_1, \dots, V_n$ intersecting A there is $k > 0$ such that $f^k(U_i \cap A) \cap V_i \neq \emptyset$ for $i = 1, \dots, n$.

Throughout this paper, by a *topological dynamical system* (or simply *TDS*) we mean a pair (X, f), where (X, d) is a compact metric space and $f \colon X \to X$ is a continuous map. If X is a singleton, then we say that (X, f) is *trivial*; if f is a homeomorphism or surjection then we say that (X, f) is *invertible* or *surjective*.

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Denote by $\mathbb{N}$, $\mathbb{N}_0, \mathbb{Z}$ and $\mathbb{R}$ the set of all positive integers, non-negative integers, integers and real numbers, respectively. The approach in [BH] can be extended as follows.

DEFINITION 1.1. Let (X, f) be a TDS, $\emptyset \neq A \subseteq X$ and $n \in \mathbb{N} \setminus \{1\}$. Define $A^{(n)} = \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in A\}$. We say that A is

(1.1.1) *transitive* if for each pair of open subsets (U, V) of X with $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$ there exists $m \in \mathbb{N}$ such that $f^m(V \cap A) \cap U \neq \emptyset$;

(1.1.2) *weakly mixing of order n* if $A^{(n)}$ is a transitive set of $(X^{(n)}, f^{(n)})$, where $f^{(n)}$ acts naturally on $X^{(n)}$ by

$$f^{(n)}(x_1, \ldots, x_n) = (f(x_1), \ldots, f(x_n));$$

(1.1.3) *weakly mixing of all orders* or simply *weakly mixing* if A is weakly mixing of order m for $m = 2, 3, \ldots$.

In the above cases we say that A is *non-trivial* if it contains at least two points.

It was proved in [BH] that an invertible TDS with positive topological entropy always has a non-trivial weakly mixing set. We will prove that in dimension one this characterization is to some extent full, that is, the following result holds.

THEOREM 1.2. *Let (G, f) be a TDS acting on a topological graph G. In that case the following statements are equivalent:*

(1.2.1) *(G, f) has positive topological entropy.*
(1.2.2) *(G, f) contains a non-trivial weakly mixing set.*
(1.2.3) *(G, f) contains a non-trivial weakly mixing set of order* 2.

By a *topological graph* we mean here a compact, connected metric space which is homeomorphic to a polyhedron (a geometric realization) of some finite one-dimensional complex (see [Mo, Chapter 0])

On the unit interval we can say even more (see Theorem 4.3): if a TDS acts on the unit interval then arbitrarily close (in the sense of Hausdorff metric) to any given weakly mixing set of order 2 we can find a weakly mixing set.

As mentioned before, there is a well-known theorem, first proved by Furstenberg, which asserts that in the case of an abstract dynamical system defined by an action of an abelian group, the property of weak mixing of order 2 implies weak mixing of all orders. This result is no longer valid for actions of non-abelian groups [G, W]. Thus, it is natural to ask whether a weakly mixing set of order 2 is weakly mixing of all orders. We will answer this question in the negative in the second part of this paper. Namely, we will provide examples showing that the concepts of weakly mixing sets of order 2 and weakly mixing sets are different.

First, following the ideas used by Glasner in his proof of [G, Theorem 4.1.2] we provide a direct construction of the following system.

EXAMPLE 1.3. *There exists a TDS (X, f) such that*

(1.3.1) *X is a weakly mixing set,*
(1.3.2) *(X, f) contains a weakly mixing set of order* 2 *but not* 3.

Next, using residual properties of a special group of homeomorphisms, we apply results of [GW] to obtain another interesting example.

EXAMPLE 1.4. *There exists a minimal invertible TDS which*

(1.4.1) *has zero topological entropy,*
(1.4.2) *contains a non-trivial weakly mixing set of order* 2*,*
(1.4.3) *contains no non-trivial weakly mixing set of order* 3*.*

This means that, there is no hope for obtaining results similar to Theorem 1.2 (or Theorem 4.3) in a general setting (or even in dimension two).

The paper is organized as follows. In Section 2, we introduce the notion of transitive sets and explore its basic properties. In Section 3, we investigate basic properties of weakly mixing sets. Among other things, we show that every non-trivial weakly mixing set of order 2 contains no isolated points; each equicontinuous TDS contains no non-trivial weakly mixing sets of order 2; a TDS with positive topological entropy contains many non-trivial weakly mixing sets; and each TDS with a non-trivial weakly mixing set has positive topological sequence entropy.

Section 4 contains the proof of Theorem 1.2 together with some other properties of weakly mixing sets specific for dimension one. In Section 5, we discuss relations between weak mixing of a factor map and weakly mixing sets in its fibers. The last two sections (Sections 6 and 7) concern the question whether a weakly mixing set of order 2 is weakly mixing of all orders (that is, weakly mixing in the sense of [BH]). We show that these concepts are different, i.e. the situation is much more complex than in the case of global weak mixing implied by weak mixing of a TDS, by providing detailed constructions of Examples 1.3 and 1.4.

2. Basic properties of transitive sets. Let (X, f) be a TDS and $\emptyset \neq K \subseteq X$ a compact subset. When $f(K) \subseteq K$, we say that (K, f) is a *subsystem* of (X, f). Let $\emptyset \neq A \subseteq X$ and $x \in X$. Put $\mathrm{Orb}^+(A, f) = \bigcup_{n\in\mathbb{N}_0} f^n(A)$, the *positive orbit of A under f*, and $\mathrm{Orb}(A, f) = \bigcup_{n\in\mathbb{Z}} f^n(A)$, the *orbit of A under f* when (X, f) is invertible. To simplify notation, we write $\mathrm{Orb}^+(x, f) = \mathrm{Orb}^+(\{x\}, f)$ and $\mathrm{Orb}(x, f) = \mathrm{Orb}(\{x\}, f)$. Obviously $(\overline{\mathrm{Orb}^+(A, f)}, f)$ (and also $(\overline{\mathrm{Orb}(A, f)}, f)$ when (X, f) is invertible) forms a subsystem of (X, f).

Let (X, f) be a TDS and $x \in X$. We say that x is a *fixed point* of (X, f) if $f(x) = x$; a *periodic point* of (X, f) if $f^n(x) = x$ for some $n \in \mathbb{N}$; a *recurrent point* of (X, f) if there exists $\{k_1 < k_2 < \cdots\} \subseteq \mathbb{N}$ such that $f^{k_n}x \to x$ as $n \to \infty$ (if and only if there exists a sequence $\{k_n\}_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $f^{k_n}x \to x$ as $n \to \infty$, if and only if $x \in \overline{\mathrm{Orb}^+(f(x), f)}$); a *non-wandering point* of (X, f) if $f^n(U) \cap U \neq \emptyset$ for some $n \in \mathbb{N}$ whenever U is a neighborhood of x and a *wandering point* of (X, f) otherwise; a *transitive point* of (X, f) if $\mathrm{Orb}^+(x, f)$ is dense in X. Denote by $\mathrm{Rec}(X, f)$, $\Omega(X, f)$ and $\mathrm{Tran}(X, f)$ the set of all recurrent points, non-wandering points and transitive points of (X, f) respectively. Recall that (X, f) is *transitive* if $f^{-n}(U) \cap V \neq \emptyset$ for some $n \in \mathbb{N}$ whenever U and V are both non-empty open subsets of X; *minimal* if $\mathrm{Tran}(X, f) = X$.

Bellow, we collect some well-known facts and easy observations.

LEMMA 2.1. *Let (X, f) be a TDS. Then*

(2.1.1) $\mathrm{Rec}(X, f) = \mathrm{Rec}(X, f^n)$ *for every $n \in \mathbb{N} \setminus \{1\}$.*

(2.1.2) *If (X, f) is transitive then $f(X) = X$ and X is either perfect (by a* perfect set *we mean a non-empty compact set without isolated points) or a periodic orbit (i.e. $X = \mathrm{Orb}^+(x, f)$ for some periodic point x of (X, f)).*

(2.1.3) *If (X, f) is invertible then (X, f) is transitive if and only if (X, f^{-1}) is transitive.*

(2.1.4) $\mathrm{Tran}(X, f) \neq \emptyset$ *does not imply the transitivity of the TDS (X, f) in general, as it may happen that* $\mathrm{Tran}(X, f) \neq \emptyset$ *even if $f(X) \subsetneq X$.*

(2.1.5) *If $f(X) = X$ or X is perfect, then* $\mathrm{Tran}(X, f) \subseteq \mathrm{Rec}(X, f)$ *and (X, f) is transitive if and only if* $\mathrm{Tran}(X, f)$ *is a dense G_δ subset of X if and only if* $\mathrm{Tran}(X, f) \neq \emptyset$.

(2.1.6) *(X, f) is minimal if and only if for each non-empty open $U \subseteq X$ there exists $n \in \mathbb{N}$ with $\bigcup_{i=0}^n f^{-i}(U) = X$ if and only if* $\mathrm{Tran}(X, f) = \mathrm{Rec}(X, f) = X$.

(2.1.7) *If (X, f) is minimal then (X, f) is transitive.*

The following result is a direct consequence of the definitions.

PROPOSITION 2.2. *Let (X, f) be a TDS and $x \in X$, $\emptyset \neq A \subseteq X_0 \subseteq X$. Then*

(2.2.1) $x \in \mathrm{Rec}(X, f)$ *if and only if $\{x\}$ is a transitive set.*

(2.2.2) *(X, f) is transitive if and only if X is a transitive set.*

(2.2.3) *If (X_0, f) is a subsystem of (X, f), then A is a transitive set with respect to (X, f) if and only if it is transitive with respect to (X_0, f).*

(2.2.4) *If A is a transitive set then $A \subseteq \Omega(X, f)$.*

(2.2.5) *If $\{X_i\}_{i\in I}$ is a family of transitive sets, where I is a totally ordered set and $X_i \subseteq X_j$ if $i \leq j$, then $\bigcup_{i\in I} X_i$ is also transitive.*

Let (X, f) be a TDS and $\emptyset \neq A,\ B \subseteq X$. Define $N(A, B) = \{n \in \mathbb{N} : f^n(A) \cap B \neq \emptyset\}$ or equivalently $N(A, B) = \{n \in \mathbb{N} : A \cap f^{-n}(B) \neq \emptyset\}$. If $A = \{x\}$, we will simply write $N(x, B)$ instead of $N(\{x\}, B)$.

PROPOSITION 2.3. *Let (X, f) be a TDS and $\emptyset \neq A \subseteq X$. Then*

(2.3.1) *A is a transitive set if and only if $N(V \cap A, U)$ is an infinite set for each pair of open sets $U, V \subseteq X$ with $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$.*
(2.3.2) *A is a transitive set if and only if $\overline{A}$ is a transitive set. Thus if A is an invariant (i.e. $f(A) \subseteq A$) transitive set then $(\overline{A}, f)$ is a transitive TDS.*
(2.3.3) *If A is a transitive set then $(\overline{\mathrm{Orb}^+(A, f)}, f)$ is transitive.*
(2.3.4) *If $A \subseteq \mathrm{Tran}(X, f) \cap \mathrm{Rec}(X, f)$ then A is a transitive set.*

Proof. (2.3.1) Assume that A is a transitive set. Fix any open sets $U, V \subseteq X$ with $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$. Then $N(V \cap A, U) \neq \emptyset$, as A is a transitive set. For each $n \in N(V \cap A, U)$, $f^{-n}(U)$ is an open subset of X with $f^{-n}(U) \cap A \neq \emptyset$ and so $N(V \cap A, f^{-n}(U)) \neq \emptyset$. In particular, there is $m > 0$ such that $n, m+n \in N(V \cap A, U)$, which implies that $N(V \cap A, U)$ is an infinite set.

(2.3.2) First, assume that $\overline{A}$ is a transitive set. Let $U, V \subseteq X$ be open sets with $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$. By the assumptions, there exists $n \in \mathbb{N}$ with $(V \cap \overline{A}) \cap f^{-n}(U) \neq \emptyset$, which implies that $(V \cap A) \cap f^{-n}(U) \neq \emptyset$ (as $V \cap f^{-n}(U)$ is an open subset of X), i.e. $N(V \cap A, U) \neq \emptyset$. Thus, A is a transitive set. A similar reasoning shows that if A is a transitive set then so is $\overline{A}$.

(2.3.3) Since we have already proved (2.3.2), it is sufficient to show that $\mathrm{Orb}^+(A, f)$ is a transitive set. Let $U, V \subseteq X$ be open sets with $U \cap \mathrm{Orb}^+(A, f) \neq \emptyset$ and $V \cap \mathrm{Orb}^+(A, f) \neq \emptyset$. There are integers $i_U, i_V \geq 0$ such that $U \cap f^{i_U}(A) \neq \emptyset$ and $V \cap f^{i_V}(A) \neq \emptyset$. Since A is a transitive set, by (2.3.1) there exists $n \in \mathbb{N}$ such that $n > i_V$ and $f^n(f^{-i_V}(V) \cap A) \cap f^{-i_U}(U) \neq \emptyset$, which implies $f^{n-i_V+i_U}(V \cap f^{i_V}(A)) \cap U \neq \emptyset$. Thus $N(V \cap \mathrm{Orb}^+(A, f), U) \supseteq N(V \cap f^{i_V}(A), U) \neq \emptyset$. That is, $\mathrm{Orb}^+(A, f)$ is a transitive set.

(2.3.4) Fix open sets $U, V \subseteq X$ such that $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$ and let $x_0 \in U \cap A \subseteq \mathrm{Tran}(X, f) \cap \mathrm{Rec}(X, f)$. Observe that there exists $n \in \mathbb{N}$ with $f^n(x_0) \in V$ (if $x_0 \notin V$ use $x_0 \in \mathrm{Tran}(X, f)$; if $x_0 \in V$ use $x_0 \in \mathrm{Rec}(X, f)$), in particular, $N(U \cap A, V) \neq \emptyset$. Thus, A is a transitive set. ∎

REMARK 2.4. The implications converse to (2.3.3) or (2.3.4) do not hold. To see this, let (X, f) be an invertible transitive TDS such that $X \setminus \mathrm{Rec}(X, f) \neq \emptyset$ (e.g. we can take the full two-sided shift over two symbols). Choose $x_1 \in X \setminus \mathrm{Rec}(X, f)$ and $x_2 \in \mathrm{Tran}(X, f)$ ($\mathrm{Tran}(X, f) \neq \emptyset$ by (2.1.5)). Define $A = \{x_1, x_2\}$. Then $(\overline{\mathrm{Orb}^+(A, f)}, f) = (X, f)$, in particular, it is a transitive TDS. However, as $x_1 \notin \mathrm{Rec}(X, f)$, there is an open set U such that $U \cap A = \{x_1\}$ and $N(x_1, U) = \emptyset$. This shows that A is not a transitive set. Additionally, since (X, f) is transitive, X is a transitive set while $X \setminus (\mathrm{Tran}(X, f) \cup \mathrm{Rec}(X, f)) \neq \emptyset$ (by (2.1.5) and (2.2.2)).

Remark 2.5. Let (X, f) be a TDS and $\emptyset \neq A \subseteq X$. By (2.3.3), if B is the maximal transitive set containing A (in the sense of set inclusion) then (B, f) will be the maximal transitive subsystem containing A (in the sense of set inclusion), whereas it may happen that if $C \supseteq A$ then C is not a transitive set. For example, let X be the one-point compactification $\mathbb{Z} \cup \{\infty\}$ of $\mathbb{Z}$ and let f act on X by $f \colon \mathbb{Z} \ni z \mapsto z + 1 \in \mathbb{Z}$, $f(\infty) = \infty$. It is easy to verify that $\{\infty\}$ is the unique transitive set in X.

For the particular case $A = \{x\}$ Proposition 2.3 gives the following:

Corollary 2.6. *Let (X, f) be a TDS and $x \in X$. Then*

(2.6.1) *$\{x\}$ is a transitive set if and only if $(\overline{\mathrm{Orb}^+(x, f)}, f)$ is transitive if and only if f is a surjection over $\overline{\mathrm{Orb}^+(x, f)}$ if and only if $x \in \mathrm{Rec}(X, f)$.*

(2.6.2) *If f is invertible then $\{x\}$ is a transitive set if and only if $\overline{\mathrm{Orb}(x, f)} = \overline{\mathrm{Orb}^+(x, f)}$ if and only if $x \in \mathrm{Tran}(\overline{\mathrm{Orb}(x, f)}, f)$.*

Proof. (2.6.1) By (2.1.2), (2.2.1) and (2.3.3), we only need to show that $x \in \mathrm{Rec}(X, f)$ when f is surjective on $\overline{\mathrm{Orb}^+(x, f)}$. But the proof is straightforward, since if $x_0 \in \overline{\mathrm{Orb}^+(x, f)}$ with $f(x_0) = x$ then $x \in \overline{\mathrm{Orb}^+(f(x), f)}$, i.e. $x \in \mathrm{Rec}(X, f)$.

(2.6.2) Assume that f is invertible. Then by (2.2.1), $\{x\}$ is a transitive set if and only if $x \in \overline{\mathrm{Orb}^+(f(x), f)}$ if and only if $f^{-1}(x) \in \overline{\mathrm{Orb}^+(x, f)}$ (f is invertible) if and only if $\overline{\mathrm{Orb}(x, f)} = \overline{\mathrm{Orb}^+(x, f)}$ if and only if $x \in \mathrm{Tran}(\overline{\mathrm{Orb}(x, f)}, f)$. ∎

Remark 2.7. It may happen that (X, f) is an invertible transitive TDS and $x \in X$ satisfies $X = \overline{\mathrm{Orb}(x, f)}$ and $x \notin \mathrm{Rec}(X, f)$. For example, this is the case for the full two-sided shift over two symbols.

For TDSs (X, f) and (Y, g), every continuous onto map $\pi \colon X \to Y$ such that $\pi \circ f = g \circ \pi$ is said to be a *factor map between (X, f) and (Y, g)*. To stress the fact that π is the factor map between these systems we write $\pi \colon (X, f) \to (Y, g)$. When such a map exists, (Y, g) is said to be a *factor* of (X, f) and (X, f) an *extension* of (Y, g). In the special case of an invertible factor map π we say that π is a *conjugacy* and that the systems $(X, f), (Y, g)$ are (*topologically*) *conjugate*.

Proposition 2.8. *Let $\pi \colon (X, f) \to (Y, g)$ be a factor map between TDSs. If $\emptyset \neq A \subseteq X$ is a transitive set then $\pi(A) \subseteq Y$ is also a transitive set.*

Proof. Let (U, V) be a pair of open subsets in Y with $U \cap \pi(A) \neq \emptyset$ and $V \cap \pi(A) \neq \emptyset$. Since A is a transitive set, there exists $n \in \mathbb{N}$ such that $f^n(\pi^{-1}(V) \cap A) \cap \pi^{-1}(U) \neq \emptyset$, which implies $g^n(V \cap \pi(A)) \cap U \neq \emptyset$. That is, $\pi(A)$ is a transitive set. ∎

3. Basic properties of weakly mixing sets. Recall that a TDS (X, f) is weakly mixing if $(X^{(2)}, f^{(2)})$ is transitive, equivalently, $(X^{(n)}, f^{(n)})$ is transitive for each $n \in \mathbb{N} \setminus \{1\}$.

By Propositions 2.2, 2.3 and 2.8, we have the following.

PROPOSITION 3.1. *Let (X, f) be a TDS and $\emptyset \neq A \subseteq X$, $n \geq k \geq 2$. Then*

(3.1.1) *A is weakly mixing of order n if and only if $\bigcap_{i=1}^{n} N(V_i \cap A, U_i) \neq \emptyset$ whenever $U_1, V_1, \ldots, U_n, V_n$ are open subsets of X with $U_i \cap A \neq \emptyset$ and $V_i \cap A \neq \emptyset$, $i = 1, \ldots, n$, if and only if $\bigcap_{i=1}^{n} N(V_i \cap A, U_i)$ is an infinite set whenever $U_1, V_1, \ldots, U_n, V_n$ are open subsets of X with $U_i \cap A \neq \emptyset$ and $V_i \cap A \neq \emptyset$, $i = 1, \ldots, n$, if and only if $\overline{A}$ is weakly mixing of order n.*

(3.1.2) *If A is a weakly mixing set of order n then A is a weakly mixing set of order k and then A is a transitive set.*

(3.1.3) *If $\{X_i\}_{i \in I}$ is a family of weakly mixing sets of order n, where I is a totally ordered set and $X_i \subseteq X_j$ if $i \leq j$, then $\bigcup_{i \in I} X_i$ is weakly mixing of order n.*

(3.1.4) *(X, f) is weakly mixing if and only if X is weakly mixing of order 2 if and only if X is weakly mixing of order n if and only if X is weakly mixing.*

PROPOSITION 3.2. *Let $\pi : (X, f) \to (Y, S)$ be a factor map between TDSs and $\emptyset \neq A \subseteq X$, $n \in \mathbb{N} \setminus \{1\}$. If A is a weakly mixing set of order n then $\pi(A) \subseteq Y$ is also a weakly mixing set of order n.*

The following fact shows that weakly mixing sets require much more complex topological structure than was allowed in the case of transitive sets.

PROPOSITION 3.3. *Let (X, f) be a TDS and $\emptyset \neq A \subseteq X$ a non-trivial weakly mixing set of order 2. Then A contains no isolated points.*

Proof. Assume on the contrary that $x \in A$ is an isolated point of A. Then there exist non-empty open subsets U_1, U_2 of X such that $U_1 \cap U_2 = \emptyset$, $U_1 \cap A = \{x\}$ and $U_2 \cap A \neq \emptyset$. Since A is a weakly mixing set of order 2, there exists $n \in \mathbb{N}$ such that

$$\emptyset \neq f^n((U_1 \times U_1) \cap A^{(2)}) \cap (U_1 \times U_2) = (\{f^n(x)\} \cap U_1) \times (\{f^n(x)\} \cap U_2),$$

which is impossible because $U_1 \cap U_2 = \emptyset$. ∎

Remark 3.4. Let (X,f) be an invertible transitive TDS with $x_1,x_2\in \mathrm{Tran}(X,f)\subseteq \mathrm{Rec}(X,f)$ and $x_1\neq x_2$. By Proposition 3.3 the set $\{x_1,x_2\}$ cannot be weakly mixing of order 2, while by (2.3.4) it is a transitive set.

Before proceeding, we restate here a version of the well-known [My, Theorem 1].

Lemma 3.5. *Let X be a perfect metric space and $R_n\subseteq X^{(r_n)}$ a subset of first category for each $n\in\mathbb{N}$. Then there exists a dense Mycielski subset K such that $(x_i)_{i=1}^{r_n}\notin R_n$ whenever $x_1,\dots,x_{r_n},n\in\mathbb{N}$, are r_n distinct elements of K.*

Then we have

Proposition 3.6. *Let (X,f) be a TDS and $A\subseteq X$ a closed non-trivial weakly mixing set of order $n\in\mathbb{N}\setminus\{1\}$. Then there exist $\delta>0$ and a dense Mycielski subset K of A such that if $x_1,\dots,x_n$ are n distinct elements of K then*

(3.1)
$$\liminf_{m\to\infty}\max_{1\le i<j\le n} d(f^m x_i,f^m x_j)=0,\qquad \limsup_{m\to\infty}\min_{1\le i<j\le n} d(f^m x_i,f^m x_j)\ge\delta.$$

Proof. Since A is a closed non-trivial weakly mixing set of order n, by Proposition 3.3, A must be perfect and so there exist pairwise disjoint open subsets $V_1,\dots,V_n$ of X with $V_i\cap A\neq\emptyset$, $i=1,\dots,n$, and $\delta\doteq\min_{1\le i<j\le n} d(\overline{V_i},\overline{V_j})>0$. For any $\epsilon>0$ and $N\in\mathbb{N}$ define

$$D_N^\epsilon=\{(x_i)_{i=1}^n\in A^{(n)}:\exists m\ge N \text{ such that } \min_{1\le i<j\le n} d(f^m x_i,f^m x_j)>\delta-\epsilon\},$$
$$A^\epsilon=\{(x_i)_{i=1}^n\in A^{(n)}:\exists m\in\mathbb{N} \text{ such that } \max_{1\le i<j\le n} d(f^m x_i,f^m x_j)<\epsilon\}.$$

It is easy to see that both D_N^ϵ and A^ϵ are open subsets of $A^{(n)}$, moreover, for each $k\in\mathbb{N}$ both $D_N^{1/k}$ and $A^{1/k}$ are dense open subsets of $A^{(n)}$ (by (3.1.1), the selection of $V_1,\dots,V_n,\delta$ and the assumption that A is a weakly mixing set of order n). Now, denote by M the set of all $(x_i)_{i=1}^n\in A^{(n)}$ satisfying (3.1). Then

$$R=\bigcap_{k\in\mathbb{N}}\Big(\bigcap_{N\in\mathbb{N}} D_N^{1/k}\cap A^{1/k}\Big),$$

and so R is a dense G_δ subset of $A^{(n)}$. Applying Lemma 3.5 to $A^{(n)}\setminus R$, we obtain a dense Mycielski subset $K\subseteq A$ such that $(x_i)_{i=1}^n\in R$ (i.e. (3.1) holds) whenever $x_1,\dots,x_n$ are n distinct elements of K. ∎

Recall that a TDS (X,f) is *equicontinuous* if for every $\epsilon>0$ there exists $\delta>0$ such that $d(f^n(x_1),f^n(x_2))<\epsilon$ for $n=0,1,\dots,$ provided that $d(x_1,x_2)<\delta$.

PROPOSITION 3.7. *Every equicontinuous TDS contains no non-trivial weakly mixing sets of order* 2.

Proof. Take any non-trivial set A and fix distinct $x, y \in A$. Define $\varepsilon = d(x, y)$ and use equicontinuity to pick δ for $\varepsilon/4$. We may assume that $\delta < \varepsilon/4$. Now, if we denote by $U_1 = B(x, \delta/2)$ the open ball with center x and radius $\delta/2$ and put $U_2 = B(y, \delta/2)$ then for every n either $f^n(U_1) \cap U_1 = \emptyset$ or $f^n(U_1) \cap U_2 = \emptyset$, as otherwise

$$d(x, y) \leq \operatorname{diam} U_1 + \operatorname{diam} U_2 + \operatorname{diam} f^n(U_1) \leq 2\delta + \varepsilon/4 < \varepsilon,$$

where $\operatorname{diam} V$ denotes the diameter of V. But $U_1 \cap A \neq \emptyset$ and $U_2 \cap A \neq \emptyset$, so A is not weakly mixing of order 2. ∎

Before proceeding, let us recall the concept of topological entropy of a given TDS. Let (X, f) be a TDS and let $\mathscr{C}$ and $\mathscr{D}$ be any finite covers of X. We define their refinement by $\mathscr{C} \vee \mathscr{D} = \{C \cap D : C \in \mathscr{C},\ D \in \mathscr{D},\ C \cap D \neq \emptyset\}$ and let $f^{-n}(\mathscr{C}) = \{f^{-n}(C) : C \in \mathscr{C}\}$ for each $n \in \mathbb{N}_0$. Fix a finite open cover $\mathscr{U}$ of X. For any $A \subseteq X$ let $r(\mathscr{U}, A)$ denote the minimum among the cardinalities of subsets of $\mathscr{U}$ that cover A. We define the *topological entropy* of $\mathscr{U}$ by

$$h_{\mathrm{top}}(f, \mathscr{U}) = \lim_{n\to\infty} \frac{1}{n} \log r\Big(\bigvee_{i=0}^{n-1} f^{-i}(\mathscr{U}), X\Big) = \inf_{n\in\mathbb{N}} \frac{1}{n} \log r\Big(\bigvee_{i=0}^{n-1} f^{-i}(\mathscr{U}), X\Big),$$

where the existence of the limit and the second identity follow from the fact that

$$\Big\{\log r\Big(\bigvee_{i=0}^{n-1} f^{-i}(\mathscr{U}), X\Big)\Big\}_{n\in\mathbb{N}}$$

is a subadditive sequence. Then the *topological entropy of* (X, f) is given by $h_{\mathrm{top}}(f, X) = \sup_{\mathscr{U}\in\mathcal{C}_X} h_{\mathrm{top}}(f, \mathscr{U})$, where $\mathcal{C}_X$ denotes the set of all possible finite open covers of X.

It was proved first in [BH] that systems (invertible or even only surjective) with positive topological entropy contain closed non-trivial weakly mixing sets (see [BH, Theorem 4.5] and remarks after it, especially on page 292). Here we sketch another proof of this fact.

THEOREM 3.8. *Let (X, f) be a TDS with positive topological entropy. Then it contains a non-trivial weakly mixing set.*

Proof. First, we assume that (X, f) is invertible. By the classical variational principle concerning topological and measure-theoretic entropy we may select an ergodic f-invariant Borel probability measure μ over X with $h_\mu(f, X) > 0$. Just as in [Z1, Lemma 4.1 and Theorem 4.4], we denote by $\pi_1 : (X, \mathcal{B}_X, \mu, f) \to (Z, \mathcal{Z}, \nu, h)$ the Pinsker factor of $(X, \mathcal{B}_X, \mu, f)$ and by $\mu = \int_Z \mu_z\, d\nu(z)$ the disintegration of μ over ν, where $\mathcal{B}_X$ is the Borel

σ-algebra of X. For each given $n \in \mathbb{N} \setminus \{1\}$, put $\lambda_n^\pi(\mu) = \int_Z \mu_z^{(n)}\, d\nu(z)$, where $\mu_z^{(n)} = \mu_z \times \cdots \times \mu_z$ (n times). Then we have

(3.8.1) $\operatorname{supp}(\mu_z)$, the support of μ_z, is not a singleton for ν-a.e. $z \in Z$,
(3.8.2) $(\operatorname{supp}(\lambda_n^\pi(\mu)), f^{(n)})$ is a transitive TDS.

Thus the proof of this case will be finished if we show that $\operatorname{supp}(\mu_z)$ is a weakly mixing set of order n for ν-a.e. $z \in Z$. In fact, as $(\operatorname{supp}(\lambda_n^\pi(\mu)), f^{(n)})$ is a transitive TDS we see that $W_n \doteq \operatorname{Tran}(\operatorname{supp}(\lambda_n^\pi(\mu)), f^{(n)}) \neq \emptyset$. By (2.3.4), for every $z \in Z$ the set $W_n^z \doteq W_n \cap (\operatorname{supp}(\mu_z))^{(n)}$ is a transitive set if it is non-empty. Observe that $1 = \lambda_n^\pi(\mu)(W_n) = \int_Z (\mu_z)^{(n)}(W_n)\, d\nu(z)$, hence $(\mu_z)^{(n)}(W_n^z) = (\mu_z)^{(n)}(W_n) = 1$ and so by (2.3.2) the set $(\operatorname{supp}(\mu_z))^{(n)} = \overline{W_n^z}$ is a transitive set for ν-a.e. $z \in Z$. In other words, $\operatorname{supp}(\mu_z)$ is a weakly mixing set of order n for ν-a.e. $z \in Z$.

Now, in general, by the well-known facts that $(\Omega(X,f), f|_{\Omega(X,f)})$ is a well-defined surjective TDS and $h_{\mathrm{top}}(f, \Omega(X,f)) = h_{\mathrm{top}}(f,X)$ (e.g. [K, Theorem 2.90]), we may assume that $f(X) = X$. Let us consider the invertible TDS $(\mathbb{X}_f, \sigma_f)$, where $\mathbb{X}_f = \{(x_1, x_2, \dots) \in \prod_1^\infty X : f(x_{n+1}) = x_n, n \in \mathbb{N}\}$ and $\sigma_f(x_1, x_2, \dots) = (f(x_1), x_1, \dots)$. Then $h_{\mathrm{top}}(\sigma_f, \mathbb{X}_f) = h_{\mathrm{top}}(f,X)$ [B, Proposition 5.2], and so, by the above discussions, there exists a closed non-trivial weakly mixing set A of $(\mathbb{X}_f, \sigma_f)$, thus $\pi(A)$ is a non-trivial weakly mixing set of (X,f) if it is not a singleton by Proposition 3.2, where the factor map $\pi : (\mathbb{X}_f, \sigma_f) \to (X,f)$, $(x_1, x_2, \dots) \mapsto x_1$, is the projection onto the first coordinate. On the other hand, by Proposition 3.6 there exist $x', x'' \in A$ such that $\limsup_{n\to\infty} d(f^n x', f^n x'') > 0$, which implies $\pi(x') \neq \pi(x'')$ (observe that if x' and x'' are contained in the same π-fiber then $\lim_{n\to\infty} d(f^n x', f^n x'') = 0$), i.e. $\pi(A)$ is not a singleton and hence a non-trivial weakly mixing set. ∎

Remark 3.9. By Theorem 3.8 we see that it may happen that a TDS contains a non-trivial weakly mixing set without admitting any weakly mixing subsystem. A particular example of such a situation is a minimal TDS with positive topological entropy which is not weakly mixing (e.g. a Toeplitz flow [Do]).

We finish this section with the following result.

Proposition 3.10. *Let (X,f) be a TDS and $A \subseteq X$ a non-trivial weakly mixing set. Then A has positive topological sequence entropy, i.e. there exist a finite open cover $\mathcal{U}$ of X and a sequence $\{t_1 < t_2 < \cdots\} \subseteq \mathbb{N}_0$ such that*

$$h_{\mathrm{top}}(f, \mathcal{U}, \{t_i\}_{i\in\mathbb{N}}) \doteq \limsup_{n\to\infty} \frac{1}{n} \log r\Big(\bigvee_{i=1}^n f^{-t_i}(\mathcal{U}), A\Big) > 0.$$

Proof. By the assumptions, there exist open sets V_1, V_2 intersecting A with $\overline{V_1} \cap \overline{V_2} = \emptyset$.

First, we claim that there exists a sequence $\{0 = t_1 < t_2 < \cdots\} \subseteq \mathbb{N}_0$ such that $A \cap \bigcap_{i=1}^n f^{-t_i}(V_{s_i}) \neq \emptyset$ (say $x_s \in A \cap \bigcap_{i=1}^n f^{-t_i}(V_{s_i})$) for each $n \in \mathbb{N}$ and every $s \doteq (s_1, \ldots, s_n) \in \{1,2\}^n$. In fact, we use induction on n. If $n = 1$ this holds obviously. Now assume that it holds for some $n = k$, $k \in \mathbb{N}$. As A is a weakly mixing set, there exists $t_{k+1} > t_k$ such that

$$t_{k+1} \in \bigcap_{s\in\{1,2\}^k} N\Big(A \cap \bigcap_{i=1}^{k} f^{-t_i}(V_{s_i}), V_1\Big) \cap \bigcap_{s\in\{1,2\}^k} N\Big(A \cap \bigcap_{i=1}^{k} f^{-t_i}(V_{s_i}), V_2\Big),$$

i.e. it also holds for $n = k+1$. This finishes the proof of the claimed property.

Now, if we put $\mathcal{U} = \{\overline{V_1}^c, \overline{V_2}^c\}$, then obviously $\mathcal{U}$ is an open cover of X, since $\overline{V_1} \cap \overline{V_2} = \emptyset$. For each $n \in \mathbb{N}$ let $A_n = \{x_s : s \in \{1,2\}^n\} \subseteq A$. Then $\# A_n = 2^n$ and each element of $\bigvee_{i=1}^n f^{-t_i}(\mathcal{U})$ contains exactly one point from A_n, which implies $r(\bigvee_{i=1}^n f^{-t_i}(\mathcal{U}), A) = 2^n$. This ends the proof, as $h_{\mathrm{top}}(f, \mathscr{U}, \{t_i\}_{i\in\mathbb{N}}) = \log 2$. ∎

Remark 3.11. We remark here that it is proved in [HLY] that any non-trivial weakly mixing set has an IP-independent subset. As that subset has positive topological sequence entropy, this can be regarded as another proof of Proposition 3.10.

4. Weakly mixing sets in one-dimensional dynamics. In this section, we shall discuss properties of dynamical systems with weakly mixing sets in one-dimensional dynamics. In particular, by some combinatorial and technical arguments, we shall give a new characterization of a dynamical system over a topological graph with positive topological entropy using the weakly mixing sets introduced.

Before proceeding, let us recall the definition of topological graph and standard one-dimensional covering relation.

Let G be a topological graph. We may regard G as a subspace of the Euclidean space $\mathbb{R}^3$ (that is, each graph is identified with some linear graph in $\mathbb{R}^3$; see [Mo, p. 22]). Moreover, we may assume that G is endowed with the taxicab metric, that is, the distance between any two points of G is equal to the length of the shortest arc in G joining these points. We say that $I \subseteq G$ is a *closed interval* if there is a homeomorphism $\varphi\colon [0,1] \to I$ such that $\varphi((0,1))$ is open in G. Let $f\colon G \to G$ be a continuous map, and let I and J be closed intervals in G. We say that I *f-covers* J (denoted by $I \overset{f}{\Longrightarrow} J$) if there exists a closed interval $K \subseteq I$ such that $f(K) = J$.

Properties of standard one-dimensional covering relations are summarized in the following lemma, which is adapted from [AdRR, p. 590].

Lemma 4.1. *Let $I, J, K, L \subseteq G$ be closed intervals and $f, g\colon G \to G$ be continuous.*

(4.1.1) *If* $I \subseteq K$, $L \subseteq J$ *and* $I \overset{f}{\Longrightarrow} J$, *then* $K \overset{f}{\Longrightarrow} L$.

(4.1.2) *If* $I \overset{f}{\Longrightarrow} J$ *and* $J \overset{g}{\Longrightarrow} K$, *then* $I \overset{g\circ f}{\Longrightarrow} K$.

(4.1.3) *If* $J \subseteq f(I)$, *and* $K_1, K_2 \subseteq J$ *are closed intervals such that* $K_1 \cap K_2$ *is at most one point, then* $I \overset{f}{\Longrightarrow} K_1$ *or* $I \overset{f}{\Longrightarrow} K_2$.

Now we are ready to prove the first main result in this paper. Before we proceed, let us comment on recent results on aspects of dynamics similar to Theorem 1.2. In [TYZ] it is proved that intrinsic topological sequence entropy tuple with length 3 implies positive topological entropy on the interval and circle [TYZ, Theorem 3.5]. Furthermore, [TYZ] proves that intrinsic topological sequence entropy tuple with length n implies positive topological entropy on any topological graph, where n depends on the structure of the graph under consideration. Finally, [TYZ] claims that $n = 3$ is enough for any topological graph (the proof is postponed to [T]).

Proof of Theorem 1.2. By Theorem 3.8 it suffices to prove (1.2.3)⇒ (1.2.1). The proof uses some technical combinatorial arguments, and we divide it into a few small steps.

Let A be a non-trivial weakly mixing set of order 2. By (3.1.1) and Proposition 3.3 we may assume that A is perfect. Note that there is an interval $I \subseteq G$ with $A \cap \operatorname{int} I \neq \emptyset$, where $\operatorname{int} I$ denotes the interior of I. We may identify I with $[0, 1]$.

If $a, b \in I$, $a < b$, then $[a, b]$ is the subinterval of I spanned by a and b, that is, $[a, b] = \{x \in I : a \leq x \leq b\}$. We define the interval (a, b) similarly. Fix any five points $p_0, \ldots, p_4 \in A \cap \operatorname{int} I$, $p_0 < \cdots < p_4$, such that each connected component of the set $\operatorname{int} I \setminus \{p_0, \ldots, p_4\}$ intersects A (as A is perfect, by the selection of I, such points exist). For sufficiently small $\varepsilon > 0$ the following intervals are well defined: $I_i = [p_i - \varepsilon, p_i + \varepsilon]$ and $J_{i+1} = (p_i + \varepsilon, p_{i+1} - \varepsilon)$, and additionally, $\operatorname{int} I_i \cap A \neq \emptyset$, $i = 0, \ldots, 4$, and $J_j \cap A \neq \emptyset$, $j = 1, \ldots, 4$.

Claim 1. *For every* $K \in \{I_1, I_2, I_3\}$ *there are* $p > 0$ *and* $L, M \in \{I_1, I_2, I_3\}$, $L \neq M$, *such that* $K \overset{f^p}{\Longrightarrow} L$ *and* $K \overset{f^p}{\Longrightarrow} M$.

We will show this for $K = I_1$. The arguments for the other two cases are identical. Define $I' = I_1 \cup J_2 \cup I_2 \cup J_3 \cup I_3$ and $\Gamma = N(A \cap \operatorname{int} I_1, J_1) \cap N(A \cap \operatorname{int} I_1, J_4)$. As A is weakly mixing of order 2, $\Gamma \neq \emptyset$. We consider the following two cases.

(1a) Assume that for every $l \in \Gamma$, we have $I' \setminus f^l(I_1) \neq \emptyset$ and fix one such l. Since $f^l(I_1)$ is connected, $I_0, I_4 \subseteq f^l(I_1)$, which by the mean value theorem implies immediately that $I_1 \overset{f^l}{\Longrightarrow} I_0$ and $I_1 \overset{f^l}{\Longrightarrow} I_4$. Again, by weak mixing of order 2 of A, there are $k, s > 0$ such that

$$f^k(A\cap \operatorname{int} I_0)\cap J_2\neq\emptyset,\quad f^k(A\cap \operatorname{int} I_0)\cap J_4\neq\emptyset,$$
$$f^s(A\cap \operatorname{int} I_4)\cap J_1\neq\emptyset,\quad f^k(A\cap \operatorname{int} I_4)\cap J_3\neq\emptyset.$$

Now, it may happen that $f^k(I_0)\supseteq I_2\cup I_3$, $f^s(I_4)\supseteq I_1\cup I_2$ or none of these situations happens (and so $f^k(I_0)\supseteq I_0\cup I_1\cup I_4$ and $f^s(I_4)\supseteq I_0\cup I_3\cup I_4$). Additionally, $I'\setminus f^k(I_0)\neq\emptyset$ and $I'\setminus f^s(I_4)\neq\emptyset$ as otherwise $f^{k+l}(I_1)\supseteq I'$ or $f^{s+l}(I_1)\supseteq I'$, which we assumed cannot happen. Then we have the implications

$$f^k(I_0)\supseteq I_2\cup I_3 \;\Rightarrow\; I_0 \overset{f^k}{\Longrightarrow} I_2 \text{ and } I_0 \overset{f^k}{\Longrightarrow} I_3,$$
$$f^s(I_4)\supseteq I_1\cup I_2 \;\Rightarrow\; I_4 \overset{f^s}{\Longrightarrow} I_1 \text{ and } I_0 \overset{f^s}{\Longrightarrow} I_2,$$

and so any of these cases ends the proof. If our setting is not covered by the above, then $f^k(I_0)\supseteq I_0\cup I_1\cup I_4$ and $f^s(I_4)\supseteq I_0\cup I_3\cup I_4$, which implies

$$I_0 \overset{f^k}{\Longrightarrow} I_0,\quad I_0 \overset{f^k}{\Longrightarrow} I_1,\quad I_0 \overset{f^k}{\Longrightarrow} I_4,$$
$$I_4 \overset{f^s}{\Longrightarrow} I_0,\quad I_4 \overset{f^s}{\Longrightarrow} I_3,\quad I_4 \overset{f^s}{\Longrightarrow} I_4.$$

By $I_0 \overset{f^k}{\Longrightarrow} I_0$ and $I_4 \overset{f^s}{\Longrightarrow} I_4$ we can take $p=l+ks$, obtaining the desired covering, since

$$I_1 \overset{f^l}{\Longrightarrow} I_0 \overset{f^{k(s-1)}}{\Longrightarrow} I_0 \overset{f^k}{\Longrightarrow} I_1,\qquad I_1 \overset{f^l}{\Longrightarrow} I_4 \overset{f^{(k-1)s}}{\Longrightarrow} I_4 \overset{f^s}{\Longrightarrow} I_3.$$

This shows that the claim holds in that case.

(1b) The remaining case is that for some $l\in\Gamma$ we have $I'\subseteq f^l(I_1)$. Put $K_1=I_1$ and $K_2=J_2\cup I_2\cup J_3\cup I_3$ and note that by (4.1.3), I_1 f^l-covers K_1 or K_2. If it covers K_2 we are done by (4.1.1), so assume that K_1 is f^l-covered by I_1. Next put $K_1'=I_1\cup J_2\cup I_2\cup J_3$ and $K_2'=I_3$. Again, if $I_1 \overset{f^l}{\Longrightarrow} K_1'$ then we are done, so assume that $I_1 \overset{f^l}{\Longrightarrow} K_2'$. But combining both cases, we see that $I_1 \overset{f^l}{\Longrightarrow} I_1$ and $I_1 \overset{f^l}{\Longrightarrow} I_3$ and so the proof of the claim is finished. ♦

Claim 2. *There are $K,L\in\{I_1,I_2,I_3\}$, $K\neq L$, and $n>0$ such that $K \overset{f^n}{\Longrightarrow} K$ and $K \overset{f^n}{\Longrightarrow} L$.*

Apply Claim 1 to the sets I_1,I_2,I_3 and assume that the integers $k,m,s>0$ obtained are such that (in any other case the claim follows)

$$I_1 \overset{f^k}{\Longrightarrow} I_2,\quad I_1 \overset{f^k}{\Longrightarrow} I_3,\quad I_2 \overset{f^m}{\Longrightarrow} I_1,\quad I_2 \overset{f^m}{\Longrightarrow} I_3,\quad I_3 \overset{f^s}{\Longrightarrow} I_1,\quad I_3 \overset{f^s}{\Longrightarrow} I_2.$$

But then

$$I_1 \overset{f^k}{\Longrightarrow} I_3 \overset{f^s}{\Longrightarrow} I_2 \overset{f^m}{\Longrightarrow} I_1 \quad\text{and}\quad I_1 \overset{f^k}{\Longrightarrow} I_2 \overset{f^m}{\Longrightarrow} I_3 \overset{f^s}{\Longrightarrow} I_2,$$

and so by (4.1.2) we see that $I_1 \overset{f^{k+m+s}}{\Longrightarrow} I_1$ and $I_1 \overset{f^{k+m+s}}{\Longrightarrow} I_2$. ♦

Claim 3. *There are* $L, M \in \{I_1, I_2, I_3\}$, $L \neq M$ *and* $l, m > 0$ *such that* $L \overset{f^l}{\Longrightarrow} L$, $L \overset{f^l}{\Longrightarrow} M$ *and* $M \overset{f^m}{\Longrightarrow} L$.

By Claim 2 there are distinct $K, K' \in \{I_1, I_2, I_3\}$ and $k > 0$ such that $K \overset{f^k}{\Longrightarrow} K$ and $K \overset{f^k}{\Longrightarrow} K'$. If there is $m > 0$ such that $K' \overset{f^m}{\Longrightarrow} K$ then it is enough to put $L = K$, $M = K'$, $l = k$ and we are done. Otherwise, by Claim 1 there are $k' > 0$ and $M \in \{I_1, I_2, I_3\} \setminus \{K, K'\}$ such that $K' \overset{f^{k'}}{\Longrightarrow} K'$ and $K' \overset{f^{k'}}{\Longrightarrow} M$. Again by Claim 1 we see that there is $m > 0$ such that $M \overset{f^m}{\Longrightarrow} K'$ or $M \overset{f^m}{\Longrightarrow} K$. If the first possibility holds then we put $L = K'$ and $l = k'$. For the second possibility observe that

$$K \overset{f^k}{\Longrightarrow} K \overset{f^k}{\Longrightarrow} K' \overset{f^{k'}}{\Longrightarrow} K' \overset{f^{k'}}{\Longrightarrow} M \overset{f^m}{\Longrightarrow} K,$$

$$K \overset{f^k}{\Longrightarrow} K' \overset{f^{k'}}{\Longrightarrow} M \overset{f^m}{\Longrightarrow} K \overset{f^k}{\Longrightarrow} K' \overset{f^{k'}}{\Longrightarrow} M,$$

so the proof of the claim is finished by putting $L = K$ and $l = 2k + 2k' + m$. ♦

To finish the proof observe that by Claim 3 we have the covering relations

$$L \overset{f^l}{\Longrightarrow} L \overset{f^l}{\Longrightarrow} M \overset{f^m}{\Longrightarrow} L, \qquad L \overset{f^l}{\Longrightarrow} M \overset{f^m}{\Longrightarrow} L \overset{f^l}{\Longrightarrow} M,$$

$$M \overset{f^m}{\Longrightarrow} L \overset{f^l}{\Longrightarrow} L \overset{f^l}{\Longrightarrow} L, \qquad M \overset{f^m}{\Longrightarrow} L \overset{f^l}{\Longrightarrow} L \overset{f^l}{\Longrightarrow} M.$$

It is well known that topological entropy must be positive in that case (f^{2l+m} has the so-called strong horseshoe [LM]). ∎

Remark 4.2. Theorem 1.2 implies that there are TDSs with positive topological sequence entropy but without non-trivial weakly mixing sets of order 2. Namely, there is a big class of Li–Yorke chaotic interval maps with positive topological sequence entropy but zero topological entropy (e.g. a subclass of the class of interval maps of type 2^∞ in the Sharkovsky ordering; for more details, e.g. see [FS, S]).

In fact, in the case of dynamical systems acting on the unit interval the situation is even more complicated, that is, arbitrarily close (in the sense of Hausdorff metric) to any given weakly mixing set of order 2 we can find a weakly mixing set.

Theorem 4.3. *Let* $([0,1], f)$ *be a TDS with* A *a closed weakly mixing set of order* 2. *Then for every* $\varepsilon > 0$ *there is a closed weakly mixing set* D *such that* $\mathcal{H}_d(A, D) < \varepsilon$, *where* $\mathcal{H}_d(A, D)$ *denotes the Hausdorff distance between* A *and* D.

Proof. Obviously, we only need to consider the case when A is not a singleton.

Fix any $\varepsilon > 0$ and take any $m > 0$ with $1/2^m < \min\{\varepsilon/2, (1/4)\operatorname{diam} A\}$. Divide $[0,1]$ into 2^m intervals of the form $[j/2^m, (j+1)/2^m]$, $j = 0, \dots, 2^m-1$. Let $I_0, \dots, I_{k+1}$ be intervals whose interiors intersect A, numbered with respect to increasing values of j. Dividing I_0, I_{k+1} if necessary, we may assume that $I_0 \cup I_1$ and $I_k \cup I_{k+1}$ are connected. Additionally, note that $k \geq 2$. Since A is perfect, we may divide these intervals as much as we want and so without loss of generality we may assume that $k = 3^{k_0} + 1$ for some $k_0 > 0$. For $i = 0, \dots, k-1$ fix any closed interval $J_i \subseteq \operatorname{int} I_{i+1}$ such that $\operatorname{int} J_i \cap A \neq \emptyset$. Then for every i there is l_i such that $f^{l_i}(J_i) \cap I_0 \neq \emptyset$ and $f^{l_i}(J_i) \cap I_{k+1} \neq \emptyset$, which in turn implies that

$$I_1 \cup \dots \cup I_k \subseteq f^{l_i}(J_i).$$

But then we see that $J_i \overset{f^{l_i}}{\Longrightarrow} J_j$ for any i, j and so

$$J_i \overset{f^{l_i}}{\Longrightarrow} J_{i+1\,(\mathrm{mod}\,k)} \xRightarrow{f^{l_{i+1\,(\mathrm{mod}\,k)}}} \cdots \xRightarrow{f^{l_{i+k-2\,(\mathrm{mod}\,k)}}} J_{i+k-1\,(\mathrm{mod}\,k)} \xRightarrow{f^{l_{i+k-1\,(\mathrm{mod}\,k)}}} J_j.$$

By Lemma 4.1 for $m = l_1 + \dots + l_k$ we have $J_i \overset{f^m}{\Longrightarrow} J_j$ for any $i, j = 0, \dots, k-1$. By standard arguments (see e.g. [Du]) there is a closed set $\Lambda \subseteq J_0 \cup \dots \cup J_{k-1}$ invariant for f^m (i.e. $f^m(\Lambda) \subseteq \Lambda$) and a factor map $\pi\colon (\Lambda, f^m) \to (\{0, \dots, k-1\}^{\mathbb{N}}, \sigma_k)$ between TDSs, where σ_k is the standard shift transformation over $\{0, \dots, k-1\}^{\mathbb{N}}$. Furthermore, we may assume that π is at most 2-to-1 and there are at most countably many non-singleton fibers. But there is an uncountable family of minimal weakly mixing subshifts on exactly k symbols with positive topological entropy, say extensions of the Chacón flow [BK, Theorem 2] (systems in [BK] are constructed over the alphabet $\{1, 2, \dots, 3^{k_0}, s\}$; see [BK, p. 125]). Then at least one of these systems is covered exactly 1-to-1 by π and so there is a minimal subsystem (Γ, f^m) of (Λ, f^m) conjugate to the above-mentioned symbolic system. But then Γ is a weakly mixing set and additionally $\Gamma \cap J_i \neq \emptyset$ for $i = 0, \dots, k-1$ since by the construction $\pi^{-1}(\{x\}) \subseteq J_i$ for any sequence $x \in \{0, \dots, k-1\}^{\mathbb{N}}$ such that $x_1 = i$. Now, observe that the sets I_i have diameters smaller than $\varepsilon/2$, they cover A and $I_0 \cup I_1$, $I_k \cup I_{k+1}$ are connected; therefore $B(\Gamma, \varepsilon) \supseteq A$, where $B(\Gamma, \varepsilon) = \{x \in [0,1] : d(x, \Gamma) < \varepsilon\}$. The inclusion $\Gamma \subseteq B(A, \varepsilon)$ is obvious by the choice of the sequence I_i and so $\mathcal{H}_d(\Gamma, A) < \varepsilon$. This ends the proof. ∎

5. Fibers of weakly mixing extensions. In this section, we shall discuss the weak mixing properties of fibers of weakly mixing extensions between TDSs under some necessary assumptions.

Let $\pi\colon (X, f) \to (Y, g)$ be a factor map between TDSs. Recall that π is *weakly mixing of order* $n \in \mathbb{N}\setminus\{1\}$ if the TDS $(R_\pi^{(n)}, f^{(n)})$ is transitive, where

$R_\pi^{(n)} = \{(x_1, \ldots, x_n) \in X^{(n)} : \pi(x_1) = \cdots = \pi(x_n)\}$; and *weakly mixing of all orders* if it is weakly mixing of order m for every $m \in \mathbb{N} \setminus \{1\}$.

Recall that a map is *open* if the image of any open set is open; and *semiopen* if the image of any open set has a non-empty interior. Examples of semiopen but not open maps are piecewise linear maps on the unit interval.

The following result [O, Theorem 15.1] highlights a nice property of open maps (see also [Z2, Lemma 3.6]).

LEMMA 5.1. *Let $\pi\colon X \to Y$ be an open continuous map between topological spaces, where X has a countable basis and Y is a complete metric space. Suppose that $E \subseteq X$ is a residual set. Then there exists a residual (and so dense) subset V of Y such that $E \cap \pi^{-1}(y)$ is residual in $\pi^{-1}(y)$ for each $y \in V$.*

Let (X, f) be a TDS and $\emptyset \neq A \subseteq X$. Given $n \in \mathbb{N} \setminus \{1\}$ define $\Delta_n(A) = \{(x_1, \ldots, x_n) \in A^{(n)} : x_1 = \cdots = x_n\}$. With the help of Lemma 5.1, we obtain the following:

THEOREM 5.2. *Let $\pi\colon (X, f) \to (Y, g)$ be an open factor map between TDSs which is weakly mixing of order $n \in \mathbb{N} \setminus \{1\}$. Then one of the following holds:*

(5.2.1) *π is a homeomorphism.*
(5.2.2) *There is a residual set $V \subseteq Y$ such that $\pi^{-1}(y)$ is a non-trivial weakly mixing set of order n (and hence perfect) for every $y \in V$.*

Proof. By assumptions, using (2.1.2) and (2.1.5) one deduces that $\mathrm{Tran}(R_\pi^{(n)}, f^{(n)})$ is residual in $R_\pi^{(n)}$. As $\pi : X \to Y$ is open, it is easy to check that the map $\pi^{(n)} : R_\pi^{(n)} \to Y, (x_1, \ldots, x_n) \mapsto \pi(x_1)$, is also open and so by Lemma 5.1 there exists a residual set $V \subseteq Y$ such that for each $y \in V$ the following set is residual in $\prod_{j=1}^n \pi^{-1}(y)$:

$$(\pi^{(n)})^{-1}(y) \cap \mathrm{Tran}(R_\pi^{(n)}, f^{(n)}) = \Big(\prod_{j=1}^n \pi^{-1}(y)\Big) \cap \mathrm{Tran}(R_\pi^{(n)}, f^{(n)}).$$

If π is a homeomorphism then we are done. In the other case, none of the fibers $\pi^{-1}(y)$ for $y \in V$ is a singleton, since there is at least one non-singleton fiber for π and so $\mathrm{Tran}(R_\pi^{(n)}, f^{(n)}) \cap \Delta_n(X) = \emptyset$. Now, fix any $y \in V$ and consider any non-empty open sets $U_1, \ldots, U_n, V_1, \ldots, V_n$ intersecting $\pi^{-1}(y)$. Observe that (by the residual property of the subset) there is

$$(z_1, \ldots, z_n) \in \Big(\prod_{j=1}^n U_j\Big) \cap \Big(\prod_{j=1}^n \pi^{-1}(y)\Big) \cap \mathrm{Tran}(R_\pi^{(n)}, f^{(n)})$$

and moreover, there is $k \in \mathbb{N}_0$ such that $f^k(z_i) \in V_i$ for $i = 1, \ldots, n$. This just means that $\pi^{-1}(y)$ is a non-trivial weakly mixing set of order n (and so is perfect by Proposition 3.3), thus the proof is finished. ■

Another problem is what conditions are sufficient to imply weak mixing of a factor map between TDSs. The following result provides a partial answer.

THEOREM 5.3. *Let $\pi : (X, f) \to (Y, g)$ be an open factor map between TDSs and $n \in \mathbb{N} \setminus \{1\}$. Then* (5.3.1)$\Rightarrow$(5.3.2)$\Rightarrow$(5.3.3), *where*

(5.3.1) *π is weakly mixing of order n.*
(5.3.2) *The TDS (Y, g) is transitive and there exists a residual subset $V \subseteq Y$ such that $\pi^{-1}(y)$ is a weakly mixing set of order n for each $y \in V$.*
(5.3.3) *The TDS (Y, g) is transitive and $\pi^{-1}(y)$ is a weakly mixing set of order n for some $y \in \mathrm{Tran}(Y, g)$.*

Moreover, if $f(X) = X$ and (Y, g) is invertible then (5.3.3)$\Rightarrow$(5.3.1).

Proof. First, we aim to prove (5.3.1)$\Rightarrow$(5.3.2). Obviously, when π is weakly mixing of order n, then the TDS (Y, g), as a factor of $(R_\pi^{(n)}, f^{(n)})$, is also transitive. If π is not a homeomorphism then the conclusion follows from (5.2.2). Now if π is a homeomorphism then, by (2.1.5) and (2.2.1), it is easy to check that, for $V \doteq \mathrm{Tran}(Y, g)$, V is residual in Y and for every $y \in V$ the fiber $\pi^{-1}(y) \subseteq \pi^{-1}(V) = \mathrm{Tran}(X, f) \subseteq \mathrm{Rec}(X, f)$ is a singleton and hence a weakly mixing set of order n.

The direction of (5.3.2)$\Rightarrow$(5.3.3) is straightforward, since by (2.1.5) the set $V \cap \mathrm{Tran}(Y, g)$ is non-empty (in fact, it is dense in Y).

Now it suffices to prove (5.3.3)$\Rightarrow$(5.3.1) under the assumptions that $f(X) = X$ and the TDS (Y, g) is invertible. Fix any $y_0 \in \mathrm{Tran}(Y, g)$ such that $\pi^{-1}(y_0)$ is a weakly mixing set of order n. Consider any non-empty open subsets $U_1, \ldots, U_n$ and $V_1, \ldots, V_n$ of X with $(\prod_{i=1}^n U_i) \cap R_\pi^{(n)} \neq \emptyset$ and $(\prod_{i=1}^n V_i) \cap R_\pi^{(n)} \neq \emptyset$; equivalently, $\bigcap_{i=1}^n \pi(U_i)$ and $\bigcap_{i=1}^n \pi(V_i)$ are both non-empty open subsets of Y (by the assumption that $\pi : X \to Y$ is open). As $y_0 \in \mathrm{Tran}(Y, g)$, there exist $k_U, k_V \in \mathbb{N}_0$ such that $g^{k_U}(y_0) \in \bigcap_{i=1}^n \pi(U_i)$ and $g^{k_V}(y_0) \in \bigcap_{i=1}^n \pi(V_i)$. Note that (by the assumption that $f(X) = X$ and $g\colon Y \to Y$ is invertible)

$$\pi(U_i) = \pi(f^{k_U}(f^{-k_U}(U_i))) = g^{k_U}(\pi(f^{-k_U}(U_i))),$$

and so $y_0 \in \pi(f^{-k_U}(U_i))$. In other words $\pi^{-1}(y_0) \cap f^{-k_U}(U_i) \neq \emptyset$, and by the same arguments $\pi^{-1}(y_0) \cap f^{-k_V}(V_i) \neq \emptyset$, $i = 1, \ldots, n$. Since $\pi^{-1}(y_0)$ is a weakly mixing set of order n, by (3.1.1), there exists $k \geq k_U + k_V$ such that, for each $i = 1, \ldots, n$, $f^k(\pi^{-1}(y_0) \cap f^{-k_V}(V_i)) \cap f^{-k_U}(U_i) \neq \emptyset$, and hence

$$(f^{(n)})^{-k_U - k + k_V}\Big(\Big(\prod_{i=1}^n U_i\Big) \cap R_\pi^{(n)}\Big) \cap \Big(\prod_{i=1}^n V_i\Big) \cap R_\pi^{(n)} \neq \emptyset.$$

Thus the TDS $(R_\pi^{(n)}, f^{(n)})$ is transitive, i.e. π is weakly mixing of order n. ∎

Remark 5.4. Let $\pi\colon (X,f)\to(Y,g)$ be a factor map between TDSs. Observe that both (X,f) and (Y,g) are factors of $(R_\pi^{(n)}, f^{(n)})$ by means of $R_\pi^{(n)}\to X$, $(x_1,\dots,x_n)\mapsto x_1$, and $R_\pi^{(n)}\to Y$, $(x_1,\dots,x_n)\mapsto \pi(x_1)$, respectively. In particular this implies that if π is weakly mixing of order n for some $n\in\mathbb{N}\setminus\{1\}$ then both (X,f) and (Y,g) are transitive TDSs and hence by (2.1.2) we have $f(X)=X$ and $g(Y)=Y$. In particular, the implication (5.3.3)⇒(5.3.1) cannot be satisfied if f is not surjective.

The following result is just an easy observation.

Proposition 5.5. *Let $\pi\colon (X,f)\to(Y,g)$ be a factor map between TDSs. If the TDS (X,f) is weakly mixing then*

(5.5.1) *The TDS (Y,g) is weakly mixing.*
(5.5.2) *For any $u,v\in Y$ and any non-empty open sets U,U',V,V' of X with $U\cap\pi^{-1}(u)\neq\emptyset, U'\cap\pi^{-1}(u)\neq\emptyset$ and $V\cap\pi^{-1}(v)\neq\emptyset, V'\cap\pi^{-1}(v)\neq\emptyset$ there is $k\in\mathbb{N}_0$ such that $f^k(U)\cap U'\neq\emptyset$ and $f^k(V)\cap V'\neq\emptyset$.*

Generally speaking, condition (5.5.2) represents some kind of synchronization over fibers. The next theorem shows that under some additional assumptions about the factor map these necessary conditions become sufficient.

Theorem 5.6. *Let $\pi\colon (X,f)\to(Y,g)$ be a semiopen factor map between TDSs which satisfies* (5.5.1) *and* (5.5.2). *If there is a dense set $V\subseteq Y$ such that*

$$f^k(\pi^{-1}(z))=\pi^{-1}(g^k(z)) \tag{5.1}$$

for every $z\in V$ and each $k\in\mathbb{N}_0$, then the TDS (X,f) is weakly mixing.

Proof. Fix any non-empty open sets $U_1,U_2,V_1,V_2\subseteq X$. Denote by U_i' and V_i' the interiors of $\pi(U_i)$ and $\pi(V_i)$, respectively, for $i=1,2$ and note that all these sets are non-empty since f is semiopen. The map g is weakly mixing, so there are $u,v\in Y$ and $s\in\mathbb{N}_0$ such that $u\in U_1'\cap g^{-s}(V_1')$ and $v\in U_2'\cap g^{-s}(V_2')$. We may also assume that $u,v\in V$, since V is dense and g is continuous.

Take any $z\in V_1$ such that $\pi(z)=g^s(u)$. Then by (5.1) it follows that

$$z\in\pi^{-1}(g^s(u))=f^s(\pi^{-1}(u)),$$

so there is $x'\in\pi^{-1}(u)$ such that $f^s(x')\in V_1$. Additionally, there is $x\in\pi^{-1}(u)\cap U_1$. By the same argument, there are $y,y'\in\pi^{-1}(v)$ such that $y\in U_2$ and $f^s(y')\in V_2$. There are open sets $W_1\ni x'$, $W_2\ni y'$ such that $f^s(W_1)\subseteq V_1$ and $f^s(W_2)\subseteq V_2$. By (5.5.2) there is $k\in\mathbb{N}_0$ such that $f^k(U_1)\cap W_1\neq\emptyset$ and $f^k(U_2)\cap W_2\neq\emptyset$. But then $f^{k+s}(U_1)\cap V_1\neq\emptyset$ and $f^{k+s}(U_2)\cap V_2\neq\emptyset$, which ends the proof. ∎

Let (X, f) be a TDS. We say that (X, f) is *mixing* if $N(U, V)$ is co-finite (i.e. $\mathbb{N}_0 \setminus N(U, V)$ is a finite subset) whenever U and V are both non-empty open subsets of X. Obviously, each mixing TDS is weakly mixing and the product TDS of a mixing TDS with a transitive TDS is still transitive.

If $X = \mathscr{A}^{\mathbb{Z}}$ or $X = \mathscr{A}^{\mathbb{N}}$, where $\mathscr{A}$ is a finite set, then we endow $\mathscr{A}$ with discrete topology and X with product topology. The standard map on X is the so-called *shift map* defined by $\sigma(x)_i = x_{i+1}$ for every $i \in \mathbb{Z}$ or $i \in \mathbb{N}$, respectively.

EXAMPLE 5.7. Let (X, f) and (Y, g) be surjective TDSs. We have natural factorization of $\pi\colon (Z, F) \to (Y, g)$, where $(Z, F) = (X \times Y, f \times g)$. In this situation condition (5.1) is trivially fulfilled. Consider the following particular cases of f, g:

(5.7.1) If (X, f) is weakly mixing and (Y, g) is an odometer, then (5.5.2) holds but (5.5.1) does not. Obviously (Z, F) is not weakly mixing.

(5.7.2) Consider thick sets $P_1, P_2 \subseteq \mathbb{N}$ such that $P_1 \cap P_2 = \emptyset$ and let $\Sigma_2 = \{0, 1\}^{\mathbb{Z}}$ with the shift map σ acting on it. For $P \subseteq \mathbb{N}$ define

$$\Lambda_P = \{x \in \Sigma_2 : \text{if } x_i = x_j = 1 \text{ then } |i - j| \in P \cup \{0\}\}$$

and put $X = \Lambda_{P_1}$, $Y = \Lambda_{P_2}$, $f = \sigma|_X$, $g = \sigma|_Y$. By [LZ] we see that both (X, f) and (Y, g) are weakly mixing, and so (5.5.1) holds. But (Z, F) is not even transitive, since if we write sequences $x \in \Lambda_{P_1}$, $y \in \Lambda_{P_2}$ one over the other then we can see the symbol 1 in the first sequence over the symbol 1 in the second sequence at most once (in other words, there is at most one i such that $x_i = y_i = 1$).

(5.7.3) If (X, f) is mixing and (Y, g) is weakly mixing then all assumptions of Theorem 5.6 are satisfied and (Z, F) is weakly mixing.

6. Construction of Example 1.3. In this section we show that it may happen that there exists a weakly mixing set of order 2 but not order 3 (in particular, the notion of weakly mixing set of order n is more general than that of weakly mixing set and so essentially extends the approach introduced in [BH]). The main technique used in our examples follows the ideas used by Glasner in his proof of [G, Theorem 4.1.2].

In fact, in Example 6.1 we will show more than is stated in Example 1.3, in particular the TDS (X, T) will be not only weakly mixing but mixing, etc. Before proceeding to the construction, let us recall some basic definitions from symbolic dynamics.

Let $\mathscr{A}$ be a finite set (an alphabet). By a *word* (*over* $\mathscr{A}$), we mean any finite sequence $u = u_0, \dots, u_{n-1}$, $n \geq 1$, where $u_i \in \mathscr{A}$. The length of u is denoted by $|u| = n$ and the set of all words is denoted by $\mathscr{A}^+$ (note that according to our definition, every word has a positive length). If $x \in \mathscr{A}^{\mathbb{N}_0}$

and $0 \le i < j$ then by $x_{[i,j]}$ we mean the sequence $x_i, x_{i+1}, \dots, x_j$. For simplicity, we use the notation $x_{[i,j)} = x_{[i,j-1]}$. We will also write $u_1 \dots u_n$ instead of $u_1, \dots, u_n$ to denote words. The same notation is also used for $x \in \mathscr{A}^{\mathbb{Z}}$. If $a_1 \dots a_m \in \mathscr{A}^+$ then we define the *cylinder set*

$$[a_1 \dots a_m] = \{x \in \mathscr{A}^{\mathbb{N}_0} : x_{[0,m)} = a_1 \dots a_m\}.$$

It is well known that the cylinder sets form a neighborhood basis for the space $\mathscr{A}^{\mathbb{N}_0}$.

Now we are ready to provide the example announced before.

Example 6.1 (Extended Example 1.3). *There is a mixing TDS (X, T) with the following properties:*

(6.1.1) *there is an open factor map $\pi\colon (X, T) \to (Y, G)$ which is weakly mixing of order* 2 *but not order* 3*,*
(6.1.2) *there is a set $A \subseteq X$ which is weakly mixing of order* 2 *but not of order* 3*.*

Proof. Let $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in the plane (endowed with the metric induced by $|\cdot|$ from $\mathbb{C}$), let $R\colon \mathbb{S}^1 \to \mathbb{S}^1$ be an irrational rotation and $c\colon \mathbb{S}^1 \to \mathbb{S}^1$, $z \mapsto \overline{z}$, the conjugation map.

Let $S\colon \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism such that it has two fixed points (each of them has one side attracting, one side repelling) and $S(-z) = -S(z)$ for every $z \in \mathbb{S}^1$. For example, first we set $S(x) = \frac{1}{2}\varphi(2x)$ for $x \in \left[0, \frac{1}{2}\right]$ and $S(x) = \frac{1}{2} + \frac{1}{2}\varphi(2x-1)$ for $x \in \left[\frac{1}{2}, 1\right]$, where $\varphi\colon [0,1] \to [0,1]$ is a homeomorphism with $\varphi(x) < x$ whenever $x \in (0,1)$, say $\varphi(x) = x^2$; next, we identify S with its lift to $\mathbb{S}^1$ obtained by $e^{2\pi i x} \mapsto e^{2\pi i S(x)}$.

Let $\mathscr{F} = \{F_0, F_1, F_2, F_3, F_4, F_5\}$ where $F_0 = \mathrm{id}_{\mathbb{S}^1}$, $F_1 = R$, $F_2 = R^{-1}$, $F_3 = S$, $F_4 = S^{-1}$ and $F_5 = c$. Let $X = \Sigma_6 \times \mathbb{S}^1$ (endowed with the product metric given by the maximum of the distance on each coordinate) where $\Sigma_6 = \mathscr{A}^{\mathbb{N}_0}$, $\mathscr{A} = \{0, \dots, 5\}$, and let $T\colon X \to X$ be defined by

$$T(\omega, x) = (\sigma(\omega), F_{\omega_0}(x))$$

with σ the standard shift transformation over Σ_6. Note that X is compact, T is continuous and $\pi\colon (X, T) \to (\Sigma_6, \sigma)$, $\pi(\omega, x) = \omega$ is an open factor map.

For any letter $a \in \mathscr{A}$ let $\bar{a}$ be the replacement of a by the second element of the respective pair $(0,0), (1,2), (3,4), (5,5)$, for example $\overline{4} = 3$. We extend this definition to words, putting $\overline{w_0 \dots w_n} = \overline{w_n} \dots \overline{w_0}$. Note that if $F_w = F_{w_{|w|-1}} \circ \dots \circ F_{w_1} \circ F_{w_0}$ is a composition of maps indexed by symbols of the word w then $F_{w\overline{w}} = F_{\overline{w}} \circ F_w = \mathrm{id}$.

First we show that (X, T) is mixing. Fix non-empty basic sets of the topology of X, that is, sets of the form $[A] \times U$, $[B] \times V$, where $U, V \subseteq \mathbb{S}^1$ are non-empty open sets, and $A, B \in \mathscr{A}^+$ are any finite words over $\mathscr{A}$. If we fix any $x \in U$ then there is $N > 0$ such that $R^N(x) \in V$. But then, for any

$n > N + 2|A|$ (where $|A|$ denotes the length of A), it is enough to define

$$\tau = A\overline{A}1^N 0^{n-N-2|A|} B0^\infty$$

and then $(\tau, x) \in [A] \times U$, while

$$T^n(\tau, x) = (\sigma^n(\tau), R^N(x)) = (B0^\infty, R^N(x)) \in [B] \times V,$$

that is, $T^n([A] \times U) \cap ([B] \times V) \neq \emptyset$, so indeed T is mixing.

Now, we are ready to show that (6.1.1) is satisfied. First observe that for any open sets $U_1, U_2, V_1, V_2 \subseteq \mathbb{S}^1$ there is a word w such that $F_w(U_1) \cap U_2 \neq \emptyset$ and $F_w(V_1) \cap V_2 \neq \emptyset$. To see this, consider first two particular cases of composition of some powers of R and S.

If we take any two distinct points $y, z \in \mathbb{S}^1$ with $|y - z| < 2$ (i.e. y, z are not antipodal), then we can find a word $v = 1^k$ for some k (i.e. $F_v = R^k$), so that both $\hat{y} = F_v(y)$ and $\hat{z} = F_v(z)$ lie in the same connected component of $\mathbb{S}^1$ with fixed points of S removed, and then $\lim_{n\to\infty} |S^n(\hat{y}) - S^n(\hat{z})| = 0$.

Furthermore, for any $\varepsilon > 0$ and any $r < 2$ there is $\delta > 0$ such that if $y, z \in \mathbb{S}^1$, $y \neq z$ and $|y - z| < \delta$ then we can find a word $v' = 1^s$ such that $r - \varepsilon < |S^n(\hat{y}) - S^n(\hat{z})| < r + \varepsilon$ for some $n > 0$, where again $\hat{y} = F_{v'}(y), \hat{z} = F_{v'}(z)$ (it is enough to have exactly one fixed point between $\hat{y}$ and $\hat{z}$; then one point, say $\hat{y}$, will be attracted to the fixed point at distance less than $\varepsilon/2$, while the other point, $\hat{z}$, will be repelled sufficiently close to the position chosen by us, once $\hat{z}$ lies in a good position close to the fixed point, which is possible by the construction of $\hat{z}$). By the above remarks we see that for any non-empty open sets $U_1, U_2, V_1, V_2 \subseteq \mathbb{S}^1$ there is a map F obtained as a result of the composition of some sequence of the maps S, R and c (it may happen that we must change the ordering of points of the pair on $\mathbb{S}^1$ and so sometimes we need to apply also c) such that $F(U_1) \cap U_2 \neq \emptyset$, $F(V_1) \cap V_2 \neq \emptyset$.

Fix any two open sets $U, V \subseteq X^{(2)}$ intersecting $R_\pi^{(2)}$ (recall that the definition of $R_\pi^{(n)}$ as well as definition of weak mixing of factors were introduced at the beginning of Section 5). If $(p, q) \in R_\pi^{(2)}$ then $\pi(p) = \pi(q)$ and so there are non-empty open sets U_1, U_2, V_1, V_2 and words $A, B \in \mathscr{A}^+$ such that $[A] \times U_1 \times [A] \times V_1 \subseteq U$ and $[B] \times U_2 \times [B] \times V_2 \subseteq V$. Let F be the above-mentioned composition of a sequence of S, R, c such that $F(U_1) \cap U_2 \neq \emptyset$, $F(V_1) \cap V_2 \neq \emptyset$. Let $w \in \mathscr{A}^+$ be a word such that $F_w = F_{w_{|w|-1}} \circ \cdots \circ F_{w_0} = F$ and let $\varsigma = A\overline{A}wB0^\infty$. If we fix any $x \in U_1 \cap F^{-1}(U_2)$, $y \in V_1 \cap F^{-1}(V_2)$ and put $m = |w| + 2|A|$ then $\sigma^m(\varsigma) \in [B]$ and $T^m(\varsigma, x) = (\sigma^m(\varsigma), F(x))$, $T^m(\varsigma, y) = (\sigma^m(\varsigma), F(y))$, so

$$((\varsigma, x), (\varsigma, y)) \in U \quad \text{and} \quad (T^{(2)})^m((\varsigma, x), (\varsigma, y)) \in [B] \times U_2 \times [B] \times V_2 \subseteq V.$$

This shows that the system $(R_\pi^{(2)}, T^{(2)})$ is transitive; in other words, π is weakly mixing of order 2.

To finish the proof of (6.1.1) we must show that π is not weakly mixing of order 3. Set

$$W = \{(x,y,z) \in (\mathbb{S}^1)^{(3)} : x,y,z \text{ do not all lie in a closed semicircle}\}.$$

Observe that W is a non-empty open subset of $(\mathbb{S}^1)^{(3)}$ and $S^{(3)}(W) = W$, $R^{(3)}(W) = W$, $c^{(3)}(W) = W$. Furthermore, since

$$R_\pi^{(3)} = \{(q,x,q,y,q,z) : q \in \Sigma_6,\, x,y,z \in \mathbb{S}^1\},$$

we see that

$$\hat{W} = \{(q,x,q,y,q,z) : q \in \Sigma_6,\, (x,y,z) \in W\}$$

is a non-empty open subset of $R_\pi^{(3)}$ and $T^{(3)}(\hat{W}) \subseteq \hat{W}$. But it is also clear that $R_\pi^{(3)} \setminus \overline{\hat{W}} \neq \emptyset$ and so $(R_\pi^{(3)}, T^{(3)})$ is not transitive. This ends the proof of (6.1.1).

Condition (6.1.2) follows almost directly from previous observations. Let $u_1, u_2, \ldots$ be all possible words over $\mathscr{A}$ and define the following sequence of words:

$$w_1 = u_1, \quad w_2 = w_1\overline{w_1}u_2w_1, \quad w_{n+1} = w_n\overline{w_n}u_{n+1}w_n$$

and let ω be the limit of the sequence w_n (note that w_n is a prefix of w_{n+1} so ω is well defined). Define $D = \{\omega\} \times \mathbb{S}^1$. We see that D is not a weakly mixing set of order 3 since, for the above-defined non-empty open subset $\hat{W}$ of $R_\pi^{(3)}$, both $\hat{W} \cap D^{(3)}$ and $D^{(3)} \setminus \overline{\hat{W}}$ are non-empty and $T^{(3)}(\hat{W}) \subseteq \hat{W}$. We are going to show that D is weakly mixing of order 2. To prove this, fix any non-empty open sets $U_1, U_2, V_1, V_2 \subseteq \mathbb{S}^1$ and prefix B of ω. Then $[B] \times U_1$, $[B] \times U_2$, $[B] \times V_1$, $[B] \times V_2$ are open sets intersecting D (and every open set intersecting D contains a subset of that form). The same way as previously, let v represent a map F (concatenation of a sequence of maps with elements S, R, c) such that $F(U_1) \cap U_2 \neq \emptyset$ and $F(V_1) \cap V_2 \neq \emptyset$. Let n be such that B is a prefix of w_n and $u_{n+1} = v0^k$ for some $k \geq 0$. Then the word sB is a prefix of ω, where $s = w_n\overline{w_n}v0^k$. Write $m = |s|$ and fix any $y \in F^{-1}(U_2) \cap U_1$, $z \in F^{-1}(V_2) \cap V_1$. Then $(\omega, y) \in ([B] \times U_1) \cap D$, $(\omega, z) \in ([B] \times V_1) \cap D$ and

$$T^m(\omega, y) = (\sigma^m(\omega), F(y)) = (B\ldots, F(y)) \in [B] \times U_2,$$
$$T^m(\omega, z) = (\sigma^m(\omega), F(z)) = (B\ldots, F(z)) \in [B] \times V_2.$$

Thus, D is weakly mixing of order 2 and so the proof is finished. ∎

7. Construction of Example 1.4. In this section, we are going to present an example of a minimal invertible TDS with zero topological entropy which contains non-trivial weakly mixing sets of order 2 and does not contain any non-trivial weakly mixing set of order 3. While it is in some aspects stronger than Example 6.1, the construction is less transparent.

Before going further, we must present some preliminary facts and definitions.

Let Y be a compact metric space with metric d. Denote by $\mathfrak{H}(Y)$ the topological group of all homeomorphisms of Y equipped with the following (complete) metric:

$$d(g,h) = \sup_{y\in Y} d(g(y), h(y)) + \sup_{y\in Y} d(g^{-1}(y), h^{-1}(y)).$$

Let $\mathfrak{g} \subseteq \mathfrak{H}(Y)$ be a subgroup. We say that $(Y, \mathfrak{g})$ is *transitive* if for any non-empty open subsets U and V of Y there exists $g \in \mathfrak{g}$ with $U \cap gV \neq \emptyset$; *weakly mixing of order* $n \in \mathbb{N} \setminus \{1\}$ if $(Y^{(n)}, \mathfrak{g})$ (defined naturally as the action of the same element on each coordinate) is transitive; and *minimal* if $\{gy : g \in \mathfrak{g}\}$ is dense in Y for every $y \in Y$. It is easy to check that $(Y, \mathfrak{g})$ is minimal if and only if for each non-empty open subset U of Y there exists $\{g_1, \ldots, g_n\} \subseteq \mathfrak{g}$, $n \in \mathbb{N}$, such that $\bigcup_{i=1}^n g_i U = Y$, and $(Y, \mathfrak{g})$ is transitive if and only if $\{gy : g \in \mathfrak{g}\}$ is dense in Y for some $y \in Y$.

The following fact must be known but we cannot provide any reference (we present its proof for completeness).

Proposition 7.1. *Let (Y, g) be an invertible TDS and set $\mathfrak{g} = \{g^n : n \in \mathbb{Z}\}$. Then*

(7.1.1) *(Y, g) is minimal if and only if $(Y, \mathfrak{g})$ is minimal.*
(7.1.2) *If Y is perfect then (Y, g) is transitive if and only if $(Y, \mathfrak{g})$ is transitive.*

Proof. (7.1.1) $(Y, \mathfrak{g})$ is minimal if and only if for each non-empty open subset U of Y there exists $\{g_1, \ldots, g_n\} \subseteq \mathfrak{g}$, $n \in \mathbb{N}$, such that $\bigcup_{i=1}^n g_i U = Y$ if and only if for each non-empty open subset U of Y there exists $m \in \mathbb{N}$ such that $\bigcup_{j=0}^m g^{-j} U = Y$, if and only if (Y, g) is minimal (using (2.1.6)).

(7.1.2) Assume that $(Y, \mathfrak{g})$ is transitive, that is, there exists $y \in Y$ such that $\{y, g^{\pm 1}(y), g^{\pm 2}(y), \ldots\}$ is dense in Y. As Y is perfect, there exists a sequence $\{n_i\}_{i\in\mathbb{N}}$ in $\mathbb{Z}$ such that $|n_1| < |n_2| < \cdots$ and $g^{n_i}(y) \to g^{-1}(y)$ as $i \to \infty$. By choosing a subsequence we may assume $n_1 < n_2 < \cdots$ or $n_1 > n_2 > \cdots$. If $n_1 < n_2 < \cdots$ then $g^{-1}(y) \in \overline{\{y, g(y), g^2(y), \ldots\}}$ and so $\overline{\{y, g^{\pm 1}(y), g^{\pm 2}(y), \ldots\}} = \overline{\{y, g(y), g^2(y), \ldots\}}$. Thus (Y, g) is transitive (using (2.1.5)). If $n_1 > n_2 > \cdots$ then, by a similar reasoning, (Y, g^{-1}) is transitive and so (Y, g) is transitive (using (2.1.3)).

The converse implication in (7.1.2) follows just by the definition. ∎

Remark 7.2. It may happen that (Y, g) is an invertible TDS which is not transitive, whereas $(Y, \{g^n : n \in \mathbb{Z}\})$ is transitive (thus Y must contain isolated points by (7.1.2)). The TDS constructed in Remark 2.5 has property.

Now let (Z,σ) be an invertible TDS. Let $X = Z\times Y$ and define $\mathfrak{g}_\sigma \subseteq \mathfrak{H}(X)$ in the following way. An element of $\mathfrak{H}(X)$ belongs to $\mathfrak{g}_\sigma$ if and only if it is the form of $g^{-1}\circ(\sigma\times\mathrm{id})\circ g$, where g is given by $g(z,y) = (z, g_z(y))$ for some continuous map $Z\to\mathfrak{g}$, $z\mapsto g_z$, and $\sigma\times\mathrm{id}$ is given by $(\sigma\times\mathrm{id})(z,y) = (\sigma(z),y)$. Clearly, $(X,f)\to(Z,\sigma)$, $(z,y)\mapsto z$, is an open factor map between TDSs for each $f\in\mathfrak{g}_\sigma$. Thus elements $f\in\mathfrak{g}_\sigma$ are homeomorphisms of the form $f(z,y) = (\sigma(z),(g^{-1}_{\sigma(z)}\circ g_z)(y))$ and so $f^k(z,y) = (\sigma^k(z),(g^{-1}_{\sigma^k(z)}\circ g_z)(y))$.

With the help of Proposition 7.1 we obtain directly the following result which combines Theorems 1 and 4 of [GW].

Theorem 7.3. *Let Y be a compact metric space, $\mathfrak{g}\subseteq\mathfrak{H}(Y)$ a pathwise connected subgroup and (Z,σ) an invertible minimal TDS with infinitely many points.*

(7.3.1) *If $(Y,\mathfrak{g})$ is minimal then there exists a residual subset $\mathfrak{R}\subseteq\overline{\mathfrak{g}_\sigma}$ such that if $f\in\mathfrak{R}$ then $(Z\times Y,f)$ is minimal.*

(7.3.2) *If $(Y,\mathfrak{g})$ is weakly mixing of order 2 then there exists a residual subset $\mathfrak{R}\subseteq\overline{\mathfrak{g}_\sigma}$ such that the factor map $\pi\colon (Z\times Y,f)\to(Z,\sigma)$ is weakly mixing of order 2 for every $f\in\mathfrak{R}$.*

Remark 7.4. If the assumptions of (7.3.2) are satisfied then Z (hence $R^{(2)}_\pi$) is perfect.

We also need the following simple fact.

Lemma 7.5. *Let Y be a compact metric space, $\mathfrak{g}\subseteq\mathfrak{H}(Y)$ a subgroup and (Z,σ) an invertible TDS. Put $X = Z\times Y$. If (X,f) is transitive for some $f\in\overline{\mathfrak{g}_\sigma}$ then $(Y,\mathfrak{g})$ is also transitive.*

Proof. Fix $f\in\overline{\mathfrak{g}_\sigma}$ such that (X,f) is a transitive TDS. Now, let U_1 and U_2 be non-empty open subsets of Y. There exists $n\in\mathbb{N}$ such that $W \doteq f^n(Z\times U_1)\cap(Z\times U_2)\neq\emptyset$ and W is also open, since f is a homeomorphism. Now if we fix any $x\in(Z\times U_1)\cap f^{-n}(Z\times U_2)$ then $\tilde f^n(x)\in W\subseteq Z\times U_2$ provided that $\tilde f$ is sufficiently close to f. In particular, there is $\tilde f\in\mathfrak{g}_\sigma$ such that $\tilde f^n(Z\times U_1)\cap(Z\times U_2)\neq\emptyset$. But by the definition $\tilde f = g^{-1}\circ(\sigma\times\mathrm{id})\circ g$, where g is given by $g(z,y) = (z,g_z(y))$ for some continuous map $Z\to\mathfrak{g}$, $z\mapsto g_z$. Then for some $(z,y)\in Z\times U_1$ we have $(\sigma^n(z),(g^{-1}_{\sigma^n(z)}\circ g_z)(y))\in Z\times U_2$. In particular, $(g^{-1}_{\sigma^n(z)}\circ g_z)(U_1)\cap U_2\neq\emptyset$ for some $z\in Z$, which shows that $(Y,\mathfrak{g})$ is transitive, because $g^{-1}_{\sigma^n(z)}\circ g_z\in\mathfrak{g}$. ∎

Before we go further we must construct a special group of homeomorphisms. It will be used later as an ingredient of more advanced constructions (in particular, it satisfies the assumptions of Theorem 7.3).

Example 7.6. *There exist a perfect space Y and a pathwise connected subgroup $\mathfrak{g} \subseteq \mathfrak{H}(Y)$ such that $(Y, \mathfrak{g})$ is minimal and weakly mixing of order* 2 *but not* 3.

Proof. Let $Y = \mathbb{S}^1$ and let R_a, $a \in [0, 1)$, be the family of all possible rotations of Y. Let T_a be the map defined by $T_a(x) = (1 - a)x + ax^2$ where $a, x \in [0, 1]$. We identify every T_a with its lift to $\mathbb{S}^1$ obtained by identifying the endpoints of the interval. Then $\{T_a\}_{a \in [0,1]}$ forms a path in $\mathfrak{H}(\mathbb{S}^1)$. Observe that if we fix any three distinct points and enumerate them in clockwise direction, then after application of R_a or T_a their order will be preserved. Furthermore, T_a is a map with exactly one fixed point, which is one side attracting, one side repelling. Let $\mathfrak{g} \subseteq \mathfrak{H}(\mathbb{S}^1)$ be the group generated by the above-defined maps R_a and T_a.

The minimality of $(Y, \mathfrak{g})$ is straightforward, as (Y, R_a) is minimal for each irrational a. We also see that $\mathfrak{g}$ is pathwise connected. Namely, if we have a composition $f_n \circ \cdots \circ f_1 \in \mathfrak{g}$ (each f_i belongs to the above-mentioned two classes of homeomorphisms R_a, T_a) then we can construct a path to $f_{n-1} \circ \cdots \circ f_1$ by flattening f_n to the identity by going with a to zero. So that way we can go to the identity and from the identity to anything we want in $\mathfrak{g}$ joining together a finite number of paths.

Now we finish our proof by showing that $(Y, \mathfrak{g})$ is weakly mixing of order 2 but not 3. Weak mixing of order 2 is obvious since we can use T_a and R_a to change the distance between two points to be close to an arbitrary (allowed) number and place these points anywhere we want on $\mathbb{S}^1$. To show that $(Y, \mathfrak{g})$ is not weakly mixing of order 3, fix any three disjoint connected open sets U_1, U_2, U_3 and fix one point in each of them, say x_1, x_2, x_3 respectively. Assume that if we start at x_1 and move clockwise on the circle, then we will first meet x_2 and next x_3 (call the described situation the *clockwise ordering* of a sequence on $\mathbb{S}^1$). For any other points from U_1, U_2, U_3 the ordering is always clockwise, since these connected sets are disjoint. Additionally, if we apply any $f \in \mathfrak{g}$ to them then the ordering of $g(x_1), g(x_2), g(x_3)$ will remain clockwise. This means that when $g(x_1) \in U_1$ and $g(x_3) \in U_2$ then $g(x_2) \notin U_3$ as otherwise we have a contradiction with clockwise ordering. Thus, $(Y, \mathfrak{g})$ is not weakly mixing of order 3. ∎

Finally, we provide the last ingredient we will need in our construction.

Proposition 7.7. *Let $(Y, \mathfrak{g})$ be constructed in Example* 7.6 *and (Z, σ) an invertible minimal TDS with infinitely many points. Define $X = Z \times Y$. Then there exists a residual subset $\mathfrak{R} \subseteq \overline{\mathfrak{g}_\sigma}$ such that for every $f \in \mathfrak{R}$ the system (X, f) is minimal and the factor map $\pi \colon (X, f) \to (Z, \sigma)$, $(z, y) \mapsto z$, is open and weakly mixing of order* 2 *but not* 3.

Proof. Note that the group from Example 7.6 satisfies the assumptions of Theorem 7.3, so there exists a residual subset $\mathfrak{R} \subseteq \overline{\mathfrak{g}_\sigma}$ such that for every $f \in \mathfrak{R}$, the TDS (X, f) is minimal and π is an open factor map which is weakly mixing of order 2.

Now we shall finish our proof by showing that π is not weakly mixing of order 3. Assume to the contrary that $(R_\pi^{(3)}, f^{(3)})$ is transitive. Then $R_\pi^{(3)}$ is perfect by (2.1.2) since it is infinite and may be homeomorphically identified with $Z' \times Y'$ where $Z' = \{(z, z, z) : z \in Z\}$ and $Y' = Y^{(3)}$. Using the same identification, $f^{(3)}$ may be viewed as an element of the group $\overline{\mathfrak{g}'_{\sigma'}}$, where $\sigma' \colon Z' \to Z'$, $(z, z, z) \mapsto (\sigma z, \sigma z, \sigma z)$, and $\mathfrak{g}' = \{g^{(3)} : g \in \mathfrak{g}\}$. By Lemma 7.5 we see that $(Y', \mathfrak{g}')$ is transitive, which equivalently means that $(Y, \mathfrak{g})$ is weakly mixing of order 3. But this is impossible by the construction of $(Y, \mathfrak{g})$ in Example 7.6. ∎

Now we are ready to present the construction of our second example.

Construction of Example 1.4. Let (Z, σ) be any invertible minimal TDS with infinitely many points which contains no non-trivial weakly mixing sets of order 2, for example, any irrational rotation over the circle. By Proposition 7.7 there exists an open factor map $\pi \colon (X, f) \to (Z, \sigma)$ between minimal invertible TDSs such that π is weakly mixing of order 2 but not 3 (thus, π is not a homeomorphism), where $X = Z \times Y$ and $f \in \overline{\mathfrak{g}_\sigma}$ with $(Y, \mathfrak{g})$ constructed in Example 7.6.

By Theorem 5.2, there exists $z_0 \in Z$ such that $\pi^{-1}(z_0)$ is a non-trivial weakly mixing set of order 2. Now we claim that (X, f) contains no non-trivial weakly mixing set of order 3. This will finish the proof, since in that case by Theorem 3.8, (X, f) has zero topological entropy.

Assume to the contrary that $\emptyset \neq A \subseteq X$ is a non-trivial weakly mixing set of order 3 (by (3.1.1) we may assume that A is closed, hence perfect by Proposition 3.3). Note that by Proposition 3.2, $\pi(A)$ is singleton, because by the assumptions (Z, σ) does not have non-trivial weakly mixing sets of order 3. Thus, there exist three distinct points (z, y_i), $i = 1, 2, 3$, from A such that $y_1, y_2, y_3 \in \mathbb{S}^1$ are clockwise ordered. By the assumption that A is weakly mixing of order 3, we can require that, for some $n \in \mathbb{N}$, $f^n(z, y_1)$ is sufficiently close to (z, y_1), $f^n(z, y_2)$ is sufficiently close to (z, y_3) and $f^n(z, y_3)$ is sufficiently close to (z, y_2); in particular, by the ordering of y_1, y_2 and y_3 we may assume that, for $f^n(z, y_i) \doteq (\sigma^n(z), y_i^*), i = 1, 2, 3$, the sequence y_1^*, y_3^*, y_2^* is clockwise ordered. Recall that for the $(Y, \mathfrak{g})$ constructed, each element of $\mathfrak{g}$ preserves the order of any given distinct three points from Y, and so it is not hard to check that each element of $\mathfrak{g}_\sigma$ (and hence each element of $\overline{\mathfrak{g}_\sigma}$, including f) preserves the order of any given triple $(x_1, x_2, x_3) \in X^{(3)}$ whenever the second coordinates of x_1, x_2, x_3 are pairwise distinct. In particular, y_1^*, y_2^*, y_3^* and y_1, y_2, y_3 must be in the same (clockwise) ordering which is a contradiction. ■

REMARK 7.8. Example 1.4 shows that in general, there is no chance to obtain results similar to Theorem 1.2 (even in the class of dynamical systems acting on two-dimensional compact manifolds).

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