STUDIA MATHEMATICA 202 (3) (2011)

Subalgebras generated by extreme points in Fourier–Stieltjes algebras of locally compact groups

by

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Abstract. Let G be a locally compact group, G^* be the set of all extreme points of the set of normalized continuous positive definite functions of G, and $a(G)$ be the closed subalgebra generated by G^* in $B(G)$. When G is abelian, G^* is the set of Dirac measures of the dual group $\hat{G}$, and $a(G)$ can be identified as $l^1(\hat{G})$. We study the properties of $a(G)$, particularly its spectrum and its dual von Neumann algebra.

1. Introduction. Let G be a locally compact group, and let $A(G)$, $B(G)$ and $VN(G)$ be the Fourier algebra, the Fourier–Stieltjes algebra and the group von Neumann algebra of G, respectively, as defined by Eymard [8]. If G is abelian, then $A(G)$ can be identified as $L^1(\hat{G})$ via the Fourier transform, $VN(G)$ can be identified as $L^\infty(\hat{G})$ via the adjoint of Fourier transform, and $B(G)$ can be identified as $M(\hat{G})$ via the Fourier–Stieltjes transform, where $\hat{G}$ is the dual group of G.

Akemann and Walter [1] first studied G^*, the set of all extreme points in the set of all continuous positive definite functions on G with norm one. See [6] for references on positive definite functions of G. This object is also studied by A. T.-M. Lau in [11]. If G is amenable (see [17]), it is proved that the convex hull of G^* is weak*-dense in the set of means on $UCB(\hat{G})$ (= norm closure of $A(G) \cdot VN(G)$). In [16], P. F. Mah and T. Miao showed that for a [SIN]-group G, G^* and $A(G)$ are disjoint if and only if G is non-compact. This object was later studied by the author (see [4], [5]).

The main purpose of this paper is to study G^* from other points of view. For a locally compact abelian group G, G^* can be viewed as the set of all Dirac measures on $\hat{G}$. We define $a(G)$, the algebra generated by G^* in $B(G)$, as a non-commutative analogue of $l^1(\hat{G})$ and prove that $\sigma(a(G))$ has a natural semigroup structure. The main results are as follows:

2010 *Mathematics Subject Classification*: Primary 43A30, 43A35, 43A40; Secondary 46J10.
Key words and phrases: Fourier–Stieltjes algebra, locally compact group, extreme point, spectrums.

DOI: 10.4064/sm202-3-5

We show that if G_1 and G_2 are locally compact groups and $a(G_1)$ and $a(G_2)$ are isometrically isomorphic, then the unitary parts of their spectra are either topologically isomorphic or anti-isomorphic. It is a natural question to ask when $\sigma(a(G))$ is a group. If G is a [Moore]-group, then $a(G)$ is the Fourier algebra of G^{ap}, where G^{ap} is the almost periodic compactification of G. In this case, $\sigma(a(G))$ is just G^{ap}. We show that $\sigma(a(G))$ is a group only if G is a [Moore]-group. Finally, we observe that if G is a discrete abelian group, then $l^1(\hat{G})$ characterizes G. We prove a non-commutative analogue of this phenomenon: if G is an [AR]-group, then $a(G)$ characterizes G.

2. Some preliminaries. Let E be a Banach space. Throughout this paper, S_E will denote the boundary of the unit ball of E respectively. Let K be a subset of E. We denote by $\mathcal{E}(K)$ the set of all extreme points of K, and by $\mathrm{co}(K)$ the algebraic convex hull of K. Let E' be the Banach dual space of E, which consists of all bounded linear functionals on E.

In this paper, all groups will be assumed to be locally compact, and G will denote a locally compact group. Let f be a function on G and $y \in G$. We define the left and right translates of f through y by

$$L_y f(x) = f(y^{-1}x), \qquad R_y f(x) = f(xy).$$

We also write ${}_x f$ and f_x for the functions $f(x\cdot)$ and $f(\cdot x)$, respectively.

Let Σ_G be the class of all unitary equivalence classes of unitary representations of G, and let $\lambda_2 : G \to B(L^2(G))$, $[\lambda_2(x)(f)](y) := f(x^{-1}y)$ $(x, y \in G,\ f \in L^2(G))$, be the *left regular representation* of G. We will also denote by $\hat{G}$ the set of all unitary equivalence classes of irreducible unitary representations of G. If G is abelian, $\hat{G}$ is just the dual group of G.

For any $f \in L^1(G)$, define

$$\|f\|_{C^*(G)} := \sup_{\pi \in \hat{G}} \|\pi(f)\|.$$

It is easily seen that $\|\cdot\|_{C^*(G)}$ is a C^*-norm on $L^1(G)$. Let $C^*(G)$ be the completion of $L^1(G)$ under $\|\cdot\|_{C^*(G)}$. Then $C^*(G)$ is called the *full group C^*-algebra* or simply the *group C^*-algebra* of G. Let $B(G) := \{x \mapsto \langle \pi(x)\xi, \eta\rangle : \pi \in \Sigma_G,\ \xi, \eta \in \mathcal{H}_\pi\}$ be the *Fourier–Stieltjes algebra* of G. $B(G)$ is a commutative Banach algebra with pointwise multiplication and its norm is given by

$$\|u\|_{B(G)} = \sup\Big\{\Big|\int uf\Big| : f \in L^1(G),\ \|f\|_{C^*(G)} \le 1\Big\}.$$

Let $A(G) := \{x \mapsto \langle \lambda_2(x)\xi, \eta\rangle : \xi, \eta \in L^2(G)\}$ be the *Fourier algebra* of G. It is well-known that $A(G)$ is a closed ideal of $B(G)$.

Recall that the involution on $L^1(G)$ is given by the following formula:

$$f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})} \quad \text{a.e.} \quad (f \in L^1(G)).$$

Let $P(G)$ be the set of all continuous positive definite functions on G, i.e.,

$$P(G) := \Big\{\phi \in B(G) : \int (f^* * f)\phi \geq 0 \text{ for any } f \in L^1(G)\Big\}.$$

It can be shown that $P(G) = \{\langle \pi(\cdot)\xi, \xi\rangle : \pi \in \Sigma_G,\ \xi \in \mathcal{H}_\pi\}$ and $\phi(e) = \|\phi\|_{B(G)}$. See [6] for details.

Let $VN(G)$ be the von Neumann algebra generated by the image of λ_2 in $B(L^2(G))$. It is called the *group von Neumann algebra* of G. For any $f \in L^1(G)$, define

$$\|f\|_{C_r^*(G)} := \|\lambda_2(f)\|.$$

It is easily seen that $\|\cdot\|_{C_r^*(G)}$ is a C^*-norm on $L^1(G)$. Let $C_r^*(G)$ be the completion of $L^1(G)$ under $\|\cdot\|_{C_r^*(G)}$. Then $C_r^*(G)$ is called the *reduced group C^*-algebra* of G. It is proved by Eymard [8] that $A(G)' = VN(G)$. For $u \in A(G)$ and $T \in VN(G)$, define $u\cdot T \in VN(G)$ by $\langle u\cdot T, v\rangle = \langle T, uv\rangle$, $v \in A(G)$.

Suppose that π is a unitary representation of G. Let $F_\pi(G) = \text{span}\,\{x \mapsto \langle \pi(x)\xi, \eta\rangle : \xi, \eta \in \mathcal{H}_\pi\}$. Then $A_\pi(G)$, the *Fourier space associated to* π, is defined to be the closure of $F_\pi(G)$ in the Banach space $B(G)$. For any representation π of G, define $VN_\pi(G)$ to be the von Neumann algebra generated by $\pi(G)$ (or $\pi(L^1(G))$) in $\mathcal{L}(\mathcal{H}_\pi)$. We have $A_\pi(G)' = VN_\pi(G)$. If $\pi = \lambda_2$, then $A_\pi(G) = A(G) = F_\pi(G)$ and $VN_\pi(G) = VN(G)$. For each $u \in A_\pi(G)$, there exist nets (ξ_n) and (η_n) in $\mathcal{H}_\pi$ such that

$$u(x) = \sum_{n=1}^{\infty}\langle \pi(x)\xi_n, \eta_n\rangle \quad\text{and}\quad \|u\| = \sum_{n=1}^{\infty}\|\xi_n\|\,\|\eta_n\|.$$

See [2] and [8] for more details.

3. Semigroup structure of the spectrum of $a(G)$. In this section, we will study the semigroup structure of the spectrum of $a(G)$. We start with the definition of G^*, which will play an important role throughout this paper. Let $P_1(G) = S_{B(G)} \cap P(G)$. In other words,

$$P_1(G) = \{\langle \pi(\cdot)\xi, \xi\rangle : \pi \in \Sigma_G,\ \xi \in \mathcal{H}_\pi,\ \|\xi\| = 1\}.$$

Let $G^* = \mathcal{E}(P_1(G))$, and let $\widetilde{G}$ be the semigroup generated by G^* in $B(G)$. The sets G^* and $\widetilde{G}$ are equipped with the relative weak* topology inherited from $B(G)$. We shall denote the elements in G^* by g^*, h^* or k^*.

REMARKS 3.1.

(a) If G is abelian, then $G^* = \widetilde{G} = \hat{G}$.
(b) We have $G^* = \{x \mapsto \langle \pi(x)\xi, \xi\rangle : \pi \in \hat{G},\ \xi \in \mathcal{H}_\pi,\ \|\xi\| = 1\}$. Hence, G^* is non-empty as $\hat{G}$ is non-empty.

(c) G^* separates the points of G. That is, if x and y are distinct points of G, there is an element $g^* \in G^*$ such that $g^*(x) \neq g^*(y)$ (see [9, Theorem 3.34]).
(d) Actually, it is proved in [1] that the following statements are equivalent:

- G is abelian.
- For every $g^* \in G^*$, we have $1/g^*(\cdot) \in P_1(G)$.
- G^*, equipped with pointwise multiplication, is a group.

Let $a_0(G)$ be the closure of the span of G^* in $B(G)$, and let $a(G)$ be the closed subalgebra generated by $a_0(G)$ in $B(G)$. We call $a(G)$ the *little Fourier algebra* of G. Denote by $vn_0(G)$ and $vn(G)$ the dual Banach spaces of $a_0(G)$ and $a(G)$, respectively. We call $vn(G)$ the *little von Neumann algebra* of G. Then the norm closure of the span of $\widetilde{G}$ in $B(G)$ is $a(G)$. Recall that $\bar{\pi}$ is the contragredient of π (for details, see [9, Chapter 3]). Note that $\bar{\pi}$ is irreducible for any irreducible representation π of G. It follows that $a(G)$ is a Banach *-algebra where the involution is given by complex conjugation. Furthermore, we can show that $a(G)$ is semisimple as G^* separates the points of G.

PROPOSITION 3.2. *Let $\pi_a = \bigoplus_{\pi\in\hat{G}} \pi$. Then $a_0(G) = A_{\pi_a}(G)$. Hence, $vn_0(G) = VN_{\pi_a}(G)$. In particular, $vn_0(G)$ is a von Neumann algebra.*

Proof. Let $\mathfrak{F}$ be the set of all unitary equivalence classes of finite direct sums of irreducible representations of G. It is clear that $\operatorname{span}(G^*) = \{x \mapsto \langle \pi(x)\xi, \eta\rangle : \pi \in \mathfrak{F}, \xi, \eta \in \mathcal{H}_\pi\}$. Suppose that $\phi \in A_{\pi_a}(G)$ is such that $\phi(x) = \langle \pi_a(x)\xi, \xi\rangle$ for some $\xi \in \mathcal{H}_{\pi_a}$. For any $\epsilon > 0$, there exists $\xi_0 \in \mathcal{H}_\pi$ for some $\pi \in \mathfrak{F}$ such that $\|\xi - \xi_0\| < \epsilon$. For any $f \in C^*(G)$,

$$\begin{aligned}|\langle \pi_a(f)\xi, \xi\rangle - \langle \pi(f)\xi_0, \xi_0\rangle| &= |\langle \pi_a(f)\xi, \xi\rangle - \langle \pi_a(f)\xi_0, \xi_0\rangle| \\ &\le |\langle \pi_a(f)\xi, \xi - \xi_0\rangle| + |\langle \pi_a(f)(\xi - \xi_0), \xi_0\rangle| \le 2\|f\|_{C^*}\|\xi\|\epsilon.\end{aligned}$$

Therefore, $\|\langle \pi_a(\cdot)\xi, \xi\rangle - \langle \pi(\cdot)\xi_0, \xi_0\rangle\|_{B(G)} \le \epsilon$. The result follows. ∎

For the definitions of direct sums and internal tensor products of unitary representations of G, we refer the reader to [9, Chapters 3 and 7].

Let $\pi_a^{(n)} = \bigotimes_{i=1}^n \pi_a$ and $\sigma = \bigoplus_{n=1}^\infty \pi_a^{(n)}$. It is straightforward to show that $a(G) = A_\sigma(G)$ and $vn(G) = VN_\sigma(G)$. Hence, $vn(G)$ is a von Neumann algebra.

A Banach space X has the *Radon–Nikodym property* (RNP) if, for every bounded subset C of X and $\epsilon > 0$, there is some $x \in C$ such that x does not lie in the norm closure of $\operatorname{co}[C \setminus (x + \{y \in X : \|y\| \le \epsilon\})]$.

REMARK 3.3. If G is a compact group, then $B(G)$ has RNP. In fact, $B(G)$ has RNP if and only if $B(G) = a_0(G)$ (see [3, Theorem 5], [19, Theorem 4.2], [13, Theorem 4.5] and [14]).

Let $A_{\mathcal{F}}(G)$ be the $\|\cdot\|_{B(G)}$-closure of $\{x \mapsto \langle \pi(x)\xi, \eta\rangle : \pi$ is a finite-dimensional representation of G, $\xi, \eta \in \mathcal{H}_\pi\}$. Let $\hat{G}_{\mathcal{F}}$ be the set of all finite-dimensional irreducible representations of G, and $\pi_F = \bigoplus_{\pi \in \hat{G}_{\mathcal{F}}} \pi$. Then $A_{\mathcal{F}}(G) = A_{\pi_F}(G) \subseteq a_0(G)$.

A [*Moore*]-*group* is a locally compact group such that all its irreducible unitary representations are finite-dimensional.

REMARKS 3.4.

(1) If G is abelian, then $a_0(G) = a(G) \cong l^1(\hat{G})$ and $vn_0(G) = vn(G) \cong l^\infty(\hat{G})$.
(2) If G is compact, then every representation of G is a direct sum of copies of irreducible representations, hence $a_0(G) = B(G) = a(G)$.
(3) If G is a [Moore]-group, it is clear that $a_0(G) = a(G) = A_{\mathcal{F}}(G)$.
(4) More generally, if $B(G)$ has RNP, then $a_0(G) = B(G) = a(G)$.
(5) If G is the "$ax + b$"-group, then $a_0(G) = A_{\mathcal{F}}(G) \oplus A(G)$, which is an algebra since $A(G)$ is an ideal in $a_0(G)$. Thus $a_0(G) = a(G)$.

Let A be a commutative Banach algebra. The *spectrum* of A, written as $\sigma(A)$, is the set of all non-zero multiplicative linear functionals on A.

From now on, π will be a unitary representation of G such that $A_\pi(G)$ is an algebra.

If $A_\pi(G)$ is a unital algebra, then it is easy to see that

$$A_\pi(G) = A_\pi(G) \cdot A_\pi(G) = \text{norm-cl}(\text{span}(A_\pi(G) \cdot A_\pi(G))).$$

Therefore, $A_\pi(G) = A_{\pi\otimes\pi}(G)$, and hence π and $\pi \otimes \pi$ are quasi-equivalent (see [2]). By a result in [7, Chapter 4], there is an isomorphism $\Phi : VN_\pi(G) \to VN_{\pi\otimes\pi}(G)$ such that

$$\Phi(\pi(g)) = (\pi \otimes \pi)(g) \quad \text{for any } g \in G.$$

Moreover, we have

$$\langle u, x\rangle_{(A_\pi(G), VN_\pi(G))} = \langle u, \Phi(x)\rangle_{(A_{\pi\otimes\pi}(G), VN_{\pi\otimes\pi}(G))}$$

for any $u \in A_\pi(G)$ and $x \in VN_\pi(G)$ (see [2]). It is easy to see that the isomorphism with the above properties is unique.

For any $x \in VN_\pi(G)$, $\pi \otimes \pi(x)$ is defined to be $\Phi(x)$. It is an operator on $H_\pi \otimes H_\pi$ since it is an element of $VN_{\pi\otimes\pi}(G)$. Since $\pi \otimes \pi(x)$ and $\pi(x) \otimes \pi(x)$ are operators on $H_\pi \otimes H_\pi$, it makes sense to ask if they are equal.

The following lemma is a generalization of [20, Theorem 1(ii)].

LEMMA 3.5. *If $A_\pi(G)$ is unital, then*

$$\sigma(A_\pi(G)) := \{x \in VN_\pi(G) \setminus \{0\} : \pi \otimes \pi(x) = \pi(x) \otimes \pi(x)\}.$$

Proof. Let $u_i = \langle \pi(\cdot)\xi_i, \eta_i\rangle \in A_\pi(G)$ where $i = 1, 2$, and let $f = u_1u_2$. Then $f(x) = \langle \pi \otimes \pi(x)\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2\rangle$ for any $x \in G$. Thus we have

$$\langle f, x\rangle = \langle f, \Phi(x)\rangle = \langle f, \pi \otimes \pi(x)\rangle = \langle \pi \otimes \pi(x)\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2\rangle.$$

If $x \in \sigma(A_\pi(G))$, then

$$\begin{aligned}\langle f, x\rangle &= \langle u_1, x\rangle\langle u_2, x\rangle = \langle \pi(x)\xi_1, \eta_1\rangle\langle \pi(x)\xi_2, \eta_2\rangle \\ &= \langle \pi(x) \otimes \pi(x)\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2\rangle.\end{aligned}$$

Therefore,

$$\langle \pi \otimes \pi(x)\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2\rangle = \langle \pi(x) \otimes \pi(x)\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2\rangle.$$

Conversely, suppose that $x \in VN_\pi(G) \setminus \{0\}$ and $\pi(x) \otimes \pi(x) = \pi \otimes \pi(x)$. Then we have

$$\begin{aligned}\langle u_1, x\rangle\langle u_2, x\rangle &= \langle \pi(x)\xi_1, \eta_1\rangle\langle \pi(x)\xi_2, \eta_2\rangle \\ &= \langle \pi(x) \otimes \pi(x)\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2\rangle = \langle f, x\rangle.\end{aligned}$$

So, $x \in \sigma(A_\pi(G))$. ∎

For any $u \in A_\pi(G)$ and $T \in VN_\pi(G)$, define $T_l(u)(x) = \langle \pi(x) \cdot T, u\rangle$.

Lemma 3.6. *We have $T_l(u)(x) = \langle T, {}_xu\rangle$. If $A_\pi(G)$ is unital, then $T_l(1)(x) \equiv \langle T, 1\rangle$.*

Proof. If $u \in A_\pi(G)$ and $u(x) = \sum_{n=1}^{\infty}\langle \pi(x)\xi_n, \eta_n\rangle$ for some $\xi_n, \eta_n \in \mathcal{H}_\pi$, then

$$(u \cdot \pi(x))(y) = \sum_{n=1}^{\infty}\langle \pi(y)\xi_n, \pi(x)^*\eta_n\rangle = \sum_{n=1}^{\infty}\langle \pi(xy)\xi_n, \eta_n\rangle = {}_xu(y)$$

for any $x, y \in G$. ∎

Lemma 3.7. *$T_l(u) \in A_\pi(G)$ for each $u \in A_\pi(G)$ and $T \in VN_\pi(G)$.*

Proof. $T_l(u)(x) = \langle \pi(x) \cdot T, u\rangle = \langle \pi(x), T \cdot u\rangle = (T \cdot u)(x)$. ∎

Lemma 3.8. *If $T \in \sigma(A_\pi(G))$, then $T_l : A_\pi(G) \to A_\pi(G)$ is a homomorphism.*

Proof. If $u, v \in A_\pi(G)$, then

$$T_l(u \cdot v)(x) = \langle T, {}_x(uv)\rangle = \langle T, {}_xu\, {}_xv)\rangle = \langle T, {}_xu\rangle\langle T, {}_xv\rangle = T_l(u)(x)T_l(v)(x).\ ∎$$

For any $S, T \in VN_\pi(G)$, define $S \circ T \in VN_\pi(G)$ by $\langle S \circ T, u\rangle = \langle S, T_l(u)\rangle$ for all $u \in A_\pi(G)$.

Proposition 3.9. *If $S, T \in VN_\pi(G)$, then $S \circ T = S \cdot T$ and $(S \cdot T)_l(u) = T_l(S_l(u))$ for all $u \in A_\pi(G)$.*

Proof. By definition, the first equality holds clearly if $S = \pi(x)$ for some $x \in G$. The rest follows from the weak* density of $\mathrm{span}(\pi(G))$ in $VN_\pi(G)$. The second equality is straightforward. ∎

Given a function $u : G \to \mathbb{C}$, let $\tilde{u} : G \to \mathbb{C}$ be the function defined by $\tilde{u}(x) = u(x^{-1})$.

Proposition 3.10. *If $\sigma(A_\pi(G)) \cup \{0\}$ is equipped with multiplication and involution inherited from the von Neumann algebra $VN_\pi(G)$, then it is a *-semitopological semigroup. In addition, if $A_\pi(G)$ is unital and $\sigma(A_\pi(G))$ is equipped with multiplication and involution inherited from $VN_\pi(G)$, then it is a compact *-semitopological semigroup.*

Proof. If $T, S \in \sigma(A_\pi(G))$ and $u, v \in A_\pi(G)$, then

$$\begin{aligned}\langle T \cdot S, uv\rangle &= \langle T, S_l(uv)\rangle = \langle T, S_l(u)S_l(v)\rangle \\ &= \langle T, S_l(u)\rangle\langle T, S_l(v)\rangle = \langle T \cdot S, u\rangle\langle T \cdot S, v\rangle.\end{aligned}$$

On the other hand, we have

$$\langle T^*, uv\rangle = \langle T, \tilde{uv}\rangle = \langle T, \tilde{u}\tilde{v}\rangle = \langle T^*, u\rangle\langle T^*, v\rangle,$$

so $T^* \in \sigma(A_\pi(G))$. Suppose that $A_\pi(G)$ is unital. Now $\langle T, 1\rangle = 1 = \langle S, 1\rangle$, so $\langle T \cdot S, 1\rangle = \langle T, S_l(1)\rangle = \langle T, 1\rangle = 1$. It follows that $T \cdot S \neq 0$. Hence, $T \cdot S \in \sigma(A_\pi(G))$. Since multiplication in a von Neumann algebra is separately weak*-continuous, we conclude that these are semitopological semigroups. ∎

Corollary 3.11. *$\sigma(a(G))$ is a compact *-semitopological semigroup if it is equipped with multiplication and involution inherited from $vn(G)$.*

Suppose that $\phi \in l^\infty(G)$ satisfies

$$\phi f = f \quad \text{for any } f \in l^1(G).$$

Then, obviously, ϕ is the constant one function. We now have the following proposition which is a non-commutative analogue of this observation:

Proposition 3.12. *Let T be a non-zero element in $vn(G)$. Then the following statements are equivalent:*

(a) *$Tu = u$ for all $u \in a(G)$.*
(b) *$T = \sigma(e)$.*

Proof. (b)⇒(a) is clear. Suppose that (a) holds. We have $[T_l(u)](x) = (Tu)(x) = u(x)$. For any $S \in vn(G)$, we obtain $\langle S \cdot T, u\rangle = \langle S, T_l(u)\rangle = \langle S, u\rangle$. Hence, $S \cdot T = S$ for all $S \in vn(G)$. Therefore, $T = \sigma(e)$. ∎

Write $\sigma_{\mathrm{u}}(A_\pi(G))$ ($\sigma_{\mathrm{inv}}(A_\pi(G))$) for the set of all unitary (resp. invertible) elements in $\sigma(A_\pi(G))$. Clearly, $\sigma_{\mathrm{u}}(A_\pi(G))$ and $\sigma_{\mathrm{inv}}(A_\pi(G))$ are semitopological groups if equipped with the relative weak* topology of $VN_\pi(G)$.

Theorem 3.13. *Let π_1 and π_2 be unitary representations of G_1 and G_2, respectively. If $A_{\pi_1}(G_1)$ and $A_{\pi_2}(G_2)$ are isometrically isomorphic, then there is a homeomorphism $\phi : \sigma(A_{\pi_1}(G_1)) \to \sigma(A_{\pi_2}(G_2))$ such that:*

(a) $\phi(T^*) = \phi(T)^*$ *for any* $T \in \sigma(A_{\pi_1}(G_1))$*;*
(b) *for each* $T, S \in \sigma(A_{\pi_1}(G_1))$*, either*

$$\phi(T \cdot S) = \phi(T)\phi(S) \quad or \quad \phi(T \cdot S) = \phi(S)\phi(T);$$

(c) ϕ *is either a* **-isomorphism or a* **-anti-isomorphism from* $\sigma_{\mathrm{u}}(A_{\pi_1}(G_1))$ *onto* $\sigma_{\mathrm{u}}(A_{\pi_2}(G_2))$*.*

Proof. Step 1: *We construct a Jordan* **-isomorphism* Φ *between* $VN_{\pi_1}(G_1)$ *and* $VN_{\pi_2}(G_2)$*.* Let $\psi : A_{\pi_2}(G_2) \to A_{\pi_1}(G_1)$ be an isometric isomorphism. It is straightforward to show that $U = \psi^*(\pi_2(e)) \in \sigma(A_{\pi_2}(G_2))$. We have $V = U^* \in \sigma(A_{\pi_2}(G_2))$ by Proposition 3.10. By Lemma 3.8, $V_l : A_{\pi_2}(G_2) \to A_{\pi_2}(G_2)$ is a homomorphism. Since V is unitary, it is easy to see that V_l is in fact an isometric isomorphism. It follows that $\psi \circ V_l : A_{\pi_2}(G_2) \to A_{\pi_1}(G_1)$ is an isometric isomorphism. Let $\Phi = (\psi \circ V_l)^*$. Then Φ is an isometry from $VN_{\pi_1}(G_1)$ onto $VN_{\pi_2}(G_2)$. Note that

$$\langle \Phi(\pi_1(e_1)), f\rangle = \langle \psi^*(\pi(e_1)), V_l(f)\rangle = \langle U, V_l(f)\rangle = \langle \pi_2(e), f\rangle$$

for any $f \in A_{\pi_1}(G_1)$. Therefore, Φ preserves units and hence is a Jordan *-isomorphism by [10, Theorem 7].

Step 2: *Let* ϕ *be the restriction of* Φ *to* $\sigma(A_{\pi_1}(G_1))$*. Then* ϕ *is a homeomorphism from* $\sigma(A_{\pi_1}(G_1))$ *onto* $\sigma(A_{\pi_2}(G_2))$*.* We show that ϕ satisfies (a) and (b). If $TS = ST$, then (b) holds, as Jordan *-isomorphisms preserve commutativity. Otherwise, we have

$$\phi(T)\phi(S) + \phi(S)\phi(T) = \phi(ST) + \phi(TS).$$

Suppose that (b) does not hold. Then $\phi(T)\phi(S)$, $\phi(S)\phi(T)$, $\phi(ST)$ and $\phi(TS)$ are pairwise distinct, hence linearly independent, in $\sigma(A_{\pi_2}(G_2))$, which leads to a contradiction.

By [10, Theorem 10], there exist central projections $z_i \in VN_{\pi_i}(G_i)$ ($i = 1,2$) such that $\Phi = \Phi_I + \Phi_A$ and $\Phi_I : VN(G_1)z_1 \to VN(G_2)z_2$ is a *-isomorphism and $\Phi_A : VN(G_1)(\pi_1(e) - z_1) \to VN(G_2)(\pi_2(e) - z_2)$ is a *-anti-isomorphism. For each $T \in \sigma_{\mathrm{u}}(A_{\pi_1}(G_1))$, define

$$H_T = \{S \in \sigma_{\mathrm{u}}(A_{\pi_1}(G_1)) : (ST - TS)z_1 = 0\},$$
$$K_T = \{S \in \sigma_{\mathrm{u}}(A_{\pi_1}(G_1)) : (ST - TS)(\pi_2(e) - z_1) = 0\}.$$

Step 3: *We show that* H_T *and* K_T *are subgroups of* $\sigma_{\mathrm{u}}(A_{\pi_1}(G_1))$ *and* $H_T \cup K_T = \sigma_{\mathrm{u}}(A_{\pi_1}(G_1))$*.* If $S_1, S_2 \in H_T$ and $S \in \sigma_{\mathrm{u}}(A_{\pi_1}(G_1))$, then

$$SS_1S_2z_1 = S_1(SS_2)z_1 = S_1(S_2Sz_1) = S_1S_2Sz_1$$

and

$$(S_1^{-1}S - SS_1^{-1})z_1 = S_1^{-1}(S_1S - SS_1)S_1^{-1}z_1 = 0.$$

It follows that H_T is a subgroup of $\sigma_{\mathrm{u}}(A_{\pi_1}(G_1))$. Similarly, K_T is a subgroup of $\sigma_{\mathrm{u}}(A_{\pi_1}(G_1))$. Finally, if $\phi(ST) = \phi(T)\phi(S)$, then $\phi(ST-TS)z_2 = 0$ (since

Φ_I is a *-isomorphism), which implies that $(ST-TS)z_1=0$. So, $S\in H_T$. Otherwise, we have $\phi(ST)=\phi(S)\phi(T)$. It follows similarly that $S\in K_T$.

STEP 4: *Define*

$$H=\{T\in\sigma_{\mathrm{u}}(A_{\pi_1}(G_1)) : H_T=\sigma_{\mathrm{u}}(A_{\pi_1}(G_1))\},$$
$$K=\{T\in\sigma_{\mathrm{u}}(A_{\pi_1}(G_1)) : K_T=\sigma_{\mathrm{u}}(A_{\pi_1}(G_1))\}.$$

We show that either $H=\sigma_{\mathrm{u}}(A_{\pi_1}(G_1))$ *or* $K=\sigma_{\mathrm{u}}(A_{\pi_1}(G_1))$. If $S_1,S_2\in H$, then, for any $S\in\sigma_{\mathrm{u}}(A_{\pi_1}(G_1))$, we have

$$S_1S_2Sz_1=S_1(SS_2z_1)=(SS_1)S_2z_1=S(S_1S_2)z_1.$$

Thus,

$$H_{S_1S_2}=\sigma_{\mathrm{u}}(A_{\pi_1}(G_1)).$$

Also, we have

$$(S_1^{-1}S-SS_1^{-1})z_1=S_1^{-1}(S_1S-SS_1)S_1^{-1}z_1=0.$$

Consequently, $H_{S_1^{-1}}=\sigma_{\mathrm{u}}(A_{\pi_1}(G_1))$. The final assertion is clear since $H_T=\sigma_{\mathrm{u}}(A_{\pi_1}(G_1))$ or $K_T=\sigma_{\mathrm{u}}(A_{\pi_1}(G_1))$ for any $T\in\sigma_{\mathrm{u}}(A_{\pi_1}(G_1)$ (as H_T and K_T are subgroups of $\sigma_{\mathrm{u}}(A_{\pi_1}(G_1)))$.

STEP 5: *Suppose that* $H=\sigma_{\mathrm{u}}(A_{\pi_1}(G_1))$ $(K=\sigma_{\mathrm{u}}(A_{\pi_1}(G_1)))$. *We show that* ϕ *is a* **-anti-isomorphism (resp. a* **-isomorphism).* Suppose that $H=\sigma_{\mathrm{u}}(A_{\pi_1}(G_1))$. We claim that

$$\phi(S_1S_2)=\phi(S_2)\phi(S_1)\quad\text{for all } S_1,S_2\in\sigma_{\mathrm{u}}(A_{\pi_1}(G_1)).$$

If not, then $\phi(S_1S_2)=\phi(S_1)\phi(S_2)$. It follows that

$$(\phi(S_1)\phi(S_2)-\phi(S_2)\phi(S_1))(\pi_2(e)-\phi(z_1))=0.$$

But $S_1,S_2\in H$ implies that $(S_1S_2-S_2S_1)z_1=0$. So, $S_1S_2=S_2S_1$. Hence, $\phi(S_1S_2)=\phi(S_2)\phi(S_1)$. Therefore, ϕ is a *-anti-isomorphism. The other case is similar. ∎

COROLLARY 3.14. *If* $a(G_1)$ *and* $a(G_2)$ *are isometrically isomorphic, then* $\sigma_{\mathrm{u}}(a(G_1))$ *and* $\sigma_{\mathrm{u}}(a(G_2))$ *are topologically isomorphic.*

REMARK 3.15.

(a) The product discussed in Proposition 3.9 is motivated by [12, Section 5].
(b) Theorem 3.13 is a generalization of [20, Theorem 2] and its proof is inspired by [12, Theorem 5.8] and [20, Theorem 2].

4. When is the spectrum of $a(G)$ a group? In this section, we investigate when the spectrum of $a(G)$ is a group.

Let G be a non-[Moore]-group. Let $\hat{G}_{\mathcal{I}}$ be the set of all infinite-dimensional irreducible representations of G, and $\pi_I=\bigoplus_{\pi\in\hat{G}_{\mathcal{I}}}\pi$. Then $\pi_a=$

$\pi_F \oplus \pi_I$. Let $\sigma_I = \bigoplus_{n\in\mathbb{N}} \pi_F \otimes \pi_I^{\otimes n}$ where $\pi_I^{\otimes n} = \bigotimes_{i=1}^n \pi_I$. It is easy to see that $\sigma = \pi_F \oplus \sigma_I$.

Since $A_{\mathcal{F}}(G) = A_{\pi_F}(G)$ is a closed translation invariant subalgebra of $B(G)$, there exists a central projection $p_F \in W^*(G)$ such that $A_{\mathcal{F}}(G) = p_F \cdot B(G)$ where $W^*(G)$ is the enveloping von Neumann algebra of $C^*(G)$ (see [18, Lemma 2.2] for more details). Note that p_F is in the spectrum of $B(G)$, and p_F is equal to the identity element of $W^*(G)$ precisely when G is compact (see [21, Theorem 2]). The algebra $A_{\mathcal{PIF}}(G) = (1 - p_F) \cdot B(G)$ is defined and proved to be an ideal of $B(G)$ in [18, Section 2].

Lemma 4.1. *Let $z_F \in vn(G)$ be the central projection such that $A_{\mathcal{F}}(G) = z_F \cdot a(G)$. Write $a(G) = A_{\mathcal{F}}(G) \oplus A_I(G)$, where $A_I(G) = (\sigma(e) - z_F)a(G)$. Then $A_I(G)$ is the ideal generated by $A_{\pi_I}(G)$ in $a(G)$, and $A_I(G) = A_{\sigma_I}(G)$.*

Proof. Note that $B(G) = A_{\mathcal{F}}(G) \oplus A_{\mathcal{PIF}}(G)$. Thus, $a(G) = A_{\mathcal{F}}(G) \oplus (a(G) \cap A_{\mathcal{PIF}}(G))$. By uniqueness of the translation invariant complement of $A_{\mathcal{F}}(G)$ in $a(G)$, we have $a(G) \cap A_{\mathcal{PIF}}(G) = A_I(G)$ (see [2, Proposition 3.16]). Since $A_{\mathcal{PIF}}(G)$ is an ideal in $B(G)$, it follows that $A_I(G)$ is an ideal in $a(G)$. ∎

We have the following proposition that gives some criteria for the equality of $a(G)$ and $a_0(G)$, which is of independent interest:

Proposition 4.2. *The following statements are equivalent:*

(a) *$a_0(G) = a(G)$.*
(b) *$a_0(G) = A_{\pi_a\otimes\pi_a}(G)$.*
(c) *$a(G)$ has RNP.*
(d) *$A_{\pi_a\otimes\pi_a}(G)$ has RNP.*
(e) *$A_I(G)$ has RNP.*
(f) *$\pi_a \otimes \pi_a$ is completely reducible.*
(g) *$\pi \otimes \rho$ is completely reducible for any $\pi, \rho \in \hat{G}$.*
(h) *$A_{\pi_I}(G)$ is an algebra and $a_0(G)A_{\pi_I}(G) = A_{\pi_I}(G)$.*

Proof. Note that $a_0(G) \subseteq A_{\pi_a\otimes\pi_a}(G) \subseteq a(G)$ and $a(G) = A_{\mathcal{F}}(G) \oplus A_I(G)$. The result follows from [3, Theorem 3]. ∎

Remark 4.3. It follows that [14] that if $a_0(G) = a(G)$, then $a(G)$ has the weak fixed point property for non-expansive mappings. We do not know if the converse is true (see also [13]).

Note that $\sigma(A_{\mathcal{F}}(G)) = \sigma(A(G^{\mathrm{ap}})) \cong G^{\mathrm{ap}}$ where G^{ap} is the almost periodic compactification of G. If G is a [Moore]-group, then $a(G) = A_{\mathcal{F}}(G) = B(G^{\mathrm{ap}}) = A(G^{\mathrm{ap}})$. Therefore, $\sigma(a(G)) = G^{\mathrm{ap}}$ is a group. We will prove below that the converse is also true.

The following lemma is a generalization of [21, Proposition 1]; the proof is left to the reader.

LEMMA 4.4. *Let s be a non-zero element of $VN_\pi(G)$ such that $s^2 = s$. Then the following are equivalent:*

(a) $s \in \sigma(A_\pi(G))$.
(b) $s \cdot A_\pi(G)$ *is an algebra and* $(\pi(e) - s)A_\pi(G)$ *is an ideal in* $A_\pi(G)$.
(c) *The map* $A_\pi(G) \to s \cdot A_\pi(G)$, $f \mapsto s \cdot f$, *is an endomorphism.*

LEMMA 4.5. *If $A_\pi(G) = A_{\pi_1}(G) \oplus A_{\pi_2}(G)$ and $m \in \sigma(A_\pi(G))$ is invertible, then $m(A_{\pi_1}(G)) \neq 0$ and $m(A_{\pi_2}(G)) \neq 0$.*

Proof. Assume that $m(A_{\pi_1}(G)) = 0$. Let $z[\pi_1]$ be the support projection of π_1 in $VN_\pi(G)$. Then $m \in A_{\pi_1}(G)^\perp = (\pi(e) - z[\pi_1])\, VN_\pi(G)$. So, $m = (\pi(e) - z[\pi_1])m$. Hence, $\pi(e) = z[\pi_1]$. Consequently, $A_{\pi_2}(G) = 0$, which is a contradiction. ∎

LEMMA 4.6. *Let $z_F \in vn(G)$ be the central projection such that $A_{\mathcal{F}}(G) = z_F \cdot a(G)$. Then $z_F \in \sigma(a(G))$.*

Proof. Since $A_I(G)$ is an ideal in $a(G)$, by Lemma 4.4, we have $z_F \in \sigma(a(G))$. ∎

Note that $a_0(G) = \oplus_1\{A_\pi(G) : \pi \in \hat{G}\} = \oplus_1\{L^1(\mathcal{H}_\pi) : \pi \in \hat{G}\}$ (see [2]) where $L^1(\mathcal{H}_\pi)$ is the space of all trace-class operators on $\mathcal{H}_\pi$. Let $c_0(\hat{G}) := \oplus_0\{\mathcal{K}(\mathcal{H}_\pi) : \pi \in \hat{G}\}$. Then it is easy to see that the dual space of $c_0(\hat{G})$ is $a_0(G)$.

LEMMA 4.7. *The following assertions are equivalent:*

(a) G *is a* [*Moore*]*-group.*
(b) $a_0(G)$ *is an l^1-sum of finite-dimensional Banach spaces.*
(c) $c_0(\hat{G})$ *is a c_0-sum of finite-dimensional C^*-algebras.*
(d) *Every bounded linear operator* $T : c_0(G) \to a_0(G)$ *is compact.*
(e) *Every irreducible representation of* $c_0(\hat{G})$ *is finite-dimensional.*

Proof. By using [15, Theorems 3.6 and 4.1], we see the equivalence of (b)–(e). It suffices to prove that (e) implies (a). Define $\hat{\pi}_0 : c_0(\hat{G}) \to B(\mathcal{H}_{\pi_0})$, $(T_\pi)_{\pi\in\hat{G}} \mapsto T_{\pi_0}$. Let $\xi, \eta \in \mathcal{H}_{\pi_0} \setminus \{0\}$. There exists $S_{\pi_0} \in \mathcal{F}(\mathcal{H}_{\pi_0})$ such that $S_{\pi_0}(\xi) = \eta$. Now, define $T_\pi = S_{\pi_0}$ if $\pi = \pi_0$ and $T_\pi = 0$ if $\pi \neq \pi_0$. Then $\hat{\pi}_0((T_\pi)_{\pi\in\hat{G}})\xi = \eta$, and hence $\hat{\pi}_0$ is irreducible. Therefore, $\mathcal{H}_{\pi_0}$ is finite-dimensional. ∎

REMARK 4.8. A Banach space is said to have *Schur's property* if all weakly convergent sequences are norm convergent. The Banach space X is said to have the *DPP* if, for any Banach space Y, every weakly compact linear operator $u : X \to Y$ sends weakly Cauchy sequences to norm convergent sequences. Actually, by using [15, Theorems 3.6 and 4.1], we can prove that the following assertions are equivalent:

(a) G is a [Moore]-group.
(b) $a_0(G)$ has Schur's property.
(c) $a_0(G)$ has DPP.
(d) $c_0(\hat{G})$ has DPP.
(e) $ap(c_0(G)) = a_0(G)$.

THEOREM 4.9. *Let G be a locally compact group. The following statements are equivalent:*

(a) *G is a [Moore]-group.*
(b) *$\sigma(a(G))$ is a group.*
(c) *The only idempotent of $\sigma(a(G))$ is $\sigma(e)$.*
(d) *$z_F \in \sigma(a(G))$ is invertible.*
(e) *$a(G) = A_{\mathcal{F}}(G)$.*
(f) *$a_0(G) = A_{\mathcal{F}}(G)$.*

Proof. (a)⇒(b)⇒(c) and (b)⇒(d) are clear. Suppose that (b) holds. Then $z_F = \sigma(e)$. So, $a(G) = z_F \cdot a(G) = A_{\mathcal{F}}(G)$. On the other hand, suppose that (d) holds. Then $z_F(A_I(G)) \neq 0$ by Lemma 4.5. This contradicts that $A_I(G) = (\sigma(e) - z_F)a(G)$. We thus get $A_I(G) = 0$, i.e. $a(G) = A_{\mathcal{F}}(G)$. If $a(G) = A_{\mathcal{F}}(G)$, then we have $a_0(G) = A_{\mathcal{F}}(G)$ as $A_{\mathcal{F}}(G) \subseteq a_0(G)$. Finally, assume that (f) is true. Then G is a [Moore]-group by Lemma 4.7. ∎

By the result above, we see that $\sigma(a(G))$ is not always a group. We will now study the unitary (invertible) part of $\sigma(a(G))$.

Recall the following definitions: A unitary representation of G is *completely reducible* if it can be written as a direct sum of irreducibles. A locally compact group G is called an *[AR]-group* if $A(G)$ has RNP. It is proved that G is an [AR]-group if and only if its left regular representation is completely reducible (see [19] for more details).

THEOREM 4.10. *Let G be an [AR]-group. Then $\sigma_{\mathrm{u}}(a(G))$ and $\sigma_{\mathrm{inv}}(a(G))$ are topologically isomorphic to G.*

Proof. We prove the statement for $\sigma_{\mathrm{u}}(a(G))$. The case of $\sigma_{\mathrm{inv}}(a(G))$ is similar. Define $\phi : G \to \sigma_{\mathrm{u}}(a(G))$ by $x \mapsto m_x$ where $m_x(u) = u(x)$. Clearly, ϕ is continuous. Since G^* separates the points of G (see Remark 4.4), the map ϕ is injective. By assumption, $A(G) \subseteq a(G)$. Let $m \in \sigma_{\mathrm{u}}(a(G))$. Then $m|_{A(G)} \neq 0$ by Lemma 4.5. Therefore, $m|_{A(G)} \in \sigma(A(G))$. Let $u \in A(G)$ and $v \in a(G)$. Note that $A(G)$ is an ideal in $a(G)$. There exists $x_0 \in G$ such that

$$m(u)m(v) = m(uv) = u(x_0)v(x_0).$$

Pick $u_0 \in A(G)$ such that $u_0(x_0) \neq 0$. We conclude that $m(v) = v(x_0)$. Hence, ϕ is surjective. The continuity of the inverse of ϕ follows from the facts that $A(G) \subseteq a(G)$ and $\sigma(A(G))$ is topologically isomorphic to G. ∎

If G is a discrete group, then $l^1(G) = L^1(G)$ is a total invariant of G by Wendel's theorem (see [22]). We have the following non-commutative analogue of this observation.

COROLLARY 4.11. *Let G_1 and G_2 be locally compact groups such that $A(G_1)$ and $A(G_2)$ have RNP, i.e., G_1, G_2 are $[AR]$-groups. The following conditions are equivalent:*

(a) *G_1 and G_2 are topologically isomorphic.*
(b) *$a(G_1)$ and $a(G_2)$ are isometrically isomorphic.*
(c) *$\sigma_{\mathrm{u}}(G_1)$ and $\sigma_{\mathrm{u}}(G_2)$ are topologically isomorphic.*
(d) *$\sigma_{\mathrm{inv}}(G_1)$ and $\sigma_{\mathrm{inv}}(G_2)$ are topologically isomorphic.*

Proof. This follows from Corollary 3.14 and Theorem 4.10. ∎

REMARK 4.12.

(a) Part of the proof of Theorem 4.9 is inspired by the proof of [20, Lemma of Theorem 2, p. 27].
(b) The proof of Theorem 4.10 follows an idea in [21, Theorem 2].

Acknowledgments. The results presented in this paper will be part of the Ph.D. thesis of the author at the University of Alberta under the supervision of Dr. Anthony To-Ming Lau. The author is grateful to Dr. Lau for many valuable suggestions.

References

[1] C. A. Akemann and M. E. Walter, *Non-abelian Pontriagin duality*, Duke Math. J. 39 (1972), 451–463.
[2] G. Arsac, *Sur l'espace de Banach engendré par les coefficients d'une représentation unitaire*, Publ. Dép. Math. (Lyon) 13 (1976), no. 2, 1–101.
[3] A. Belanger and B. E. Forrest, *Geometric properties of coefficient function spaces determined by unitary representations of a locally compact group*, J. Math. Anal. Appl. 193 (1995), 390–405.
[4] M. Y.-H. Cheng, *A bipolar property of subgroups and translation invariant closed convex subsets*, submitted.
[5] —, *Translation invariant means on group von Neumann algebras*, submitted.
[6] A. Derighetti, *Some results on the Fourier–Stieltjes algebra of a locally compact group*, Comment. Math. Helv. 45 (1970), 219–228.
[7] J. Dixmier, *C^*-algebras*, North-Holland Math. Library 15, North-Holland, Amsterdam, 1977.
[8] P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France 92 (1964), 181–236.
[9] G. B. Folland, *A Course in Abstract Harmonic Analysis*, Stud. Adv. Math., CRC Press, Boca Raton, FL, 1995.
[10] R. V. Kadison, *Isometries of operator algebras*, Ann. of Math. 54 (1951), 325–338.
[11] A. T.-M. Lau, *Uniformly continuous functionals on the Fourier algebra of any locally compact group*, Trans. Amer. Math. Soc. 232 (1979), 39–59.

[12] A. T.-M. Lau, *The Fourier–Stieltjes algebra of a topological semigroup with involution*, Pacific J. Math. 77 (1978), 165–181.

[13] A. T.-M. Lau and M. Leinert, *Fixed point property and the Fourier algebra of a locally compact group*, Trans. Amer. Math. Soc. 360 (2008), 6389–6402.

[14] A. T.-M. Lau, P. F. Mah and A. Ülger, *Fixed point property and normal structure for Banach spaces associated to locally compact groups*, Proc. Amer. Math. Soc. 125 (1997), 2021–2027.

[15] A. T.-M. Lau and A. Ülger, *Some geometric properties on the Fourier and Fourier–Stieltjes algebras of locally compact groups, Arens regularity and related problems*, Trans. Amer. Math. Soc. 337 (1993), 321–359.

[16] P. F. Mah and T. Miao, *Extreme points of the unit ball of the Fourier–Stieltjes algebra*, Proc. Amer. Math. Soc. 128 (2000), 1097–1103.

[17] A. L. T. Paterson, *Amenability*, Math. Surveys Monogr. 29, Amer. Math. Soc., Providence, RI, 1988.

[18] V. Runde and N. Spronk, *Operator amenability of Fourier–Stieltjes algebras*, Math. Proc. Cambridge Philos. Soc. 136 (2004), 675–686.

[19] K. F. Taylor, *Geometry of the Fourier algebras and locally compact groups with atomic unitary representations*, Math. Ann. 262 (1983), 183–190.

[20] M. E. Walter, *W^*-algebras and non-abelian harmonic analysis*, J. Funct. Anal. 11 (1972), 17–38.

[21] —, *On the structure of the Fourier–Stieltjes algebra*, Pacific J. Math. 58 (1975), 267–281.

[22] J. G. Wendel, *Left centralizers and isomorphisms of group algebras*, Pacific J. Math. 2 (1952), 251–261.

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Received August 13, 2010
Revised version January 9, 2011 (6966)